The General Singularity Surface of Planar Three-Legged Platforms

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Abstract

In this paper the singular configurations of planar parallel three-legged platforms are studied. The constraints of each leg are mapped to quadric surfaces in a three-dimensional projective image space. The conditions on rank deficiency of a Jacobian representation of the corresponding velocity constraints in the image space are used to derive a general quartic surface whose points represent all possible singular platform poses. A rational parametrisation of this singularity surface is given.This quartic surface contains at least nine distinct, proper real lines. Sections of the quartic in planes parallel to a specific image space reference plane consist of conic sections and a double line. The pre-image of one of these conic sections is a curvilinear translation with point paths of the same conic type. Furthermore, the singularity locus and the corresponding locus of poles of instantaneous pencils, to which the legs must belong, are similar, and in special cases congruent.

1 Introduction

In this paper, the singular assembly configurations of a particular sub-class of planar three-legged platforms will be investigated. The three legs each join the fixed base to the moving platform by three single degree of freedom (DOF) lower kinematic pairs, one of which is actuated. In the plane, they consist of combinations of R- and P-pairs taken three-at-a-time starting with the base-fixed joint. The legs are termed RR-type because an RR passive kinematic sub-chain is what remains after the actuated joint is locked by specifying an input value. There are 6 possible RR-type legs. Given that the platform has three legs, 56 distinct RR-type platform architectures exist.

Singular assembly configurations of parallel platforms have the property that the set of joint inputs is not sufficient to define the pose. This is due to the gain, or loss of an infintessimal, or even continuous DOF. Sefrioui and Gosselin [14, 15] examined the loci of singular positions in the workspace of the platform for a fixed orientation. They observed that the loci are conic sections. Later, Collins and McCarthy [3] employed planar quaternions to obtain an algebraic expression of a quartic surface in a 3-D projective space that represents all singular poses of $R\underline{P}R$ platforms[1]. The contribution offered herein is

[1]We use the *underscore* to identify the actuated joint in a leg. The term $R\underline{P}R$ platform means it consists of three architecturally identical legs having the same actuated joint in each one.

an extension of the abovementioned work. We will proceed in an intuitive geometric way. In so doing we provide a comprehensive analysis of the quartic singularity surface and offer a valuable design visualisation tool.

2 The General Quartic Singularity Surface

A very detailed account of the kinematic mapping of planar displacements can be found in [1]. It is shown that one, two and three parameter motions in the Cartesian reference plane map to one parameter curves, two parameter surfaces and three parameter solids in the image space. Consider the possible platform motions when two of the three attachment points, connecting platform to ground via two legs, are disconnected (Figure 1. When the actuated joint in the remaining leg is locked the platform can rotate about the ungrounded R-pair, while the platform and ungrounded R-pair is free to rotate about the base-fixed R-pair. In other words, one point of the platform is constrained to move on a fixed centred circle with constant radius. The other constraint, which corresponds to PR-type and RP-type passive sub-chains, is that a point of the platform is constrained to move on a fixed line. Bottema and Roth [1] show that the surface corresponding to circular constraints is a hyperboloid of one sheet. They call the surface of linear constraints a *special* hyperboloid. However, it is shown in [6] that these constraint surfaces are hyperbolic paraboloids. The forward kinematics (FK) problem of planar three-legged platforms reduces to determining the intersections of the corresponding three quadric surfaces. The image of the workspace of RR-type platforms is obtained by determining the intersection of the constraint solids of each leg corresponding to minimum and maximum actuated joint inputs [8]. Each constraint solid contains all singularities of a given leg.

The image of every singular pose can be obtained by considering that the location of a point on the constraint surface is a function of the image space coordinates, $(x_0 : x_1 : x_2 : x_3)$ and the joint inputs. The joint input for a particular leg can be expressed as the distance between the corresponding pair of fixed and moving revolute centres. Since this distance is the radius of a circle, it is indicated by r_i, where the subscript indicates the leg, $i \in \{1, 2, 3\}$.

