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Geometry and Topology:

A Mapping of Plane Kinematics

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1 Introduction

Planar kinematic mappings are mappings of the set of all planar displacements onto the points of a three dimensional projective space with Cartesian homogeneous coordinates X_i $(i = 1, 2, 3, 4)$. It has recently been shown that this mapping has important applications in robotics, specifically, in the solution of the forward kinematics problem of planar and spatial Stewart–Gaugh–type platforms [12], [13].

In this paper, Grünwald's kinematic mapping of planar displacements will be derived. Furthermore, its application will be demonstrated by an example wherein the forward kinematics problem of a planar parallel manipulator will be solved.

2 A Kinematic Mapping of Planar Displacements

A general displacement in the plane requires three independent coordinates to fully characterise it. The position of one rigid body relative to another then is given by three numbers. Typically, a displacement is described by $D(a, b, \phi)$, where a and b are the magnitudes of the components of a position vector in the direction of linearly independent basis vectors, and ϕ is a rotation angle about some fixed axis normal to the plane. In 1911, Grünwald and Blashke independently suggested using the three numbers which describe a planar position as the coordinates of points in a three dimensional space, called the *image space* [12].

A planar motion is a continuous series of positions, hence, a complete motion in the plane is mapped to a curve of the image space. Since one, two and three degree of freedom planar motions are represented respectively by curves, surfaces, and solids in the image space the classification of planar motions can be reduced to the classification of curves, surfaces, and solids [18], [19].

It is convenient to think of the relative planar motion between two rigid bodies as the motion of a Cartesian reference coordinate system, E attached to one of the bodies, with respect the Cartesian coordinate system, Σ attached to the other, [3]. Without loss of generality, Σ may be considered as fixed while E is free to move. Then the position of a point in E relative to Σ can be given by

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}, \tag{1}$$

where

i. (x', y') are the Cartesian coordinates of a point in E.

ii. (X', Y') are the Cartesian coordinates of the same point in Σ.

iii. (a, b) are the Cartesian coordinates of the origin of E measured in Σ, ie, the components of the position vector of the origin of E in Σ.

iv. ϕ is the rotation angle measured from the X'–axis to the x'–axis, the positive sense being counter–clockwise.

Equation (1) does not represent a linear transformation. This fact is computationally inconvenient, and can be remedied by the use of Cartesian homogeneous coordinates [20]

$$x' = \frac{x}{z} \quad , \quad y' = \frac{y}{z}$$
$$X' = \frac{X}{Z} \quad , \quad Y' = \frac{Y}{Z}.$$

Substituting these homogeneous coordinates in equation (1) gives for X'

$$\frac{X}{Z} = \frac{x}{z}\cos\phi - \frac{y}{z}\sin\phi + a. \qquad (2)$$

Setting the homogenising coordinates to be equal, ie set $Z = z$ and multiplying through by z gives

$$X = x\cos\phi - y\sin\phi + az.$$

Similarly, the Y' expression becomes

$$Y = x\sin\phi + y\cos\phi + bz.$$

Thus, the following linear transformation is obtained:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad (3)$$

which may be expressed very compactly as the vector–matrix equation

$$\mathbf{X} = \mathbf{A}\mathbf{x}. \qquad (4)$$

Equation (4) represents a displacement of E with respect to Σ. If $\mathbf{A}$ is a continuous function of a parameter, such as time, then equation (4) represents a motion of E with respect to Σ.

