Kinematic Mapping of Planar Stewart-Gough Platforms

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1. PLANAR STEWART-GOUGH PLATFORMS

A planar Stewart-Gough platform (SGP) consists of a moving platform connected to a fixed base by three identical kinematic chains. Each chain is connected by three independent one degree-of-freedom (DOF) joints, one of which is active. Thus, each chain provides the control of one of three DOF of the moving platform. When the three actuators are locked the platform should be a structure, i.e., the degree-of-mobility should be nil. From a geometric perspective a planar SGP consists of three arbitrary points in a fixed base reference frame, Σ, and three arbitrary points in a moving platform reference frame, E, with each platform point a specific distance from each base point. These distances are determined by the variable joint input parameters and the particular topology of the characteristic kinematic chain.

In this paper we will deal only with lower kinematic pair joints. Since the displacements of the platform are confined to the plane, only *revolute* (R) and *prismatic* (P) pairs are considered. Each kinematic chain is described by three letters indicating the succession of joints beginning with the one at the fixed base. The possible combinations are: RRR, RPR, RRP, PRR, RPP, PRP, PPR, PPP. The last chain is excluded because three P-pairs represent three translations in the plane, which can not be independent [2]. Since each chain in a planar SGP is identical, there are seven possible topologies, illustrated in Fig. 1, each described by one of the three chains.

Since the kinematic chains for each topology are identical, we require the actuated joint to be the same in each manipulator. We will identify the active joint by underlining it. Since any of the three joints may be active, there are twenty-one possibilities. However, if the chain obtained after locking the active joint is of the PP-type, it must be eliminated [2]. Thus, there are a total of eighteen possible platforms, listed in Table I.

Table I
The 18 possible planar SGP.

$\underline{R}RR$	$\underline{R}PR$	$\underline{R}RP$	$\underline{P}RR$		$\underline{P}RP$	$\underline{P}PR$
$R\underline{R}R$	$R\underline{P}R$	$R\underline{R}P$	$P\underline{R}R$	$R\underline{P}P$		$P\underline{P}R$
$RR\underline{R}$	$RP\underline{R}$	$RR\underline{P}$	$PR\underline{R}$	$RP\underline{P}$	$PR\underline{P}$	

Examining Table I, it can be seen that there are only three types of chain when the active joint is locked. They are RR-type, PR-type and RP-type.

2. KINEMATIC MAPPING

A mapping of planar rigid-body displacements was introduced simultaneously, but independently, in 1912 first by Grünwald and then by Blaschke [1]. Three independent planar displacement parameters are mapped

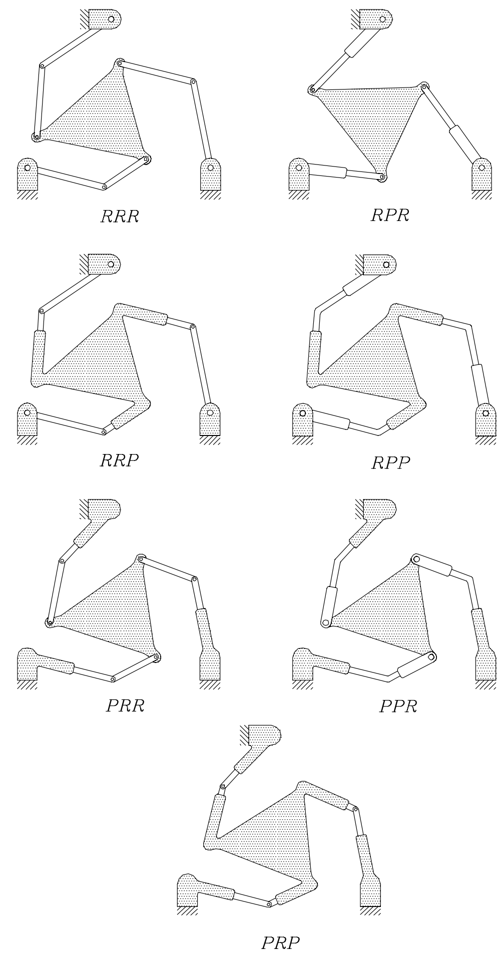

Figure 1. The seven possible topologies.

to the points of a three-dimensional projective homogeneous kinematic mapping image space. A detailed account is given in [1].

