A Generalized Profit Function

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This paper provides a generalization of the profit function by relaxing the assumption of price-taking behaviour. The resulting profit-maximization problem is analyzed, and forms of a generalized profit function together with the corresponding generalized cost function and their properties are established under alternative assumptions about the functional form of the (inverse) output demands and input supplies faced by the firm.

Key Words: imperfect competition; non-price-taking behaviour; producer theory.

JEL Classification Numbers: D21, D43, L13.

1. Introduction

As is well known by economists, the *profit function* is defined as the maximum profit that can be made by a firm that is a price-taker in all relevant markets as a function of the prices of the inputs it demands from and the outputs it supplies to those markets. This function has a number of important, somewhat intuitive properties that are artifacts of the assumption of profit maximization only: convexity, homogeneity of degree one, and continuity in all relevant prices, non-increasingness in input prices, non-decreasingness in output prices, and its derivatives being equal to the firm's associated *net* output supply functions. The latter property is known as Hotelling's lemma on account of the fact that Hotelling (1932, p. 594) was the first to articulate it.[1] The homogeneity, continuity, and monotonicity properties were first stated by Samuelson (1953–1954, p. 20)[2] and the convexity property by Gorman (1968, §3). Further historical notes on

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[1] "Just as we have a utility (or profit) function u of the quantities consumed whose derivatives are the prices, there is, dually, a function of the prices whose derivatives are the quantities consumed."

[2] "For any set of p's being given and quantities of the remaining variables being given, there will be a maximum value for ΣpX, where the summation is over the prescribed p's. This maximum value can be written as $V(p; X)$ where it is understood that no good ever has both its p and X specified. V is a continuous and homogeneous function of the first order in the p variables alone .... The vector $\partial V/\partial p$ is proportional to the X's ...," which implies that V gets smaller as p rises for any $X < 0$ and larger as p rises for any $X > 0$. Note that "inputs [are treated] as if they were negative outputs" (ibid., p. 17).

the study of the profit function, including related duality theorems and suitable functional forms, are presented by Diewert (1974, p. 141). More recent developments are discussed in Pastor et al. (2016) and Aparicio et al. (2016).

In all such studies it is taken as axiomatic that the underlying firm is a price-taker. Debreu (1959, p. 43) provides one of the more fulsome justifications for this assumption in asserting that “each producer considers prices as given because, for example, his output or input of any commodity is relatively small and he thinks his action cannot influence prices.” In the real world, of course, this is very often *not* the situation faced by firms in one or more relevant markets. Non-price-taking behaviour in an output market requires modelling the demand side of that market along with the supplies by all firms in the industry in question; non-price-taking behaviour in an input market requires modelling the supply side of that market along with the demands by all firms, including those outside the industry in question. As market prices are no longer parametric to the firm, the resulting *generalized* profit function will not depend on them, but rather on the parameters of the functional form used to specify output demands and input supplies. Certain functional forms yield generalized profit functions with properties similar to those of the profit function (stated above).

The decision problem of a profit-maximizing firm that exhibits non-price-taking behaviour in all markets in which it is either a seller or a buyer is analyzed in Section 2 The associated (form of) generalized profit function and properties thereof are established for each of three different functional forms for (inverse) output demands and input supplies in Sections 3 and 4. Section 5 concludes.

2. Theoretical Model

Suppose there are $n \geq 2$ commodity types. Starting from the "shut down" point $\mathbf{0}_n$, with no inputs demanded and no outputs supplied, production by the firm entails using up certain amounts of some types of commodities and generating certain amounts of others. Since amounts used up are draw-downs of existing stocks, they can be thought of as negative numbers; since amounts generated are additions to existing stocks, they can be thought of as positive numbers (any commodities neither used up nor generated are zeros). Reckoned in this way, a production plan $\mathbf{y} \in \mathbb{R}^n$ is simply a discrete change from $\mathbf{0}_n$ with negative components corresponding to input quantities and positive components corresponding to output quantities.

