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THE EFFECT OF DATA-SET CARDINALITY ON THE DESIGN AND STRUCTURAL ERRORS OF FOUR-BAR FUNCTION-GENERATORS

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1 Introduction

Design and structural errors are important performance indicators in the assessment and optimisation of function-generating linkages arising by means of approximate synthesis. The design error indicates the error residual incurred by a specific linkage regarding the verification of the synthesis equations. The structural error, in turn, is the difference between the prescribed linkage output and the actual generated output for a given input value [Tinubu and Gupta 1984]. From a design point of view it may be successfully argued that the structural error is the one that really matters, for it is directly related to the performance of the linkage.

The main goal of this paper is to demonstrate that, as the data-set cardinality increases, the Euclidean norms of the design and structural errors converge. The important implication is that the minimisation of the Euclidean norm of the structural error can be accomplished indirectly via the minimisation of the corresponding norm of the design error, provided that a suitably large number of input-output (I/O) pairs is prescribed. Note that the minimisation of the Euclidean norm of the design error leads to a linear least-square problem whose solution can be obtained directly [Wilde 1982], while the minimisation of the same norm of the structural error leads to a nonlinear least-squares problem, and hence, calls for an iterative solution [Tinubu and Gupta 1984].

2 Procedure

The synthesis problem of four-bar function-generators consists of determining all relevant design parameters such that the mechanism can produce a prescribed set of m input-output (I/O) pairs, $\{\psi_i, \phi_i\}_1^m$, where ψ_i and ϕ_i represent the i^{th} input and output variables, respectively, and m is the cardinality of the data-set.

Let n be the number of independent design parameters required to characterise the mechanism. For planar $RRRR$ linkages, $n = 3$ [Freudenstein 1955], while for spherical $RRRR$ linkages $n = 4$ [Hartenberg and Denavit 1964]. For spatial $RCCC$ function-generators, the issue is not as straightforward. The output of this type of linkage consists of both angular and translational displacements, although they are coupled. If we only consider the angular output, which is necessary if comparisons are to be made with the other two for generating identical functions, then $n = 4$ [Liu 1993].

Approximate synthesis problems involve sets of I/O equations such that $m > n$. If $m = n$, the problem is termed *exact synthesis* and may be considered a special case of the former [Liu and Angles 1992]. The optimisation problem of four-bar function-generators usually involves the approximate solution of an overdetermined linear system of equations with the minimum error. The I/O equations can be written in the form

$$\mathbf{Sk} \;=\; \mathbf{b}, \tag{1}$$

where $\mathbf{S}$ is the $m \times n$ *synthesis matrix*, $\mathbf{b}$ is an m-dimensional vector, whereas $\mathbf{k}$ is the n-dimensional vector of design variables, usually called the *Freudenstein parameters* as they were first introduced in [Freudenstein 1955] for the synthesis of planar four-bar linkages. Moreover, the i^{th} row of $\mathbf{S}$, $\mathbf{s}_i^T$, and the i^{th} component of $\mathbf{b}$, b_i, are functions of ψ_i and ϕ_i only. For the planar $RRRR$ mechanism:

$$\mathbf{s}_i^T \;=\; [\; 1 \quad \cos\phi_i \quad -\cos\psi_i \;], \quad i = 1, ..., m, \tag{2}$$

$$b_i \;=\; [\; \cos(\psi_i - \phi_i) \;], \quad i = 1, ..., m, \tag{3}$$

$$\mathbf{k} \;=\; [\; k_1 \quad k_2 \quad k_3 \;]^T. \tag{4}$$

For the spherical $RRRR$ mechanism:

$$\mathbf{s}_i^T \;=\; [\; 1 \quad -\cos\phi_i \quad \cos\psi_i \quad \cos\phi_i \cos\psi_i \;], \quad i = 1, ..., m, \tag{5}$$

$$b_i \;=\; [\; -\sin\psi_i \sin\phi_i \;], \quad i = 1, ..., m, \tag{6}$$

$$\mathbf{k} \;=\; [\; k_1 \quad k_2 \quad k_3 \quad k_4 \;]^T. \tag{7}$$

For the spatial $RCCC$ mechanism:

$$\mathbf{s}_i^T \;=\; [\; 1 \quad \sin\phi_i \quad \sin\psi_i \quad \sin\phi_i \sin\psi_i \;], \quad i = 1, ..., m, \tag{8}$$

