Journal of Economic Theory **101**, 585–605 (2001)
doi:10.1006/jeth.2000.2767, available online at http://www.idealibrary.com on IDEAL®

Microeconomic Foundations for the Theory of International Comparisons

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Received October 23, 1997; final version received May 22, 2000; published online July 19, 2001

This paper establishes a basis in economic theory for solving the problem of how to make comparisons of aggregate quantities and price levels across two or more geographic regions. Two new classes of relative purchasing power measures are set out and the dual relationships between them are explored. The first is a many-household analogue of the (single-household) cost-of-living index and, as such, is rooted in the theory of group cost-of-living indexes. The second is motivated by a generalization of the cost-of-living index and consists of sets of (nominal) expenditure-share deflators. A particular set of these deflators and the associated system of (real) consumption shares are shown to have definite bounds, and the latter is shown to provide rigorous exact index-number interpretations for a trio of axiomatic quantity indexes. *Journal of Economic Literature* Classification Numbers: C43, C81, F31, O18, O57. © 2001 Elsevier Science

Key Words: economic approaches to index-number theory; index numbers; multilateral comparisons; purchasing power parities.

1. INTRODUCTION

It is by now well known that the relative purchasing power of currencies cannot be approximated using exchange rates.[2] This knowledge is reflected in the widespread practice among organizations concerned with international affairs of publishing real income data calculated using one of the

[1] The author wishes to thank Erwin Diewert, Chuck Blackorby and three anonymous referees for helpful comments on earlier versions of this paper.

[2] To paraphrase Balassa [2, p. 586], if international productivity differences are greater in the production of traded goods than in the production of non-traded goods and if per-capita real incomes are taken as representative of levels of productivity, the ratio of PPP to the exchange rate will be an increasing function of relative per-capita real income. Since PPP and relative per-capita real income are not independent variables—their product being constrained to equal relative per-capita *nominal* income—this means that, although the PPP will tend to deviate from the exchange rate in a systematic manner, there is no way to provide a general characterization of the former in terms of the latter.

0022-0531/01 $35.00

two "purchasing power parity" (PPP) methods that have come to be favoured by most experts in the field of international comparisons.[3] Notwithstanding the fact that either of these methods is an enormous improvement over the exchange-rate approach, all three suffer from the same apparent lack of grounding in economic theory. This is also true of every other multilateral comparison formula developed to date. Nowhere in the relevant literature is there any strong economic justification for the use of one formula over another.

Any price index between two groups of countries is, by definition, a measure of PPP. The most commonly required indexes of this sort involve single-country groups in which commodities are valued using different currencies. Since there is no definite relationship between such a measure and the corresponding exchange rate parity—the most readily observable and seemingly appropriate proxy—how should suitable PPPs be constructed?

A natural way to begin thinking about PPPs is in terms of the cost-of-living index. The present paper shows that the interspatial interpretation of this concept can be generalized to facilitate comparisons of real private consumption among groups of countries. It should be understood that, by necessarily limiting the domain of comparison to private consumer goods and services, countries and households instead of the more general final-use commodities, geographic regions and purchasers, this approach cannot offer a complete answer to the question posed above. It is, however, an important first step towards doing so.

Two new classes of PPP measures relevant for international comparisons of income and consumption levels are set out below. The first, developed in Sections 2 and 3, is the many-household analogue of the (single-household) cost-of-living index constructed by taking a mean of order λ of the relevant single-household indexes. The resulting aggregate index is shown to have desirable properties in the most unrestricted setting, and additional desirable properties if preferences are assumed to be identical across countries.

Members of the second class of PPP measures are sets of (nominal) expenditure-share deflators, each corresponding to a system of (real) consumption shares for a bloc of countries. Such measures are instrumental to the theoretical economic analysis of the (axiomatic) multilateral comparison formulae alluded to above. Sections 4 and 5 introduce reasonable generalizations of the cost-of-living index and the Allen consumption index as a self-consistent pair within the second class. These indexes are shown to have definite bounds in terms of observable bilateral fixed-weight indexes and to be dual to the mean-of-order-one index of the first class.

[3] *Viz*., the Eltetö–Köves–Szulc (EKS) method and the Geary–Khamis (GK) method.

Using an exact index-number argument, Section 6 shows that three particular systems of axiomatic quantity indexes are justifiable in terms of specific members of the second class. The practical relevance of this result is that it provides a basis in economic theory for choosing a multilateral comparison formula.

2. BLOC-SPECIFIC PPP INDEXES

Consider a bloc comprising $n \geqslant 2$ countries indexed by the set $\mathcal{N} := \{1, ..., n\}$. Within this bloc there are $m \geqslant 2$ well-defined types of consumer goods and services. Let $\mathcal{M} := \{1, ..., m\}$ denote the general list of these commodities and let $\mathcal{M}_k \subseteq \mathcal{M}$ denote the subset that is available in country $k \in \mathcal{N}$. Every country-specific commodity list contains at least two items.[4]

In each country $k \in \mathcal{N}$, a representative household is assumed to purchase $x_{k\ell} \geqslant 0$ units of commodity $\ell \in \mathcal{M}_k$ at a price of $p_{k\ell} > 0$ country-k currency units (k\$, for short).[5] For any good or service $\ell \in \mathcal{M} \backslash \mathcal{M}_k$ that is unavailable in country k, $x_{k\ell} \equiv 0$ and there is sufficient information to estimate a reservation price $p_{k\ell} > 0$.[6] Let $\mathbf{x}_k := (x_{k1}, ..., x_{km})^\top \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ denote the representative country-k consumption bundle and let $\mathbf{p}_k := (p_{k1}, ..., p_{km})^\top \in \mathbb{R}^m_{++}$ denote the vector of country-k commodity prices. The $(n \times m)$ matrix of all commodity prices in the bloc and the corresponding matrix of representative quantities are denoted by $\mathbf{P} := (\mathbf{p}_1, ..., \mathbf{p}_n)^\top$ and $\mathbf{X} := (\mathbf{x}_1, ..., \mathbf{x}_n)^\top$, respectively.

The preferences of the kth (representative) household over the commodity space indexed by $\mathcal{M}$ are assumed to be represented by a (direct) utility function $u_k : \mathbb{R}^m_+ \backslash \{\mathbf{0}\} \to \mathbb{R}$. Household k's expenditure function $c_k : \mathbb{R}^m_{++} \times \mathcal{U}_k \to \mathbb{R}$ shows the minimum expenditure required to attain utility level v in $\mathcal{U}_k$, the range of u_k, at commodity prices $\mathbf{p} \in \mathbb{R}^m_{++}$; i.e.,

$$c_k(\mathbf{p}, v) := \min_{\mathbf{x} \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}} \{\mathbf{p}^\top \mathbf{x} : u_k(\mathbf{x}) \geqslant v\}. \tag{1}$$

Assuming that u_k is continuous and increasing (regularity conditions R1), c_k is non-decreasing, positively linearly homogeneous (PLH) and concave

[4] Formally, $|\mathcal{M}_k| \geqslant 2$.

[5] Representative households are assumed for the sake of expositional clarity. The need for their existence may be expunged by regarding the bloc as a collection of households rather than as a collection of countries. As such, $\mathcal{N}$ would index the constituent households, $\mathcal{M}_k$ would denote a household-specific commodity list, and $x_{k\ell}$ would be household k's consumption of commodity ℓ.