The image space constraint surface equation for a general RR-type leg can be written in the following way [6]:

$$\begin{aligned}[R - 2(C_1x + C_2y)]x_0^2 + [R + 2(C_1x + C_2y)]x_1^2 + 4C_0(x_2^2 + x_3^2) + \\ [4x_1(C_1y - C_2x) + 4x_2(C_2 - C_0y) + 4x_3(C_0x - C_1)]\, x_0 - \\ [4x_2(C_0x - C_1) + 4x_3(C_2 - C_0y)]\, x_1 = 0. \qquad (1)\end{aligned}$$

Points on the surface are represented by the homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$. These represent a position and orientation of the moving platform reference coordinate system, E, with respect to the fixed base frame, Σ. The coordinates of the origin of E in Σ, indicated by $O_{E/\Sigma}$, are a and b, while the orientation of E in Σ is given by the angle φ. The mapping of a pose (a, b, φ) is given by [1]

$$x_0 = 2\cos\varphi/2, \qquad (2)$$

$$x_1 = 2\sin\varphi/2, \qquad (3)$$

$$x_2 = a\sin\varphi/2 - b\cos\varphi/2, \qquad (4)$$

$$x_3 = -(a\cos\varphi/2 + b\sin\varphi/2). \qquad (5)$$

The pre-image of a point on the surface is

$$\begin{aligned}\tan\varphi/2 =& x_1/x_0,\\ a =& 2(x_1x_2 - x_0x_3)/(x_0^2 + x_1^2),\\ b =& -2(x_1x_3 + x_0x_2)/(x_0^2 + x_1^2).\end{aligned} \tag{6}$$

The parameters $[C_0 : C_1 : C_2 : C_3]$ are the homogeneous coordinates of the circle circumference upon which the moving revolute centre is bound. They are obtained from the general equation for a circle with centre coordinates $(W : X_c : Y_c)$ and radius r:

$$C_0(X^2 + Y^2) - 2C_1XW - 2C_2YW + C_3W^2, \tag{7}$$

where

$$\begin{aligned} C_0 &= \text{ arbitrary homogenising constant,}\\ C_1 &= X_c,\\ C_2 &= Y_c,\\ C_3 &= R - C_0(X^2 + Y^2),\\ R &= C_1^2 + C_2^2 - r^2 + C_0(X^2 + Y^2),\end{aligned}$$

and $(W : X : Y)$ are homogeneous coordinates of the points on the circumference. The coordinates of a point $(w : x : y)$ in the platform frame E can be mapped to the coordinates $(W : X : Y)$ of the same point in the fixed frame Σ in terms of the image space coordinates as follows:

$$\begin{bmatrix} W \\ X \\ Y \end{bmatrix} = \begin{bmatrix} x_0^2 + x_1^2 & 0 & 0 \\ 2(x_2x_0 - x_3x_1) & x_0^2 - x_1^2 & -2x_0x_1 \\ 2(x_2x_1 + x_3x_1) & 2x_0x_1 & x_0^2 - x_1^2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix}. \tag{8}$$

An equation for the constraint surface emerges when the expressions for $(W : X : Y)$ from Equation (8) are substituted into Equation (7). Without loss in generality, the special coordinate systems shown in Figure 1 can be used to simplify the expressions. The point coordinates of the base and platform points in their respective frames are listed in Table 1.

i	$F_{i/\Sigma}$	$M_{i/E}$
A	$(1:0:0)$	$(1:0:0)$
B	$(1:B_1:0)$	$(1:b_1:0)$
C	$(1:C_1:C_2)$	$(1:c_1:c_2)$

Table 1: Fixed and moving point coordinates.

When the platform moves then the coordinates (x_i) of respective poses in the image space and the radii r_i are functions of time. Differentiating the constraint surface equations with respect to time yields three equations that are linear in $\dot{x}_0$, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{R}_i$, $i \in \{A, B, C\}$. An additional constraint arising from Equations (2-5) is that

$$x_0^2 + x_1^2 = 1. \tag{9}$$

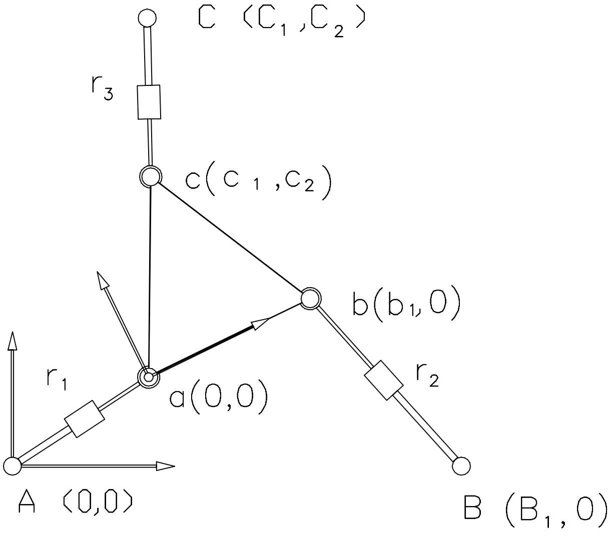

Figure 1: Special coordinate systems.