The state of technology at the time of production determines which production plans are feasible. The set of all feasible production plans from which the firm can choose is the production possibilities set $\mathcal{Y} \subseteq \mathbb{R}^n$. The *im*possibility of getting something for nothing or "no free lunch" implies that $\mathbb{R}^n_+ \backslash \{\mathbf{0}_n\} \subsetneq \mathcal{Y}$. The possibility of unbounded waste of acquired quantities at no cost or "free disposal" implies that $\mathbb{R}^n_- \subseteq \mathcal{Y}$. If $\mathcal{Y}$ is closed and satisfies free disposal, then there exists a transformation function $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(\mathbf{y}) \leq 0$ if and only if $\mathbf{y} \in \mathcal{Y}$.[3] The equation $T(\mathbf{y}) = 0$ describes the transformation frontier since $T(\mathbf{y}') > 0$ for all $\mathbf{y}' \in \mathbb{R}^n$ such that $\mathbf{y}' > \mathbf{y}$ and since, by free disposal, $T(\mathbf{y}') < 0$ for all $\mathbf{y}' \in \mathbb{R}^n$ such that $\mathbf{y}' < \mathbf{y}$. The transformation frontier is therefore a *monotonic* function.

Assuming that $T(\cdot)$ is differentiable, we can totally differentiate the associated frontier to

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[3] Or, equivalently, $T(\mathbf{y}) > 0$ if and only if $\mathbf{y} \notin \mathcal{Y}$.

get

$$\sum_{i=1}^{n} T_i(\mathbf{y}) dy_i = 0 \; .$$

For $dy_\ell \neq 0 \neq dy_k$ and $dy_j = 0 \; \forall j : \ell \neq j \neq k$, we have

$$T_\ell(\mathbf{y}) dy_\ell + T_k(\mathbf{y}) dy_k = 0 \; ,$$

which implies the marginal rate of transformation of ℓ from k:

$$MRT_{\ell k}(\mathbf{y}) := -\frac{dy_k}{dy_\ell}\bigg|_{T(\mathbf{y})=0} = \frac{T_\ell(\mathbf{y})}{T_k(\mathbf{y})} \; .$$

Other things being equal, $MRT_{\ell k}(\mathbf{y})$ is the opportunity cost of the production of ℓ in terms of k if both commodities are outputs, the marginal rate of technical substitution of ℓ for k if both commodities are inputs, and the marginal product of ℓ in the production of k if the former is an input and the latter is an output.

The price of commodity i is given as $P_i(Y_i)$, where $P_i : \mathbb{R}_{-/+} \rightarrow \mathbb{R}_{++}$ is the associated decreasing, twice-continuously differentiable, inverse total-market input-supply (if the domain of P_i is $\mathbb{R}_-$) *or* output-demand (if the domain of P_i is $\mathbb{R}_+$) function, and $Y_i - Y_i^0 \equiv \mathbf{1}_m \cdot (: y_i :)$ is *either* the negative of the quantity of input demanded, in which case $y_i < 0$ and $Y_i^0 \leq 0$, *or* the quantity of output supplied, in which case $y_i > 0$ and $Y_i^0 \equiv 0$, by the m firms in the industry.[4] Since each price is a function of the unweighted sum of quantities demanded or supplied of the associated commodity, the conjectural variation in cases where m is a finite integer greater than one is that each firm treats the inputs and outputs of the other firms as given when it chooses its own inputs and outputs. Since all these firms are identical by assumption, their choices are the same and we can focus on just one of them.

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[4] Notation: $(: y_i :) \in \mathbb{R}^m$ denotes the vector of *individual* net outputs of commodity i associated with the m firms.

The firm's profit-maximization problem is

$$\max_{\mathbf{y}} \left\{ \sum_{i=1}^{n} P_i(Y_i) y_i : T(\mathbf{y}) = 0 \right\} . \tag{1}$$

The first-order (necessary) conditions for an interior solution $\mathbf{y}^*$ to this decision problem imply

$$\frac{\left[1 + \frac{1}{\varepsilon_\ell^*} \frac{y_\ell^*}{Y_\ell^*}\right] P_\ell(Y_\ell^*)}{\left[1 + \frac{1}{\varepsilon_k^*} \frac{y_k^*}{Y_k^*}\right] P_k(Y_k^*)} = \frac{T_\ell(\mathbf{y}^*)}{T_k(\mathbf{y}^*)} \, , \, k \neq \ell,$$

where $\varepsilon_i^* := \frac{P_i(Y_i^*)}{P_i'(Y_i^*) Y_i^*} \notin (-1, 0]$ is the elasticity of the input supply (if $Y_i^* < 0$) *or* output demand (if $Y_i^* > 0$) for commodity $i = 1, \ldots, n$. Note that the left-hand side of this expression reduces to the ratio of the prices of commodities ℓ and k in the perfectly competitive case.[5]