[6] Using a hedonic price index, for example. See Griliches [11, 12] and Kravis and Lipsey [16].

in $\mathbf{p}$, increasing in v, jointly continuous in $(\mathbf{p}, v)$ and positive (regularity conditions R2).[7]

In the present context, the (Konüs [15] -type) cost-of-living index for household k is the ratio of the minimum expenditure required to attain a particular utility level under the price regimes of any two countries in the bloc:

$$r_k(\mathbf{p}_i, \mathbf{p}_j, v) := \frac{c_k(\mathbf{p}_i, v)}{c_k(\mathbf{p}_j, v)}. \tag{2}$$

The number $r_k(\mathbf{p}_i, \mathbf{p}_j, v)$ is the factor by which household k's nominal expenditure at country-i prices must be deflated in order to make it equal to the same household's nominal expenditure at country-j prices. Thus $r_k(\mathbf{p}_i, \mathbf{p}_j, v)$ is a "country-k-specific" PPP of the dimensionality $i\$/j\$$—the number of units of country i's currency per unit of country j's. There are four essential properties of the country-k PPP index that follow directly from its definition and R2: r_k is positive (P1), non-decreasing in $\mathbf{p}_i$ (P2), PLH in $\mathbf{p}_i$ (P3) and transitive with respect to $\mathbf{p}_i$ and $\mathbf{p}_j$ (P4).[8]

The primary application of a set of PPPs is in extending the usefulness of national accounts data by making economically meaningful cross-country comparisons or combinations of such data feasible. This is achieved by using the PPPs to deflate associated nominal-expenditure ratios.[9] The results of such calculations are needed for policy purposes and for the purposes of empirical economic analysis by international organizations that exist to further the collective interests of a bloc of countries.

For example, policy decisions regarding desirable levels of intra-bloc aid from "have" to "have-not" countries require a measure of each country's per-capita consumption level relative to some numéraire. None of the country-specific indexes $\{r_k\}_{k \in \mathcal{N}}$ provide an appropriate basis for such a measure. This is due to the fact that, in general, different country-specific indexes yield different sets of PPPs for the members of the same bloc, and there is no good reason to choose one country's representative household over another's to represent the bloc as a whole. What is required, then, is an index that somehow reflects the preferences of all representative

[7] A proof of this statement can be found in Diewert [5].

[8] With the singular exception of Samuelson and Swamy's [22, pp. 571–572] "dimensional invariance test," which asserts that the cost-of-living index is invariant to changes in the dimensionality and/or ordering of prices, all other properties of r_k that appear in the literature are implied by one or more of P1–P4. Dimensional invariance follows from the fact that a change in the dimensionality and/or ordering of prices imposes no restrictions *per se* on the functional form of r_k.

[9] E.g., see Eq. (22) below.

households in the relevant bloc. Under such a requirement, the purchasing power of one national currency relative to another will depend on whether some third country is a member of the same bloc. In other words, it will be "bloc specific."

One way to construct such an index is by aggregating over the n instances of the country-specific variety. Indexes defined under this approach in the intertemporal context are called group cost-of-living indexes because they measure the impact on a group of households of moving from one price regime to another and are constructed as weighted averages of single-household cost-of-living indexes. The theory of such index-number formulae was developed by Pollak [18, 19] and Diewert [7].

A natural way to aggregate over the country-k PPP indexes is to choose a real number λ and a set of weights $\mathbf{a} \in \mathcal{S}^{n-1} \cap \mathbb{R}^n_{++}$, where $\mathcal{S}^{n-1} := \{\mathbf{r} \in \mathbb{R}^n_+ : \sum r_k = 1\}$ denotes the unit simplex of dimension $n-1$, and then take a mean of order λ,

$$R_{\lambda, \mathbf{a}}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}) := M_{\lambda, \mathbf{a}}(r_1(\mathbf{p}_i, \mathbf{p}_j, v_1), ..., r_n(\mathbf{p}_i, \mathbf{p}_j, v_n)), \tag{3}$$

where the function $M_{\lambda, \mathbf{a}} : \mathbb{R}^n_{++} \to \mathbb{R}$ defined by

$$M_{\lambda, \mathbf{a}}(\mathbf{r}) := \begin{cases} \left(\sum a_k r_k^\lambda\right)^{1/\lambda} & \text{if} \quad \lambda \in \mathbb{R}\backslash\{0\} \\ \prod r_k^{a_k} & \text{if} \quad \lambda = 0 \end{cases} \tag{4}$$

is a mean of order λ and $\mathbf{v} := (v_1, ..., v_n)^\top \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ is the vector of representative base utility levels.[10] If, for each $k \in \mathcal{N}$, a_k is chosen to be the fraction of households living in country k, then $R_{\lambda, \mathbf{a}}$ is "democratic" in the sense of assigning weights to the relevant country-specific indexes that are increasing in the number of households that the indexes represent. Alternatively, if a_k is chosen to be the bloc expenditure share of the country-k households, then $R_{\lambda, \mathbf{a}}$ is "plutocratic" in the sense of giving more weight to r_ks that represent higher spending.

Since $M_{\lambda, \mathbf{a}}$ is positive and continuous, and non-decreasing and PLH in its arguments, $R_{\lambda, \mathbf{a}}$ satisfies P1–P3. For $\lambda \neq 0$, $R_{\lambda, \mathbf{a}}$ does not in general satisfy P4. It is possible, however, to modify the aggregation rule so that the resulting index is transitive for all $\lambda \in \mathbb{R}$. This can be accomplished by making the weights depend in a particular way on λ, $\mathbf{p}_j$, $\mathbf{v}$ and a vector of positive real numbers $\mathbf{h} := (h_1, ..., h_n)^\top$.

[10] Note that v_k is not necessarily equal to $v_k^* := u_k(\mathbf{x}_k)$.

THEOREM 1. *For any* $\lambda \in \mathbb{R}\backslash\{0\}$, $R_{\lambda, \mathbf{a}}$ *is transitive in* $\mathbf{p}_i$ *and* $\mathbf{p}_j$ *if and only if*

$$a_k = \alpha_k(\mathbf{p}_j, \mathbf{v}, \mathbf{h}, \lambda) := \frac{h_k[c_k(\mathbf{p}_j, v_k)]^\lambda}{\sum_\ell h_\ell[c_\ell(\mathbf{p}_j, v_\ell)]^\lambda}. \tag{5}$$

Proof. See Appendix.

In Eq. (5), since $c_k(\mathbf{p}_j, v_k)$ is a measure of the income of the single household considered to be representative of all households living in country $k \in \mathcal{N}$, it makes sense to set h_k equal to the number of country-k households and to interpret λ as a (negative) "equity weight." Thus, the bigger the country, the larger the value of h_k and the higher its weight per unit of income (in *j*\$). For $\lambda < 0$, poor countries receive a higher weight per household than rich ones. For $\lambda > 0$, rich countries receive a higher weight per household than poor ones.

Formally, then, a *mean-of-order-*λ *PPP index for country i relative to country j* is a function $R_\lambda : \mathbb{R}^{2m}_{++} \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_n \times \mathbb{R}^n_{++} \to \mathbb{R}$ with image $R_\lambda(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h})$ defined by the right-hand side of (3) with weights given by (5). As a weighted average of the r_ks, R_λ inherits the dimensionality *i*\$/*j*\$ and depends on the preferences of all the representative households in the bloc.[11] As a weighted average with carefully chosen weights, R_λ has the same desirable properties of monotonicity, homogeneity and transitivity as does r_k.[12]

The final result of this section is an axiomatic characterization of the class of mean-of-order-λ PPP indexes that is analogous to the axiomatic characterization of the cost-of-living index provided by Diewert [6, Theorem 1].

THEOREM 2. *Let* $R : \mathbb{R}^{2m+n}_{++} \times \mathbb{R}^n_+ \to \mathbb{R}$ *satisfy* P1–P4. *For some* $\mathbf{p}_\ell \in \mathbb{R}^m_{++}$, $\mathbf{v}_{-k} := (v_1, ..., v_{k-1}, v_{k+1}, ..., v_n)^\top \in \mathbb{R}^{n-1}_{++}$ *and for all* $(\mathbf{p}_i, v_k) \in \mathbb{R}^{m+1}_{++}$, $k \in \mathcal{N}$, *let* $c_k(\mathbf{p}_i, v_k) := v_k R(\mathbf{p}_i, \mathbf{p}_\ell, \mathbf{v}, \mathbf{e}_k)$, *where* $\mathbf{e}_k$ *is the n-dimensional unit column vector with* $e_{kk} = 1$. *Further, for some* $\lambda \in \mathbb{R}$ *and for all* $(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) \in \mathbb{R}^{2m+n}_{++} \times \mathbb{R}^n_+$, *let R satisfy the mean-of-order-*λ *property*

$$\text{M.} \quad R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) = M_{\lambda, \mathbf{a}}(R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{e}_1), ..., R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{e}_n))$$

$$\textit{with} \quad a_k := h_k v_k^\lambda / \textstyle\sum h_t v_t^\lambda.$$

[11] Consequently, a more accurate (and cumbersome) notation for the image of this function would be $R_\lambda(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}; u_1, ..., u_n)$.