Differentiating Equation (9) gives

$$x_0\dot{x}_0 + x_1\dot{x}_1 = 0. \qquad (10)$$

Combining these results with the other three differential equations a type of *Jacobian*, relating the time-rate of change of the joint inputs to the time-rate of change of the image space coordinates, is obtained. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} R_0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & R_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{R}_1 \\ \dot{R}_2 \\ \dot{R}_3 \end{bmatrix} = 0, \qquad (11)$$

which can be expressed more compactly as

$$\boldsymbol{A}\dot{\boldsymbol{x}} + \boldsymbol{B}\dot{\boldsymbol{r}} = 0. \qquad (12)$$

The elements of matrix $\boldsymbol{A}$ are the following coefficients of the differential equations:

$$\begin{aligned} a_{11} &= x_0, & a_{12} &= x_1, \\ a_{13} &= 0, & a_{14} &= 0, \\ a_{21} &= 0, & a_{22} &= 0, \\ a_{23} &= 4x_2, & a_{24} &= 4x_3, \\ a_{31} &= 2(b_1 - B_1)x_3, & a_{32} &= 4B_1b_1x_1 - 2(b_1 + B_1)x_2, \end{aligned}$$

$$
\begin{aligned}
a_{33} =& 4x_2 - 2(b_1 + B_1)x_1, \\
a_{34} =& 4x_3 + 2(b_1 - B_1)x_0, \\
a_{41} =& (c_1 - C_1)x_3 + (C_2 - c_1)x_2 + (C_1c_2 - C_2c_1)x_1, \\
a_{42} =& (C_1c_2 - C_2c_1)x_0 + 2(C_1c_1 + C_2c_2)x_1 - (c_1 + C_1)x_2 + (c_2 + C_2)x_3, \\
a_{43} =& (C_2 - c_2)x_0 - (C_1 + c_1)x_1 - 2x_2 \\
a_{44} =& (c_1 - C_1)x_0 - (C_2 + c_2)x_1 + 2x_3
\end{aligned} \tag{13}
$$

The platform is in a singular configuration whenever the determinant of either matrix $\boldsymbol{A}$ or $\boldsymbol{B}$, or both, vanish. For $RR-$type platforms, matrix $\boldsymbol{B}$ will always be non-singular as the leg lengths are chosen so that the fixed and moving points in a given leg are never incident. However, matrix $\boldsymbol{A}$ can, in general, be singular. The determinant of $\boldsymbol{A}$, after choosing specific design parameters b_1, c_1, c_2, B_1, C_1 and C_2, gives an implicit equation in x_i, $i \in \{0,1,2,3\}$, representing a fourth order (quartic) surface S. The expanded equation of the general singularity surface contains 44 terms:

$$
\begin{aligned}
& C_2c_1B_1x_0^3x_2 + C_1c_2b_1x_0^3x_2 - C_2c_1b_1x_0^3x_2 - C_1c_2B_1x_0^3x_2 + 2C_2c_2b_1x_0^2x_1x_2 - 2C_2c_2B_1x_0^2x_1x_2 \\
& + 2C_1c_1b_1x_0^2x_1x_2 - 2C_1c_1B_1x_0^2x_1x_2 + 2B_1b_1C_1x_0^2x_1x_2 - 2B_1b_1c_1x_0^2x_1x_2 - 2B_1b_1c_2x_0^2x_1x_3 \\
& - C_2c_1b_1x_0^2x_1x_3 + C_1c_2b_1x_0^2x_1x_3 - C_2c_1B_1x_0^2x_1x_3 + 2B_1b_1C_2x_0^2x_1x_3 + C_1c_2B_1x_0^2x_1x_3 \\
& + 2B_1c_1x_0^2x_2^2 - 2b_1C_1x_0^2x_2^2 + 2B_1c_2x_0^2x_2x_3 - 2b_1C_2x_0^2x_2x_3 + 2B_1b_1c_2x_0x_1^2x_2 - C_2c_1B_1x_0x_1^2x_2 \\
& + 2B_1b_1C_2x_0x_1^2x_2 - C_1c_2b_1x_0x_1^2x_2 + C_1c_2B_1x_0x_1^2x_2 + C_2c_1b_1x_0x_1^2x_2 + 2C_2c_2B_1x_0x_1^2x_3 \\
& + 2C_1c_1B_1x_0x_1^2x_3 + 2C_2c_2b_1x_0x_1^2x_3 + 2C_1c_1b_1x_0x_1^2x_3 - 2B_1b_1c_1x_0x_1^2x_3 - 2B_1b_1C_1x_0x_1^2x_3 \\
& - 2c_2B_1x_0x_1x_2^2 - 2C_2b_1x_0x_1x_2^2 - 2B_1c_2x_0x_1x_3^2 - 2b_1C_2x_0x_1x_3^2 - C_1c_2B_1x_1^3x_3 + C_2c_1b_1x_1^3x_3 \\
& - C_1c_2b_1x_1^3x_3 + C_2c_1B_1x_1^3x_3 + 2B_1c_2x_1^2x_2x_3 - 2b_1C_2x_1^2x_2x_3 + 2b_1C_1x_1^2x_3^2 - 2B_1c_1x_1^2x_3^2 = 0
\end{aligned} \tag{14}
$$