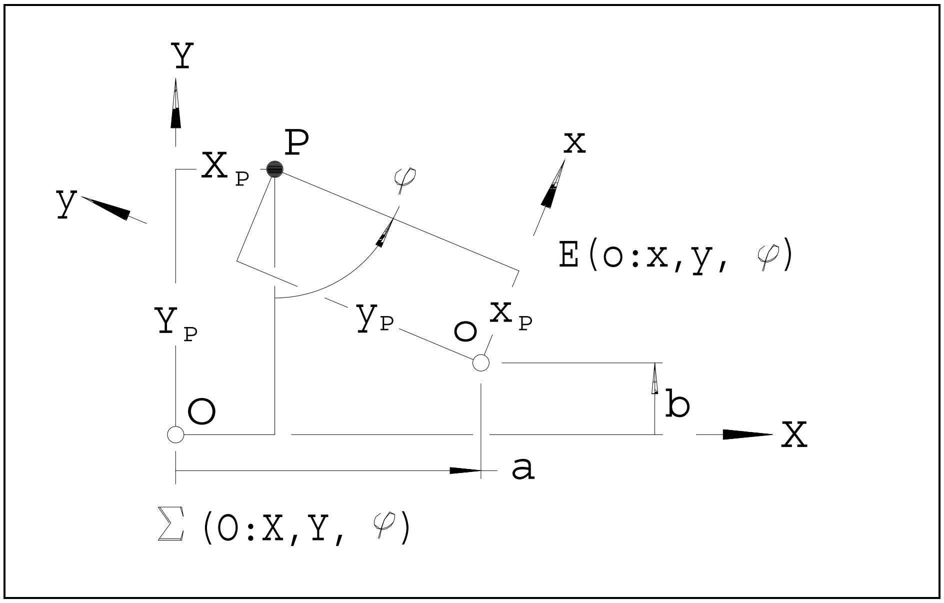

Figure 1: The pole has the same coordinates in E and Σ.

2.1 The Image Point

All general planar displacements that are not translations may be represented as a rotation through a finite angle about a fixed axis normal to the plane. Even a pure translation can be considered a rotation through an infintessimal angle about a point at infinity on a line perpendicular to the direction of the translation. The coordinates of the piercing point of this axis is the *pole* of the displacement. If E and Σ are initially coincident then after the displacement the pole has the same coordinates in both E and Σ. This is illustrated in Figure 1.

To prove that the pole is an invariant of the displacement, the eigenvalues of the 3×3 transformation matrix $\mathbf{A}$ are examined. The eigenvalue problem is stated as follows:

$$\begin{aligned} \lambda \mathbf{x} &= \mathbf{A}\mathbf{x} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} &= \mathbf{0}, \end{aligned}$$

where $\mathbf{A}$ is a square matrix, and λ is a scalar constant.

The system of equations has non–trivial solutions if, and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

The characteristic polynomial for this matrix is found by the Laplacian expansion of the above determinant,

$$\begin{aligned}(1-\lambda)([\cos\phi-\lambda][\cos\phi-\lambda]+\sin^2\phi) &= 0\\ (1-\lambda)(\lambda^2-2\lambda\cos\phi+\cos^2\phi+\sin^2\phi) &= 0\\ (1-\lambda)(\lambda^2-2\lambda\cos\phi+1) &= 0.\end{aligned}$$

The first eigenvalue is $\lambda_1 = 1$. The second and third are from

$$\begin{aligned}\lambda_{2,3} &= \frac{1}{2}(2\cos\phi \pm \sqrt{4\cos^2\phi - 4})\\ &= \cos\phi \pm \sqrt{\cos^2\phi - 1}\\ &= \cos\phi \pm \sqrt{-\sin^2\phi}\\ &= \cos\phi \pm \sin\phi\sqrt{-1}\\ &= \cos\phi \pm i\sin\phi\\ &= e^{\pm i\phi}.\end{aligned}$$

Hence, for any general planar displacement the homogeneous transformation matrix has only one real eigenvalue, $\lambda = 1$. Corresponding to this eigenvalue, the eigenvalue–matrix equation is quite similar to equation (4)

$$\mathbf{x} = \mathbf{A}\mathbf{x}.$$

Now, re–consider equation (4). It can be *de–homogenized* and expressed as

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}. \tag{5}$$

If it is true that the pole is an invariant, then its coordinates must be the same in E and in Σ, ie, $X'_p = x'_p$ and $Y'_p = y'_p$, where the subscript p denotes *pole*. Substituting these into the previous equation gives

$$\begin{bmatrix} x'_p \\ y'_p \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x'_p \\ y'_p \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}. \tag{6}$$

This is compactly expressed as

$$\mathbf{x'}_p = \mathbf{B}\mathbf{x'}_p + \mathbf{d}, \tag{7}$$

where the components of the vector $\mathbf{x}'$ are x_p and y_p, $\mathbf{B}$ is the 2×2 rotation matrix and $\mathbf{d}$ is the translation vector whose components are a and b.

