INVERSE KINEMATICS OF A PLANAR MANIPULATOR WITH HOLONOMIC HIGHER PAIRS

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Abstract. A solution for the inverse kinematics problem of a planar parallel 3-legged platform with holonomic higher pairs is presented. Kinematic mapping is used to represent distinct planar displacements of the end-effector as discrete points in the image space. Separate motion of each leg of the manipulator traces an hyperboloid of one sheet in this space. Therefore, points of intersection of the three hyperboloids represent feasible end-effector displacements. Determination of the three joint input variables required to attain a desired feasible pose, from the respective hyperboloid equations, is described and a numerical example is given.

1. Introduction

The goal of this paper is to present a novel and practical solution procedure for the inverse kinematics (IK) problem of a planar parallel manipulator with holonomic higher pairs. This procedure uses the inverse of a planar kinematic mapping introduced independently by Blaschke and Grünwald in 1911 (Bottema and Roth, 1979). Description of distinct displacements in the plane requires three independent parameters. It follows that each distinct planar displacement maps to a unique point in this three dimensional kinematic image space. We propose to extract a set of joint inputs from the pre-image of given image points which represents feasible displacements.

This particular mapping is well suited to manipulators of the type discussed in this paper (see Fig. 1) since it is independent of the geometry of the platform (Husty, 1995). This is important in our case as the platform attachment points are not fixed, but change continuously between poses. It is important to note that the pure rolling nature of the higher pairs make the manipulator in Fig. 1 markedly different from lower pair jointed Stewart-Gough (SG) type platforms because the pure rolling condition renders IK solutions completely dependent on the initial assembly configuration (IAC). As a result the kinematic analysis employed by Gosselin *et al.* (1991) and Wohlhart (1992) cannot be used.

Kinematic mapping has some recent important applications to planar robots. Various mappings are used in De Sa *et al.* (1981) to classify one parameter planar algebraic motions. The Blaschke-Grünwald kinematic mapping is used in Ravani *et al.* (1983) to study planar motion synthesis. Husty (1995) wove the same mapping into the fabric of an elegant forward kinematics (FK) solution procedure for planar 3-legged SG type platforms. It was then used to analyse the workspace of the same type of platform (Husty, 1996a). We have also been successful in using this mapping to develop a solution procedure for the FK problem of our planar parallel manipulator (Hayes *et al.*, 1996b; 1997). However, it has never, to the best of our knowledge, been applied to the IK problem.

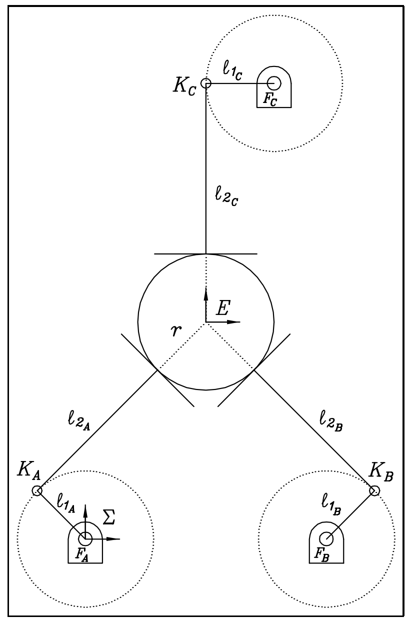

Figure 1. Planar platform.

Indeed, there exists no practical IK solution procedure for the manipulator considered here. An algorithm is offered in Agrawal *et al.* (1992), however, the authors fail to account for the orientation of the end-effector in the inertial reference frame. They use instead a relative angle which can change for certain displacements while the orientation of the end-effector remains constant. The only other algorithm, Hayes *et al.* (1996a), yields equations that are difficult to solve because they depend on a joint parameter which, it turns out, can not be directly evaluated.

2. Manipulator Description

The planar manipulator, shown in Fig. 1, consists of three closed kinematic chains. The circular disk, modelled as a pinion, rolls without slip on each of the three racks tangent to it. We call the kinematic connection between the rack and pinion a *gear* (G) *pair*. It is a *higher* kinematic pair because of the point (or line) contact between the two links. Moreover, the rolling constraints are holonomic due to the pure rolling and because the motion is planar. Hence, the constraint equations can be expressed in terms of displacement, i.e., in *integral form*. Each of the three legs connect a rack to a base point via two *revolute* (R) *pairs*. This means each closed chain is an -R-R-G-G-R-R- chain. The leg links are rigid and a rack is rigidly attached to the disk end of each second link. The R-pairs connecting two links in a leg shall be referred to as *knee joints* K_A, K_B, K_C, and are constrained to move on circles centred on the three points F_A, F_B, F_C, which are grounded to a fixed rigid base. The position and orientation of the pinion end-effector are described by reference frame E, which has its origin on the disk centre and moves with it. Frame Σ has its origin at the base of leg A and is fixed.

