MULTI-MODAL CONTINUOUS APPROXIMATE ALGEBRAIC INPUT-OUTPUT SYNTHESIS OF PLANAR FOUR-BAR FUNCTION GENERATORS

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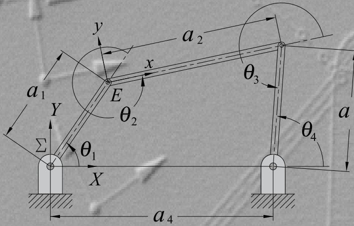
Introduction

- A new method for deriving the IO equations of planar $4R$ mechanisms was needed for the approximate synthesis of function generators for the following reasons.
  - It has been observed that as the cardinality of the prescribed discrete IO data set increases the linkages that minimise the 2-norm of the design and structural errors tend to converge to the same linkage.
    - The design error indicates the error residual incurred by a specific linkage regarding the verification of the synthesis equations.
    - The structural error is the difference between the prescribed linkage output angle and the generated output angle for a prescribed input angle value.
  - The important implication of this observation is that the minimisation of the Euclidean norm of the structural error can be accomplished indirectly via the minimisation of the corresponding norm of the design error, provided that a suitably large number of IO pairs is prescribed.

Introduction Continued

- Minimisation of the Euclidean norm of the design error leads to a linear least-squares problem whose solution can be obtained directly, while the minimisation of the same norm of the structural error leads to a nonlinear least-squares problem, requiring an iterative solution.
- All this has suggested the need to develop *Continuous Approximate Synthesis* (CAS) as an alternative to discrete approximate synthesis.
- CAS involves integrating the synthesis equations between the bounds of the minimum and maximum input angles.
- The resulting need for numerical integrators suggests representing the transcendental Freudenstein equations as algebraic ones using the tangent of the half angle substitutions.

Coordinate Systems and the IO Equations



$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{1}$$

where

$$A = A_1A_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),$$
$$B = B_1B_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4),$$
$$C = C_1C_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),$$
$$D = D_1D_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),$$
$$v_1 = \tan\frac{\theta_1}{2}, \quad v_4 = \tan\frac{\theta_4}{2}.$$

Coordinate Systems and the IO Equations

Since there are six ways to distinctly pair four quantities, there are five additional IO equations:

$$A_1B_1v_1^2v_2^2 + A_2B_2v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \tag{2}$$

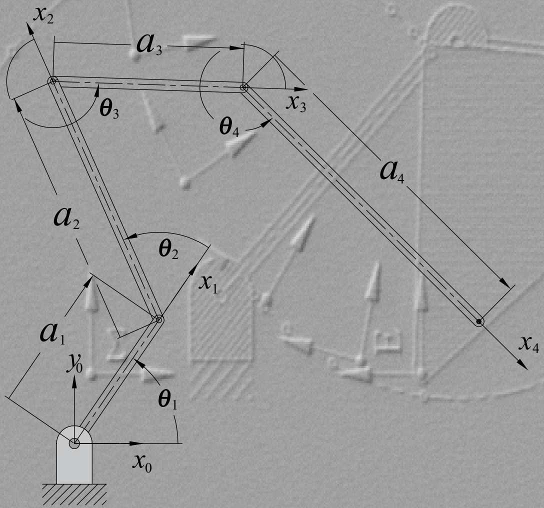
$$A_2B_1v_1^2v_3^2 + A_1B_2v_1^2 + C_1D_1v_3^2 + C_2D_2 = 0, \tag{3}$$

$$B_1C_1v_2^2v_3^2 + A_1D_2v_2^2 + A_2D_1v_3^2 - 8a_1a_3v_2v_3 + B_2C_2 = 0, \tag{4}$$

$$A_1C_1v_2^2v_4^2 + B_1D_2v_2^2 + A_2C_2v_4^2 + B_2D_1 = 0, \tag{5}$$

$$A_2C_1v_3^2v_4^2 + B_1D_1v_3^2 + A_1C_2v_4^2 - 8a_2a_4v_3v_4 + B_2D_2 = 0. \tag{6}$$