The Hessian of the Lagrange function evaluated at the aforementioned interior optimum is the n by n matrix of second-order partial derivatives defined by

$$\frac{\partial^2 \mathcal{L}^*}{\partial y_i^2} = P_i''(Y_i^*) y_i^* + 2P_i'(Y_i^*) - \lambda^* T_{ii}(\mathbf{y}^*)$$

and

$$\frac{\partial^2 \mathcal{L}^*}{\partial y_i \partial y_j} = -\lambda^* T_{ij}(\mathbf{y}^*) \, ,$$

where

$$\lambda^* = \frac{P_i'(Y_i^*) y_i^* + P_i(Y_i^*)}{T_i(\mathbf{y}^*)} \ \ \forall i$$

is the Lagrange multiplier. The second-order sufficient conditions for $(\mathbf{y}^*, \lambda^*)$ are

$$(-1)^i \begin{vmatrix} 0 & T_1^* & \cdots & T_i^* \\ T_1^* & \mathcal{L}_{11}^* & \cdots & \mathcal{L}_{1i}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_i^* & \mathcal{L}_{i1}^* & \cdots & \mathcal{L}_{ii}^* \end{vmatrix} > 0 \, , \, i = 2, \ldots, n.$$

Since the objective function of the firm's decision problem (1) is continuous when the constituent $P_i(\cdot)$ functions are continuous and since the associated constraint set given by $T(\mathbf{y}) = 0$ is a non-empty, compact-valued, continuous correspondence, the associated

5 Since $\varepsilon_i^* \to +\infty$ if $Y_i^* < 0$ and $\varepsilon_i^* \to -\infty$ if $Y_i^* > 0$.

optimum $\pi^*(\mathbf{a}, \mathbf{b})$ is a continuous function and the associated optimal choice set given as $\mathbf{y}^*$ is an upper-hemicontinuous correspondence (which, if single-valued, is a continuous function). The proof of this assertion follows directly from the Theorem of the Maximum (Berge, 1963, pp. 115–17).

Let $N := \{1, \ldots, n\}$ denote the set of commodity types. This set can be partitioned between the non-empty (sub)set of commodities used N_- and the non-empty (sub)set of commodities produced N_+ in relation to any interior solution $\mathbf{y}^*$ to the firm's profit maximization problem so that $\mathbf{y}^*_- \in \mathbb{R}^{|N_-|}_-$ is the associated vector of inputs and $\mathbf{y}^*_+ \in \mathbb{R}^{|N_+|}_+$ is the associated vector of outputs. Since the roles of different commodity types in different production plans are not necessarily pre-determined, $\{N_-, N_+\}$ is in general a *flexible* partition of N. Using this conceptualization, we can re-write the firm's profit maximization problem equivalently as

$$\max_{\mathbf{y}_+} \left\{ \sum_{i \in N_+} P_i(Y_i) y_i - c^*(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+) : T(\mathbf{y}^*_-, \mathbf{y}_+,) = 0 \right\} ,$$

where

$$c^*(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+) := \min_{\mathbf{y}_-} \left\{ - \sum_{i \in N_-} P_i(Y_i) y_i : T(\mathbf{y}_-; \mathbf{y}_+) = 0 \right\}$$

denotes the generalized cost function and $\mathbf{y}^\pm_- := \mathbf{y}_-(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+)$ denotes the solution to the defining cost minimization problem for the given vector of outputs $\mathbf{y}_+$ corresponding to some partition $\{N_-, N_+\}$ of N such that there exists at least one $\mathbf{y}_- \in \mathbb{R}^{|N_-|}_-$ which, together with $\mathbf{y}_+ \in \mathbb{R}^{|N_+|}_+$, satisfies $T(\mathbf{y}_-, \mathbf{y}_+) = 0$.