3. A Kinematic Mapping of Planar Displacements

It is convenient to think of the relative planar motion between two rigid bodies as the motion of a Cartesian reference coordinate system, E, attached to one of the bodies, with respect to the Cartesian coordinate system, Σ, attached to the other. The position of a point in E relative to Σ can be given by the homogeneous linear transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & a \\ \sin\varphi & \cos\varphi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{1}$$

where $(x/z, y/z)$ are the Cartesian coordinates of a point in E, $(X/Z, Y/Z)$ are those of the same point in Σ. The Cartesian coordinates of the origin of E measured in Σ are (a, b). The rotation angle measured from the X-axis to the x-axis is φ, the positive sense being counter-clockwise.

The kinematic mapping used here is discussed in detail by Bottema and Roth (1979), as well as by De Sa (1979) and Ravani (1982). The image of the displacement parameters (a, b, φ) under the kinematic mapping is called the *image point*. Distinct displacements have unique image points, given by

$$\begin{aligned}(X_1 : X_2 : X_3 : X_4) \;=\; & [(a\sin(\varphi/2) - b\cos(\varphi/2) : \\ & (a\cos(\varphi/2) + b\sin(\varphi/2) : \\ & 2\sin(\varphi/2) : 2\cos(\varphi/2)]. \end{aligned} \tag{2}$$

Care must be taken because the mapping is injective, not bijective: *there is at most one pre-image for each of the points in the image space*. Not every point in the image space represents a displacement in the plane.

The ungrounded R-pair in a 2R mechanism is constrained to move on a circle with a fixed centre. The image points that correspond to all possible displacements of the ungrounded link with respect to a fixed reference frame constitute a quadric hyper-surface. If the circle centre has fixed homogeneous coordinates $(X_c : Y_c : Z)$ and radius r, the *constraint hyperboloid*, H, in the image space has an equation of the form (Bottema and Roth, 1979):

$$\begin{aligned} H: \quad & 0 = z^2(X_1^2 + X_2^2) + (1/4)[(x^2+y^2) - 2C_1xz - 2C_2yz + C_3z^2]X_3^2 + \\ & (1/4)[(x^2+y^2) + 2C_1xz + 2C_2yz + C_3z^2]X_4^2 + (C_1z - x)zX_1X_3 + \\ & (C_2z - y)zX_2X_3 - (y + C_2z)zX_1X_4 + (C_1z + x)zX_2X_4 + \\ & (C_2x - C_1y)zX_3X_4, \end{aligned} \tag{3}$$

where

$$\begin{aligned} C_1 &= -X_c, \\ C_2 &= -Y_c, \\ C_3 &= (C_1^2 + C_2^2 - r^2) = (X_c^2 + Y_c^2 - r^2). \end{aligned}$$

4. An Application to the IK Problem

4.1. VIRTUAL PLATFORM

The IK problem may be stated as follows: given the position and orientation of the pinion, determine the joint input variables and corresponding assembly modes required to attain the desired pose. Since the manipulator has three degrees-of-freedom (DOF), three joint input variables are required. We select these to be the change in arclength along the rack due to the change in contact point (this choice was made to better suit computations). They are given by the three numbers $\Delta d_i = r\Delta\tau_i$, $i \in \{A, B, C\}$. The $\Delta\tau_i$ are the change between the initial and final rack angles and are independent of the choice of coordinate reference frame. The pinion radius is r. Since the racks are always in tangential contact with the disk, the change in these angles represent the change in angle of disk tangents. Because the bases are orthogonal, the change in tangent angle is the same as the change in normal angle: $\Delta\tau_i = \Delta\eta_i$.

It is important to observe that the rolling constraints impose a kinematic dependency on the IAC. Displacement analysis requires the presence of initial conditions in the kinematic closure equations. This dependency on the IAC means that analysis is not possible using the techniques employed on lower pair jointed SG type platforms by Gosselin *et al.* (1991) and Wohlhart (1992), for instance.