It is a simple matter to solve for $\mathbf{x}'_p$:

$$\begin{aligned}\mathbf{x}'_p - \mathbf{B}\mathbf{x}'_p &= \mathbf{d}\\ (\mathbf{I}-\mathbf{B})\mathbf{x}'_p &= \mathbf{d}\\ \mathbf{x}'_p &= (\mathbf{I}-\mathbf{B})^{-1}\mathbf{d}.\end{aligned}$$

The last equation may be rearranged as

$$\mathbf{x}'_p = -(\mathbf{B}-\mathbf{I})^{-1}\mathbf{d}. \tag{8}$$

These are the Cartesian coordinates of the pole.

Returning now to the eigenvalue problem,

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

Setting $\lambda = 1$, the only real eigenvalue for the matrix $\mathbf{A}$,

$$(\mathbf{A}-\mathbf{I})\mathbf{x} = \mathbf{0}.$$

The matrix $(\mathbf{A}-\mathbf{I})$ can be partitioned as

$$\begin{bmatrix} (\mathbf{B}-\mathbf{I}) & \mathbf{d} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}. \tag{9}$$

Equation (9) may be de–homogenized and expanded giving

$$(\mathbf{B}-\mathbf{I})\mathbf{x}' + \mathbf{d} = \mathbf{0}.$$

Solving for the eigenvector, $\mathbf{x}'$ yields

$$\mathbf{x}' = -(\mathbf{B}-\mathbf{I})^{-1}\mathbf{d}. \tag{10}$$

Comparing equations (8) and (10) it is seen that the eigenvector which corresponds to the sole real eigenvalue shared by all planar homogeneous displacement transformation matrices, is identical to the pole of the displacement. Since it is an eigenvector, the pole is coordinate system independent, and hence, invariant. The location of the pole of a displacement along with the rotation angle convey sufficient information to characterise the displacement. The image of the pole under the kinematic mapping is called the *image point*.

2.2 Pole Coordinates in Terms of a, b and ϕ

The pole coordinates in terms of the displacement parameters a, b and ϕ are obtained by substituting $[X_p, Y_p, Z_p]^T$ for both $[X, Y, Z]^T$ and $[x, y, z]^T$ in equation (3) and expanding to get expressions for X_p and Y_p. The X_p equation is

$$\begin{aligned} X_p &= X_p \cos\phi - Y_p \sin\phi + aZ_p \\ &= -Y_p \frac{\sin\phi}{1-\cos\phi} + \frac{aZ_p}{1-\cos\phi}. \end{aligned} \tag{11}$$

Which can be reduced with the half–angle identity $\tan(\phi/2) = \frac{1-\cos\phi}{\sin\phi}$, giving

$$X_p = -Y_p \cot(\phi/2) + aZ_p \frac{\cot(\phi/2)}{\sin\phi}. \tag{12}$$

Finally, after using the identity $\sin\phi = 2\sin(\phi/2)\cos(\phi/2)$, the following is obtained:

$$X_p = -Y_p \frac{\cos(\phi/2)}{\sin(\phi/2)} + \frac{1}{2} aZ_p \frac{1}{\sin^2(\phi/2)}. \tag{13}$$

After a similar procedure, the Y_p equation may be expressed:

$$Y_p = X_p \frac{\cos(\phi/2)}{\sin(\phi/2)} + \frac{1}{2} bZ_p \frac{1}{\sin^2(\phi/2)}. \tag{14}$$

Substituting equation (13) into (14) yields

$$\begin{aligned} Y_p &= -Y_p \frac{\cos^2(\phi/2)}{\sin^2(\phi/2)} + \frac{1}{2} aZ_p \frac{\cos(\phi/2)}{\sin^3(\phi/2)} + \frac{1}{2} bZ_p \frac{1}{\sin^2(\phi/2)} \\ &= \frac{1}{2} Z_p \left[a \frac{\cos(\phi/2)}{\sin^3(\phi/2)} + b \frac{1}{\sin^2(\phi/2)} \right] \frac{1}{1+\cot^2(\phi/2)}. \end{aligned} \tag{15}$$

Using the identity $\frac{1}{1+\cot^2(\phi/2)} = \sin^2(\phi/2)$ gives

$$Y_p = \frac{1}{2} Z_p \left[a \frac{\cos(\phi/2)}{\sin(\phi/2)} + b \right]. \tag{16}$$