This surface, derived by Collins and McCarthy [3] in implicit form using planar quaternions for $R\underline{P}R$ platforms, and alluded to by Sefrioui and Gosselin *et al* [14, 15] for a special architecture, has never been parameterized. Its geometric properties are functions of the design parameters defining the specific kinematic architecture of the platform. From the equation it is clear that the line $x_2 = x_3 = 0$ is on the surface and the intersection of S and the plane at infinity $x_0 = 0$ is

$$
\begin{aligned}
& x_0 = 0, \\
& - x_1^2x_3(x_1C_1c_2x_1 + x_1C_1c_2B_1 - x_1C_2c_1x_1 - x_1C_2c_1B_1 \\
& + 2x_3B_1c_1 - 2x_3x_1C_1 - 2B_1x_2c_2 + 2x_1x_2C_2) = 0
\end{aligned} \tag{15}
$$

These two equations represent a double line $x_0 = x_1 = 0$ and two more lines, only one of them dependent on the design parameters.

An example for a specific singularity surface is given in Figure 2. The chosen design parameters are: $B_1 = 16,\ C_1 = 8,\ C_2 = 6, b_1 = 14,\ c_1 = 7,\ c_2 = 10$.

2.1 Parametrisation

Distinct platform designs have distinct quartic singularity surfaces. For instance, Collins and McCarthy [3] give examples where the quartic degenerates into a quadric and two

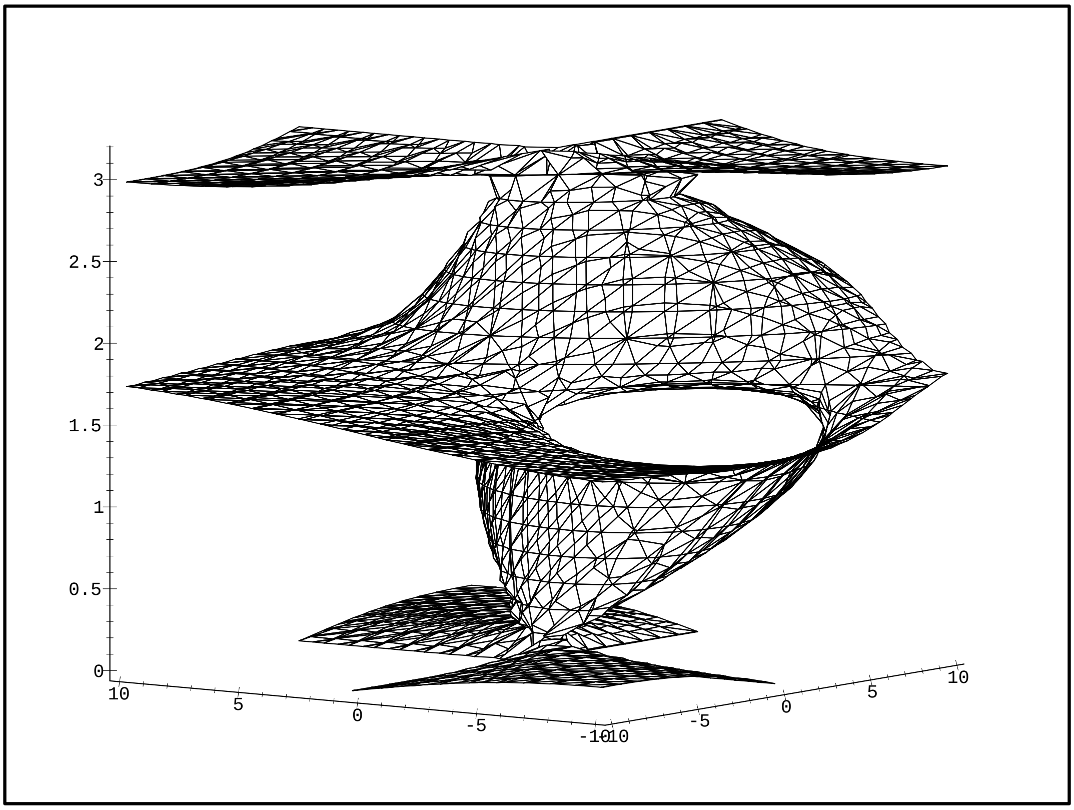

Figure 2: A quartic singularity surface.