The Actual DH Coordinate Systems



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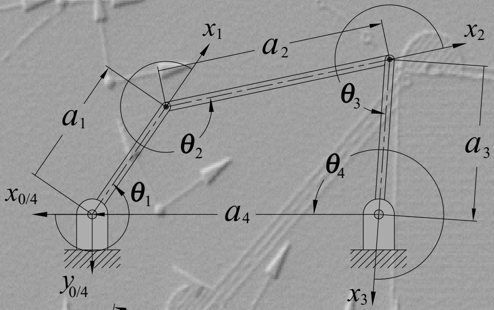
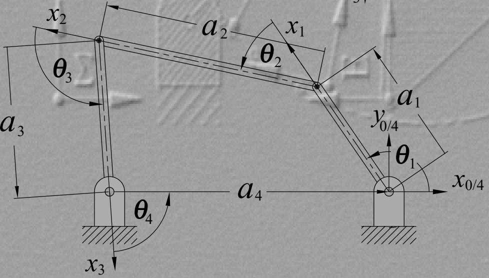
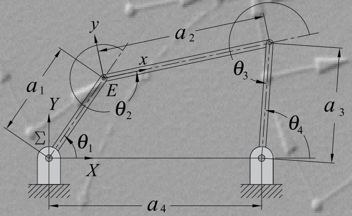
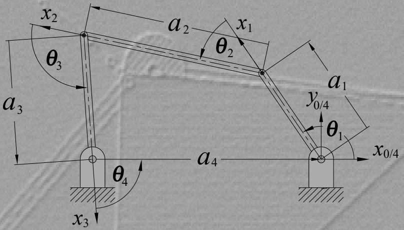
Two Ways to Close the Chain





Introducing The Real Equations





$$Av_1^2v_4^2 + Bv_1^2 + Cv_4^2 - 8a_1a_3v_1v_4 + D = 0, \tag{7}$$

where

$$\begin{aligned}
A &= A_1A_2 = (a_1 - a_2 + a_3 - a_4)(a_1 + a_2 + a_3 - a_4),\\
B &= B_1B_2 = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 - a_4),\\
C &= C_1C_2 = (a_1 - a_2 - a_3 + a_4)(a_1 + a_2 - a_3 + a_4),\\
D &= D_1D_2 = (a_1 - a_2 + a_3 + a_4)(a_1 + a_2 + a_3 + a_4),\\
v_1 &= \tan\frac{\theta_1}{2}, \quad v_4 = \tan\frac{\theta_4}{2}.
\end{aligned}$$

The Real $v_i - v_j$ Equations

The five additional IO equations are also subtly affected:

$$A_1B_2v_1^2v_2^2 + A_2B_1v_1^2 + C_1D_2v_2^2 + 8a_2a_4v_1v_2 + C_2D_1 = 0, \quad (8)$$

$$A_1B_1v_1^2v_3^2 + A_2B_2v_1^2 + C_2D_2v_3^2 + C_1D_1 = 0, \quad (9)$$

$$A_1D_2v_2^2v_3^2 + B_2C_1v_2^2 + B_1C_2v_3^2 - 8a_1a_3v_2v_3 + A_2D_1 = 0, \quad (10)$$

$$A_1C_1v_2^2v_4^2 + B_2D_2v_2^2 + A_2C_2v_4^2 + B_1D_1 = 0, \quad (11)$$

$$A_1C_2v_3^2v_4^2 + B_1D_2v_3^2 + A_2C_1v_4^2 + 8a_2a_4v_3v_4 + B_2D_1 = 0. \quad (12)$$