Z_p is the homogenising coordinate and its value is arbitrary. Without loss of generality, let

$$Z_p = \sin(\phi/2).$$

Substituting this back into equation (16) yields the Y–pole coordinate in terms of a, b, and ϕ:

$$Y_p = \frac{1}{2} a \cos(\phi/2) + \frac{1}{2} b \sin(\phi/2). \tag{17}$$

The X–pole component is determined in a similar fashion. Substitute the expression for the Y pole, equation (17) into (13) to get

$$\begin{aligned} X_p &= -\left[\frac{1}{2}a\cos(\phi/2) + \frac{1}{2}b\sin(\phi/2)\right]\frac{\cos\phi/2}{\sin\phi/2} + \frac{1}{2}a\frac{1}{\sin\phi/2} \\ &= -\frac{1}{2}a\frac{\cos^2(\phi/2)}{\sin^2(\phi/2)} + \frac{1}{2}a\frac{1}{\sin(\phi/2)} - \frac{1}{2}b\cos(\phi/2) \\ &= \frac{1}{2}a\frac{1}{\sin(\phi/2)}\left[1-\cos^2(\phi/2)\right] - b\cos(\phi/2). \end{aligned}$$

Finally, the identity $\sin^2\phi = 1 - \cos^2\phi$ is employed to put the X pole component in the form

$$X_p = \frac{1}{2}a\sin(\phi/2) - \frac{1}{2}b\cos(\phi/2). \tag{18}$$

So, the homogeneous coordinates of the pole, which are identical in each of the two coordinate systems, in terms of the three displacement parameters a, b and ϕ are given by

$$\begin{aligned} x_p = \; X_p &= \frac{1}{2}a\sin(\phi/2) - \frac{1}{2}b\cos(\phi/2) \\ y_p = \; Y_p &= \frac{1}{2}a\cos(\phi/2) + \frac{1}{2}b\sin(\phi/2) \\ z_p = \; Z_p &= \sin\phi/2. \end{aligned} \tag{19}$$

2.3 The Image Point and Image Space

Many mappings can be defined that map a position (a, b, ϕ) of the moving coordinate system E with respect to the fixed system Σ in the plane to a point described by the homogeneous coordinates $(X_1 : X_2 : X_3 : X_4)$ of a three dimensional projective *image space*, Σ'. The mapping used here is as follows:

$$(X_1 : X_2 : X_3 : X_4) = (X_p : Y_p : Z_p : \tau Z_p) \tag{20}$$

Where

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) &\neq (0 : 0 : 0 : 0) \\ \tau &= \cot(\phi/2) \\ 0 \leq \phi &< 2\pi \end{aligned}$$

And $X_p : Y_p : Z_p$ depend on (a, b, ϕ) as given by the set of equations 19. This point is called the *image point* of the position (a, b, ϕ). The image point is given by

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) = \; &[(a\sin(\phi/2) - b\cos(\phi/2) : \\ &(a\cos(\phi/2) + b\sin(\phi/2) : \\ &2\sin(\phi/2) : 2\cos(\phi/2)]. \end{aligned} \tag{21}$$

By virtue of the relationships expressed in (21), the linear transformation operator, the matrix **A** from equation (4) may be expressed in terms of the homogeneous coordinates of the image space, Σ'. Recall that

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix}.$$

A_{11} and A_{22} may be re–expressed using the identities $\cos^2(\phi/2) = (1+\cos\phi)/2$ and $\sin^2(\phi/2) = (1-\cos\phi)/2$. This gives

$$\begin{aligned} X_4^2 - X_3^2 &= (2\cos(\phi/2))^2 - (2\sin(\phi/2))^2 \\ &= \frac{4+\cos\phi)}{4} - \frac{4-\cos\phi)}{4} \\ &= 4\cos\phi. \end{aligned} \tag{22}$$

A_{12} and A_{21} are related by $A_{12} = -A_{21}$. A_{12} may be obtained from

$$2X_3X_4 = 2\left[(2\sin(\phi/2))(2\cos(\phi/2))\right]. \tag{23}$$

The identity

$$2\sin(\phi/2) = \frac{\sin\phi}{\cos(\phi/2)}$$

is used to get

$$2\left[\frac{\sin\phi}{\cos(\phi/2)}2\cos(\phi/2)\right] = 4\sin\phi. \tag{24}$$

