SOLVING THE FORWARD KINEMATICS OF A PLANAR 3-LEGGED PLATFORM WITH HOLONOMIC HIGHER PAIRS

M.J.D. Hayes[1], M.L. Husty[2], and P.J. Zsombor-Murray[1]

[1] McGill University, Dep't. of Mech. Eng. & Centre for Intelligent Machines, 817 r. Sherbrooke O., Montréal, Québec, Canada.
[2] Montan University Leoben, Inst. of Applied Geometry, Fr. Josefstr.18, A-8700 Leoben, Austria.
johnh@cim.mcgill.ca, husty@unileoben.ac.at, paul@cim.mcgill.ca

Abstract. In this paper kinematic mapping is used to solve the forward kinematic problem of a planar parallel 3-legged platform with holonomic higher pairs. The end effector is a circular disk which rolls without slip along the straight lines of the non–grounded rigid links of each of three 2R legs. The R-pair joining the grounded and non-grounded link in each leg is called a knee joint. The straight lines and circular disk are modelled as three racks on a common pinion. One notes that motions of a planar rigid body wherein a point moves on a circle maps to a hyperboloid in a 3–dimensional projective kinematic image space. However, the geometry of this manipulator does not easily reveal such points. Let the joint inputs be the change in arclength on each rack. For every input set, the knee joints determine the vertices of a triangle which we call the virtual platform. Clearly, these vertices move on circles. To express the motion of the knee joints in the pinion reference coordinate system in terms of the input parameters, fix the pinion coordinate system and observe that the relative motion of a rack with respect to the pinion is a Frenet-Serret motion with the initial contact point moving on an involute of the pinion. The link lengths and the initial assembly configuration are all known, hence a complete description of the motion of the knee joint is obtained. This one parameter set of positions for each of the three knee joints can then be used as inputs for the kinematic mapping, which reduces the problem to determining the intersections of three hyperboloids. A numerical example is given.

1. Introduction

It has recently been shown that kinematic mapping has important applications in planar robot kinematics [6, 3]. The goal of this paper is to present a practical solution procedure for the forward kinematics (FK) problem of a planar Stewart–Gough (SG) type platform with holonomic higher pairs. This procedure uses kinematic mapping and the fact that displacements with one point bound to a circle map to hyperboloidal surfaces in the image space. Furthermore, this mapping is independent of the geometry of the platform.

A kinematic mapping procedure to solve the FK of a higher pair jointed SG type platform was used in [3], but it assumed a priori knowledge of the platform orientation. This requirement can render the solution procedure somewhat impractical. Algebraic approaches were successfully used in [8, 2] to obtain the FK solutions of lower pair jointed planar three legged platforms. But, these procedures require that the platform geometry be constant, i.e., the platform attachment points remain at a fixed distance relative to each other. This is not the case for manipulators with higher pairs of the type considered here. The planar manipulator, shown in Fig. 1, consists of three closed kinematic chains. The disk, modelled as a pinion gear, rolls without slip on each of the three racks tangent to it. The rolling constraints are holonomic due to the pure rolling and because the motion is planar, hence the constraint equations can be expressed in terms of displacement, i.e., in *integral form*. Each of the three legs connect a rack to a base point via two revolute (R) pairs. The leg links are rigid and a rack is rigidly attached to the disk end of each second link. The R–pairs connecting two links in a leg shall be referred to as *knee joints* A, B and C, and are constrained to move on circles centred on the three base points A_0, B_0, C_0, which are fixed to a rigid base. Joint, link parameters, vectors and transformations are identified by left and right sub and superscripts. The generic parameter

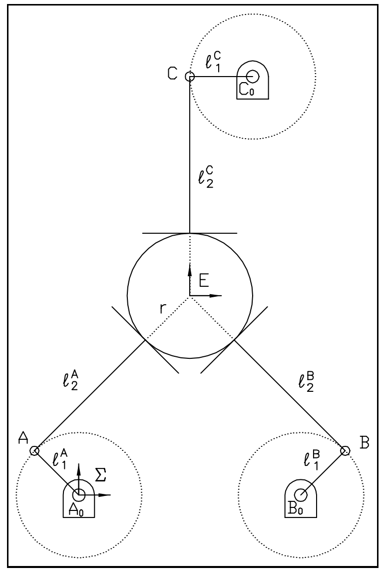

Fig. 1 Planar platform.

$$^{k}\Psi_i^j$$

is identified as follows: the right sub–script i, $i \in \{1, 2, 3\}$ identifies the joint number. For each manipulator leg, the joint number at the connection between the first link and the base is 1. Between the first and second links is 2. The higher pair between link 2 and the disk is 3. The right super–script, j, $j \in \{A, B, C\}$ denotes a particular manipulator leg. The left super–script, k, $k \in \{\Sigma, 0, 1, 2, E\}$ refers to the reference frame in which the variable is represented. The left super–script is omitted for frame invariant parameters. For instance, the scalar quantity l_i^j is the length of link i in leg j.