To solve the problem using the inverse of the procedure in Husty (1995) and Hayes *et al.* (1997), fixed points in E which move on fixed circles in Σ are required. In order to establish these points we must first define the *platform* (or *end-effector*). The only points bound to move on fixed circles in Σ are points on the first link in each leg, which are connected to the fixed base by an R-pair. We will use the centres of the three knee joints, K_A, K_B, and K_C. Now, consider a *virtual platform* (VP) formed by the triangle whose vertices are the three knee joints expressed relative to the disk frame E: $K_{A/E}$, $K_{B/E}$, $K_{C/E}$ (see Fig. 2). For a given assembly configuration, these *virtual platform points* (VPP) are fixed relative to each other, but change from pose to pose. Although the VP geometry changes continuously during platform motion, for any given displacement it can be considered a rigid body, since we are only interested in the correspondence between an initial and a final position. Hence, the VPP meet the requirements of being points in E which move on fixed circles in Σ.

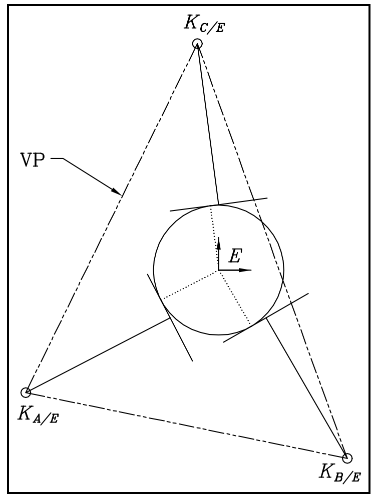

Figure 2. VP for a set of inputs.

4.2. INVOLUTE INPUTS

The next task is to develop expressions for the VPP so they can be used as inputs to the kinematic mapping. We require expressions for the VPP in terms of the joint input variables, $\Delta\tau_i$. Consider, for now, only leg A in Fig. 3 and observe that the knee joint K_A, which has a fixed position in the reference frame attached to the rack, R_A, moves on a circle in the fixed frame, Σ. But, it also has a relative motion in the moving disk frame E. What is required is a description of that motion in terms of the joint inputs. This turns out to be straightforward: fix the disk and observe that the relative motion of the rack with respect to E is pure rolling with the original contact point moving on an involute of the pinion (Husty, 1996b). Because there is a bijective correspondence between positions of a given rack point on its involute and knee joint positions, we now have a complete description (in terms of the change in rack tangent angle) of the motion of the knee joints with respect to both the moving frame, E, and the fixed frame, Σ. Due to their positional dependence on an involute, we call these one parameter sets of knee joint positions *involute inputs.*

The motion of the knee joints of the remaining two legs must be the same type as that of leg A relative to E, but the starting points of the involutes are different. Thus, for every set of three joint input parameters one obtains a set of three VPP expressed in E. With the VPP transformed to involute inputs the kinematic mapping can be used.

We will now show how the involute inputs can be derived. Fig. 3 shows the reference coordinate systems used to transform the position of the knee joint from the moving rack reference frame, R_A, to the relatively fixed pinion reference frame, E. The origin of R_A moves along its involute and R'_A gives the new position of R_A after a rotation $\Delta\tau_A$. The intermediate system, E'_A, whose utility is discussed below, is fixed relative to E. For each leg, E'_i is rotated from E through $\theta_i = (5\pi/4), (7\pi/4), (\pi/2)$ for $i \in \{A, B, C\}$. Examining Fig. 3, it is easy to see that for each leg the required transformations to take the coordinates of the knee joint K_i from frame R'_i to frame E are

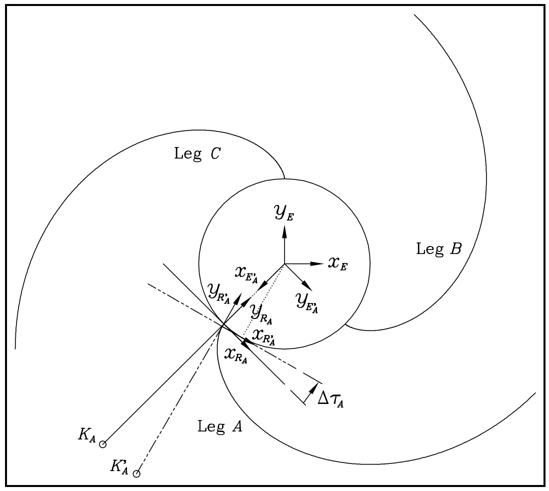

Figure 3. Leg A reference frames.

$$\begin{aligned}\mathbf{T}_{R'_i/E} &= \mathbf{T}_{E'_i/E}\mathbf{T}_{R'_i/E'_i}\\ &= \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s\Delta\tau_i & -c\Delta\tau_i & r(c\Delta\tau_i + \Delta\tau_i s\Delta\tau_i) \\ c\Delta\tau_i & -s\Delta\tau_i & r(s\Delta\tau_i - \Delta\tau_i c\Delta\tau_i) \\ 0 & 0 & 1 \end{bmatrix},\end{aligned}$$

where $c = \cos$, and $s = \sin$.

