Grassmannian Reduction of Quadratic Forms

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1. INTRODUCTION

One may efficiently obtain principal axis direction and centre of a conic on five given points by expanding three subdeterminants derived from the singular matrix of the conic equation. Computation entails solution of a quadratic equation, in $\cos^2\phi$, where ϕ is a principal axis rotation angle with respect to the frame of the original points, and linear ones in s_0 and t_0 where these are the respective translations to centre the origin of the new, aligned frame on the conic. This closed form solution is coded in an algorithm that needs only 122 arithmetic operations; no trigonometric ones.

Analysis of quadratic forms plays an important rôle in engineering. For example, one may wish to design an elliptical trammel to guide a manipulator smoothly through five desired points. However it is not the intention here to dwell on applications and the interested reader may find some in a brief discourse by Sawyer[1]. Suffice it to say that linear algebra courses, *e.g.*, Anton[2], deal with the subject and treat the reduction of the two-variable equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \qquad (1)$$

to standard form

$$a'x^2 + b'y^2 + f' = 0 \qquad (2)$$

Without elaboration, the usual approach follows these steps.

- Orthogonal matrix diagonalization, determination of eigenvalues and application of the two-dimensional *Principal Axis Theorem* yield the conic axis direction by eliminating the coefficient c in Eq. 1.
- Elimination of coefficients d and e in Eq. 1 produces the two required translations.

Note that, in [2], the diagonalization step is done numerically and the problem starts with Eq. 1 to obtain Eq. 2. On the other hand we begin with five given points to arrive at essentially the same result without ever bothering with the coefficients of Eq. 1. Furthermore the reader will see immediately that a', b', f' can be computed, from the results obtained, with three 2×2 determinants.

2. THE GRASSMANNIAN

English language literature on Hermann Grassmann is scarce but his work has, with the dawning of symbolic computation, aroused interest recently[3]. Furthermore his theory of determinants was applied to the analysis of conics by Askwith[4]. Let us introduce this by finding the equation of the circle on three given points $P_i(x_i, y_i)$, $i = 1, 2, 3$. Consider the following singular 4×4 determinant.

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad (3)$$

When this is expanded, the 3×3 coefficient cofactors may be regarded as four circle *coordinates.*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) - \begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix} x$$
$$+ \begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix} y - \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix} = 0 \qquad (4)$$

3. THE GENERAL QUADRATIC FORM

$$\begin{vmatrix} x^2 & y^2 & xy & x & y & 1 \\ x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4y_4 & x_4 & y_4 & 1 \\ x_5^2 & y_5^2 & x_5y_5 & x_5 & y_5 & 1 \end{vmatrix} = 0 \qquad (5)$$

Extracting the symbolic coefficients of *this* equation needs six 5×5 determinants and produces an expression with far too many terms. Such a futile exercise will not be attempted.

4. BREAKING DOWN THE PROBLEM

The Rotation

This is obtained by expanding only the cofactor of xy which vanishes when the coordinate axes are aligned with those of the conic. The problem is further simplified by choosing a convenient frame for P_i. The one where $x_1 = y_1 = x_2 = 0$ is chosen to begin with. There is no loss in generality. This frame is placed to produce the following 5×5 determinant.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ a_{23}^2 & a_{24}^2 & a_{23} & a_{24} & 1 \\ a_{i1} & a_{i2} & a_{i3} & a_{i4} & 1 \end{vmatrix} xy = 0 \qquad (6)$$

$a_{23} = y_2\sin\phi$, $a_{24} = y_2\cos\phi$, $a_{i1} = a_{i3}^2$, $a_{i2} = a_{i4}^2$, $a_{i3} = x_i\cos\phi + y_i\sin\phi$ and $a_{i4} = y_i\cos\phi - x_i\sin\phi$, $i = 3, 4, 5$. With copious help from *Maple* V^{TM}, a quadratic equation in $\cos^2\phi$ emerges.

$$A\cos^4\phi + B\cos^2\phi + C = 0$$

The coefficients A, B, C are not too daunting as may be seen in Section **5.**.

The Translations

After performing the rotation $P_i(x_i, y_i) \Rightarrow P_i(s_i, t_i)$ indicated by ϕ, the five points in the new s-t frame satisfy Eq. 6 and now only four of the five given points are needed to get s_0 along s and t_0 along t, the respective translations to put the translated frame origin on the conic centre. This is much easier than computing ϕ. It is represented by the two singular determinants below.

$$\begin{vmatrix} s_0^2 & 0 & 0 & 1 \\ (s_2-s_0)^2 & t_2^2 & t_2 & 1 \\ (s_3-s_0)^2 & t_3^2 & t_3 & 1 \\ (s_4-s_0)^2 & t_4^2 & t_4 & 1 \end{vmatrix} s=0 \tag{7}$$

$$\begin{vmatrix} 0 & t_0^2 & 0 & 1 \\ s_2^2 & (t_2-t_0)^2 & s_2 & 1 \\ s_3^2 & (t_3-t_0)^2 & s_3 & 1 \\ s_4^2 & (t_4-t_0)^2 & s_4 & 1 \end{vmatrix} t=0 \tag{8}$$

5. THE ALGORITHM

It is hoped that the reader will forgive, in the interests of brevity and an example of a working program, a coded listing. The seven non-trivial given point coordinates are supplied to produce $\cos\phi$, $\sin\phi$, s_0 and t_0. `I` is the only "tweak" required to resolve the possibility that the sign combination of $\cos\phi$ and $\sin\phi$ reflects the desired axis direction about a coordinates axis. The quadratic in $\cos^2\phi$ poses an ambiguity in $\pm\phi$. Therefore `I=-1` will correct an unsuccessful try with `I=1`. The test example is shown in Fig. 1.