[12] Note that the generalized mean-of-order-zero PPP index defined by (3) alone also has these properties.

Then, for all $k \in \mathcal{N}$, c_k is an expenditure function that satisfies R2 *and the money-metric utility scaling property*

$$c_k(\mathbf{p}_j, v_k) = v_k \qquad \forall v_k \in \mathbb{R}_{++}, \tag{6}$$

and R is the mean-of-order-λ PPP index for country i relative to country j corresponding to the preferences that are dual to $\{c_k\}_{k \in \mathcal{N}}$. Conversely, given expenditure functions $c_k : \mathbb{R}^{m+1}_{++} \to \mathbb{R}$ that satisfy R2 *and* (6) *and given some $\lambda \in \mathbb{R}$, $R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) := M_{\lambda, \mathbf{a}}(r_1(\mathbf{p}_i, \mathbf{p}_j, v_1), ..., r_n(\mathbf{p}_i, \mathbf{p}_j, v_n))$ with $a_k := h_k[c_k(\mathbf{p}_j, v_k)]^\lambda / \sum_t h_t[c_t(\mathbf{p}_j, v_t)]^\lambda$ satisfies M,* P1–P4 *and*

$$c_k(\mathbf{p}_i, v_k) = v_k R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{e}_k). \tag{7}$$

Proof. See Appendix.

3. PLUTOCRATIC AND DEMOCRATIC PPP INDEXES

Prais [21] was the first to note that official group cost-of-living indexes like the Consumer Price Index assign an implicit weight to each constituent household's consumption pattern that is proportional to its total expenditure. He called such indexes "plutocratic" and suggested an alternative "democratic" variety that treats all households equally. Pollak [18] formalized these concepts by extending the theory of the (single-household) cost-of-living index to groups. The present section interprets this extended theory in terms of the mean-of-order-λ class of bloc-specific PPP indexes defined above and compares the results with those obtained by Diewert [7] in the intertemporal context.

Under the maintained international-comparisons interpretation, Pollak's *Scitovsky group cost-of-living index* becomes the (*Prais–Pollak*) *plutocratic PPP index*[13] and is defined as the ratio of the minimum bloc expenditure required to attain per-household utility levels $\mathbf{v}$ at country-i prices to that required at country-j prices:

$$R_{PP}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) := \frac{\sum_k h_k c_k(\mathbf{p}_i, v_k)}{\sum_\ell h_\ell c_\ell(\mathbf{p}_j, v_\ell)}. \tag{8}$$

Multiplying and dividing the numerator on the right-hand side of this expression by $c_k(\mathbf{p}_j, v_k)$ reveals that the plutocratic PPP index is an expenditure-share-weighted average of the corresponding country-specific PPP indexes; i.e.,

$$R_{PP}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) = \sum_k \alpha_k(\mathbf{p}_j, \mathbf{v}, \mathbf{h}, 1)\, r_k(\mathbf{p}_i, \mathbf{p}_j, v_k), \tag{9}$$

where $\alpha_k(\mathbf{p}_j, \mathbf{v}, \mathbf{h}, 1)$ is country-k's share of (possibly hypothetical) bloc expenditure at prices $\mathbf{p}_j$ and utility levels $\mathbf{v}$.

Pollak's *democratic group cost-of-living index* becomes the *additive democratic PPP index*[13] and is defined as a household-share-weighted average of the corresponding country-specific PPP indexes

$$R_{AD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) := \sum_k \bar{h}_k r_k(\mathbf{p}_i, \mathbf{p}_j, v_k), \tag{10}$$

where $\bar{h}_k := h_k / \mathbf{1}_n^\top \mathbf{h}$ is the fraction of bloc households living in country k, $\mathbf{1}_n$ being the n-dimensional (column) vector of ones. As shown in Diewert [7], a second type of democratic index can be constructed by replacing the arithmetic average in (10) by its geometric counterpart. The result of doing so is called the *multiplicative democratic PPP index*[13] *for country i relative to country j*:

$$R_{MD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) := \prod_k [r_k(\mathbf{p}_i, \mathbf{p}_j, v_k)]^{\bar{h}_k}. \tag{11}$$

In both sorts of democratic PPP index, the use of household-share weights has the effect of counting every household equally. In contrast, by implicitly weighting each household-specific PPP index by its total expenditure, the plutocratic variety counts every dollar of consumption spending equally.

Since by (5) $\alpha_k(\mathbf{p}_j, \mathbf{v}, \mathbf{h}, \lambda) = \bar{h}_k$ when $\lambda = 0$, it is clear from their respective definitions that the multiplicative democratic PPP index is the same as the mean-of-order-zero PPP index:

$$R_{MD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) \equiv R_0(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}).$$

Similarly, the plutocratic PPP index is the same as the mean-of-order-one PPP index[14]:

$$R_{PP}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) \equiv R_1(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}).$$

The additive democratic PPP index, however, is not a member of the mean-of-order-λ class of bloc-specific PPP indexes. By Theorem 1 and since $\alpha_k(\mathbf{p}_j, \mathbf{v}, \mathbf{h}, \lambda) \neq \bar{h}_k$ when $\lambda = 1$, R_{AD} is not transitive with respect to $\mathbf{p}_i$ and $\mathbf{p}_j$. Worse still, it does not even satisfy the weaker property of "country reversal:"

[13] This term with "cost-of-living" substituted for "PPP" is due to Diewert [7].

[14] Since $R_1 \equiv R_{PP}$ and $R_0 \equiv R_{MD}$, Theorem 2 is a generalization of Diewert's [7, Theorems 6 and 10] separate axiomatic characterizations of the plutocratic and multiplicative democratic PPP indexes.

$$\begin{aligned} R_{AD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) &\geqslant R_{MD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) \\ &= \frac{1}{R_{MD}(\mathbf{p}_j, \mathbf{p}_i, \mathbf{v}, \mathbf{h})} \\ &\geqslant \frac{1}{R_{AD}(\mathbf{p}_j, \mathbf{p}_i, \mathbf{v}, \mathbf{h})}, \end{aligned} \tag{12}$$

where each of the two inequalities follows by the Theorem of the Arithmetic and Geometric Means[15] and the equality follows by the transitivity and positivity properties of R_{MD}. The existence of an alternative to R_{AD} that is both democratic and transitive eliminates the need to consider this index any further.

The next theorem shows that the translog PPP index (defined below) is exact for the multiplicative democratic PPP index evaluated at a particular vector of base utility levels when preferences are dual to a translog expenditure function.