A_{13} is obtained from

$$\begin{aligned} 2(X_1X_3 + X_2X_4) &= 2[(a\sin(\phi/2) - b\cos(\phi/2))(2\sin(\phi/2)) \\ &\quad +(a\cos(\phi/2) + b\sin(\phi/2))(2\cos(\phi/2))] \\ &= 4(a\sin^2(\phi/2) - b\cos(\phi/2)\sin(\phi/2)) + 4(a\cos^2(\phi/2) + b\cos(\phi/2)\sin(\phi/2)) \\ &= 4a. \end{aligned} \tag{25}$$

A_{23} is obtained from

$$\begin{aligned} 2(X_2X_3 - X_1X_4) &= 2[(a\cos(\phi/2) + b\sin(\phi/2))(2\sin(\phi/2)) \\ &\quad -(a\sin(\phi/2) - b\cos(\phi/2))(2\cos(\phi/2))] \\ &= 4(a\cos(\phi/2)\sin(\phi/2) + b\sin^2(\phi/2)) - 4(a\cos(\phi/2)\sin(\phi/2) - b\cos^2(\phi/2)) \\ &= 4b. \end{aligned} \tag{26}$$

A_{33} is obtained from

$$\begin{aligned} X_3^2 + X_4^2 &= (2\sin(\phi/2))^2 + (2\cos(\phi/2))^2 \\ &= 4. \end{aligned} \tag{27}$$

Notice that 4 is a factor common to all non zero terms of $\mathbf{A}$. Since homogeneous coordinates are used

$$\mathbf{X} = \mathbf{A}\mathbf{x} = 4\mathbf{A}\mathbf{x}.$$

Because, in general

$$(x : y : z) \quad = \quad (\mu x : \mu y : \mu z).$$

This is due to the definition of homogeneous coordinates:

$$x' = \frac{x}{z} = \frac{\mu x}{\mu z} \quad ; \quad y' = \frac{y}{z} = \frac{\mu y}{\mu z}.$$

So, equation (3) may be re–expressed using the homogeneous coordinates of the image space. This means that we now have a linear transformation to express a position of E with respect to Σ in terms of the image point as given by (21):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (X_4^2 - X_3^2) & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & (X_4^2 - X_3^2) & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & (X_4^2 + X_3^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{28}$$

Since equation (28) is a linear transformation, then for each unique displacement described by (a, b, ϕ) there is a corresponding point in the image space. From equation (21), the inverse mapping is obtained. That is, for a given point of the image space, the displacement parameters are obtained from

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2). \end{aligned} \tag{29}$$

3 An Application for Planar Parallel Manipulators

Consider the planar manipulator shown in Fig. 2. It consists of three closed kinematic chains. The three base points A_0, B_0, C_0, which are rigidly fixed, are connected to the vertices A, B, C, of the triangular end–effector via three legs. Each leg consists of two rigid links l_{i1} and l_{i2}, $i \in \{A, B, C\}$.

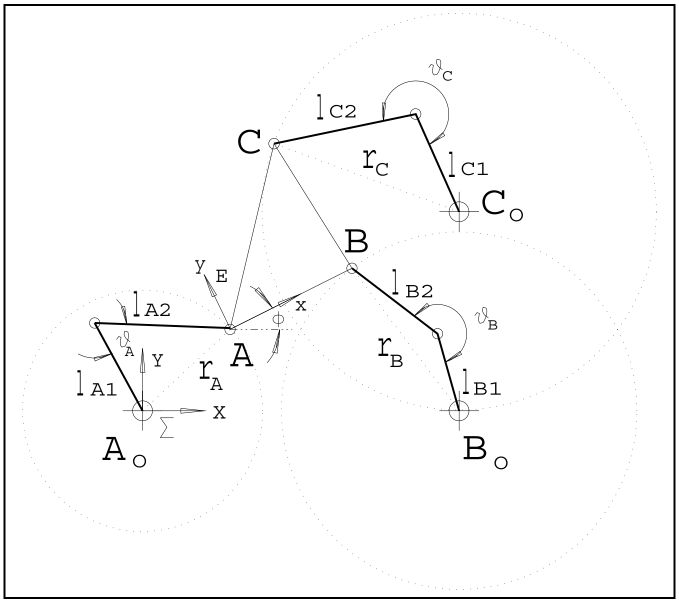

Figure 2: A planar manipulator with three DOF.