planes. However, the general constraint surface equation is derived in its implicit form only. A parametrisation is quite useful for computer generated plots of projections of the surface. Furthermore having a parametric representation points on the singularity surface can be computed without solving polynomial equations which is a big advantage compared to the implicit representation of Equation (14).

It can be easily verified that the quartic singularity surface contains the real line of intersection of the hyperplanes $x_2 = x_3 = 0$. Thus, the surface can be parametrised by considering the intersection curves of the quartic with the family of planes perpendicular to this line. It has been shown by Collins and McCarthy [3] that the intersection curves in hyperplanes where $x_1 = const.$ split into a conic section and a double line. This double line is incident on the line of intersection of the hyperplanes $x_0 = x_1 = 0$. It is well known that points on this line have no pre-image that is a finite real displacement [1, 6, 7, 8]. Clearly, the point on the real line $x_2 = x_3 = 0$ that intersects every cutting plane $x_1 = const.$ is a point on the quartic. The remaining points on the conic are a one parameter set of points. This means the singularity surface can be parametrised rationally (*i.e. with the absence of radical factors*) with an angle, to move along the line $x_2 = x_3 = 0$ (this is because the only way to satisfy the system consisting of Equations (4) and (5) is if displacement parameters a and b both vanish), and a parameter giving the incidence of a polar pencil whose apex is the point on $x_2 = x_3 = 0$, and the conic in that plane.

To obtain the parametric equations substitute in Equation 14 for the x_3 coordinate simply $p\ x_2$ ($p \in (-\infty, \infty)$). The resulting equation is quadratic in x_2. Then solve the equation for x_2. One solution is always $x_2 = 0$ because the apex of the polar pencil of lines is always on the conic. The other solution yields the parametric equation for x_2. The complete parametric representation of the singularity surface with parameters p, u is then given by

$$
\begin{aligned}
x_0 &= \frac{1-u^2}{1+u^2} \\
x_1 &= \frac{2u}{1+u^2} \\
x_2 &= \frac{1}{2}\frac{Au^6+Bu^5+Cu^4+Du^3+Eu^2+Fu+G}{(Hu^4+Iu^3+Ju^2+Ku+L)(u^2+1)} \\
x_3 &= p\ x_2, \qquad\qquad p, u \in (-\infty, \infty)
\end{aligned}
\tag{16}
$$

where the following abbreviations have been used:

$$
\begin{aligned}
A :=& \, C_2\,c_1\,B_1 - C_1\,c_2\,B_1 + C_1\,c_2\,b_1 - C_2\,c_1\,b_1 \\
B :=& \, 4\,C_2\,c_2\,B_1 - 2\,C_1\,c_2\,p\,b_1 - 4\,B_1\,b_1\,C_1 - 4\,B_1\,b_1\,p\,C_2 + 2\,C_2\,c_1\,p\,b_1 - 4\,C_2\,c_2\,b_1 \\
& + 2\,C_2\,c_1\,p\,B_1 + 4\,C_1\,c_1\,B_1 - 4\,C_1\,c_1\,b_1 + 4\,B_1\,b_1\,p\,c_2 + 4\,B_1\,b_1\,c_1 - 2\,C_1\,c_2\,p\,B_1 \\
C :=& \, 8\,C_1\,c_1\,p\,B_1 + 7\,C_2\,c_1\,b_1 + 8\,B_1\,b_1\,c_2 - 7\,C_2\,c_1\,B_1 - 8\,B_1\,b_1\,p\,c_1 + 8\,C_2\,c_2\,p\,b_1 \\
& + 7\,C_1\,c_2\,B_1 + 8\,C_2\,c_2\,p\,B_1 + 8\,B_1\,b_1\,C_2 + 8\,C_1\,c_1\,p\,b_1 - 7\,C_1\,c_2\,b_1 - 8\,B_1\,b_1\,p\,C_1 \\
D :=& \, -12\,C_2\,c_1\,p\,B_1 + 8\,B_1\,b_1\,p\,C_2 - 8\,C_2\,c_2\,B_1 + 12\,C_1\,c_2\,p\,b_1 + 8\,B_1\,b_1\,C_1 + 12\,C_1\,c_2\,pB_1 \\
& - 12\,C_2\,c_1\,p\,b_1 - 8\,B_1\,b_1\,c_1 - 8\,B_1\,b_1\,p\,c_2 + 8\,C_1\,c_1\,b_1 + 8\,C_2\,c_2\,b_1 - 8\,C_1\,c_1\,B_1 \\
E :=& \, 7\,C_2\,c_1\,B_1 - 8\,B_1\,b_1\,c_2 - 7\,C_2\,c_1\,b_1 - 7\,C_1\,c_2\,B_1 + 8\,B_1\,b_1\,p\,c_1 - 8\,C_2\,c_2\,p\,B_1 \\
& - 8\,C_1\,c_1\,p\,b_1 - 8\,C_1\,c_1\,p\,B_1 - 8\,C_2\,c_2\,p\,b_1 - 8\,B_1\,b_1\,C_2 + 7\,C_1\,c_2\,b_1 + 8\,B_1\,b_1\,p\,C_1 \\
F :=& \, 4\,C_2\,c_2\,B_1 - 2\,C_1\,c_2\,p\,b_1 - 4\,B_1\,b_1\,C_1 - 4\,B_1\,b_1\,p\,C_2 + 2\,C_2\,c_1\,p\,b_1 - 4\,C_2\,c_2\,b_1 \\
& + 2\,C_2\,c_1\,p\,B_1 + 4\,C_1\,c_1\,B_1 - 4\,C_1\,c_1\,b_1 + 4\,B_1\,b_1\,p\,c_2 + 4\,B_1\,b_1\,c_1 - 2\,C_1\,c_2\,p\,B_1 \\
G :=& \, C_2\,c_1\,B_1 + C_2\,c_1\,b_1 + C_1\,c_2\,B_1 - C_1\,c_2\,b_1 \\
H :=& \, (B_1\,p\,c_2 - b_1\,p\,C_2 - b_1\,C_1 + B_1\,c_1 \\
I :=& \, 2\,p^2\,b_1\,C_2 + 2\,c_2\,B_1 + 2\,C_2\,b_1 + 2\,B_1\,p^2\,c_2 \\
J :=& \, 2\,B_1\,p\,c_2 - 2\,B_1\,c_1 + 4\,p^2\,b_1\,C_1 - 4\,B_1\,p^2\,c_1 + 2\,b_1\,C_1 - 2\,b_1\,p\,C_2 \\
K :=& \, -2\,C_2\,b_1 - 2\,c_2\,B_1 - 2\,B_1\,p^2\,c_2 - 2\,p^2\,b_1\,C_2 \\
L :=& \, 2\,B_1\,p\,c_2 - 2\,b_1\,p\,C_2 - 2\,b_1\,C_1 + 2\,B_1\,c_1.
\end{aligned}
\tag{17}
$$

3 At Least 9 Proper Lines Are Contained on the Quartic

The quartic singularity surface contains sections in planes parallel to $x_1 = 0$ that are either ellipses, hyperbolas, parabolas or pairs of lines, together with the double-line $x_0 = x_1 = 0$. The singular motions corresponding to these curves are translations by virtue of the fact

x_1 is constant, meaning that the rotation angle, φ, is also constant. The real straight lines are the images of singular rectilinear translations, while the ellipses and hyperbolas are singular curvilinear translations.

Three real lines correspond to the condition where pairs of leg directions are incident. This happens when one side of the platform (for example $\overline{ab}$) is incident with the corresponding side of the base ($\overline{AB}$). This means that there exists a one parametric rectilinear translation which is always singular (because two leg directions are incident). This one parameter set has a line image in the kinematic image space. In the plane spanned by this line and the double line at infinity there has to be a fouth line, because every plane intersects S in a fourth order curve. In the case at hand this curve has to split into four lines. This gives in total six lines (which are always real!). There are three more lines, which belong to one parameter families of rotations that correspond to the three rotations about the three base points when one of the legs has zero length. [2] Note that we have shown that the suface contains four more ideal lines (one double) in the plane at infinity. It is an open question what is the upper bound of proper lines on a general singularity surface.