The reference frame E has its origin on the disk centre and moves with the disk. Frame Σ has its origin at the base of leg A and is fixed. In the home, or zero position shown in Fig. 1, the basis directions of Σ and E are parallel. By changing the location of the contact point between the rack and disk, the disk end–effector can be brought to any position with any desired orientation within the physical limits of its workspace. Thus, the manipulator has three degrees of freedom.

2. A Kinematic Mapping of Planar Displacements

A general displacement in the plane requires three independent coordinates to fully characterise it. It is convenient to think of the relative planar motion between two rigid bodies as the motion of a Cartesian reference coordinate system E attached to one of the bodies, with respect to the Cartesian coordinate system Σ attached to the other, [1]. Without loss of generality, Σ may be considered as fixed while E is free to move. Then the position of a point in E relative to Σ can be given by the homogeneous linear transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{1}$$

where $(x/z, y/z)$ are the Cartesian coordinates of a point in E, $(X/Z, Y/Z)$ are those of the same point in Σ. (a, b) are the Cartesian coordinates of the origin of E measured in Σ, and ϕ is the rotation angle measured from the X–axis to the x–axis, the positive sense being counter–clockwise.

All general planar displacements may be represented by a single rotation through a finite angle about a fixed axis normal to the plane. The piercing point of this axis is the *pole* of the displacement. If frames E and Σ are initially coincident then the pole is the unique point whose coordinates, in both E and Σ, remain unaltered by a given displacement of E. The mapping used here takes a displacement pole point, given by the homogeneous point coordinates $(X_p : Y_p : Z_p)$, to a point $(X_1 : X_2 : X_3 : X_4)$ in a 3–dimensional homogeneous projective image space, Σ'. It is expressed as [1, 6, 7]:

$$(X_1 : X_2 : X_3 : X_4) = (X_p : Y_p : Z_p : \tau Z_p), \tag{2}$$

where

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) &\neq (0 : 0 : 0 : 0) \\ \tau &= \cot(\phi/2) \\ 0 \leq \phi &< 2\pi. \end{aligned}$$

The image of the pole coordinates of the displacement under the kinematic mapping is called the *image point*. Distinct displacements have unique image points. The image point is given by

$$\begin{aligned} (X_1 : X_2 : X_3 : X_4) = {} & [(a\sin(\phi/2) - b\cos(\phi/2) : \\ & (a\cos(\phi/2) + b\sin(\phi/2) : \\ & 2\sin(\phi/2) : 2\cos(\phi/2)] \end{aligned} \tag{3}$$

By virtue of the relationships expressed in equation (3), the transformation matrix from equation (1) may be expressed in terms of the homogeneous coordinates of the image space, Σ'. This means that we now have a linear transformation to express a displacement of E with respect to Σ in terms of the image point:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (X_4^2 - X_3^2) & -2X_3X_4 & 2(X_1X_3 + X_2X_4) \\ 2X_3X_4 & (X_4^2 - X_3^2) & 2(X_2X_3 - X_1X_4) \\ 0 & 0 & (X_4^2 + X_3^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (4)$$

Since equation (4) is a linear transformation, for each unique displacement described by (a, b, ϕ) there is a corresponding point in the image space. So, the inverse mapping can be obtained from equation (3). For a given point of the image space, the displacement parameters are:

$$\begin{aligned} \tan(\phi/2) &= X_3/X_4 \\ a &= 2(X_1X_3 + X_2X_4)/(X_3^2 + X_4^2) \\ b &= 2(X_2X_3 - X_1X_4)/(X_3^2 + X_4^2). \end{aligned} \quad (5)$$

Clearly, any image point with $X_3 = X_4 = 0$ does not represent a displacement of E. From equation (5), this condition renders ϕ indeterminate and places a and b on the line at infinity.