THEOREM 3. *Let $c_T : \mathbb{R}^{m+1}_{++} \to \mathbb{R}_{++}$ be a general translog expenditure function defined by*

$$\begin{aligned} \ln c_T(\mathbf{p}, \mu) := \alpha + \sum \beta_\ell \ln p_\ell + \beta_0 \ln \mu + \tfrac{1}{2} \sum \sum \gamma_{t\ell} \ln p_t \ln p_\ell \\ + \sum \gamma_{0\ell} \ln p_\ell \ln \mu + \tfrac{1}{2} \gamma_{00} (\ln \mu)^2, \end{aligned} \tag{13}$$

$\alpha \in \mathbb{R}_+$, $\beta_\ell \in \mathbb{R}_+$ *for all* $\ell \in \mathscr{M} \cup \{0\} =: \mathscr{L}$, $\sum \beta_\ell = 1$, $\gamma_{t\ell} = \gamma_{\ell t} \in \mathbb{R}_+$ *for all* $(t, \ell) \in \mathscr{L} \times \mathscr{L}$, *and* $\sum_\ell \gamma_{t\ell} = 0$ *for all* $t \in \mathscr{L}$. *For all* $k \in \mathscr{N}$, *suppose that* $(\mathbf{p}_k, \mu_k^*) \in \mathbb{R}^{m+1}_{++}$ *and* $\mathbf{x}_k := \nabla_{\mathbf{p}} c_T(\mathbf{p}_k, \mu_k^*)$. *Then*

$$R_{MD}(\mathbf{p}_i, \mathbf{p}_j, \mu_{ij}^*, \ldots, \mu_{ij}^*, \mathbf{h}) = \frac{c_T(\mathbf{p}_i, \mu_{ij}^*)}{c_T(\mathbf{p}_j, \mu_{ij}^*)} = \rho_T(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j), \tag{14}$$

where

$$\rho_T(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) := \prod_{\ell \in \mathscr{M}} \left(\frac{p_{i\ell}}{p_{j\ell}} \right)^{(\omega_{i\ell} + \omega_{j\ell})/2} \tag{15}$$

is the country-j translog PPP index, $\omega_{k\ell} := p_{k\ell} x_{k\ell} / \mathbf{p}_k^\top \mathbf{x}_k$ *is the* ℓ*th household-k expenditure share and* $\mu_{ij}^* := (\mu_i^* \mu_j^*)^{1/2}$.[16]

[15] See Hardy *et al.* [14, pp. 16–21] for a general statement and proof of this result.

[16] Since R_{MD} reduces to r when households have identical tastes, this theorem is formally equivalent to Diewert's [3, Theorem 2.16] result for a single household.

An obvious weakness of this result is its dependence on preferences being identical across households—a weakness not shared by the intertemporal analogue as comparison with Diewert [7, Theorem 9] reveals. A similar asymmetry of outcomes is evident in the relative applicability of certain bounds on the plutocratic and additive democratic indexes: The bounding theorems for these indexes in the intertemporal context[17] have no direct counterparts in the international one. All of these asymmetries are due to the fact that in the latter context it is impossible to observe each household's consumption bundle at the prices faced by at least one other household. By contrast, the data available in the former context include (in principle) the consumption bundles of each household at the prices faced by all households during at least two periods of time; i.e., for each period $t \in \mathscr{T} \subseteq \mathbb{N}$ and for each representative group-k household ($k \in \mathscr{N}$), there is a vector $\mathbf{x}_k^t \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$ of quantities purchased at prices $\mathbf{p}^t \in \mathbb{R}_{++}^m$. Direct application of Diewert's [7] Theorem 9 to the international context would require knowledge of $\mathbf{x}_k^j$ and $\mathbf{x}_k^i$ for some $(i, j) \in \mathscr{N} \times \mathscr{N}$ and for all $k \in \mathscr{N}$. Theorem 3 above acquires this knowledge by assuming identical preferences so that $\mathbf{x}_k^j \equiv \mathbf{x}^j$ for all $(j, k) \in \mathscr{N} \times \mathscr{N}$.

4. THE MULTILATERAL-KONÜS PPP INDEX

Section 2 above introduced a new class of bloc-specific PPP indexes constructed by taking a mean of order λ of the associated country-specific indexes. The remainder of the paper pursues alternative multilateral approaches to the construction of bloc-specific indexes with the object of providing a theoretical basis for certain systems of axiomatic quantity indexes. In the present section, a reasonable generalization of r_k is used to specify a PPP index that relates the general or average price level of each country to that of the bloc as a whole. The same index is derived indirectly in the next section by generalizing a quantity counterpart to r_k, and then both types of indexes are shown to be dual to the plutocratic PPP index.

To begin with, let $\gamma_k \in \mathbb{R}_{++}$ denote the price of a unit of country k's currency (1 k\$) in terms of the currency units of some numéraire country.[18] Consequently, the ratio γ_k/γ_i (i\$/k\$) is country i's exchange rate with respect to country k. Throughout the rest of the paper, $\bar{\mathbf{p}}_k := \gamma_k \mathbf{p}_k$ denotes the vector of numéraire-country-denominated country-k commodity prices

[17] See Diewert [7, Theorem 4 and pp. 33–34]. For a more descriptive derivation of the Paasche and Laspeyres bounds on the plutocratic cost-of-living index, see Fisher and Griliches [9, Section II] or Fisher and Shell [10, pp. 166–168].

[18] Note that the numéraire country is not required to be a member of the bloc.

and $\overline{\mathbf{P}} := (\bar{\mathbf{p}}_1, ..., \bar{\mathbf{p}}_n)^\top$ denotes the associated matrix. Use of such a normalization allows the summation of household expenditure functions as in the following definition of country i's share of (possibly hypothetical) bloc expenditure at utility levels $\mathbf{v}$:

$$s_i(\overline{\mathbf{P}}, \mathbf{v}, \mathbf{h}) := \frac{h_i c_i(\bar{\mathbf{p}}_i, v_i)}{\sum h_k c_k(\bar{\mathbf{p}}_k, v_k)}. \tag{16}$$

Likewise, it facilitates the definition of the bloc expenditure function[19]:

$$C(\overline{\mathbf{P}}, \mathbf{v}, \mathbf{h}) = \sum h_k c_k(\bar{\mathbf{p}}_k, v_k). \tag{17}$$

The substitution of C for c_k in (2) gives rise to a logical bloc-specific counterpart to the (Konüs-type) country-k PPP index. Specifically, the *multilateral-Konüs* (MK) *PPP index for country i relative to the bloc as a whole* is defined as the ratio of the minimum expenditure required to attain utility levels $\mathbf{v}^* := (v_1^*, ..., v_n^*)^\top$, $v_k^* := u_k(\mathbf{x}_k)$, when every representative household faces the prices of country i to the minimum expenditure required to attain the same utility levels when each household faces the prices of its home country:

$$D_{MK,\,i}(\overline{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := \frac{C(\bar{\mathbf{p}}_i \mathbf{1}_n^\top, \mathbf{v}^*, \mathbf{h})}{C(\overline{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})} \tag{18}$$

$$= \frac{\sum_j h_j c_j(\bar{\mathbf{p}}_i, v_j^*)}{\sum h_k c_k(\bar{\mathbf{p}}_k, v_k^*)}, \qquad \text{by (17).} \tag{19}$$

The number $D_{MK,\,i}(\overline{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})$ may be interpreted as the factor by which cost-minimizing bloc expenditure at country-i prices and actual utility levels must be deflated in order to make it equal to nominal bloc expenditure. Thus the numerator of (19) is the sum of (hypothetical) bloc expenditures when the jth household ($j \in \mathcal{N}$) faces the prices of country i, $\bar{\mathbf{p}}_i$, and its utility level is held constant at the actual value v_j^*.

Use of (16) and (2) with $v := v_j^*$ in conjunction with (19) reveals that the MK PPP index is an expenditure-share-weighted sum of country-specific PPP indexes; i.e.,

$$D_{MK,\,i}(\overline{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) = \sum_j s_j(\overline{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})\, r_j(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, v_j^*). \tag{20}$$

[19] Note that this is *not* a Scitovsky expenditure function since prices are, in general, unequal across countries.

Following directly from this fact is a corollary to Pollak's [17, p. 11] well-known bounding theorem for $r_j(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, v_j^*)$ establishing bounds on $D_{MK,i}$.