4 Similar and Similarly Placed Conics

Two conic sections will be similar and similarly placed if the coefficients of corresponding quadratic terms in their point equations are the same up to a constant multiplier [13]. The quadratic terms of two arbitrary, possibly distinct, conics can be written as

$$ax^2 + 2bxy + cy^2 \quad \text{and} \quad a'x^2 + 2b'xy + c'y^2. \tag{18}$$

If they are similar and similarly placed then the following relation among the coefficients must be satisfied:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}. \tag{19}$$

The platform is in a singular pose whenever the three legs are in a pencil. Consider the locus of pencil apices that correspond to a particular orientation of the platform. For each apex, there is a singular platform position. We consider now a curvilinear motion which keeps the platform in a singular position and a fixed orientation. This motion corresponds in the image space to one of the conic sections of S with a plane $x_1 = const$. The locus of singular platform positions is represented by the motion of the origin of the moving reference frame E. For every platform orientation there is a pair of loci, the first being the locus of apices of leg pencils and the other the path of the origin in the curvilinear motion. These two loci are related. In fact, they are similar and similarly placed conics.

This suprising result is easily proved by directly comparing the ratios of the coefficients of the respective quadratic forms. With the aid of a computer algebra system (in this case *Maple V* Release 4) it can be shown that the general locus of pencil apices is:

$$\begin{gathered}4uC_2b_1(1-u^2)X^2 + \Big[C_1b_1(u^4-6u^2+1) + B_1c_1(6u^2-1-u^4) + 4uB_1c_2(1-u^2)\Big]Y^2 + \\ \Big[(4uC_1(u^2-1) + C_2(6u^2-1-u^4))b_1 + 4uB_1c_1(1-u^2) + B_1c_2(u^4-6u^2+1)\Big]XY.\end{gathered} \tag{20}$$

[2] These rotations have no practical meaning because a zero leg length is mechanically not possible.

The corresponding locus of singular platform positions is:

$$\begin{aligned}
&4uC_2b_1(1+u^2-u^4-u^6)X^2+\Big[C_1b_1(1-4u^2-10u^4-4u^6+u^8)+\\
&B_1c_1(4u^2+10u^4+4u^6-u^8-1)+4uB_1c_2(1+u^2-u^4-u^6)\Big]\,Y^2+\\
&\Big[\Big(4uC_1(-1-u^2+u^4+u^6)-C_2(1-4u^2-10u^4-4u^6+u^8)\Big)\,b_1+\\
&4uB_1c_1(1+u^2-u^4-u^6)+B_1c_2(1-4u^2-10u^4-4u^6+u^8)\Big]\,XY.
\end{aligned} \tag{21}$$

In both Equations (20) and (21) the following substitutions have been made:

$$\cos\frac{\varphi}{2}=\frac{1-u^2}{1+u^2},\quad \sin\frac{\varphi}{2}=\frac{2u}{1+u^2},\quad \text{where } u=\tan\frac{\varphi}{4}. \tag{22}$$

The ratio of the coefficients of X^2 of Equation (20) to Equation (21) is

$$\frac{1}{(1+u^2)^2} \tag{23}$$

It is easy to verify that this is the same ratio for the coefficients of Y^2 and of the cross-term XY. This proves that all pairs of conics of pencil apices and singular positions are similar and similarly placed. The conics are the same up to the multiplicative factor represented by Equation (23). This tells us that the factor does not depend on the design parameters. It only depends on the rotation angle. A remarkable result! Figure 3 shows platform, base and leges of the example used above in one singular position. The other plotted positions of the platform (without the corresponding legs) are positions during a curvilinear translation which keeps the platform in singular positions.

5 Next Singularity

Given a certain position of the platform the designer could be interested in answering the question: what is the next singularity to the given position? The difficulty that arises now is how to define the meaning of "next". We will use in the following a definition of distance which comes from the geometry of the kinematic image space. With this definition a notion of a shortest distance is available. The investigation will show that the used concept is very natural from geometric and kinematic point of view but is quite limited from mechanical standpoint.

In Bottema-Roth ([1], p. 397pp.) it is shown that kinematic mapping induces naturally a geometry in the image space. This geometry is a borderline case of elliptic geometry and Blaschke [2] has therfore called it quasi-elliptic. Within this geometry a distance between two points P_i, $i=1,2$ in the image space is defined in the following way: both points represent displacements which have rotation angles ϕ_1 and ϕ_2 (see Equations (2)-(5)). The distance between P_1 and P_2 is now defined: $\overline{P_1P_2}=\phi_2-\phi_2$. It is clear that this definition does not give a metric in the kinematic image space because two positions that differ only by a translation will have zero distance in the image space. But their kinematic images are different. They only have the same x_1 coordinate which essentially represents the rotation angles. Blaschke introduces for points having this property a "replacement distance" which is defined as the euclidean distance of their projections in the $x_1=0$