The leg links are joined to the base, end–effector, and each other by revolute joints, each with a full 360° range of rotation. By changing the relative angle between the two links in each leg, the triangular end–effector can be brought to any position with any desired orientation within the physical limits of its workspace. Thus, the manipulator has three degrees of freedom (DOF).

Since the manipulator has three DOF, three inputs are required, one for each DOF. It is convenient to use the relative angles between the links of each leg as the inputs. These angles are:

$$\begin{bmatrix} \vartheta_A \\ \vartheta_B \\ \vartheta_C \end{bmatrix} \tag{30}$$

Where the angle is measured from link l_{i1} to l_{i2}, with counter–clock–wise (CCW) considered the positive sense.

There are two main classes of problem in the kinematic analysis of robots. The first is the *forward kinematic problem* which may be stated as follows:

> *Given the input parameters, one for each degree of freedom, what is the position and orientation of the end–effector?*

The second is the *inverse kinematic problem* which may be stated as follows:

> *Given the position and orientation of the end–effector, what are the required joint angles and link lengths.*

The inverse kinematics problem of manipulators of the type in Fig. 2 is easily solved and is considered as 'trivial' [8]. On the other hand, the forward kinematics problem usually requires numerical approaches, such as the Secant or Newton–Raphson methods to obtain solutions. This is due to the largely economic problem of redundant actuation. That is, only one actuator is required per DOF. In this case, there are nine joints, three per leg, however, only three require actuation. Because of this, only three inputs may be specified. This situation invariably leads to a system of equations with far more unknowns than equations. It has recently been discovered in [12], [13] that the solution to the forward kinematics problem of parallel manipulators may be obtained using kinematic mapping. An original example is presented below.

3.1 The Forward Kinematics Problem

The forward kinematics problem of the manipulator shown in Fig. 2 can be stated in the following way: Given the coordinates of the three base points A_0, B_0, C_0 in an arbitrary fixed coordinate system, Σ, the coordinates of the vertices A, B, C of the triangular end–effector in an arbitrary coordinate system, E which moves with the end–effector, the fixed lengths of each link, l_{ij}, $i \in \{A, B, C\}$ and $j \in \{1, 2, 3\}$, and given the three input angles, $\vartheta_1, \vartheta_2, \vartheta_3$ which were described earlier, find the position(s) of of the end–effector such that the vertices A, B, C can be joined to the base points A_0, B_0, C_0 with legs of the given lengths and relative angles.

To obtain the solutions for a given set of input angles, begin by removing the end–effector connections with legs B and C. Observe that the point A is constrained to move on a circle with A_0 as its centre and radius r_A determined by the law of cosines,

$$r_A^2 = l_{A1}^2 + l_{A2}^2 - 2l_{A1}l_{A2}\cos\vartheta_A.$$

Furthermore, the end–effector can rotate about A. Since this is a two parameter motion it must correspond to a two parameter set of points in the image space. This set of image points is a surface, called a *constraint surface*, H. The equation of H is found using equation (28) and the fact that the moving point A is bound to a circle. The general homogeneous equation of this circle is determined as follows: A circle with a centre described by the homogeneous coordinates $(X_c : Y_c : Z)$ and radius r has an equation

$$(X - X_cZ)^2 + (Y - Y_cZ)^2 - r^2Z^2 = 0. \tag{31}$$

Expanded, this becomes

$$X^2 + Y^2 - 2XX_cZ - 2YY_cZ + X_c^2Z^2 + Y_c^2Z^2 - r^2Z^2 = 0. \tag{32}$$

We can set

$$\begin{aligned} C_1 &= -X_c \\ C_2 &= -Y_c \\ C_3 &= X_c^2 + Y_c^2 - r^2, \end{aligned}$$