THEOREM 4. *For all* $i \in \mathcal{N}$,

$$\sum_j s_j^* \min_{\ell \in \mathcal{M}} \left\{ \frac{\bar{p}_{i\ell}}{\bar{p}_{j\ell}} \right\} \leqslant D_{MK,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) \leqslant \sum_j s_j^* \theta(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, \mathbf{x}_j), \tag{21}$$

where $s_j^* := s_j(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})$ *denotes the actual bloc expenditure share for country* j *and* $\theta(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, \mathbf{x}_j) \equiv \mathbf{x}_j^\top \bar{\mathbf{p}}_i / \mathbf{x}_j^\top \bar{\mathbf{p}}_j$ *is the country-j Laspeyres PPP index.*

5. THE MULTILATERAL-ALLEN CONSUMPTION-SHARE SYSTEM

A quantity counterpart to the cost-of-living index can be obtained by using r_k as a deflator for household k's expenditure ratio between two different price-utility situations $(\mathbf{p}_i, v_i)$ and $(\mathbf{p}_j, v_j)$. More precisely, the implicit country-k (Konüs-type real) consumption index is defined as

$$\tilde{q}_k(\mathbf{p}_i, \mathbf{p}_j, v_i, v_j, v) := \frac{c_k(\mathbf{p}_i, v_i)}{c_k(\mathbf{p}_j, v_j)} \Bigg/ r_k(\mathbf{p}_i, \mathbf{p}_j, v). \tag{22}$$

The result of substituting for $r_k(\mathbf{p}_i, \mathbf{p}_j, v)$ using (2) and setting $k := i$, $v_i := v_i^*$, $v_j := v_i$ and v equal to either v_i or v_i^* is the country-i Allen [1, p. 199] consumption index at prices $\mathbf{p} := \mathbf{p}_i$ or $\mathbf{p} := \mathbf{p}_j$:

$$q_i(\mathbf{p}, v_i^*, v_i) := \frac{c_i(\mathbf{p}, v_i^*)}{c_i(\mathbf{p}, v_i)}. \tag{23}$$

The number $q_i(\mathbf{p}, v_i^*, v_i)$ is a measure of household i's consumption at its actual utility level v_i^* relative to that at some reference utility level v_i evaluated using prices $\mathbf{p}$.[20]

If v_i is chosen so that $c_i(\mathbf{p}, v_i) = c_j(\mathbf{p}, v_j^*)$ for some $j \in \mathcal{N}$, then q_i becomes a per-household consumption index for country i relative to country j:

$$q_{ij}(\mathbf{p}, v_i^*, v_j^*) := \frac{c_i(\mathbf{p}, v_i^*)}{c_j(\mathbf{p}, v_j^*)}. \tag{24}$$

[20] Since c_i is increasing in v for $\mathbf{p}$ constant, $c_i(\mathbf{p}, v)$ is a (money-metric) utility function representing the same preferences as u_i. Thus q_i measures the welfare change experienced by household i in moving from $(\mathbf{p}, v_i)$ to $(\mathbf{p}, v_i^*)$.

A natural way to generalize this bilateral country-specific measure into a multilateral bloc-specific one is to use it as the basic building block of a system of consumption shares:

$$\frac{h_i q_{ij}(\mathbf{p}, v_i^*, v_j^*)}{\sum_k h_k q_{kj}(\mathbf{p}, v_k^*, v_j^*)} = \frac{h_i c_i(\mathbf{p}, v_i^*)/c_j(\mathbf{p}, v_j^*)}{\sum_k h_k c_k(\mathbf{p}, v_k^*)/c_j(\mathbf{p}, v_j^*)}. \tag{25}$$

Thus the *multilateral-Allen* (MA) *consumption share for country i* is defined as the ratio of the minimum country-*i* expenditure required to attain representative-household utility level v_i^* at reference prices $\mathbf{p}$ to the minimum bloc expenditure required to attain representative-household utility levels $\mathbf{v}^*$ at the same prices:

$$S_{MA,\,i}(\mathbf{p}, \mathbf{v}^*, \mathbf{h}) := \frac{h_i c_i(\mathbf{p}, v_i^*)}{\sum_k h_k c_k(\mathbf{p}, v_k^*)}. \tag{26}$$

The number $S_{MA,\,i}(\mathbf{p}, \mathbf{v}^*, \mathbf{h})$ is the fraction of total bloc expenditure that would be attributable to country-*i* households at reference prices $\mathbf{p}$.

The data set $\mathbf{P}$ admits n possible choices for the reference-price vector $\mathbf{p}$ in (26). If the country-*i* price vector $\mathbf{p}_i$ is chosen, $S_{MA,\,i}(\mathbf{p}_i, \mathbf{v}^*, \mathbf{h})$ is called the MA *own-price* consumption share for country *i*. In general, $\mathbf{S}_{MA}(\mathbf{P}, \mathbf{v}^*, \mathbf{h}) := [S_{MA,\,1}(\mathbf{p}_1, \mathbf{v}^*, \mathbf{h}), ..., S_{MA,\,n}(\mathbf{p}_n, \mathbf{v}^*, \mathbf{h})]^\top$ is only a *quasi*-consumption-share system since its components do not necessarily sum to unity.

The *MA own-price expenditure-share deflator for country i* is a bloc-specific PPP index $\tilde{D}_{MA,\,i}$ defined implicitly by

$$\tilde{D}_{MA,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})\, S_{MA,\,i}(\bar{\mathbf{p}}_i, \mathbf{v}^*, \mathbf{h}) = s_i(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}). \tag{27}$$

The number $\tilde{D}_{MA,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})$ is the amount by which country *i*'s actual expenditure share must be deflated in order to make it equal to the same country's MA own-price consumption share. Since $\mathbf{s}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := [s_1(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}), ..., s_n(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})]^\top \in \mathcal{S}^{n-1}$,

$$\sum \tilde{D}_{MA,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})\, S_{MA,\,i}(\bar{\mathbf{p}}_i, \mathbf{v}^*, \mathbf{h}) = 1. \tag{28}$$

Using the definitions of $\tilde{D}_{MA,\,i}$, $S_{MA,\,i}$ and s_i, $\tilde{D}_{MA,\,i}$ can be shown to be equal to the MK PPP index $D_{MK,\,i}$:

$$\tilde{D}_{MA,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) = D_{MK,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}). \tag{29}$$

Thus the MA own-price consumption-share system and the MK PPP index are completely consistent with one another. This fact together with (20) implies a corollary to Pollak's [17, p. 11] bounding theorem for $r_j(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, v_j^*)$ establishing bounds on $S_{MA,\,i}$.

THEOREM 5. *For all $i \in \mathcal{N}$,*

$$\left\{\sum_j [\theta(h_i\mathbf{x}_i, h_j\mathbf{x}_j, \bar{\mathbf{p}}_i)]^{-1}\right\}^{-1} \leqslant S_{MA,i}(\bar{\mathbf{p}}_i, \mathbf{v}^*, \mathbf{h})$$
$$\leqslant \left\{\sum_j \left[\frac{s_i^*}{s_j^*} \max_{\ell \in \mathcal{M}} \left\{\frac{\bar{p}_{j\ell}}{\bar{p}_{i\ell}}\right\}\right]^{-1}\right\}^{-1}, \tag{30}$$

where $\theta(h_i\mathbf{x}_i, h_j\mathbf{x}_j, \bar{\mathbf{p}}_i) \equiv \bar{\mathbf{p}}_i^\top(h_i\mathbf{x}_i)/\bar{\mathbf{p}}_i^\top(h_j\mathbf{x}_j)$ is the country-j Paasche consumption index.