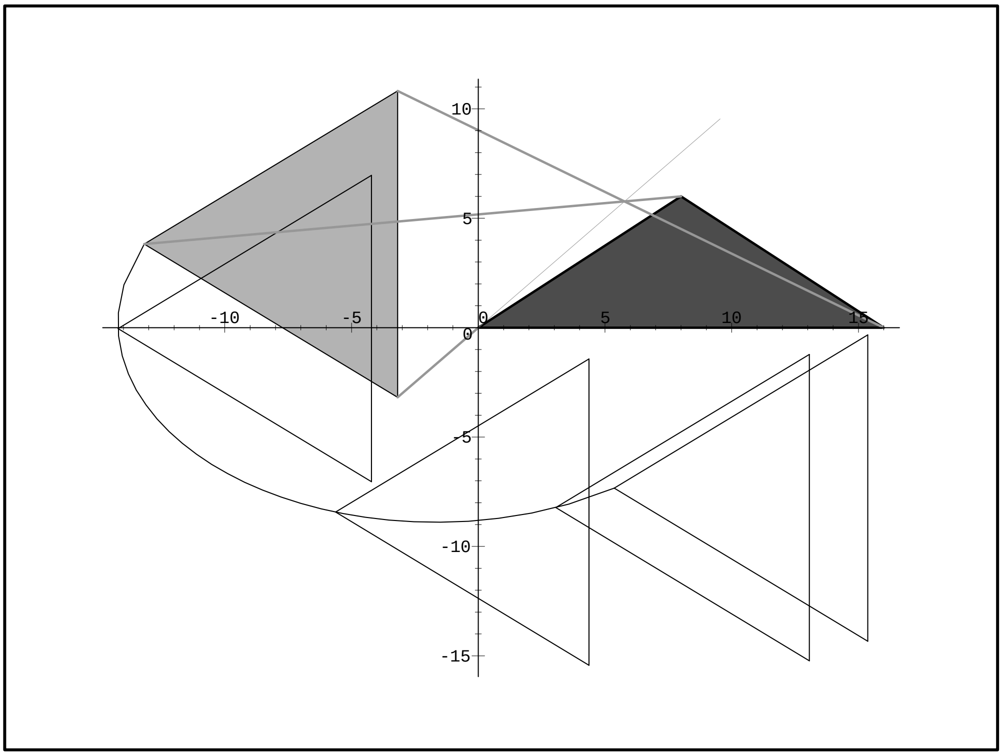

Figure 3: Singular Curvilinear Translation.

plane. From kinematic point of view this replacement distance is linked to the distance of the origins of the two frames belonging to the translations.

From the geometry of the image space the above question has two answers: The next singularity is either the shortest pure rotation or the shortest pure translation. Both concepts will be employed in the following subsections.

5.1 Next Singularity by Rotation

It is shown by Blaschke [2] resp. Bottema and Roth [1] that a pure rotation maps to a straight line in the image space. A rotation that contains the given position will map to a line that contains the kinematic image of that position. As the surface S is of fourth order each line l will intersect S in general in four points. These four points may be real or complex. When all four points are complex, then the platform can fully rotate about one point of the base system. This point would belong to the dexterous workspace of the platform. More likely the line l will have real intersections with S. Then the intersection point, which has the smallest difference in the x_1 image space coordinate with respect to the image of the given position, represents the next singularity which can be reached by a pure rotation. A more detailed explanation with examples is given in [9].

5.2 Next Singularity by Translation

When a pure translation is considered, then we have to apply the replacement distance to find the shortest distance to the next singularity. All positions which are translationally congruent to a given position map to a plane $x_1 = const.$ of the image space. By definition of the replacement distance in this plane we have usual Euclidean geometry. As the singularity surface will intersect the plane $x_1 = const.$ in a conic section, the whole problem comes down to find the shortest (Euclidean) distance between a given point and a conic section in the plane. A possible algorithm to find the shortest distance is to construct a circle centered at the given point with variable radius. Then determine the radius in such a way that the circle is tangent to the conic section (a detailed computation with examples is given in [9]).

6 Conclusion

Kinematic image space was used to derive the singularity surface of planar three-legged platforms. Close geometric inspection of the implicit surface equation revealed an algorithm to compute a parametric representation of this surface. Sections of the singularity surface in planes parallel to a specific image space reference plane consist of conic sections and a double line. The corresponding curvilinear translations have the property that paths of points and instanteneous centers ar similar and similarily placed conic sections. The geometry of the kinematic image space was used to present a concept of finding the next singularity of a given pose of the platform.

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