and substitute these constants back into equation (32) to get

$$X^2 + Y^2 + 2C_1XZ + 2C_2YZ + C_3Z^2 = 0. \tag{33}$$

Substituting the expressions for X, Y, Z from equation (28) into equation (33) gives the quadric surface

$$\begin{aligned} H: \quad & 0 = z^2(X_1^2 + X_2^2) + (1/4)[(x^2 + y^2) - 2C_1xz - 2C_2yz + C_3z^2]X_3^2 + \\ & (1/4)[(x^2 + y^2) + 2C_1xz + 2C_2yz + C_3z^2]X_4^2 + (C_1z - x)zX_1X_3 + \\ & (C_2z - y)zX_2X_3 - (y + C_2z)zX_1X_4 + (C_1z + x)zX_2X_4 + (C_2x - C_1y)zX_3X_4. \end{aligned} \tag{34}$$

It is shown in [3] that this constraint surface, which corresponds to a displacement for which one point stays on a circle, is a hyperboloid that contains the points $J_1(1 : i : 0 : 0)$ and $J_2(1 : -i : 0 : 0)$. Any point with $X_3 = X_4 = 0$ can not be mapped to a displacement of the plane. It can be seen from equation (21) that this condition requires $\vartheta = 0°$ and $\vartheta = 180°$ simultaneously. When the other two points B and C are opened in turn, three hyperboloidal surfaces are generated, H_A, H_B, and H_C, which correspond to the complete range of possible motions around the points still connected. The points of intersection of H_A, H_B and H_C represent the positions of the end–effector where its three vertices are on their respective circles. Therefore, these points of intersection constitute the solution(s) to the forward kinematics problem. It is to be noted that three second order surfaces in three dimensional space can intersect in, at most, eight points. However, all hyperboloidal constraint surfaces corresponding to planar displacements with one point that moves on a circle contain the points J_1 and J_2 and hence, these two points are always in the solution set. But, since these points do not correspond to real displacements, they must be disregarded. Therefore, there is a maximum of six real solutions to the forward kinematics problem for manipulators of this type.

	x	y		x	y
A_0	0	0	A	0	0
B_0	8	0	B	3	0
C_0	8	5	C	3	4

$\vartheta_A = 90°$	$l_{A1} = 2$	$l_{A2} = \sqrt{12}$	$r_A = 4$
$\vartheta_B = 90°$	$l_{B1} = 3$	$l_{B2} = 4$	$r_B = 5$
$\vartheta_C = -90°$	$l_{C1} = 4$	$l_{C2} = 3$	$r_C = 5$

Table 1: Input parameters.

3.2 Example

Table 1 gives the coordinates of the base points A_0, B_0, C_0 in the fixed frame Σ with origin at A_0, the coordinates of the end–effector vertices A, B, C in the moving frame E, with origin at vertex A, along with the input angles ϑ_i, fixed link lengths l_{ij}, and radii r_i.

Substituting the data from Table 1 into equation (34) gives the following three constraint surfaces in the image space:

$$H_A: \quad X_1^2 + X_2^2 - 4X_3^2 - 4X_4^2 = 0 \tag{35}$$

$$H_B: \quad X_1^2 + X_2^2 - 11X_1X_3 - 5X_2X_4 + 24X_3^2 = 0 \tag{36}$$

$$H_C: \quad X_1^2 + X_2^2 - 11X_1X_3 - 9X_2X_3 + X_1X_4 - 5X_2X_4 + 17X_3X_4 + \frac{177}{4}X_3^2 + \frac{1}{4}X_4^2 = 0. \tag{37}$$

Since X_4 is the homogenising coordinate, its value is arbitrary, hence it is set $X_4 = 1$. The set of three equations $H_A = 0, H_B = 0, H_C = 0$ can now be solved for the variables X_1, X_2, X_3. The following solutions are obtained:

$$\begin{aligned}
S_1: &\quad X_1 = -2.742268, X_2 = 0.071634, X_3 = -0.938771 \\
S_2: &\quad X_1 = -1.416436, X_2 = 1.448375, X_3 = 0.161308 \\
S_3: &\quad X_1 = 1.741723 - 0.494410i, X_2 = 1.290283 + 0.721925i, X_3 = -0.105133 - 0.166950i \\
S_4: &\quad X_1 = 1.741723 + 0.494410i, X_2 = 1.290283 - 0.721925i, X_3 = -0.105133 + 0.166950i \\
S_5: &\quad X_1 = 1.873596 + 0.128223i, X_2 = -0.748438 + 0.720142i, X_3 = 0.191145 - 0.390727i \\
S_6: &\quad X_1 = 1.873596 - 0.128223i, X_2 = -0.748438 - 0.720142i, X_3 = 0.191145 + 0.390727i.
\end{aligned}$$