Continuous Approximate $v_4 = f(v_1)$ Synthesis

Square the v_1-v_4 IO equation, Equation (7) and separate into coefficient array **c** and variable, or *synthesis*, array **s**:

$$\mathbf{c} = \begin{bmatrix} A^2 \\ 2AB \\ B^2 \\ -16Aa_1a_3 \\ -16Ba_1a_3 \\ 2AC \\ 64a_1^2a_3^2 + 2AD + 2BC \\ 2BD \\ -16Ca_1a_3 \\ -16Da_1a_3 \\ C^2 \\ 2CD \\ D^2 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} v_1^4v_4^4 \\ v_1^4v_4^2 \\ v_1^4 \\ v_1^3v_4^3 \\ v_1^3v_4 \\ v_1^2v_4^4 \\ v_1^2v_4^2 \\ v_1^2 \\ v_1v_4^3 \\ v_1v_4 \\ v_4^4 \\ v_4^2 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1^4f(v_1)^4 \\ v_1^4f(v_1)^2 \\ v_1^4 \\ v_1^3f(v_1)^3 \\ v_1^3f(v_1) \\ v_1^2f(v_1)^4 \\ v_1^2f(v_1)^2 \\ v_1^2 \\ v_1f(v_1)^3 \\ v_1f(v_1) \\ f(v_1)^4 \\ f(v_1)^2 \\ 1 \end{bmatrix}.$$

Continuous Approximate $v_4 = f(v_1)$ Synthesis

The infinite cardinality of the IO data set of precision points in a prescribed range, given an input angle and the corresponding output angle that satisfies the prescribed function, can be obtained as definite integration of the synthesis arrays between the respective input angle parameter limits

$$\min_{(a_1,a_2,a_3,a_4)\in\mathbb{R}} \left(\mathbf{c}^T \int_{v_{1_{min}}}^{v_{1_{max}}} \mathbf{s}(v_1, f(v_1))dv_1 \right).$$

The multi-modal, or multi-function, function generation synthesis is

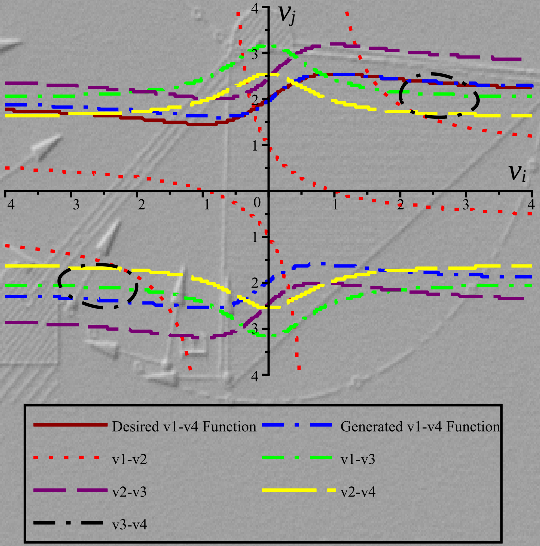
$$\min_{(a_1,a_2,a_3,a_4)\in\mathbb{R}} \left(\mathbf{c}_1^T \int_{v_{1_{min_1}}}^{v_{1_{max_1}}} \mathbf{s}_1(v_1, f_1(v_1))dv_1 + \mathbf{c}_2^T \int_{v_{1_{min_2}}}^{v_{1_{max_2}}} \mathbf{s}_2(v_1, f_2(v_1))dv_1 \right).$$

Multi-modal Synthesis

Let the prescribed $v_4 = f(v_1)$ function be

$$v_4 = f(v_1) = 2+\tan\left(\frac{v_1}{{v_1}^2+1}\right).$$

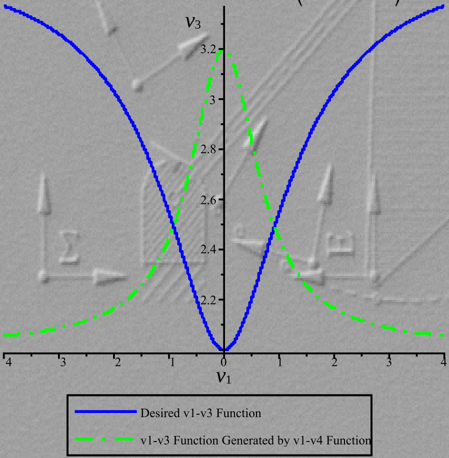
This function exactly generates five additional $v_j = f(v_i)$ functions given the link lengths identified to approximately generate the prescribed function:



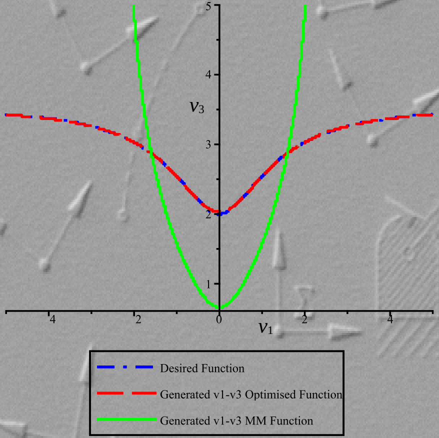
Multi-modal Synthesis

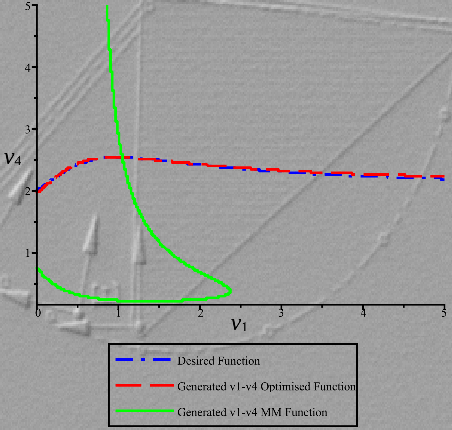
Suppose we additionally wish to generate a different $v_3 = f(v_1)$ function:

$$v_3 = f(v_1) = 2 + \tan\left(\frac{v_1^2}{{v_1}^2 + 1}\right)$$



Multi-modal Synthesis





Multi-modal Synthesis: Polynomial Interpolation

- Clearly, the second, third, fourth, et c., functions need to be constrained with respect to the five precise functions generated by the link lengths that approximately generate the first prescribed function.
- Enter: polynomial interpolation.
- If we wish to generate a different, though heavily constrained, $v_3 = g(v_1)$ function we can specify a generatable function that is an interpolant of the one determined by the specified $v_4 = f(v_1)$ function.
- To do this we arbitrarily elect to use Lagrange polynomial interpolation.
- The first step is to solve the v_1-v_3 IO equation imposed by the generated $v_4 = f(v_1)$ function.
- This yields the exact $v_3 = g(v_1)$ function generated by the identified a_i link lengths that approximately satisfy the specified $v_4 = f(v_1)$ function.

Multi-modal Synthesis: Polynomial Interpolation

- Select n (v_1, v_3) IO pairs from the exact $v_3 = g(v_1)$ function generated by the identified a_i to use as inputs for the Lagrange polynomial interpolation formula.
- In general, this method takes the n points in an arbitrary x-y plane, with no two x_i the same and returns a polynomial of degree at most

$$d \leq n - 1.$$

- The Lagrange polynomial interpolant is a linear combination

$$L(x) = \sum_{i=1}^{n} y_i l_i(x)$$

of Lagrange basis polynomials

$$l_i(x) = \prod_{\substack{1 \leq m \leq n \\ m \neq i}} \frac{x - x_m}{x_i - x_m} = \left(\frac{x - x_1}{x_i - x_1}\right)\left(\frac{x - x_2}{x_i - x_2}\right)\cdots\left(\frac{x - x_n}{x_i - x_n}\right)$$