$$b_i \;=\; [\; \cos\psi_i \cos\phi_i \;], \quad i = 1, ..., m, \tag{9}$$

$$\mathbf{k} \;=\; [\; k_1 \quad k_2 \quad k_3 \quad k_4 \;]^T. \tag{10}$$

These synthesis equations are linear in the components of $\mathbf{k}$. This matrix form has obvious representational advantages, but more importantly, it allows us to determine values of the I/O dial zeros, α and β, that will best condition the synthesis matrix, $\mathbf{S}$ [Liu and Angeles 1993]. Here, we regard the I/O pairs as a set of incremental angular changes, $\{\Delta\psi_i \; \Delta\phi_i\}_1^m$. The I/O data set is then

$$\psi_i = \alpha + \Delta\psi_i, \quad \phi_i = \beta + \Delta\phi_i \quad i = 1, \ldots, m. \tag{11}$$

The Nelder-Mead *downhill simplex algorithm in multi-dimensions* [Liu and Angeles 1993] is employed to estimate the optimal values for α and β. It should be mentioned that, while changing the dial zeros of the I/O angles improves the condition number, κ, of planar $RRRR$, spherical $RRRR$ and spatial $RCCC$ linkages, this method does not always work for spatial $RSSR$ linkages [Liu and Angeles 1993].

When $m > n$ there is, in general, no $\mathbf{k}$ which will exactly satisfy all the equations. There are two well established indicators to assess the approximation error, namely the design and structural errors. We define the design error vector $\mathbf{d}$ as

$$\mathbf{d} \;\equiv\; \mathbf{Sk} - \mathbf{b}. \tag{12}$$

The Freudenstein parameters, $\mathbf{k}$, may be optimised by minimising the Euclidean norm of $\mathbf{d}$. The scalar objective function is

$$z \;\equiv\; \frac{1}{2}(\mathbf{d}^T \mathbf{W} \mathbf{d}), \tag{13}$$

which must be minimised over $\mathbf{k}$. The scalar quantity $\mathbf{d}^T\mathbf{W}\mathbf{d}$ is the weighted Euclidean norm of $\mathbf{d}$. The matrix $\mathbf{W}$ is a diagonal matrix of positive weighting factors, which can be used to make some of the data points affect the minimisation more, or less, than others, depending on their relative importance to the design. For the sake of simplicity $\mathbf{W}$ will be set equal to the identity matrix in this article, $\mathbf{d}^T\mathbf{W}\mathbf{d}$ being indicated by $\|\mathbf{d}\|_2$.

The quantity $\|\mathbf{d}\|_2$ can be minimised, in a least squares sense, very efficiently by transforming $\mathbf{S}$ using Householder reflections [Golub and Van Loan 1989], the Moore-Penrose generalised inverse thus not being explicitly computed. Design error minimisation is therefore a linear problem; a desirable trait, indeed. Unfortunately, as a performance indicator, the design error is not directly related to the I/O performance of the function-generator.

Alternatively we may approach the optimisation problem by minimising the same norm of the structural error. Since this error is defined as the difference between the generated and prescribed outputs for a given input, it is directly related to function-generator performance. Let the structural error vector $\mathbf{s}$ be defined as

$$\mathbf{s} \equiv \begin{bmatrix} \varphi_1 - \phi_1 \\ \vdots \\ \varphi_m - \phi_m \end{bmatrix}, \tag{14}$$

where φ_i is the generated value of the output ϕ attained at $\psi = \psi_i$, and ϕ_i is, as defined earlier, the prescribed value of the output angle at $\psi = \psi_i$. It can be shown that the structural and design errors are related by

$$\mathbf{d} = \mathbf{d}(\mathbf{s}) \equiv \mathbf{S}\mathbf{k} - \mathbf{b}, \tag{15}$$

where $\mathbf{d}$ is a nonlinear function of $\mathbf{s}$ [Tinubu and Gupta 1984]. Hence, it is evident that minimising $\|\mathbf{d}\|_2$ is not equivalent to minimising the Euclidean norm of the structural error, $\|\mathbf{s}\|_2$.

To minimise the Euclidean norm of this error, the iterative Gauss-Newton procedure is employed. The conditions under which the procedure converges in the neighbourhood of a minimum are discussed in [Dahlquist and Björck 1969]. In this case, the scalar objective function to be minimised over $\mathbf{k}$ is

$$\zeta \equiv \frac{1}{2}(\mathbf{s}^T\mathbf{W}\mathbf{s}). \tag{16}$$

Here, again, $\mathbf{W}$ is set equal to the identity matrix, the weighted Euclidean norm being indicated by $\|\mathbf{s}\|_2$.