The geometrical significance of $\mathbf{T}_{R'_i/E'_i}$ is seen when each column is examined. The first column is the direction of the disk tangent in E'_i (the direction of the x-axis of frame R'_i). The second column is the direction in E'_i (towards the centre of the pinion) of the normal at the new contact point. The third column is the position of the origin of frame R'_i on the involute, expressed again in E'_i. The remaining transformation, $\mathbf{T}_{E'_i/E}$, depends on the angle between the x-axis of frame E and the rack normal in the home position, shown in Fig. 1.

The knee joints, shown in Fig. 1, all have the same coordinates in their respective R_i frames:

$$\mathbf{k}_{i/R_i} = \begin{bmatrix} 0 \\ -\ell_{2_i} \\ 1 \end{bmatrix}.$$

Once the arclength parameters (joint inputs), $\Delta\tau_i$, are given, the coordinates of the knee joints (involute inputs) in frame E, $\mathbf{k}_{i/E}$, are easily determined by left multiplying the $\mathbf{k}_{i/R'_i}$ with the appropriate $\mathbf{T}_{R'_i/E}$,

$$\mathbf{k}_{i/E} \quad = \quad \mathbf{T}_{R'_i/E}\mathbf{k}_{i/R'_i}. \tag{4}$$

5. Solution Procedure

Recall that the goal is to determine the inputs required to attain a desired feasible end-effector pose. In our solution procedure we first examine the constraint hyperboloids for each leg, given by equation (3). The image point $(X_1 : X_2 : X_3 : X_4)$ is fixed by the given pinion displacement parameters (a, b, φ). Furthermore, the constants (C_1, C_2, C_3) are known because the circle centres and radii are all specified. This leaves the three homogeneous VPP coordinates $(x : y : z)$ as unknowns. Thus, we have three hyperboloid equations and nine unknowns:

$$H_i = f_i(x_i, y_i, z_i), \qquad i \in \{A, B, C\}. \tag{5}$$

Since no practical design requires the VP to have points on the line at infinity, $\mathcal{L}_\infty$, we can safely set $z_i = 1$, and reduce the quantity of unknowns to six:

$$H_i = f_i(x_i, y_i), \qquad i \in \{A, B, C\}. \tag{6}$$

At least three more equations are required. Consider the involute input equations (4). With $z_i = 1$, these are a set of six equations expressing the knee joint coordinates in the moving frame, E, in terms of the three unknown rack tangent angle inputs, $\Delta\tau_i$. This gives nine equations and nine unknowns, coming in independent sets of three. That is, $x_i, y_i, \Delta\tau_i$ can be solved independently for each $i \in \{A, B, C\}$:

$$\left.\begin{array}{r} H_i = f_i(x_i, y_i) \\ x_i = g_i(\Delta\tau_i) \\ y_i = h_i(\Delta\tau_i) \end{array}\right\}, \qquad i \in \{A, B, C\} \tag{7}$$

where f_i is a function in the two variables x_i and y_i, which are themselves single variable functions g_i and h_i, respectively, in terms of $\Delta\tau_i$.

Substituting the expressions for $x_i = g(\Delta\tau_i)$ and $y_i = h(\Delta\tau_i)$ into H_i gives the single variable function

$$\begin{aligned} H_i(\Delta\tau_i) \quad = \quad & a_0 + a_1\Delta\tau_i + a_2(\Delta\tau_i)^2 + a_3\cos\Delta\tau_i + a_4\sin\Delta\tau_i + \\ & \Delta\tau_i(a_5\cos\Delta\tau_i + a_6\sin\Delta\tau_i), \end{aligned} \tag{8}$$

where the a_i are coefficients in the field of real numbers. Solve $H_i(\Delta\tau_i)$ for $\Delta\tau_i$, and use this value to determine x_i and y_i from the g_i and h_i. This immediately yields the knee joint coordinates in the moving pinion frame. $H_i(\Delta\tau_i)$ represents a quadric hyper-surface. Because of its quadratic nature we can expect at most two real solutions for the change in rack tangent angle $\Delta\tau_i$. It is a simple matter of plane trigonometry to extract the assembly configuration(s) from the nine parameters, x_i, y_i, and $\Delta\tau_i$. The solutions between legs are decoupled, so we have an upper bound on the number of solutions to the IK problem given by

$$2^n, \tag{9}$$

where n is the number of -R-R-G- legs.

