The Mathematics of Foreign Exchange

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April 29, 2021

It is well known that, since the supply of the currency of one country is the demand for the currency of another and vice versa, either may be treated as the quantity and the other as the price in a standard neoclassical model of the market for foreign exchange between the two countries. What is less well known is how the shape of the one curve is related to the shape of the equivalent other. Haberler (1936; 1949) and Machlup (1939; 1950) articulated this relationship to some extent, but neither they nor anyone else appears to have developed fully the particulars of the mathematics behind it. The present paper attempts to fill that lacuna.

Key Words: foreign exchange; supply and demand; functional-form relationships.

JEL Classification Numbers: A2, C02, D41, F31.

Declarations of Interest: None.

1. Introduction

There have been two main approaches to the development of a theory of short-run exchange-rate determination.[1] The more recent is the asset-market approach, which was first suggested in the 1970s and views exchange rates as being determined "in terms of stocks of currencies relative to the willingness of people to hold these stocks. Several variants of stock-based theories of exchange rates have been developed [over the] years, where these theories differ primarily in the range of different assets that are considered, and in how quickly product prices can adjust to changes in exchange rates" (Levi, 2009, p. 187).

The older traditional flow approach, also called the balance of payments view or the exchange-market approach, begins with the (irrefutable) fact that real-world (flexible) exchange rates are determined in foreign exchange markets subject to the forces of supply and demand (Gandolfo, 2001, p. 226). These forces result from the various components of the balance of payments, which account for flows of goods and services and financial capital and transfers across national borders, as well as from speculation about future exchange-rate movements. It is in specifying such supplies and demands that problems with the flow approach arise. The simplest way to do so is to follow Levi (2009, p. 166) and assume that their slopes depend on the

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[1] A third approach, purchasing power parity (PPP) theory, in either its absolute or relative version, "is put forward as a *long-run* theory of the equilibrium exchange rate, in the sense that in the short run there may be marked deviations from PPP which, however, set into motion forces capable of bringing the exchange rate back to its PPP value in the long term" (Gandolfo, 2001, p. 224).

effects of exchange rates on values of imports and exports, respectively, and that each of the other components of the balance of payments as well as speculation "can be considered as shifting the supply or demand curve." In other words, *ceteris paribus*, the quantity supplied of a country's currency for foreign-exchange purposes depends on the values of its imports at different values of the exchange rate measured in terms of foreign-currency units, the quantity demanded of the country's currency for foreign-exchange purposes depends on the values of its exports at different values of the exchange rate measured in terms of foreign-currency units, and all other influences on these quantities supplied or demanded are independent of the exchange rate.

"Since the supply of one currency constitutes the demand for the other and vice versa, we may treat either of them as a commodity [i.e., quantity] and the other as money [i.e., price]" (Haberler, 1936, p. 19). This is true because a willingness to buy a currency for foreign-exchange purposes at a particular exchange rate must be accompanied by a willingness to sell the relevant other currency for the same purposes at the reciprocal of the particular exchange rate, which is of course measured in terms of units of the desired currency. There are therefore two equivalent ways to view a given market for foreign exchange between two countries with different currencies. And as Machlup (1939, p. 376) said, "It is not difficult, for example, to translate the demand for dollars on the Paris market into a supply of francs on the combined foreign exchange market, and likewise the supply of dollars on the Paris market into a demand for francs on the combined foreign exchange market." Amusingly I am sure to most current university instructors, he went on to say:

> It is a good undergraduate exercise to practice such a translation: starting from a demand curve for dollars in terms of francs the amounts of dollars are shown by the horizontal axis (x), the amounts of francs offered in exchange for these dollars are shown by the rectangle (xy); this gives a supply curve of francs for dollars where the abscissae correspond to the

values of the rectangles in the original demand curve, while the ordinates on the new supply curve, i.e. the prices of francs in terms of dollars, correspond to the quotient of the abscissae divided by the values of the rectangles $\left(\frac{x}{xy}\right)$ of the original demand curve. The analogous calculation has to be done in order to transform the original supply curve of dollars in terms of francs into a demand curve for francs in terms of dollars. This sounds complicated—yet every sophomore ought to be able to do it, or he has never grasped the meaning of demand and supply curves. [Ibid.] [2]