We start with an initial guess for the Freudenstein parameters that minimise the Euclidean norm of the design error, and modify the guess until the normality condition,

$$\frac{\partial\zeta}{\partial\mathbf{k}} = \mathbf{0}, \tag{17}$$

is satisfied to a specified tolerance, ϵ, such that

$$\frac{\partial\zeta}{\partial\mathbf{k}} < \epsilon, \quad \text{for } \epsilon > 0. \tag{18}$$

We do not actually evaluate the normal equations, since they are typically ill-conditioned. Rather, we proceed in the following way: the i^{th} I/O equation is a function of ψ_i, ϕ_i and the Freudenstein parameters, $\mathbf{k}$, and may be written as

$$f_i(\psi_i, \phi_i; \mathbf{k}) = 0. \tag{19}$$

The Jacobian of $\mathbf{f}$ with respect to the vector of output values, $\boldsymbol{\phi}$, is the following diagonal matrix:

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\phi}} = \operatorname{diag}\left(\frac{\partial f_1}{\partial \phi_1}, \ldots, \frac{\partial f_m}{\partial \phi_m}\right) \equiv \mathbf{D}. \tag{20}$$

If we regard Eq. (19) as a function of only ϕ_i we can write

$$\boldsymbol{\phi}(\mathbf{k}) = \boldsymbol{\phi}. \tag{21}$$

However, we want

$$\boldsymbol{\phi}(\mathbf{k}) = \boldsymbol{\varphi}. \tag{22}$$

Assume we have an approximation to $\mathbf{k}_{\text{opt}}$, which we call $\mathbf{k}^{\nu}$, obtained from the ν^{th} iteration. We now require a correction vector, $\Delta\mathbf{k}$, so that

$$\boldsymbol{\phi}(\mathbf{k}^{\nu} + \Delta\mathbf{k}) = \boldsymbol{\varphi}. \tag{23}$$

It can be shown [Dahlquist and Björck 1969], after expanding the left-hand side of Eq. (23) in series, and ignoring higher order terms, that

$$\boldsymbol{\phi}(\mathbf{k}^{\nu}) - \boldsymbol{\varphi} = \mathbf{D}^{-1}\mathbf{S}\Delta\mathbf{k}, \tag{24}$$

the left-hand side of Eq. (24) being $-\mathbf{s}^{\nu}$. Now we find $\Delta\mathbf{k}$ as the least-square approximation of Eq. (24). It can be proven that $\Delta\mathbf{k} \approx \mathbf{0}$ implies $\partial\zeta/\partial\mathbf{k} \approx \mathbf{0}$, which means that we can satisfy the normality condition without evaluating it explicitly.

We show with one example below that, as the cardinality m of the data points increases, the design and structural errors converge.

3 Example

We synthesise here a planar *RRRR*, a spherical *RRRR* and a spatial *RCCC* four-bar mechanism to generate a quadratic function for an input range of $0^\circ \le \Delta\psi \le 60^\circ$, namely,

$$\Delta\phi_i = \frac{9\Delta\psi_i^2}{8\pi}. \tag{25}$$

For each mechanism the I/O dial zeros (α, β) are selected to minimise the condition number κ of $\mathbf{S}$ for each data-set [Liu and Angeles 1993]. Then both the design and structural errors are determined for the linkages that minimise the respective Euclidean norms for data-sets with cardinalities of $m = \{10, 40, 70, \text{ and } 100\}$. These results are listed in Tables 1–4. Finally the structural errors, corresponding to $m = 40$, of the linkages that minimise the Euclidean norms of the design and structural errors are graphically displayed in Fig. 1.

Table 1: Results for $m = 10$.

	Planar $RRRR$	Spherical $RRRR$	Spatial $RCCC$
α_{opt} (deg.)	123.8668	43.3182	-46.6817
β_{opt} (deg.)	91.7157	89.5221	-0.4781
κ_{opt}	33.2974	200.5262	200.5262
$\lVert\mathbf{d}\rVert_2$	7.273×10^{-3}	7.60×10^{-4}	7.60×10^{-4}
$\lVert\mathbf{s}\rVert_2$	5.965×10^{-3}	4.17×10^{-4}	4.17×10^{-4}

Table 2: Results for $m = 40$.