6. Example

Table 1 gives the manipulator's initial assembly configuration (IAC). The $x_{F_i/\Sigma}$ and $y_{F_i/\Sigma}$ are the coordinates of the base of each leg expressed in the fixed frame, Σ. The initial rack normal angles in the moving frame, E, are $\eta_{i/E}$. The relative angles between the first link and base, and between the second and first links are $\theta_{i_1/0}$ and $\theta_{i_2/1}$, respectively. The location of the contact point along a rack measured in the corresponding rack frame, R_i, is d_{i_3/R_i}. The link lengths, in generic units, are: $r = 4$; $\ell_{i_1} = 4$; $\ell_{i_2} = 10$. Note that in Fig. 1 the link reference frames, assigned using the Denavit-Hartenberg convention, are not shown so as to avoid clutter.

TABLE 1. Initial assembly configuration (IAC).

i	$x_{F_i/\Sigma}$	$y_{F_i/\Sigma}$	$\eta_{i/E}$	$\vartheta_{i_1/0}$	$\vartheta_{i_2/1}$	d_{i_3/R_i}
A	0	0	225°	135°	270°	0
B	$10\sqrt{2}$	0	315°	45°	90°	0
C	$5\sqrt{2}+4$	$9\sqrt{2}+14$	90°	180°	90°	0

The desired pose of the end-effector and the corresponding image point are:

$$\begin{bmatrix} a \\ b \\ \varphi \text{ (deg.)} \end{bmatrix} = \begin{bmatrix} 9.429 \\ 11.845 \\ 3.716^\circ \end{bmatrix}, \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -11.503 \\ 9.807 \\ 0.065 \\ 1.999 \end{bmatrix}.$$

After making the appropriate substitutions, the following three univariate functions are obtained.

$$\begin{aligned} H_A(\Delta\tau_A) &= 408.489 + 16(\Delta\tau_A)^2 - 422.777\cos\Delta\tau_A - 19.875\sin\Delta\tau_A + \\ &\quad \Delta\tau_A(5.678\cos\Delta\tau_A - 120.793\sin\Delta\tau_A), \\ H_B(\Delta\tau_B) &= 341.710 + 16(\Delta\tau_B)^2 - 317.434\cos\Delta\tau_B + 161.514\sin\Delta\tau_B - \\ &\quad \Delta\tau_B(46.147\cos\Delta\tau_B + 90.694\sin\Delta\tau_B), \\ H_C(\Delta\tau_C) &= 405.104 + 16(\Delta\tau_C) - 413.715\cos\Delta\tau_C + 72.949\sin\Delta\tau_C - \\ &\quad \Delta\tau_C(20.843\cos\Delta\tau_C + 118.204\sin\Delta\tau_C). \end{aligned}$$

The values of $\Delta\tau_i$ from each solution are used to evaluate equations (4), giving the corresponding knee joint coordinates (x_i, y_i). These are listed in Table 2. The relative link angles for each assembly configuration are determined using plane trigonometry and the given position and orientation of the pinion end-effector (i.e., the moving frame, E). Fig. 4 illustrates one of the eight real assembly configurations.

TABLE 2. Change in rack tangent angle and corresponding knee joint coordinates.

$\Delta\tau_A$	$x_{K_A/E}$	$y_{K_A/E}$	$\Delta\tau_B$	$x_{K_B/E}$	$y_{K_B/E}$	$\Delta\tau_C$	$x_{K_C/E}$	$y_{K_C/E}$
-17.5°	-11.845	-7.548	-15°	7.907	-11.601	7.5°	-1.308	13.949
25.04°	-6.422	-12.563	-55.07°	0.783	-14.554	-35.46°	6.106	12.839

7. Conclusions

The solution to the IK problem is fundamental to further investigation of this type of manipulator. Kinematic mapping has proven to be a useful tool in our solution. To use the mapping, the knee joint positions are expressed as one-parameter motions of initial rack contact points along involutes of the pinion. The constraint hyperboloid equations in the image space, along with the knee joint position equations, are used to determine the joint input variables required to attain the desired pose. The upper bound on the number of real solutions is 2^n.

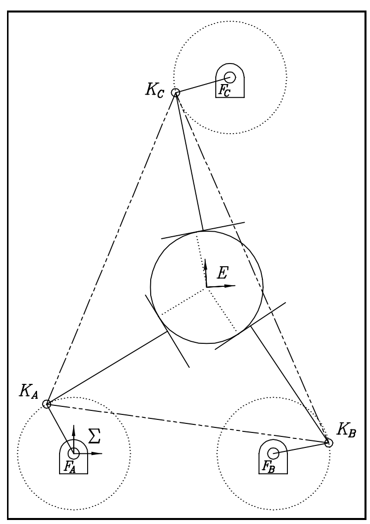

Figure 4. One solution.

8. Acknowledgements

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