Such an exercise of translating a demand or supply curve of one currency in terms of another into the equivalent supply or demand curve of the second currency in terms of the first suggests that the shapes of the two curves are related in a particular way. Haberler (1936, p. 20) would seem to have been the first to suggest that there might "be a point at which the supply curve curls backwards and slopes upwards to the left." He also noted that "the supply and demand curve are not symmetrical" and "[t]he point where the supply curve turns to the left corresponds to that point in the demand curve when the area of the inscribed rectangle begins to diminish." The latter claim is demonstrated graphically in relation to a straight-line demand curve (for francs in terms of dollars) in Haberler (1949; pp. 204–5). Machlup (1939, p. 384) also discussed the possibility of a backward-bending supply curve and noted that it "cannot rise backward more sharply" than shown in his "Figure 2" since "[t]he slope of the negative inclination must at every point be greater than that of the rectangular hyperbola passing through that point." This claim is illustrated more clearly in relation to a gently curved (convex-to-the-origin) demand curve (for lire in terms of dollars) in Machlup (1950; Fig. 1).

More recent treatments of the traditional flow approach discuss in elasticity terms the shapes of one pair of supply and demand curves, but make no reference to the equivalent other.

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2 "Professor Machlup … says that every undergraduate ought to know how that is to be done. But experience shows that he is too optimistic in making that assumption" (Haberler, 1949, p. 204).

Gandolfo (2001, §7.3) and Levi (2009, Ch. 8) in particular also discuss the possibility of a downward-sloping supply curve with justifications rooted in the fact that "for currencies we plot values (price × quantity) on the horizontal axis, whereas we normally [for non-currency goods] plot just physical quantities" (Levi, 2009, p. 178). Both provide diagrams illustrating the stability-of-equilibrium implications of downward-sloping supply with Gandolfo (2001, Fig. 7.1b) replicating (without attribution) the aforementioned backward-bending form derived by Haberler (1949).

An exception to the no-reference-to-the-equivalent-other norm is OpenStax College (2018), which asserts that, "In foreign exchange markets, demand and supply become closely interrelated, because a person or firm who demands one currency must at the same time supply another currency—and vice versa" (ibid., §29.1). Subsequently, however, there is no mention of how the assumed shape of one curve affects the shape of the other. And although the two-graph diagram illustrating "Demand and Supply for the U.S. Dollar and Mexican Peso Exchange Rate" (ibid., §29.2) portrays the corresponding equilibrium points correctly, it errs in its portrayal of corresponding non-equilibrium points.[3]

What has been missing from the literature, then, is a thorough analysis of the relationship between an assumed functional form of a country's foreign-exchange supply or demand and that of another country's foreign-exchange demand or supply. Section 2 specifies this relationship in general mathematical terms and then analyzes and provides illustrations of five different pairs of closed-form specifications of supply-and-demand or demand-and-supply. Section 3 concludes

[3] For instance, since $(7, 9)$ is very clearly a point on the supply curve of U.S. dollars, $xy = 63$ together with $\frac{x}{xy} \approx 0.111$ should be a point on the demand curve for Mexican pesos, which it most definitely is not!

after arguing that such specifications provide a more feasible and expeditious basis for empirical estimations than do those that might be derived from the underlying foreign-exchange demand behaviour of domestic and foreign agents (mostly large firms) for the purposes of international trade and foreign direct and portfolio investment abroad.

2. Mathematical Analysis

For the sake of simplicity, consider a world comprising just two countries, each with its own currency, and a single, competitive, free market within which one currency can be exchanged for the other. Taking the perspective of one country or the other, the market quantity supplied or demanded of the domestic currency is some non-decreasing or non-increasing function $\varphi : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ of the exchange rate e measured in units of foreign currency per unit of domestic currency.[4] The inverse correspondence $\varphi^{-1}(\cdot)$ can be used as the price[5] in converting a market quantity of domestic currency units Q into the equivalent quantity of foreign currency units Q^*:

$$Q^* = eQ = \varphi^{-1}(Q)Q =: \xi(Q) \ . \tag{1}$$

Clearly, the inverse correspondence $\xi^{-1}(\cdot)$ converts the quantity Q^* into the quantity Q and must be decomposable into the inverse (or reciprocal) exchange rate e^{-1} times Q^*:

$$Q = \xi^{-1}(Q^*) = e^{-1}Q^* \ . \tag{2}$$

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4 That is, the market for foreign exchange is assumed to be characterized by an upward-sloping supply curve together with a downward-sloping demand curve.