Note that $\mathbf{y}^\pm_- \equiv \mathbf{y}^*_-$ at $\mathbf{y}_+ = \mathbf{y}^*_+$. Note also that $c^* : \mathbb{R}^{n-N\setminus\{n\}}_+ \times \mathbb{R}^{n-N\setminus\{n\}}_+ \times \mathbb{R}^{N\setminus\{n\}} \to \mathbb{R}$ is a *variadic* function—i.e., one that accepts a variable number of arguments.[6] The domain of the

[6] Variadic functions are a common feature of modern computer programming languages. See, for example, the pages

$\mathbf{y}_+$ argument is any vector of real numbers of dimension 1 through $n-1$; the domain of each of the $\mathbf{a}_-$ and $\mathbf{b}_-$ arguments is any vector of non-negative real numbers of dimension n minus the dimension of the associated $\mathbf{y}_+$.

In order to derive more-precise profit- and cost-function analogues and establish their properties, it is necessary to specify the other side of each of the markets in which our m-firm industry participates. Towards this end, the following two sections employ, in turn, three simple functional forms used frequently in applied market analysis.

3. Log-Linearity

Assuming that the inverse total-market input-supply and output-demand functions are log-linear, we have

$$P_i(Y_i; a_i, b_i) = \begin{cases} b_i\,(-Y_i)^{a_i} & \text{if } Y_i \leq 0 \\ b_i Y_i^{-a_i} & \text{if } Y_i \geq 0 \end{cases}$$

or, equivalently,

$$P_i(Y_i; a_i, b_i) = b_i\,[Y_i \operatorname{sgn} Y_i]^{-a_i \operatorname{sgn} Y_i} \ ,$$

where $b_i > 0$ and $1 > a_i \geq 0$ for all $i \in \{1, \ldots, n\}$.[7] These functions are isoelastic since

$$\varepsilon_i := \frac{P_i(Y_i)}{P_i'(Y_i)Y_i} = \frac{b_i\,[Y_i \operatorname{sgn} Y_i]^{-a_i \operatorname{sgn} Y_i}}{-a_i\,(\operatorname{sgn} Y_i)\,b_i\,[Y_i \operatorname{sgn} Y_i]^{-a_i \operatorname{sgn} Y_i - 1}\,(\operatorname{sgn} Y_i)\,Y_i} = \frac{-1}{a_i \operatorname{sgn} Y_i}$$

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at https://www.gnu.org/savannah-checkouts/gnu/libc/manual/html_node/Variadic-Functions.html of the GNU C Library (glibc) manual.

[7] The sign or signum function of a real number x is defined as

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \ .$$

does not depend on the magnitude of Y_i. The associated generalized profit function is

$$\pi^*_{LL}(\mathbf{a},\mathbf{b}) \equiv \sum_{i=1}^{n} b_i \Phi_i(y_i^*; a_i) y_i^* \ ,$$

where

$$\Phi_i(y_i^*; a_i) := \left[\left(m y_i^* + Y_i^0\right)(\operatorname{sgn} y_i^*)\right]^{-a_i \operatorname{sgn} y_i^*} \ ,$$

and the associated generalized cost function is

$$c^*_{LL}(\mathbf{a}_-,\mathbf{b}_-,\mathbf{y}_+) \equiv \sum_{i\in N_-} b_i \Phi_i(y_i^+; a_i)\left(-y_i^+\right) \ ,$$

where

$$\Phi_i(y_i^+; a_i) := \left[m\left(-y_i^+\right) + \left(-Y_i^0\right)\right]^{a_i} \ .$$

By the envelope theorem,

$$\frac{\partial \pi^*_{LL}(\mathbf{a},\mathbf{b})}{\partial b_i} = \Phi_i(y_i^*; a_i) y_i^* \ ,$$

which is equal to y_i^* if $a_i = 0$, the case of perfectly elastic input supply or output demand. Other properties of $\pi^*_{LL}(\mathbf{a},\mathbf{b})$ are convexity in the scale parameters $\mathbf{b}$, non-increasingness in the scale parameters associated with inputs, and non-decreasingness in the scale parameters associated with outputs. Thus, for fixed $\mathbf{a}$, $\pi^*_{LL}(\mathbf{a},\mathbf{b})$ exhibits a modified version of Hotelling's lemma as well as all other properties of the profit function with the exception of homogeneity. Precise statements and proofs of these properties can be found in the appendix.