Multi-modal Synthesis: Planar 4R Example

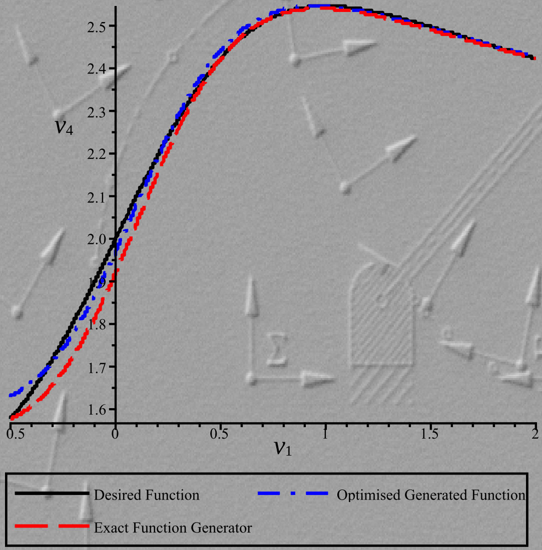
- The primary function we wish to generate with a rigid planar 4R closed kinematic chain is

$$v_4 = 2 + \tan\left(\frac{v_1}{v_1^2 + 1}\right),$$
$$-\frac{1}{2} < v_1 < 2.$$

- The continuous approximate synthesis algorithm described earlier converges to the link lengths:

Link length	a_1	a_2	a_3	a_4
Rational	$-\frac{13077}{78259}$	$\frac{45079}{42170}$	$\frac{27203}{20556}$	$\frac{101727}{110482}$
Floating point	-0.167098992	1.068982689	1.323360576	0.920756322
Normalised	-0.1814801460	1.160983273	1.437253857	1

Multi-modal Synthesis: Planar 4R Example



Structural error

Exact synthesis:
0.024159094

Continuous synthesis:
-0.002471306

Multi-modal Synthesis: Secondary Function

- The corresponding v_1-v_3 function exactly generated by the identified link lengths is obtained from the v_1-v_3 IO equation using the a_i from the $v_4 = f(v_1)$ continuous approximate synthesis step is

$$v_3 = \pm \frac{11268158900\sqrt{\left(v_1^2 + \frac{28145}{62561}\right)\left(v_1^2 + \frac{43467}{38278}\right)}}{5593605380v_1^2 + 2516456313}$$

- We will use Lagrange polynomial interpolation to obtain a different, but constrained function using $n = 4$ points on this v_1-v_3 curve:

$$(v_1, v_3) = \left(-\frac{1}{2}, \frac{62167}{21933}\right), \left(\frac{1}{4}, \frac{80364}{26089}\right), \left(\frac{3}{5}, \frac{64227}{23462}\right), \left(\frac{11}{10}, \frac{39821}{16629}\right).$$

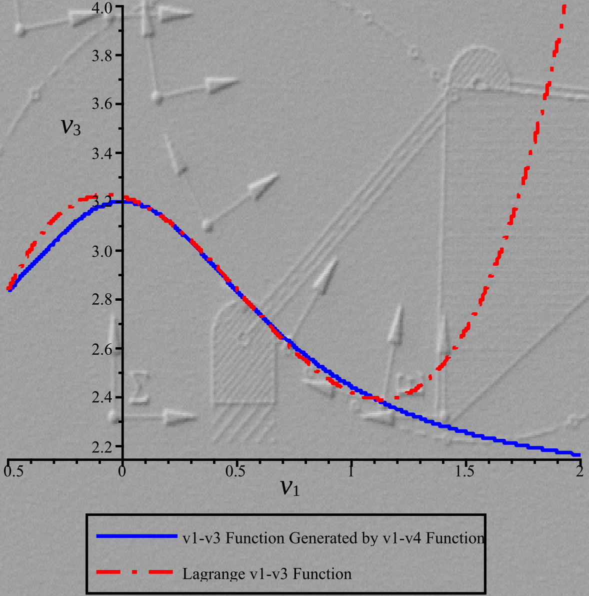
Multi-modal Synthesis: Secondary Function

- The resulting degree 3 Lagrange polynomial $v_3 = g(v_1)$ function is

$$
\begin{aligned}
v_3 &= \frac{1401524525646276756 50}{14611549916120684996 7}v_1^3 - \frac{14850063812931730926 5}{974103327741378999 78}v_1^2 \\
&\quad - \frac{136182081139230857387}{584461996644827399868}v_1 + \frac{570102429959436714 17}{1771096959529779999 6}.
\end{aligned}
$$

- Let this be the specified secondary function.