The ungrounded R–pair in a 2R mechanism is constrained to move on a circle with a fixed centre. The image points that correspond to all possible displacements of the ungrounded link with respect to a fixed reference frame constitute a quadric surface. It can be derived as follows: the equation of the circle with radius r centred on the homogeneous coordinates $(X_c : Y_c : Z)$ is of the form

$$(X - X_cZ)^2 + (Y - Y_cZ)^2 - r^2Z^2 = 0. \quad (6)$$

Expanded, this becomes

$$X^2 + Y^2 - 2XX_cZ - 2YY_cZ + X_c^2Z^2 + Y_c^2Z^2 - r^2Z^2 = 0. \quad (7)$$

Setting

$$\begin{aligned} C_1 &= -X_c, \\ C_2 &= -Y_c, \\ C_3 &= (C_1^2 + C_2^2 - r^2) = (X_c^2 + Y_c^2 - r^2), \end{aligned}$$

yields

$$X^2 + Y^2 + 2C_1XZ + 2C_2YZ + C_3Z^2 = 0. \quad (8)$$

Substituting the expressions for X, Y, Z from equation (4) into equation (8) produces the quadric surface equation

$$H: \quad 0 = z^2(X_1^2 + X_2^2) + (1/4)[(x^2 + y^2) - 2C_1xz - 2C_2yz + C_3z^2]X_3^2 + (1/4)[(x^2 + y^2) + 2C_1xz + 2C_2yz + C_3z^2]X_4^2 + (C_1z - x)zX_1X_3 + (C_2z - y)zX_2X_3 - (y + C_2z)zX_1X_4 + (C_1z + x)zX_2X_4 + (C_2x - C_1y)zX_3X_4. \tag{9}$$

3. An Application to the FK Problem

The FK problem is conventionally expressed as a transformation of the position and orientation of the end effector from a joint space representation to a Cartesian space representation. That is, given a set of n joint variables, one per degree of freedom, determine the position and orientation of the end effector with respect to a non–moving reference coordinate system. To employ the same mapping procedure used in [6, 3], platform points which move on circles are required. The only such points are on the knee joints, A, B, and C (see Fig. 1). In this case the disk is the platform. In order to use the knee joints as platform points consider a *virtual platform* (VP), formed by the triangle whose vertices are the three knee joints expressed in the disk frame E (see Fig. 2). For a given assembly configuration, these *virtual platform points* (VPP) are fixed relative to each other, but change from pose to pose. Hence, the VP geometry changes continuously during platform motion, but for any given displacement the VP can be considered a rigid body.

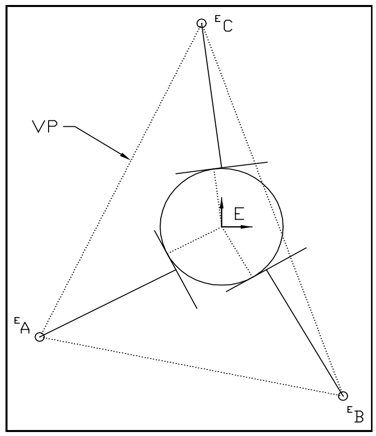

Fig. 2. A VP.

It is important to note that the pure rolling nature of the higher pairs make the manipulator in Fig. 1 markedly different from lower pair jointed SG platforms because the pure rolling condition renders FK solutions completely dependent on the initial assembly configuration (IAC). Moreover, the FK analysis cannot be reduced to the lower pair SG case because there exists no such equivalent mechanism which can exactly reproduce a rack–and–pinion motion (see Hunt, p.106 [4]). Hence, the methods in [2, 8] cannot be used.

3.1 Involute Inputs

We select as our three variable joint inputs the change in arclength along the rack, or disk, due to the change of the initial contact point. They are given by the three numbers $\Delta d_3^j = r\Delta t^j$. The Δt^j are the angles between the initial and final rack positions and r is the disk radius. Since the racks are always in tangential contact with the disk, the change in these angles represent the change in angle of disk tangents. Because of the orthogonal bases, $\Delta t = \Delta n$, the change in tangent angle is the same as the change in normal angle.

Initially, consider only leg A in Fig. 3. Observe that the knee joint A, which has a fixed position in the reference frame, R, attached to the rack, moves on a circle. But, it also has a relative motion in the moving disk frame, E. What is required is a description of that motion in terms of the joint inputs. This turns out to be straightforward: fix the the disk and observe that the relative motion of the rack with respect to E is pure rolling. The rolling is a Frenet–Serret motion ([1], p. 301) with the original contact point moving on an involute of the disk. This gives a complete description of the motion of the knee joints with respect to E. We can now track the motion of this point in both E and Σ respectively. This one parameter set of knee joint positions can be used as an input for the kinematic mapping because they determine the VPP. Due to their positional dependence on the involute, we define these knee joint positions as *involute inputs*.