Similarly,

$$\frac{\partial c^*_{LL}(\mathbf{a}_-,\mathbf{b}_-,\mathbf{y}_+)}{\partial b_i} = \Phi_i(y_i^+; a_i)\left(-y_i^+\right) \ ,$$

which is equal to $-y_i^+ > 0$ if $a_i = 0$, the case of perfectly elastic input supply, and $c^*_{LL}(\mathbf{a}_-,\mathbf{b}_-,\mathbf{y}_+)$ is concave and non-decreasing in the scale parameters $\mathbf{b}_-$ since $\pi^*_{LL}(\mathbf{a},\cdot)$ is a sum of n functions that are necessarily convex and $-c^*_{LL}(\mathbf{a}_-,\cdot,\mathbf{y}_+)$ is a sum of $|N_-| < n$ of those functions. Thus, for fixed $\mathbf{a}_-$ and $\mathbf{y}_+$, $c^*_{LL}(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+)$ exhibits a modified version of Shephard's lemma as well as all other properties of the cost function with the exception of homogeneity (of degree one).

4. Linearity or Semi-Log-Linearity

Assuming instead that the inverse total-market input-supply and output-demand functions are linear or semi-log-linear, we have, respectively,

$$P_i(Y_i; a_i, b_i) = \begin{cases} b_i - a_i Y_i & \text{if } Y_i < \frac{b_i}{a_i} \\ 0 & \text{if } Y_i \geq \frac{b_i}{a_i} \end{cases}$$

or

$$P_i(Y_i; a_i, b_i) = \begin{cases} b_i - (\operatorname{sgn} Y_i)\, a_i \ln(Y_i \operatorname{sgn} Y_i) & \text{if } Y_i < 0 \text{ or } 0 < Y_i < \exp \frac{b_i}{a_i} \\ 0 & \text{if } Y_i \geq \exp \frac{b_i}{a_i} \end{cases} \quad ,$$

where $b_i \geq 0$ and $a_i \geq 0$ for all $i \in \{1, \ldots, n\}$. These functions have non-constant elasticities given as

$$\varepsilon_i := \frac{P_i(Y_i)}{P_i'(Y_i)Y_i} = \frac{b_i - a_i Y_i}{-a_i Y_i} = 1 - \frac{b_i}{a_i Y_i}$$

in the former case and

$$\varepsilon_i := \frac{P_i(Y_i)}{P_i'(Y_i)Y_i} = \frac{b_i - (\operatorname{sgn} Y_i)\, a_i \ln(Y_i \operatorname{sgn} Y_i)}{\frac{-(\operatorname{sgn} Y_i) a_i}{Y_i \operatorname{sgn} Y_i} (\operatorname{sgn} Y_i)\, Y_i} = \ln(Y_i \operatorname{sgn} Y_i) - \frac{b_i}{a_i} \operatorname{sgn} Y_i$$

in the latter. The associated generalized profit function is

$$\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}) \equiv \sum_{i=1}^{n} \left[b_i - a_i \phi_i(y_i^*)\right] y_i^* \, ,$$

where

$$\phi_i(y_i^*) := m y_i^* + Y_i^0$$

in the case of linearity or

$$\phi_i(y_i^*) := (\operatorname{sgn} y_i^*) \ln \left(\left[m y_i^* + Y_i^0 \right] \operatorname{sgn} y_i^* \right)$$

in the case of semi-log-linearity. The associated generalized cost function is

$$c^*_{\boldsymbol{\phi}}(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+) \equiv \sum_{i \in N_-} \left[b_i - a_i \phi_i(y_i^+)\right] \left(-y_i^+\right) \ ,$$

where

$$\phi_i(y_i^+) := m y_i^+ + Y_i^0$$

in the case of linearity or

$$\phi_i(y_i^+) := -\ln\left(m\left[-y_i^+\right] + \left[-Y_i^0\right]\right)$$

in the case of semi-log-linearity.

By the envelope theorem,

$$\frac{\partial \pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})}{\partial b_i} = y_i^* \ .$$

Other properties of $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})$ are convexity and homogeneity of degree one in the slope *and* scale parameters $(\mathbf{a}, \mathbf{b})$, non-increasingness in the scale parameters associated with inputs, non-decreasingness in the scale parameters associated with outputs, non-decreasingness in the slope parameters associated with inputs, and non-increasingness in the slope parameters associated with outputs. Thus, for fixed $\mathbf{a}$, $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})$ exhibits Hotelling's lemma and the monotonicity properties of the profit function; it also exhibits the opposite monotonicity properties for fixed $\mathbf{b}$ and the convexity and homogeneity properties in relation to both sets of parameters together. Precise statements and proofs of these properties can be found in the appendix.