By rearranging the terms of the definition of the MA own-price consumption shares multiplied and divided by their sum, the former can be re-expressed as the product of the latter and the harmonic mean of the associated national expenditure ratios, each deflated by the corresponding plutocratic PPP index:

$$S_{MA,i}(\bar{\mathbf{p}}_i, \mathbf{v}^*, \mathbf{h})$$
$$= \left\{\sum_k \left[\frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)}{h_k c_k(\bar{\mathbf{p}}_k, v_k^*)} \Big/ R_{PP}(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})\right]^{-1}\right\}^{-1} \sum S_{MA,k}(\bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h}). \tag{31}$$

Dividing both sides of (31) by $\sum S_{MA,k}(\bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})$ reveals that the *normalized* MA own-price consumption-share system is the system of consumption shares that is dual to the plutocratic PPP index:

$$S_{PP,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := \frac{S_{MA,i}(\bar{\mathbf{p}}_i, \mathbf{v}^*, \mathbf{h})}{\sum S_{MA,k}(\bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})} \tag{32}$$
$$= \left\{\sum_k \left[\frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)}{h_k c_k(\bar{\mathbf{p}}_k, v_k^*)} \Big/ R_{PP}(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})\right]^{-1}\right\}^{-1}. \tag{33}$$

THEOREM 6. *For any R satisfying* P1–P4, *the right-hand side of* (33) *with $R_{PP} := R$ defines a system of consumption shares $\mathbf{S}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := [S_1(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}), ..., S_n(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})]^\top \in \mathcal{S}^{n-1}$ that is continuous in $(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})$ and homogeneous of degree zero in each of $\bar{\mathbf{p}}_1, ..., \bar{\mathbf{p}}_{n-1}$ and $\bar{\mathbf{p}}_n$.*

Proof. See Appendix.

COROLLARY 1. *The ratio of per-household consumption shares for countries i and j is equal to the per-household consumption index for country i relative to country j obtained by using R to deflate the per-household expenditure ratio between the two countries:*

$$\frac{S_i(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})/h_i}{S_j(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h})/h_j} = \frac{c_i(\mathbf{p}_i, v_i^*)}{c_j(\mathbf{p}_j, v_j^*)} \Big/ R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}^*, \mathbf{h}) \tag{34}$$

$$=: \tilde{Q}(\mathbf{p}_i, \mathbf{p}_j, v_i^*, v_j^*, \mathbf{v}^*, \mathbf{h}). \tag{35}$$

Proof. See Appendix.

Thus, under P1–P4, any bloc-specific PPP index R and its quantity counterpart $\tilde{Q}$ are dual to some system of consumption shares $\mathbf{S}$ and its PPP counterpart $\tilde{\mathbf{D}}$, the ith element of which is defined as the amount by which the actual expenditure share s_i^* must be deflated in order to make it equal to S_i. In particular, the plutocratic PPP index and its quantity counterpart $\tilde{Q}_{PP}$ defined by the right-hand side of (34) with $R := R_{PP}$ are dual to the MA consumption-share system and the MK PPP index.

6. THE EXACT APPROACH

As with all economic approaches to index number theory, the exact approach is based on the assumption of utility-maximizing behaviour. Further, it is assumed that household preferences can be adequately approximated by a certain class of utility functions. In the present section, this assumption is used to develop a novel exact index-number argument that shows that the novel quantity approach of the preceding section justifies three particular systems of axiomatic quantity indexes.

In addition to satisfying R1, suppose that u_k is independent of k and (positively) homothetic. Consequently, household preferences can be represented by a PLH utility function u and the associated expenditure function can be written as

$$c(\mathbf{p}, \mu) = \mu\pi(\mathbf{p}), \tag{36}$$

where $\pi(\mathbf{p}) := \min_{\mathbf{x}}\{\mathbf{p}^\top \mathbf{x} : u(\mathbf{x}) \geqslant 1\}$ is the unit expenditure function for u.

A bilateral axiomatic per-household (real) consumption index for country i relative to country j is a real-valued function ϕ of the observed price and (per-household) quantity data for the two countries. Such an index is defined to be *exact* for a PLH utility function u if, for every $(\mathbf{p}_i, \mathbf{x}_i)$ and $(\mathbf{p}_j, \mathbf{x}_j)$ such that $\mathbf{p}_k^\top \mathbf{x}_k = c(\mathbf{p}_k, u(\mathbf{x}_k))$, $k \in \{i, j\}$,

$$\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) = \frac{u(\mathbf{x}_i)}{u(\mathbf{x}_j)}. \tag{37}$$

Diewert [4, p. 181] noted that the price and quantity vectors in this equation are not completely independent variables since $(\mathbf{p}_k, \mathbf{x}_k)_{k \in \{i, j\}}$ is assumed to be consistent with expenditure-minimizing behaviour.

Given $c_k := c$ for all $k \in \mathcal{N}$, the MA consumption share for country i is defined as

$$\hat{S}_{MA,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h}) := \frac{h_i c(\mathbf{p}, \mu_i^*)}{\sum h_j c(\mathbf{p}, \mu_j^*)}, \tag{38}$$

where $\boldsymbol{\mu}^* := (\mu_1^*, ..., \mu_n^*)^\top$ and $\mu_k^* := u(\mathbf{x}_k)$. Dividing the numerator and denominator of the right-hand side of this expression by $c(\mathbf{p}, \mu_k^*)$ for some $k \in \mathcal{N}$ and substituting for $c(\mathbf{p}, \mu_\ell^*)$ using (36) with $\mu := u(\mathbf{x}_\ell)$ for all $\ell \in \mathcal{N}$ yields

$$\hat{S}_{MA,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h}) = \frac{h_i \dfrac{u(\mathbf{x}_i)}{u(\mathbf{x}_k)}}{\sum_j h_j \dfrac{u(\mathbf{x}_j)}{u(\mathbf{x}_k)}}. \tag{39}$$

Now, by (37),

$$\hat{S}_{MA,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h}) = \frac{h_i \phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_j h_j \phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)} =: \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}). \tag{40}$$

Thus, given a bilateral axiomatic per-household consumption index that is exact for a PLH utility function representing the (homothetic) preferences of the (identical) representative households, the ith country-k star-system consumption share $\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})$ is a direct approximation for the MA consumption index for country i.

Substituting for $c(\mathbf{p}, \mu_\ell^*)$ in (38) using (36) with $\mu := u(\mathbf{x}_\ell)$ for all $\ell \in \mathcal{N}$, dividing the resulting numerator and denominator by $u(\mathbf{x}_i)$, and then invoking (37) yields

$$\hat{S}_{MA,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h}) = h_i \left\{ \sum_j h_j [\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)]^{-1} \right\}^{-1}. \tag{41}$$

Dividing each side of this equation by the corresponding sum over the is yields

$$\begin{aligned}\hat{S}_{PP,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h}) &:= \frac{\hat{S}_{MA,i}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h})}{\sum \hat{S}_{MA,k}(\mathbf{p}, \boldsymbol{\mu}^*, \mathbf{h})} \\ &= \frac{h_i \{\sum_j h_j [\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)]^{-1}\}^{-1}}{\sum_k h_k \{\sum_\ell h_\ell [\phi(\mathbf{p}_k, \mathbf{p}_\ell, \mathbf{x}_k, \mathbf{x}_\ell)]^{-1}\}^{-1}} \\ &=: \sigma_{OS,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}). \end{aligned} \tag{42}$$

Thus, given a bilateral axiomatic per-household consumption index that is exact for a PLH utility function representing the (homothetic) preferences of the (identical) representative households, the ith own-share of bloc consumption $\sigma_{OS,\,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})$[21] is a direct approximation for the plutocratic consumption index for country i.