However, three involute inputs are required. The motion of the knee joints of the remaining two legs must be the same type as that of leg A relative to E, but the starting points of the involutes are different. This is achieved by left multiplying the transformation matrix with the inverse of the rotation which has to take place to attain the leg's new point of contact, assuming that all positions can be transformed back to the IAC shown in Fig. 1. For every set

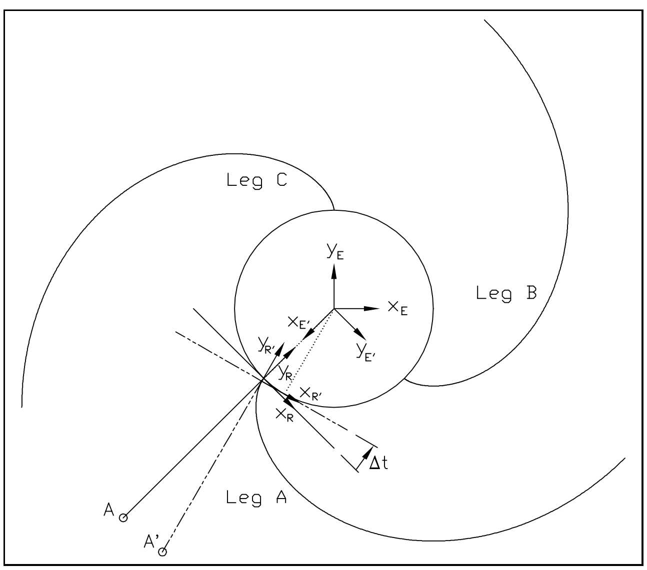

Fig. 3. Reference systems in leg A after a rotation Δt.

of three joint input parameters one obtains a set of three VPP expressed in E. Now the kinematic mapping from [6, 3] becomes applicable.

To obtain the solutions for a given set of inputs, begin by removing the disk connections with legs B and C. Observe that the higher pairs are locked in the corresponding VP configuration by virtue of the specified input parameters. That is, there can be no relative motion between the disk and the rack because that would change the relative positions of the knee joints. The knee joint, ${}^{E}A$, is constrained to move on a circle with centre A_0 and radius l_1^A. Furthermore, the VP can rotate about ${}^{E}A$. All poses of this virtual rigid body correspond to the image points on the constraint hyperboloid, H, given by equation (9).

When the other two points B and C are analysed in turn, three hyperboloidal surfaces are generated, H_A, H_B, and H_C, which correspond to the complete range of possible displacements around the points still connected. The points of intersection of H_A, H_B, and H_C represent the positions of the VP where its three knee joints are on their respective circles. Therefore, these points of intersection constitute the solution(s) to the FK problem. However, it is shown in [1] that all such hyperboloids contain the isotropic points $J_1(1 : i : 0 : 0)$ and $J_2(1 : -i : 0 : 0)$ of the kinematic image space, which correspond to no real displacement. These two points are, therefore, always in the solution set and must be discarded. Thus, there are a maximum of six real solutions to the FK problem for manipulators of this type in general, which confirms result established in [5].

Fig. 3 shows the reference coordinate systems used to transform the position of the knee joint from the moving rack reference frame, R, to the relatively fixed pinion frame, E. The origin of R moves along the involute. R' gives the new position of R after a rotation Δt. E' is an intermediate relatively fixed system. It is rotated from E through $\theta^j = (5\pi/4), (7\pi/4), (\pi/2)$ for $j \in \{A, B, C\}$.

Examining Fig. 3, it can be seen that for each leg the required transformations to take the position of the knee joint of leg j in frame R'^j to frame E are

$$
\begin{aligned}
{}^{E}\mathbf{T}^{j}_{R'} &= {}^{E}\mathbf{T}^{j}_{E'}\,{}^{E'}\mathbf{T}^{j}_{R'} \\
&= \begin{bmatrix} c\theta^j & -s\theta^j & 0 \\ s\theta^j & c\theta^j & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s\Delta t & -c\Delta t & r(c\Delta t + \Delta t s\Delta t) \\ c\Delta t & -s\Delta t & r(s\Delta t - \Delta t c\Delta t) \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

where $c = \cos$, and $s = \sin$.