5 That is, willingness to accept or pay for the marginal unit on the part of sellers or buyers, respectively.

The inverse exchange rate is given by a correspondence $\psi^{-1}(\cdot)$ defined by

$$e^{-1} = \frac{\xi^{-1}(Q^*)}{Q^*} \ , \tag{3}$$

which means that the market quantity demanded or supplied of foreign currency is given by $\psi(e^{-1})$, which we would like to be a function. Whether it is or not depends on the assumed functional form of $\varphi(\cdot)$ as we will see below. Note that

$$\varphi^{-1}(Q)\psi^{-1}(Q^*) = ee^{-1} \equiv 1 \ . \tag{4}$$

Assuming that $\varphi^{-1}(\cdot)$ is a differentiable function, so is $\xi(\cdot)$ and the elasticity of supply of or demand for domestic currency is

$$\eta := \frac{e}{Q\frac{de}{dQ}} = \frac{\varphi^{-1}(Q)}{Q\varphi^{-1\prime}(Q)} = \frac{\varphi^{-1}(Q)}{\xi'(Q) - \varphi^{-1}(Q)} \ . \tag{5}$$

Assuming that $\psi^{-1}(\cdot)$ is a differentiable function, so is $\xi^{-1}(\cdot)$ and the elasticity of demand for or supply of foreign currency is

$$\varepsilon := \frac{e^{-1}}{Q^*\frac{de^{-1}}{dQ^*}} = \frac{\psi^{-1}(Q^*)}{Q^*\psi^{-1\prime}(Q^*)} = \frac{\psi^{-1}(Q^*)}{\xi^{-1\prime}(Q^*) - \psi^{-1}(Q^*)} \ . \tag{6}$$

Since $\xi^{-1\prime}(Q^*) = \frac{1}{\xi'(Q)}$ and $\psi^{-1}(Q^*) = \frac{1}{\varphi^{-1}(Q)}$, we have

$$\varepsilon = \frac{\frac{1}{\varphi^{-1}(Q)}}{\frac{1}{\xi'(Q)} - \frac{1}{\varphi^{-1}(Q)}} = \frac{\xi'(Q)}{\varphi^{-1}(Q) - \xi'(Q)} \ . \tag{7}$$

Summing the two elasticities as functions of Q yields

$$\eta + \varepsilon = -1 \ . \tag{8}$$

Note that this result is the same as that inferred by Haberler (1949, p. 205) from a geometric analysis of the affine demand case.[6] The foregoing constitutes a more rigorous proof of it.

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[6] Since Haberler (1949, n. 1) "follow[s] the usual procedure of taking η as positive (although the slope of the demand curve is conventionally called negative, [he felt the need] to use the same convention for the supply elasticity. It follows

The algebra of the affine demand case proceeds from assuming that

$$\varphi^{-1}(Q) = \alpha - \beta Q \,, \tag{9}$$

where $\alpha > 0$ and $\beta \geq 0$ are constants. This assumption implies that

$$\xi(Q) = \alpha Q - \beta Q^2 \tag{10}$$

so that

$$\eta = 1 - \frac{\alpha}{\beta Q} \,, \tag{11}$$

$$\varepsilon = \frac{\alpha}{\beta Q} - 2 \,, \tag{12}$$

and

$$-\xi(Q) + Q^* = \beta Q^2 - \alpha Q + Q^* = 0 \,. \tag{13}$$

Solving this (latter) equation for Q using the quadratic formula gives us

$$\xi^{-1}(Q^*) = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta Q^*}}{2\beta} \tag{14}$$

and then

$$\psi^{-1}(Q^*) = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta Q^*}}{2\beta Q^*} \,, \tag{15}$$

which is not a function—its inverse is, however, albeit one that is non-monotonic with a unique global maximum at $e^{-1} = 2/\alpha$. In the related special case of perfectly elastic demand (or supply), $\beta = 0$ so that $\varphi^{-1}(Q) = \alpha$, $-\alpha Q + Q^* = 0$, and $\psi^{-1}(Q^*) = a^{-1}$.