By Theorem 6, the right-hand side of (33) with $R_{PP} := R_{0,\,\mathbf{1}_n/n}$ defines a system of consumption shares that is dual to the generalized mean-of-order-zero PPP index with equal weights:

$$S_{0,\,\mathbf{1}_n/n,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := \left\{ \sum_k \left[\frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)}{h_k c_k(\bar{\mathbf{p}}_k, v_k^*)} \Big/ R_{0,\,\mathbf{1}_n/n}(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h}) \right]^{-1} \right\}^{-1}. \quad (43)$$

Substituting for $R_{0,\,\mathbf{1}_n/n}$ using (3), (4) and (2), and rearranging terms,

$$S_{0,\,\mathbf{1}_n/n,\,i}(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) = \frac{h_i \prod_\ell \left[\dfrac{c_i(\bar{\mathbf{p}}_i, v_i^*)}{c_\ell(\bar{\mathbf{p}}_i, v_\ell^*)} \right]^{1/n}}{\sum_k h_k \prod_\ell \left[\dfrac{c_k(\bar{\mathbf{p}}_k, v_k^*)}{c_\ell(\bar{\mathbf{p}}_k, v_\ell^*)} \right]^{1/n}}. \quad (44)$$

Assuming identical homothetic preferences and then invoking (37) yields

$$\hat{S}_{0,\,\mathbf{1}_n/n,\,i}(\bar{\mathbf{P}}, \boldsymbol{\mu}^*, \mathbf{h}) = \frac{h_i \prod_\ell [\phi(\mathbf{p}_i, \mathbf{p}_\ell, \mathbf{x}_i, \mathbf{x}_\ell)]^{1/n}}{\sum_k h_k \prod_\ell [\phi(\mathbf{p}_k, \mathbf{p}_\ell, \mathbf{x}_k, \mathbf{x}_\ell)]^{1/n}} := \sigma_{EKS,\,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}). \quad (45)$$

Thus, given a bilateral axiomatic per-household consumption index that is exact for a PLH utility function representing the (homothetic) preferences of the (identical) representative households, the ith Eltetö–Köves–Szulc (EKS) consumption share $\sigma_{EKS,\,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})$ is a direct approximation for the generalized mean-of-order-zero consumption index for country i with equal weights.

The significance of the preceding results is that they provide rigorous exact index-number interpretations for three different systems of axiomatic quantity indexes. Hence they represent the first justifications grounded in economic theory for the use of particular (practical) multilateral comparison methods. Interestingly, Diewert [8, Propositions 8, 13 and 14] asserted the superiority of the same three methods (in relation to five others) using his multilateral test approach.

[21] This index-number formula is due to Diewert [8, p. 25].

7. CONCLUDING REMARKS

The foregoing provides the first formal extension of the theory of the cost-of-living index into the realm of multilateral international comparisons. Such comparisons can be made from the viewpoint of an individual or from that of a group. Those that reflect the perspective of an individual who is seen to represent everyone in her country of residence are called “country specific.” Those that reflect the perspective of a group that includes a representative from each country being compared are called “bloc specific” in recognition of the fact that they depend on which countries are members of the bloc in question. The paper introduces this terminology with the object of re-emphasizing the practical importance of the associated conceptual distinctions.

Different classes of bloc-specific indexes can be distinguished by the types of comparisons they facilitate. The dual relationships among four of these—two comprising indexes of relative purchasing power and two comprising indexes of real consumption—were introduced above. Within this novel framework, (i) the plutocratic and multiplicative democratic PPP indexes were shown to belong to a particular “mean-of-order-λ” class of PPP indexes with good axiomatic properties, (ii) novel multilateral analogues to the Konüs PPP index and the Allen consumption index were each shown to have definite bounds and to be mutually consistent with the plutocratic PPP index, and (iii) the star, own-share and EKS systems of axiomatic quantity indexes were shown to be justifiable.

APPENDIX

Proof of Theorem 1.

$$R_{\lambda,\mathbf{a}}(\mathbf{p}_i, \mathbf{p}_t, \mathbf{v})\, R_{\lambda,\mathbf{a}}(\mathbf{p}_t, \mathbf{p}_j, \mathbf{v}) = R_{\lambda,\mathbf{a}}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v})$$

$$\Leftrightarrow \sum_k a_k(\mathbf{p}_i, \mathbf{p}_t, \mathbf{v}, \lambda) \left[\frac{c_k(\mathbf{p}_i, v_k)}{c_k(\mathbf{p}_t, v_k)}\right]^\lambda \cdot \sum_k a_k(\mathbf{p}_t, \mathbf{p}_j, \mathbf{v}, \lambda) \left[\frac{c_k(\mathbf{p}_t, v_k)}{c_k(\mathbf{p}_j, v_k)}\right]^\lambda$$

$$= \sum_k a_k(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \lambda) \left[\frac{c_k(\mathbf{p}_i, v_k)}{c_k(\mathbf{p}_j, v_k)}\right]^\lambda$$

$$\Leftrightarrow \sum_k \left\{\frac{\sum_\ell h_\ell[c_\ell(\mathbf{p}_t, v_\ell)]^\lambda}{h_k[c_k(\mathbf{p}_t, v_k)]^\lambda}\right\} a_k(\mathbf{p}_i, \mathbf{p}_t, \mathbf{v}, \lambda)\, h_k[c_k(\mathbf{p}_i, v_k)]^\lambda$$

$$\cdot \sum_k \left\{\frac{\sum_\ell h_\ell[c_\ell(\mathbf{p}_j, v_\ell)]^\lambda}{h_k[c_k(\mathbf{p}_j, v_k)]^\lambda}\right\} a_k(\mathbf{p}_t, \mathbf{p}_j, \mathbf{v}, \lambda)\, h_k[c_k(\mathbf{p}_t, v_k)]^\lambda$$

$$= \sum_{\ell} h_{\ell}[c_{\ell}(\mathbf{p}_t, v_{\ell})]^{\lambda}$$

$$\cdot \sum_{k} \left\{ \frac{\sum_{\ell} h_{\ell}[c_{\ell}(\mathbf{p}_j, v_{\ell})]^{\lambda}}{h_k[c_k(\mathbf{p}_j, v_k)]^{\lambda}} \right\} a_k(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \lambda)\, h_k[c_k(\mathbf{p}_i, v_k)]^{\lambda}$$

$$\Leftrightarrow a_k(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \lambda) = \frac{h_k[c_k(\mathbf{p}_j, v_k)]^{\lambda}}{\sum_{\ell} h_{\ell}[c_{\ell}(\mathbf{p}_j, v_{\ell})]^{\lambda}}. \quad \blacksquare$$

Proof of Theorem 2. For $\lambda = 0$,

$$\begin{aligned} &R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) \\ &\quad = \prod_k [R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{e}_k)]^{\bar{h}_k}, \qquad \text{by M} \\ &\quad = \prod_k \left[\frac{v_k R(\mathbf{p}_i, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{e}_k)}{v_k R(\mathbf{p}_j, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{e}_k)} \right]^{\bar{h}_k} \qquad \text{since } v_k > 0 \text{ and } R \text{ is pos. and trans.} \\ &\quad = \prod_k \left[\frac{c_k(\mathbf{p}_i, v_k)}{c_k(\mathbf{p}_j, v_k)} \right]^{\bar{h}_k}, \qquad \text{by the definition of } c_k \\ &\quad =: R_0(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}). \end{aligned}$$

For $\lambda \neq 0$,

$$\begin{aligned} R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}) &= \frac{R(\mathbf{p}_i, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{h})}{R(\mathbf{p}_j, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{h})} \qquad \text{since } R \text{ is positive and transitive} \\ &= \left\{ \frac{\sum_k h_k[v_k R(\mathbf{p}_i, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{e}_k)]^{\lambda}}{\sum_t h_t[v_t R(\mathbf{p}_j, \mathbf{p}_{\ell}, \mathbf{v}, \mathbf{e}_t)]^{\lambda}} \right\}^{1/\lambda}, \qquad \text{by M} \\ &= \left\{ \frac{\sum_k h_k[c_k(\mathbf{p}_i, v_k)]^{\lambda}}{\sum_t h_t[c_t(\mathbf{p}_j, v_t)]^{\lambda}} \right\}^{1/\lambda}, \qquad \text{by the definition of } c_k \\ &= \left\{ \sum_k \frac{h_k[c_k(\mathbf{p}_j, v_k)]^{\lambda}}{\sum_t h_t[c_t(\mathbf{p}_j, v_t)]^{\lambda}} \left[\frac{c_k(\mathbf{p}_i, v_k)}{c_k(\mathbf{p}_j, v_k)} \right]^{\lambda} \right\}^{1/\lambda} \\ &=: R_{\lambda}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}, \mathbf{h}). \end{aligned}$$

The remainder of the proof is straightforward.