The geometrical significance of ${}^{E'}\mathbf{T}^{j}_{R'}$ is seen when the individual columns are examined. The first column is the direction of the disk tangent in E' (the direction of the x–axis of frame R'). The second column is the direction in E' (towards the centre) of the normal at the new contact point. The third column is the position of the origin of frame R' on the involute, expressed again in

E'. The remaining transformation, ${}^E\mathbf{T}^j_{E'}$, depends on the angle between the x–axis of frame E and the rack normal in the home position.

Without loss in generality, all positions of each knee joint may use the involutes of the disk generated by the motions of the origins of the R^j to express knee joint loci in E. For the manipulator shown in Fig. 1, all knee joints have the same coordinates in their respective R^j frames:

$$ {}^R\boldsymbol{j} = \begin{bmatrix} 0 \\ -l_2^j \\ 1 \end{bmatrix}. $$

Once the arclength parameters (joint inputs), Δt^j, are given, the coordinates of the knee joints (involute inputs) in frame E, ${}^E\boldsymbol{j}$, are easily determined by left multiplying the ${}^{R'}\boldsymbol{j}$ with the appropriate ${}^E\mathbf{T}^j_{R'}$,

$$ {}^E\boldsymbol{j} = {}^E\mathbf{T}^j_{R'}\,{}^{R'}\boldsymbol{j}. \tag{10} $$

4. Example

Table 1 gives the coordinates of the base points A_0, B_0, C_0 in the fixed frame Σ with origin at A_0, the arclength parameters, and the corresponding knee joint positions in E (VPP), given by equation (10). The link lengths, in generic units, are: $r = 4$, $l_1^j = 4$, $l_2^j = 10$, and the IAC are ${}^En_0^j = (225°, 315°, 90°)$, $d_{30}^j = (0,0,0)$, ${}^0\vartheta_{10}^j = (135°, 45°, 180°)$, and ${}^1\vartheta_{20}^j = (270°, 90°, 90°)$, where $j \in \{A, B, C\}$. Note, in Fig. 1 the link reference frames are not shown in order to avoid clutter. These frames were assigned using the Denavit–Hartenberg convention.

Table 1. Base points in Σ, joint inputs, and VPP.

j	${}^\Sigma x$	${}^\Sigma y$	Δt^j	${}^E x$	${}^E y$
A	0	0	$-17.5°$	-11.85401931	-7.548168766
B	$10\sqrt{2}$	0	$-15°$	7.906899696	-11.60075686
C	$5\sqrt{2}+4$	$9\sqrt{2}+14$	$7.5°$	-1.308247378	13.94857141

Setting $z = 1$ in equation (9), which can always be done as no practical VP will have a vertex on the line at infinity $(x : y : 0)$, and then substituting the VPP from Table 1 into equation (9) gives expressions for the three constraint hyperboloids in the image space. X_4 is the homogenising coordinate in the image space, hence $X_4 = 0$ represents the plane at infinity, but also corresponds to VP rotations of $\phi = 180°$. Therefore, $X4 = 0$ is a practical concern, unlike the case of $z = 0$. However, this condition gives only the solutions J_1 and J_2. After having checked this, we can normalise the image space homogeneous coordinates by setting $X_4 = 1$. Solving the resulting set of three equations, $H_A = 0, H_B = 0, H_C = 0$, for X_1, X_2, X_3 gives two real and two pairs of complex conjugate solutions. The two real solutions are:

$$S_1: \quad X_1 = -4.724652386, X_2 = 4.561069802, X_3 = -0.05146192114,$$
$$S_2: \quad X_1 = -5.754360118, X_2 = 4.906081896, X_3 = 0.03244152899.$$

Note that the solution set always contains an even number of real solutions because complex solutions arise in conjugate pairs. Fig. 4 is a view of the resulting hyperboloids where one of the intersections is visible. The position and orientation of the disk corresponding to each real solution in terms of the displacement parameters (a, b, ϕ) can be found by substituting the solutions for X_1, X_2, X_3, along with $X_4 = 1$ into the set of equations (5). The resulting pair of real displacement parameters are given in Table 2.

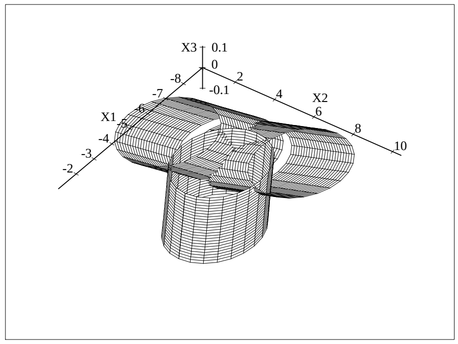

Fig. 4. The constraint hyperboloids .