Similarly,

$$\frac{\partial c^*_{\boldsymbol{\phi}}(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+)}{\partial b_i} = -y_i^+$$

and $c^*_{\boldsymbol{\phi}}(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+)$ is concave and homogeneous of degree one in the slope *and* scale parameters $(\mathbf{a}_-, \mathbf{b}_-)$, non-decreasing in $\mathbf{b}_-$, and non-increasing in $\mathbf{a}_-$.[8] Thus, for fixed $\mathbf{a}_-$ and $\mathbf{y}_+$,

[8] Like the case of $c^*_{LL}(\cdot)$, the properties of $c^*_{\boldsymbol{\phi}}(\cdot)$ hold because the associated profit function is a sum of n functions

$c^*_\phi(\mathbf{a}_-, \mathbf{b}_-, \mathbf{y}_+)$ exhibits Shephard's lemma and the monotonicity properties of the cost function; it also exhibits the opposite monotonicity properties for fixed $\mathbf{b}_-$ and the concavity and homogeneity properties in relation to both sets of parameters together.

5. Conclusion

For $\mathbf{a} = \mathbf{0}_{(n)}$, the inverse total-market input-supply and output-demand functions for all three cases discussed in the preceding two sections reduce to $P_i(Y_i; a_i, b_i) = b_i$, which corresponds to perfectly elastic supply/demand ($\varepsilon_i = \pm\infty$) for each commodity $i \in \{1, \ldots, n\}$ used/produced (bought/sold) by our m-firm industry. Consequently,

$$\pi^*_{LL}(\mathbf{0}, \mathbf{b}) \equiv \pi^*_\phi(\mathbf{0}, \mathbf{b}) \equiv \pi(\mathbf{b}) \ ,$$

where $\pi(\cdot)$ is the profit function and $\mathbf{b}$ is the vector of competitive prices taken as given by each firm, which shows that $\pi^*_{LL}(\cdot)$ and $\pi^*_\phi(\cdot)$ are consistent generalizations. In addition,

$$c^*_{LL}(\mathbf{0}, \mathbf{b}_-, \mathbf{y}_+) \equiv c^*_\phi(\mathbf{0}, \mathbf{b}_-, \mathbf{y}_+) \equiv c(\mathbf{b}_-, \mathbf{y}_+) \ ,$$

where $c(\cdot)$ is the cost function and $\mathbf{b}_-$ is the vector of competitive input prices taken as given by each firm, which shows that $c^*_{LL}(\cdot)$ and $c^*_\phi(\cdot)$ are consistent generalizations.

The properties of these generalizations for $\mathbf{a} \neq \mathbf{0}$ differ somewhat from each other as well as from those of the profit/cost function, however. For $\mathbf{a}$ fixed, the isoelastic, log-linear form exhibits a modified version of Hotelling's/Shephard's lemma and the monotonicity and convexity/concavity properties of the profit/cost function, but not homogeneity of degree one, whereas the non-constant-elasticity, linear or semi-log-linear forms exhibit Hotelling's/Shephard's

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that are necessarily convex and homogeneous of degree one and $-c^*_\phi(\cdot)$ is a sum of $|N_-| < n$ of those functions.

lemma and the monotonicity properties of the profit/cost function, but neither convexity/concavity nor homogeneity (in $\mathbf{b}$). Although there are undoubtedly other forms that are consistent generalizations of the profit/cost function—including some based on different conjecture variations from the one assumed herein—none are likely to exhibit *all* of their properties. The choice of functional form for possible empirical purposes therefore boils down to data issues concerning the estimability of the supply and demand sides of the markets in which the industry under examination demands and supplies, respectively.