We now give a brief summary of the relevant details. A general displacement of one rigid-body with respect to another in the plane can be conveniently described as the relative displacement of two coordinate reference frames Σ and E. Without loss in generality, Σ may be considered as fixed while E is free to move. The image point of a displacement of E in Σ is given by

$$(X_1 : X_2 : X_3 : X_4) = (a\sin(\phi/2) - b\cos(\phi/2) : a\cos(\phi/2) + b\sin(\phi/2) : 2\sin(\phi/2) : 2\cos(\phi/2)), \quad (1)$$

where (a, b) are the coordinates of the origin of frame E in Σ, and ϕ is the orientation of E in Σ.

Using Eq. (1) we can transform the coordinates of a point $(x : y : z)$ in E to those of the same point

$(X : Y : Z)$ in Σ:

$$
\begin{aligned}
X &= (X_4^2 - X_3^2)x - 2X_3X_4y + 2(X_1X_3 + X_2X_4)z \\
Y &= 2X_3X_4x + (X_4^2 - X_3^2)y + 2(X_2X_3 - X_1X_4)z \\
Z &= (X_4^2 + X_3^2)z. \qquad (2)
\end{aligned}
$$

Consider the motion of a fixed point in E that is constrained to move on a fixed circle in Σ,

$$K_0(X^2 + Y^2) - 2K_1XZ - 2K_2YZ + K_3Z^2 = 0, \quad (3)$$

where $[K_0 : K_1 : K_2 : K_3]$ are the *circle coordinates*. The image points will lie on a hyperboloid in the image space having the equation

$$
\begin{aligned}
H: \;& K_0z^2(X_1^2 + X_2^2) + (1/4)\left[K_0(1 - z^2)(x^2 + y^2) + \right. \\
& \left. 2z(K_1x + K_2y) + Rz^2\right] X_3^2 + (1/4)\left[Rz^2 + \right. \\
& \left. K_0(1 - z^2)(x^2 + y^2) - 2z(K_1x + K_2y)\right] X_4^2 - \\
& (K_1z^2 + K_0xz)X_1X_3 + (K_2z^2 - K_0yz)X_1X_4 - \\
& (K_0yz + K_2z^2)X_2X_3 + (K_0xz - K_1z^2)X_2X_4 + \\
& (K_1yz - K_2xz)X_3X_4 = 0, \qquad (4)
\end{aligned}
$$

where $K_1 = X_c$, $K_2 = Y_c$, $K_3 = R - K_0(x^2 + y^2)$, and $R = X_c^2 + Y_c^2 - r^2 + K_0(x^2 + y^2)$ (the last two substitutions are made to reduce the quantity of terms in the equations that follow), with X_c and Y_c being the coordinates of the circle centre of radius r, and K_0 is an arbitrary homogenising constant. If $K_0 = 0$ we obtain a line, which is a real degenerate circle, with *line coordinates* $[L_1 : L_2 : L_3] = [-2K_1 : -2K_2 : K_3]$.