Multi-modal Synthesis: Secondary Function



Multi-modal Synthesis: Results

- The $v_4 = f(v_1)$ and $v_3 = g(v_1)$ are used to generate the respective synthesis equations with variable angle parameters expressed as v_1 and $f(v_1)$ in the first, and v_1 and $g(v_1)$ in the second.
- The two synthesis equations are squared, then the coefficients and variables are separated in the arrays $\mathbf{c}_1$, $\mathbf{s}_1(v_1, f(v_1))$, $\mathbf{c}_2$, and $\mathbf{s}_2(v_1, g(v_1))$.
- We then ask Maple to evaluate

$$\min_{(a_1,a_2,a_3,a_4)\in\mathbb{R}} \left(\mathbf{c}_1^T \int_{v_1=-\frac{1}{2}}^{v_1=2} \mathbf{s}_1(v_1, f(v_1))dv_1 + \mathbf{c}_2^T \int_{v_1=-0.1}^{v_1=1.25} \mathbf{s}_2(v_1, g(v_1))dv_1 \right)$$

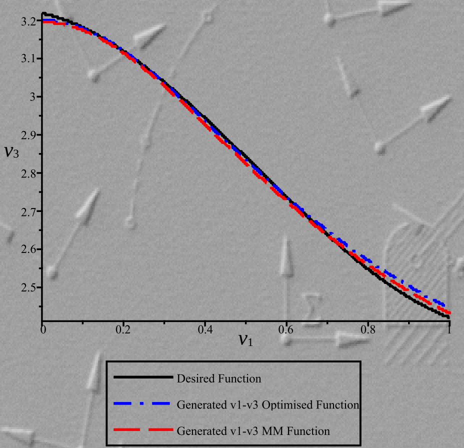
Multi-modal Synthesis: Results

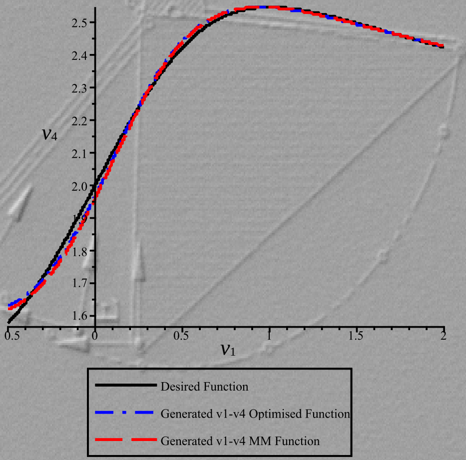
- The optimisation converges to the link lengths:

Link length	a_1	a_2	a_3	a_4
Rational	$-\frac{84421}{571159}$	$\frac{20733}{22295}$	$\frac{24253}{21126}$	$\frac{22053}{27487}$
Floating point	-0.1478064777	0.9299394483	1.148016662	0.8023065449
Normalised	-0.1842269375	1.159082466	1.430895297	1
CAS normalised	-0.1814801460	1.160983273	1.437253857	1

	Structural error
$v_4 = f(v_1)$ only	-0.002471306
$v_4 = f(v_1)$ multi-modal	0.009542948
$v_3 = g(v_1)$ only	0.005358289
$v_3 = g(v_1)$ multi-modal	0.004161159

Multi-modal Synthesis: Results





Conclusions

- We have examined the feasibility of generating more than one arbitrary function between joint angle pairs.
- Identifying the link lengths that generate one prescribed $v_j = f(v_i)$ function with our continuous approximate synthesis method imposes five additional exact functions between the five other angle pairs.
- We have shown that it is computationally possible to perturb one, or possibly more, of these exact functions and approximately generate two (modestly) unrelated functions with continuous structural errors of less than 1% over the entire input angle ranges.
- These results suggest that it is worthwhile to pursue this investigation.