Appendix

Properties of $\pi^*_{LL}(\mathbf{a},\mathbf{b})$: (i) Non-decreasing/increasing in the scale parameters associated with outputs/inputs. If $b_i' \geq b_i$ for all outputs and $b_j' \leq b_j$ for all inputs, then $\pi^*_{LL}(\mathbf{a},\mathbf{b}') \geq \pi^*_{LL}(\mathbf{a},\mathbf{b})$. Proof: Let $\mathbf{y}^*$ and $\mathbf{y}'$ be profit-maximizing at $\mathbf{b}$ and $\mathbf{b}'$, respectively, so that $\pi^*_{LL}(\mathbf{a},\mathbf{b}) \equiv \sum_{i=1}^n b_i\Phi_i(y_i^*;a_i)y_i^*$ and $\pi^*_{LL}(\mathbf{a},\mathbf{b}') \equiv \sum_{i=1}^n b_i'\Phi_i(y_i';a_i)y_i'$. Since $T(\mathbf{y}^*) = T(\mathbf{y}') = 0$, $\mathbf{y}^*$ is feasible but not necessarily optimal at $\mathbf{b}'$ so that

$$\sum_{i=1}^n b_i'\Phi_i(y_i';a_i)y_i' \geq \sum_{i=1}^n b_i'\Phi_i(y_i^*;a_i)y_i^* \,.$$

Since $b_i' \geq b_i$ for all i for which $y_i \geq 0$ and $b_i' \leq b_i$ for all i for which $y_i \leq 0$,

$$\sum_{i=1}^n b_i'\Phi_i(y_i^*;a_i)y_i^* \geq \sum_{i=1}^n b_i\Phi_i(y_i^*;a_i)y_i^* \,.\text{ Q.E.D.}$$

(ii) Convex in $\mathbf{b}$. Let $\mathbf{b}'' := t\mathbf{b} + (1-t)\,\mathbf{b}'$ for some $t \in [0,1]$. Then $\pi^*_{LL}(\mathbf{a},\mathbf{b}'') \leq t\pi^*_{LL}(\mathbf{a},\mathbf{b}) + (1-t)\,\pi^*_{LL}(\mathbf{a},\mathbf{b}')$. Proof: Let $\mathbf{y}^*$, $\mathbf{y}'$, and $\mathbf{y}''$ be profit-maximizing at $\mathbf{b}$, $\mathbf{b}'$, and $\mathbf{b}''$, respectively, so that

$$\begin{aligned}
\pi^*_{LL}(\mathbf{a},\mathbf{b}'') &\equiv \sum_{i=1}^n b_i''\Phi_i(y_i'';a_i)y_i'' \\
&= \sum_{i=1}^n \left[tb_i + (1-t)\,b_i'\right]\Phi_i(y_i'';a_i)y_i'' \\
&= t\sum_{i=1}^n b_i\Phi_i(y_i'';a_i)y_i'' + (1-t)\sum_{i=1}^n b_i'\Phi_i(y_i'';a_i)y_i'' \\
&\leq t\sum_{i=1}^n b_i\Phi_i(y_i^*;a_i)y_i^* + (1-t)\sum_{i=1}^n b_i'\Phi_i(y_i';a_i)y_i'
\end{aligned}$$

since $\mathbf{y}''$ is feasible but not necessarily optimal at either $\mathbf{b}$ or $\mathbf{b}'$. Q.E.D.

Properties of $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})$: (i) Non-decreasing/increasing in the scale parameters associated with outputs/inputs. If $b'_i \geq b_i$ for all outputs and $b'_j \leq b_j$ for all inputs, then $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}') \geq \pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})$. Proof: Let $\mathbf{y}^*$ and $\mathbf{y}'$ be profit-maximizing at $\mathbf{b}$ and $\mathbf{b}'$, respectively, so that $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}) \equiv \sum_{i=1}^n [b_i - a_i\phi_i(y_i^*)]\, y_i^*$ and $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}') \equiv \sum_{i=1}^n [b'_i - a_i\phi_i(y'_i)]\, y'_i$. Since $T(\mathbf{y}^*) = T(\mathbf{y}') = 0$, $\mathbf{y}^*$ is feasible but not necessarily optimal at $\mathbf{b}'$ so that

$$\sum_{i=1}^n [b'_i - a_i\phi_i(y'_i)]\, y'_i \geq \sum_{i=1}^n [b'_i - a_i\phi_i(y_i^*)]\, y_i^* \, .$$

Since $b'_i \geq b_i$ for all i for which $y_i \geq 0$ and $b'_i \leq b_i$ for all i for which $y_i \leq 0$,

$$\sum_{i=1}^n [b'_i - a_i\phi_i(y_i^*)]\, y_i^* \geq \sum_{i=1}^n [b_i - a_i\phi_i(y_i^*)]\, y_i^* \, . \text{ Q.E.D.}$$