Programmed Example

```
100 INPUT Y2,X3,Y3,X4,Y4,X5,Y5,I:Y23=Y2-Y3:
    Y24=Y2-Y4:Y25=Y2-Y5:REM End of setup
110 Q1=X3*X4*(X4*(Y5-Y3)+X5*(Y3-Y4)+X3
    *(Y4-Y5):Q2=X5*(Q1+Y3*Y4*(X3*Y24-X4*Y23))
120 Q3=Y5*(X4*Y3*(X5*Y23-X3*Y25)+X3*Y4
    *(X4*Y25-X5*Y24)):Q=2*Y2*(Q2+Q3)
130 R=Y2*(X4*X5*Y3*Y23*(X4-X5)+X3*(X5*Y4*Y24
    *(X5-X3)+X4*Y25*(X3-X4)))
140 P=-2*R:QQ=Q*Q:A=P*P+QQ:B=QQ-2*P*R:C=R*R
    A2=2*A:IF A=0 THEN STOP
150 D=B*B-4*A*C:IF D<0 THEN STOP
160 DR=SQR(D):CP=SQR((B+DR)/A2): SP=I*SQR(1
    -CP*CP):REM End of rotation routine
170 S2=Y2*SP:T2=Y2*CP:S3=X3*CP+Y3*SP:T3=Y3
    *CP-X3*SP:S4=X4*CP+Y4*SP:T4=Y4*CP-X4*SP
180 REM End of rotated coord. transformation
190 S01=S2*T3*T4*(T4-T3):S02=S3*T2*T4*(T2-T4)
    S03=S4*T2*T3*(T3-T2)
200 DS0=2*(S01+S02+S03):IF DS0=0 THEN STOP
210 NS0=S2*S01+S3*S02+S4*S03:S0=NS0/DS0
    REM End of s-axial displacement routine
220 T01=S2*S4*T3*(S4-S2):T02=S2*S3*T4*(S2-S3)
    T03=S3*S4*T2*(S3-S4)
230 DT0=2*(T01+T02+T03):IF DT0=0 THEN STOP
240 NT0=T3*T01+T4*T02+T2*T03:T0=NT0/DT0
    REM End of t-axial displacement routine
250 PRINT CP,SP,S0,T0:END
Ok
run
? 7.209,9.391,8.05,-4.919,0.894,7.155,3.578,1
0.8943964 0.4472752 4.000085 2.999937
Break in 250
```

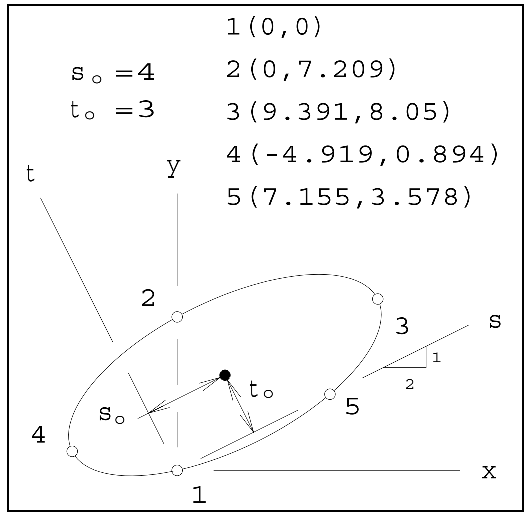

Figure 1. Reduction of an ellipse

6. CONCLUSION

Geometric thinking, a hallmark of engineering design, is exemplified by careful choice of initial conditions when formulating a problem, in this case a good coordinate frame. All the theory presented above is pure classical geometry too; largely unknown or forgotten. Anton[2] mentions advanced methods to do the rotation, Lagarange's and Kronecker's reductions, but there is no word of Grassmann. As methods of symbolic computation, which are still quite primitive in this regard, develop, the design engineer will become more a designer of algorithms than a designer of specific solutions which will fall into the domain of engineering technicians skilled in the *use* of advanced software. Similarly, advances in symbolic software will inevitably ressurect the dormant methods of Grassmann and his successors whose results were eclipsed by the "new age" of quantum and relativistic mechanics. These results have lain fallow awaiting application in a future with appropriate computational tools. One may recall that numerical analysis remained an impractical mathematical curiosity until automatic numerical computation established its importance in the '60s and '70s. We toilers in classical mechanics can look forward to the next millenium with great confidence that new tools will enhance, not eliminate, our jobs, whether in practice or academe.

REFERENCES

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[4] E.H. Askwith: "The Analytic Geometry of the Conic Sections", 3rd Ed., A.&G. Black, Ltd., 1927.