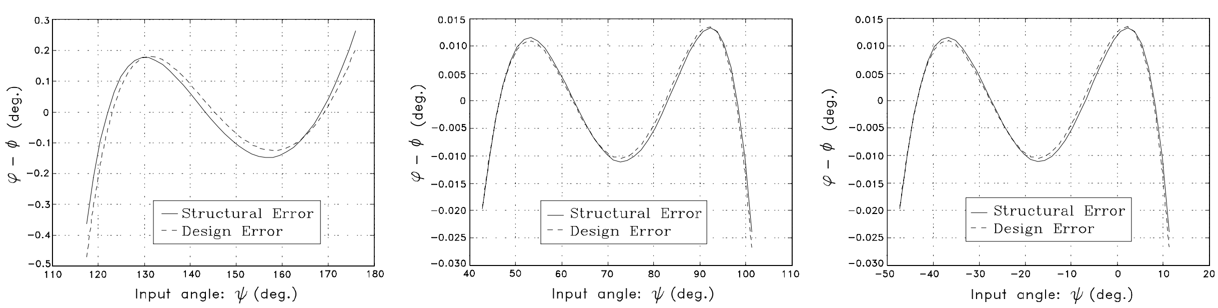
	Planar $RRRR$	Spherical $RRRR$	Spatial $RCCC$
α_{opt} (deg.)	117.4593	42.7696	-47.2301
β_{opt} (deg.)	89.4020	88.8964	-1.1037
κ_{opt}	32.5549	203.0317	203.0317
$\lVert\mathbf{d}\rVert_2$	1.571×10^{-2}	1.887×10^{-3}	1.887×10^{-3}
$\lVert\mathbf{s}\rVert_2$	1.502×10^{-2}	1.057×10^{-3}	1.057×10^{-3}

Table 3: Results for $m = 70$.

	Planar $RRRR$	Spherical $RRRR$	Spatial $RCCC$
α_{opt} (deg.)	116.4699	42.7014	-47.2987
β_{opt} (deg.)	89.0488	88.8045	-1.1956
κ_{opt}	32.5242	204.7696	204.7696
$\lVert\mathbf{d}\rVert_2$	2.088×10^{-2}	2.536×10^{-3}	2.536×10^{-3}
$\lVert\mathbf{s}\rVert_2$	2.040×10^{-2}	1.423×10^{-3}	1.423×10^{-3}

Table 4: Results for $m = 100$.

	Planar $RRRR$	Spherical $RRRR$	Spatial $RCCC$
α_{opt} (deg.)	116.0679	42.6740	-47.3261
β_{opt} (deg.)	88.9057	88.7674	-1.2326
κ_{opt}	32.5170	205.5603	205.5603
$\lVert\mathbf{d}\rVert_2$	2.499×10^{-2}	3.047×10^{-3}	3.047×10^{-3}
$\lVert\mathbf{s}\rVert_2$	2.464×10^{-2}	1.712×10^{-3}	1.712×10^{-3}



(i) *(ii)* *(iii)*

Figure 1. Structural error comparison for *(i)* planar, *(ii)* spherical $RRRR$ and *(iii)* spatial $RCCC$ mechanisms minimising $\lVert\mathbf{s}\rVert_2$ & $\lVert\mathbf{d}\rVert_2$.

4 Discussion and Conclusions

Examining Tables 1–4, it can be seen that $\lVert\mathbf{d}\rVert_2$ and $\lVert\mathbf{s}\rVert_2$ increase with m for each mechanism. The trend for the planar $RRRR$ is towards convergence. It is interesting to note that the error results are identical for the spherical $RRRR$ and the spatial $RCCC$ linkages, except that α_{opt} and β_{opt} are different. In a sense, this is not surprising because of the symmetrical nature of the function in the $\psi - \phi$ plane. Moreover, the synthesis equations for these two linkages are, with the exception of sign, trigonometric complements in the form considered in this article. However, compared to the planar $RRRR$, we see the errors converge near $m = 40$, but then diverge again for higher values of m. Fig. 1 shows the close agreement of the respective structural error curves for $m = 40$. In all cases treated, a number of prescribed I/O values of at least $m = 10$ is sufficient for the minimisation of the Euclidean norm of the design error to lead to the same norm of the structural error within a reasonable difference.

These results support our hypothesis that for a suitably large data-set cardinality linkage optimisation using design and structural error based objective functions result in virtually identical function-generating mechanisms. The obvious weakness is that the cardinality of the data-set for which convergence is obtained is not known *a priori*. Nonetheless, further pursuit of this result is worthwhile because of its computational simplicity.

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