(ii) Non-increasing/decreasing in the slope parameters associated with outputs/inputs. If $a'_i \geq a_i$ for all outputs and $a'_j \leq a_j$ for all inputs, then $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}', \mathbf{b}) \leq \pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})$. Proof: Let $\mathbf{y}^*$ and $\mathbf{y}'$ be profit-maximizing at $\mathbf{a}$ and $\mathbf{a}'$, respectively, so that $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}) \equiv \sum_{i=1}^n [b_i - a_i\phi_i(y_i^*)]\, y_i^*$ and $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}', \mathbf{b}) \equiv \sum_{i=1}^n [b_i - a'_i\phi_i(y'_i)]\, y'_i$. Since $a'_i \geq a_i$ for all i for which $y_i \geq 0$ and $a'_i \leq a_i$ for all i for which $y_i \leq 0$,

$$\sum_{i=1}^n [b_i - a'_i\phi_i(y'_i)]\, y'_i \leq \sum_{i=1}^n [b_i - a_i\phi_i(y'_i)]\, y'_i \, .$$

Since $T(\mathbf{y}^*) = T(\mathbf{y}') = 0$, $\mathbf{y}'$ is feasible but not necessarily optimal at $\mathbf{a}$ so that

$$\sum_{i=1}^n [b_i - a_i\phi_i(y'_i)]\, y'_i \leq \sum_{i=1}^n [b_i - a_i\phi_i(y_i^*)]\, y_i^* \, . \text{ Q.E.D.}$$

(iii) Convex in $(\mathbf{a}, \mathbf{b})$. Let $(\mathbf{a}'', \mathbf{b}'') := t\,(\mathbf{a}, \mathbf{b}) + (1-t)\,(\mathbf{a}', \mathbf{b}')$ for some $t \in [0, 1]$. Then $\pi^*_{\boldsymbol{\phi}}(\mathbf{a}'', \mathbf{b}'') \leq t\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b}) + (1-t)\,\pi^*_{\boldsymbol{\phi}}(\mathbf{a}', \mathbf{b}')$. Proof: Let $\mathbf{y}^*$, $\mathbf{y}'$, and $\mathbf{y}''$ be profit-maximizing at

$(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}', \mathbf{b}')$, and $(\mathbf{a}'', \mathbf{b}'')$, respectively, so that

$$\begin{aligned}
\pi^*_{\boldsymbol{\phi}}(\mathbf{a}'', \mathbf{b}'') &\equiv \sum_{i=1}^{n} \left[b''_i - a''_i \phi_i(y''_i)\right] y''_i \\
&= \sum_{i=1}^{n} \left\{\left[t b_i + (1-t)\, b'_i\right] - \left[t a_i + (1-t)\, a'_i\right] \phi_i(y''_i)\right\} y''_i \\
&= t \sum_{i=1}^{n} \left[b_i - a_i \phi_i(y''_i)\right] y''_i + (1-t) \sum_{i=1}^{n} \left[b'_i - a'_i \phi_i(y''_i)\right] y''_i \\
&\leq t \sum_{i=1}^{n} \left[b_i - a_i \phi_i(y''_i)\right] y^*_i + (1-t) \sum_{i=1}^{n} \left[b'_i - a'_i \phi_i(y''_i)\right] y'_i
\end{aligned}$$

since $\mathbf{y}''$ is feasible but not necessarily optimal at either $(\mathbf{a}, \mathbf{b})$ or $(\mathbf{a}', \mathbf{b}')$. Q.E.D. (iv) Homogenous of degree one in $(\mathbf{a}, \mathbf{b})$. $\pi^*_{\boldsymbol{\phi}}(t\mathbf{a}, t\mathbf{b}) = t\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})\ \forall t \geq 0$. Proof:

$$\pi^*_{\boldsymbol{\phi}}(t\mathbf{a}, t\mathbf{b}) \equiv \sum_{i=1}^{n} \left[t b_i - t a_i \phi_i(y^*_i)\right] y^*_i = t \sum_{i=1}^{n} \left[b_i - a_i \phi_i(y^*_i)\right] y^*_i = t\pi^*_{\boldsymbol{\phi}}(\mathbf{a}, \mathbf{b})\ .\ \text{Q.E.D.}$$

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