Table 2. Displacement parameters.

	a	b	ϕ (deg.)
Sol'n 1	9.583039940	8.956143130	-5.891904208
Sol'n 2	9.428879858	11.81460751	3.716222033

Expressed relative to the disk frame, E, the inputs in Table 1, whether given as Δt^j, or Δd_3^j, reveal the geometry of the VP shown in Fig. 2. The origin of E is on the disk centre. Once the orientation and position of the VP, and hence E, are known it is a simple matter of plane trigonometry to determine the relative link angles for the assembly configuration that correspond to the solution. Fig. 5 illustrates the two real solutions, where the vertices of the VP are on their respective circles.

5. Conclusions

A solution for the FK problem of platforms of the type in Fig. 1 using kinematic mapping has been presented. The involute inputs to the mapping are the knee joint positions expressed as one parameter motions of initial rack contact points along involutes of the disk. In the example, only two real solutions were found. However, there can be as many as three pairs. This solution is fundamental to any further investigation of this type of platform.

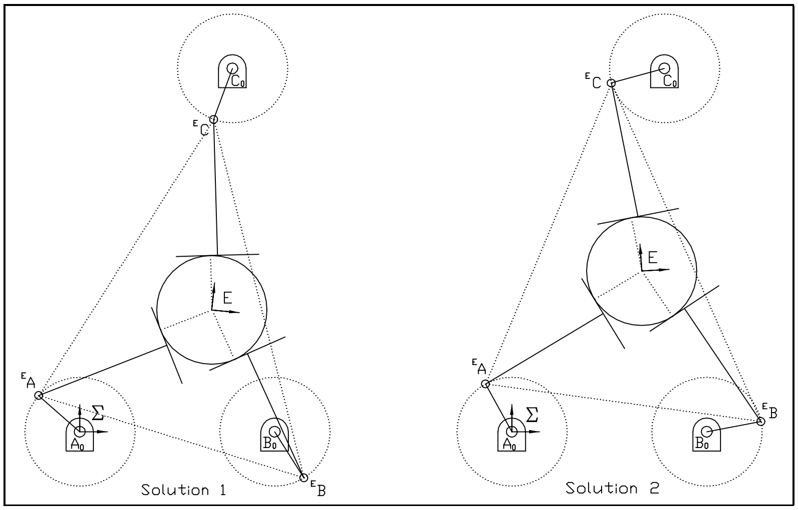

Fig. 5. The two real solutions.

References

1. Bottema, O. & Roth, B., *Theoretical Kinematics*, Dover Publications, Inc., New York, 1990.
2. Gosselin, C., Sefrioui, J., "Polynomial Solutions for the Direct Kinematics Problem of Planar Parallel Three–Degree–of–Freedom Parallel Manipulators", *Proc. 5th Int. Conf. on Adv. Rob. (ICAR)*, Pisa, Italy, pp. 1124–1129, 1991.
3. Hayes, M.J.D., Zsombor–Murray, P., "Kinematic Mapping of 3–legged Planar Platforms With Holonomic Higher Pairs ", *Recent Advances in Robotic Kinematics*, eds. Lenarčič, J., Parenti–Castelli, V., Kluwer Academic Publishers, Dordrecht, pp. 421–430, June 1996.
4. Hunt, K.H., *Kinematic Geometry of Mechanisms*, Clarendon Press, Oxford, England, 1978.
5. Hunt, K.H., 1983, "Structural Kinematics of In–Parallel–Actuated robot arms", *ASME J. of Mech., Trans. and Automation in Design*, Vol. 105, No. 4, pp. 705–712.
6. Husty, M.L., "Kinematic Mapping of Planar Three–Legged Platforms", *Proc. 15th Canadian Congress of Applied Mechanics, CANCAM 1995*, Vol. 2, pp. 876-877, 1995.
7. De Sa, S., Roth, B., "Kinematic Mappings. Part 1: Classification of Algebraic Motions in the Plane", ASME, *J. of Mech. Design, Vol. 103, pp. 585-591*, July, 1981.
8. Wohlhart, K., "Direct Kinematic Solution of the General Planar Stewart Platform", *Proc. of the Int. Conf. on Comp. Int. Manu.*, Zakopane, pp. 403–411, March, 1992.