Thus, there are two real solutions, two sets of complex conjugate solutions for a total of six solutions, as expected, since two of the possible eight are the solutions J_1 and J_2. Back substitution

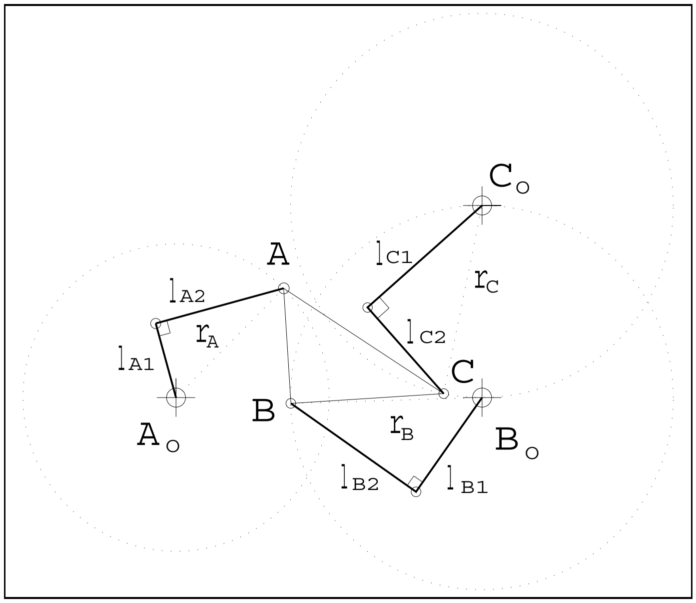

Figure 3: Solution 1.

of the solutions into equations (35), (36), and (37) verifies the two real solutions. The position and orientation of the end–effector corresponding to each of these solutions in terms of the displacement parameters a, b, and ϕ can be found by substituting the solutions for X_1, X_2, X_3, along with $X_4 = 1$ into equations (29). The subsequent two sets of displacement parameters are given in Table 2. The two real solutions are illustrated in Figures 3 and 4.

	Sol'n 1	Sol'n 2
a	2.812957	2.377911
b	2.843813	3.216448
ϕ (deg.)	-86.382243	18.326665

Table 2: Two real positions and orientations.

4 Conclusions

Grünwald's kinematic mapping for planar displacements has been presented. An important application of this mapping is the solution of the forward kinematics problem of planar Stewart–Gough–type manipulators.

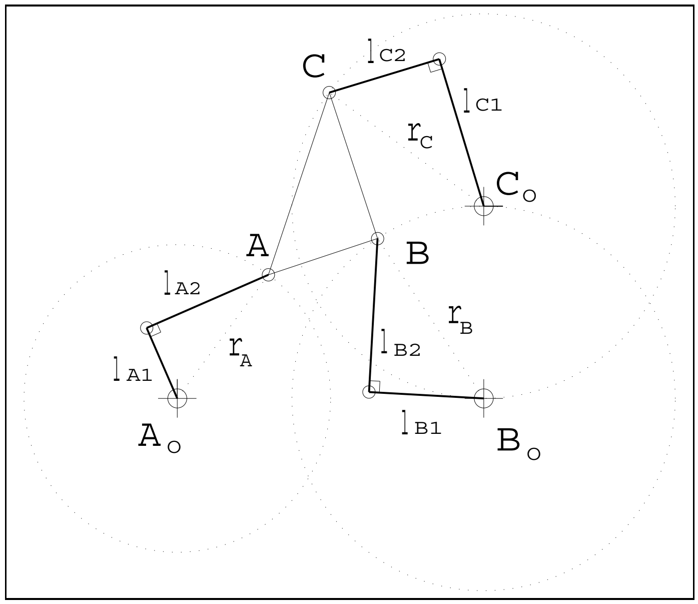

Figure 4: Solution 2.

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