Proof of Theorem 6. Since both R and c_k ($k \in \mathcal{N}$) are positive and continuous in their respective arguments, so is $\mathbf{S}$. By the positivity and transitivity properties of R,

$$\sum S_i(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := \sum_i \left\{ \sum_k \left[\frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)/h_k c_k(\bar{\mathbf{p}}_k, v_k^*)}{R(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})} \right]^{-1} \right\}^{-1}$$
$$= \sum_i h_i c_i(\bar{\mathbf{p}}_i, v_i^*) \left\{ \sum_k h_k c_k(\bar{\mathbf{p}}_k, v_k^*) \frac{R(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, \mathbf{v}^*, \mathbf{h})}{R(\bar{\mathbf{p}}_j, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})} \right\}^{-1}$$
$$= \sum_i \frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)}{R(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_j, \mathbf{v}^*, \mathbf{h})} \left\{ \sum_k \frac{h_k c_k(\bar{\mathbf{p}}_k, v_k^*)}{R(\bar{\mathbf{p}}_j, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})} \right\}^{-1}$$
$$= 1.$$

Since $\bar{\mathbf{p}}_k := \gamma_k \mathbf{p}_k$, R is homogeneous of degree minus one in $\mathbf{p}_k$, and R and c_i are each PLH in $\mathbf{p}_i$,

$$S_i(\bar{\mathbf{P}}, \mathbf{v}^*, \mathbf{h}) := \left\{ \sum_k \left[\frac{h_i c_i(\bar{\mathbf{p}}_i, v_i^*)/h_k c_k(\bar{\mathbf{p}}_k, v_k^*)}{R(\bar{\mathbf{p}}_i, \bar{\mathbf{p}}_k, \mathbf{v}^*, \mathbf{h})} \right]^{-1} \right\}^{-1}$$
$$= \left\{ \sum_k \left[\frac{h_i c_i(\gamma_i \mathbf{p}_i, v_i^*)/h_k c_k(\gamma_k \mathbf{p}_k, v_k^*)}{R(\gamma_i \mathbf{p}_i, \gamma_k \mathbf{p}_k, \mathbf{v}^*, \mathbf{h})} \right]^{-1} \right\}^{-1}$$
$$= \left\{ \sum_k \left[\frac{h_i c_i(\mathbf{p}_i, v_i^*)}{h_k c_k(\mathbf{p}_k, v_k^*)} \bigg/ R(\mathbf{p}_i, \mathbf{p}_k, \mathbf{v}^*, \mathbf{h}) \right]^{-1} \right\}^{-1}$$
$$= S_i(\mathbf{P}, \mathbf{v}^*, \mathbf{h}). \quad \blacksquare$$

Proof of Corollary 1. By the positivity and transitivity properties of R,

$$\frac{S_i(\mathbf{P}, \mathbf{v}^*, \mathbf{h})/h_i}{S_j(\mathbf{P}, \mathbf{v}^*, \mathbf{h})/h_j} := \frac{c_i(\mathbf{p}_i, v_i^*) \sum_k h_k c_k(\mathbf{p}_k, v_k^*) R(\mathbf{p}_j, \mathbf{p}_k, \mathbf{v}^*, \mathbf{h})}{c_j(\mathbf{p}_j, v_j^*) \sum_k h_k c_k(\mathbf{p}_k, v_k^*) R(\mathbf{p}_i, \mathbf{p}_k, \mathbf{v}^*, \mathbf{h})}$$
$$= \frac{c_i(\mathbf{p}_i, v_i^*)}{c_j(\mathbf{p}_j, v_j^*)} \frac{\sum_k h_k c_k(\mathbf{p}_k, v_k^*) R(\mathbf{p}_i, \mathbf{p}_k, \mathbf{v}^*, \mathbf{h})/R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}^*, \mathbf{h})}{\sum_k h_k c_k(\mathbf{p}_k, v_k^*) R(\mathbf{p}_i, \mathbf{p}_k, \mathbf{v}^*, \mathbf{h})}$$
$$= \frac{c_i(\mathbf{p}_i, v_i^*)}{c_j(\mathbf{p}_j, v_j^*)} \bigg/ R(\mathbf{p}_i, \mathbf{p}_j, \mathbf{v}^*, \mathbf{h}). \quad \blacksquare$$

REFERENCES

1. R. G. D. Allen, The economic theory of index numbers, *Economica* **16** (1949), 197–203.
2. B. Balassa, The purchasing-power parity doctrine: a reappraisal, *J. Polit. Econ.* **72** (1964), 584–596.
3. W. E. Diewert, Exact and superlative index numbers, *J. Econometrics* **4** (1976), 115–145.
4. W. E. Diewert, The economic theory of index numbers: a survey, *in* "Essays in the Theory and Measurement of Consumer Behaviour in Honour of Sir Richard Stone" (A. Deaton, Ed.), pp. 163–208, Cambridge University Press, London, 1981.
5. W. E. Diewert, Duality approaches to microeconomic theory, *in* "Handbook of Mathematical Economics" (K. J. Arrow and M. D. Intriligator, Eds.), Vol. 2, pp. 535–599, North-Holland, Amsterdam, 1982.

6. W. E. Diewert, The theory of the cost-of-living index and the measurement of welfare change, *in* "Price Level Measurement: Proceedings from a Conference Sponsored by Statistics Canada" (W. E. Diewert and C. Montmarquette, Eds.), pp. 163–233, Minister of Supply and Services, Ottawa, 1983.
7. W. E. Diewert, Group cost-of-living indexes: approximations and axiomatics, *Meth. Oper. Res.* **48** (1984), 23–45.
8. W. E. Diewert, "Microeconomic Approaches to the Theory of International Comparisons," Discussion Paper 86-31, Department of Economics, University of British Columbia, 1986.
9. F. M. Fisher and Z. Griliches, Aggregate price indices, new goods, and generics, *Quart. J. Econ.* **110** (1995), 229–244.
10. F. M. Fisher and K. Shell, "Economic Analysis of Production Price Indexes," Cambridge University Press, Cambridge, UK, 1998.
11. Z. Griliches, Hedonic price indexes for automobiles: an econometric analysis of quality change, *in* "The Price Statistics of the Federal Government," General Series, No. 73, National Bureau of Economic Research, New York, 1961. [As reprinted in Griliches, 1971]
12. Z. Griliches, Hedonic price indexes revisited: some notes on the state of the art, *in* "Proceedings of the Business and Economics Statistics Section," pp. 324–332, American Statistical Association, Washington, 1967. [As reprinted in Griliches, 1971]
13. Z. Griliches, "Price Indexes and Quality Change: Studies in New Methods of Measurement," Harvard University Press, Cambridge, MA, 1971.
14. G. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities," 2nd ed., Cambridge University Press, Cambridge, UK, 1952.
15. A. A. Konüs, The problem of the true index of the cost of living, *Econometrica* **7** (1939, translation, 1924), 10–29.
16. I. B. Kravis and R. E. Lipsey, International price comparisons by regression methods, *in* "Price Competitiveness in World Trade," Chap. 5, NBER, New York, 1971. [As reprinted in Griliches, 1971]
17. R. A. Pollak, "The Theory of the Cost-of-Living Index," Research Discussion Paper 11, Research Division, Office of Prices and Living Conditions, U. S. Bureau of Labor Statistics, Washington, 1971. [As reprinted in Pollak, 1989]
18. R. A. Pollak, Group cost-of-living indexes, *Amer. Econ. Rev.* **70** (1980), 273–278. [As reprinted in Pollak, 1989]
19. R. A. Pollak, The social cost-of-living index, *J. Public Econ.* **15** (1981), 311–336. [As reprinted in Pollak, 1989]
20. R. A. Pollak, "The Theory of the Cost-of-Living Index," Oxford University Press, New York, 1989.
21. S. Prais, Whose cost of living? *Rev. Econ. Stud.* **26** (1959), 126–134.
22. P. A. Samuelson and S. Swamy, Invariant economic index numbers and canonical duality: survey and synthesis, *Amer. Econ. Rev.* **64** (1974), 566–593.