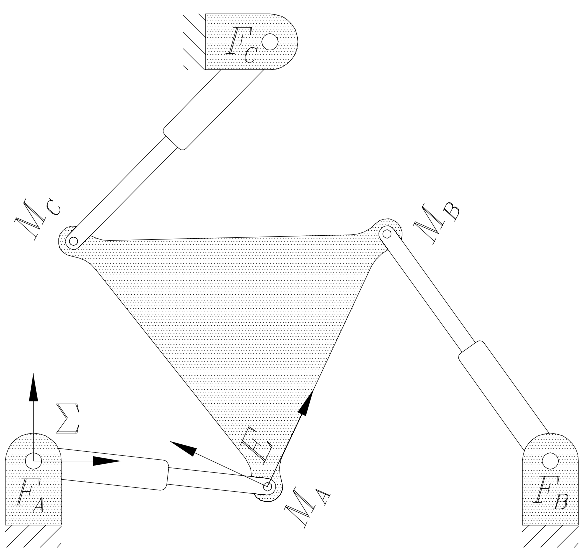

Figure 2. The fixed and moving frames.

Consider the RPR SGP shown in Fig. 2. The three chains are identified with subscripts A, B, C. Let the actuated joint in each leg be the P-pair, making this an RR-type SGP. Opening the platform connections at points M_B and M_C we see that point M_A remains on a circle around F_A with radius r_A (the prismatic input for leg A), while frame E can still rotate about M_A. Opening the connections of the other two platform points, in turn, we obtain three hyperboloid equations, H_A, H_B and H_C, representing the possible displacements of E about the base point still connected. For the frames shown in Fig. 2 we have the following homogeneous coordinates for the fixed and moving points: $F_{A/\Sigma} = (0 : 0 : 1)$, $F_{B/\Sigma} = (B_1 : 0 : 1)$, $F_{C/\Sigma} = (C_1 : C_2 : 1)$, $M_{A/E} = (0 : 0 : 1)$, $M_{B/E} = (b_1 : 0 : 1)$, $M_{C/E} = (c_1 : c_2 : 1)$. Substituting these quantities, along with the three inputs, into Eq. (4) gives three specific hyperboloid equations.

Platform rotations of $\phi = \pi$ (half-turns) signifies that $X_4 = 0$. However, this condition means there are no real common intersections for any set of three constraint hyperboloids. After having checked this, we normalise the image space coordinates by setting $X_4 = 1$. Next, we subtract H_B from H_A and H_C from H_A, giving two equations linear in X_1 and X_2. The resulting expressions for X_1 and X_2 are substituted into H_C, yielding a univariate sextic polynomial in X_3. In the general case, i.e., leaving K_0 arbitrary, the univariate has 3613 terms.

3. THE FORWARD KINEMATICS PROBLEM

The forward kinematics (FK) problem involves determining all possible poses of the moving platform for a given set of actuator inputs. The FK problem of all planar SGP, listed in Table I, can be solved by finding the roots of the univariate polynomial. The remaining image space coordinates are linearly dependent on each root and give the intersection points of the three hyperboloids. Each point of intersection is the image of a pose of the moving platform such that the platform points are on their respective circles, hence they represent the solutions to the FK problem.

Examining Fig. 1, it is easy to see that RR-type platforms require a fixed point in E to move on a fixed non-degenerate real circle in Σ. Thus, we can set $K_0 = 1$ in the hyperboloid equations which reduces the number of terms in the univariate to 694.

The PR-types require a fixed point in E to move on a fixed line in Σ, and the RP-types require a fixed line in E to move on a fixed point in Σ. These two types are inversions: one can be obtained from the other by changing the roles of E and Σ. Because a line is a degenerate circle we may still express the problem as a fixed point in a moving frame (either E or Σ, depending whether the platform is PR or RP) constrained to be on a fixed circle in a fixed frame. Thus, we set $K_0 = 0$ and use K_1, K_2, K_3 as images of the line coordinates. The number of terms in the univariate then reduces to 26.

Finally, we must deal with the fact that the solution for RP-type platforms gives the pose $(a', b'; \phi')$ of the *base* frame, Σ, with respect to the moving frame, E. However, we require the pose $(a, b; \phi)$ of E in Σ. It is easy to show that $\phi = -\phi'$. We then obtain (a, b) with a coordinate transformation using ϕ as the rotation angle.

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