The "augmented power" (supply or demand) case assumes that

$$\varphi^{-1}(Q) = \alpha Q^{-1} + \beta Q^{\gamma} \,, \tag{16}$$

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that ε is positive when the supply curve is negatively inclined and negative when it is positively inclined." Therefore, the statement of his result in terms of the present notation is $(-\eta) + (-\varepsilon) = 1$.

where α, $\beta \neq 0$, and γ are constants, which implies that

$$\xi(Q) = \alpha + \beta Q^{\gamma+1} \tag{17}$$

and then

$$\eta = \frac{\alpha + \beta Q^{\gamma+1}}{\beta\gamma Q^{\gamma+1} - \alpha} \,, \tag{18}$$

$$\varepsilon = \frac{\beta Q^{\gamma+1}\left(\gamma+1\right)}{\alpha - \beta\gamma Q^{\gamma+1}} \,, \tag{19}$$

and

$$\psi^{-1}(Q^*) = (Q^*)^{-1}\left(\frac{Q^* - \alpha}{\beta}\right)^{\frac{1}{\gamma+1}} \,. \tag{20}$$

In the related special case of isoelastic demand, $\alpha = 0$, $\beta > 0$, and $\gamma < 0$ so that $\varphi^{-1}(Q) = \beta Q^{\gamma}$ and

$$\psi^{-1}(Q^*) = \left[\beta\left(Q^*\right)^{\gamma}\right]^{-\frac{1}{\gamma+1}} \,, \tag{21}$$

which is an increasing (isoelastic inverse supply) function over all $Q^* > 0$ if and only if $\gamma > -1$. In the related special case of linear supply, $\alpha = 0$, $\beta > 0$, and $\gamma = 1$ so that $\varphi^{-1}(Q) = \beta Q$ and $\psi^{-1}(Q^*) = (\beta Q^*)^{-\frac{1}{2}}$, which is a *non*-affine (isoelastic inverse demand) function.

The "depressed quadratic" supply case assumes that

$$\varphi^{-1}(Q) = \alpha + \beta Q^2 \,, \tag{22}$$

where α and $\beta > 0$ are constants, which implies that

$$\xi(Q) = \alpha Q + \beta Q^3 \tag{23}$$

so that

$$\eta = \frac{\alpha + \beta Q^2}{2\beta Q^2} \,, \tag{24}$$

$$\varepsilon = \frac{\alpha + 3\beta Q^2}{-2\beta Q^2} \,, \tag{25}$$

and

$$\beta Q^3 + \alpha Q - Q^* = 0 \,. \tag{26}$$

Solving this ("depressed cubic") equation for Q using del Ferro's (circa 1515) formula[7] gives us

$$\xi^{-1}(Q^*) = \sqrt[3]{\sqrt{\left(\frac{Q^*}{2\beta}\right)^2 + \left(\frac{\alpha}{3\beta}\right)^3} + \frac{Q^*}{2\beta}} - \sqrt[3]{\sqrt{\left(\frac{Q^*}{2\beta}\right)^2 + \left(\frac{\alpha}{3\beta}\right)^3} - \frac{Q^*}{2\beta}} \,, \tag{27}$$

which is an increasing, strictly concave, non-negative, real-valued (inverse demand) function over all Q^* such that

$$27\beta^3 \left(Q^*\right)^2 + 4\alpha^3\beta^2 \geq 0 \tag{28}$$

or, equivalently,

$$Q^* \geq \begin{cases} 2\sqrt{\frac{-\alpha^3}{27\beta}} > 0 & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha \geq 0 \end{cases} \,. \tag{29}$$

The "generalized depressed cubic" supply case assumes that

$$\varphi^{-1}(Q) = \alpha Q^\gamma + \beta Q^{2\gamma+1} \,, \tag{30}$$

where α, $\beta > 0$, and $\gamma \in \mathbb{N}$ are constants,[8] which implies that

$$\xi(Q) = \alpha Q^{\gamma+1} + \beta Q^{2(\gamma+1)} \tag{31}$$

so that

$$\eta = \frac{\alpha + \beta Q^{\gamma+1}}{\gamma + \beta\left(2\gamma+1\right) Q^{\gamma+1}} \,, \tag{32}$$

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[7] See Contreras (2015, pp. 25–26).

[8] Note that $\gamma = 0$ together with $\beta \leq 0$ and $\alpha > 0$ would render this case to be that of affine demand above.

$$\varepsilon = \frac{-(\alpha+\gamma) - 2\beta(\gamma+1)\,Q^{\gamma+1}}{\gamma + \beta(2\gamma+1)\,Q^{\gamma+1}}\ , \tag{33}$$

and

$$\beta Q^{2(\gamma+1)} + \alpha Q^{\gamma+1} - Q^* = 0\ . \tag{34}$$

Solving this equation for $Z := Q^{\gamma+1}$ using the quadratic formula gives us

$$\xi^{-1}(Q^*) = \sqrt[\gamma+1]{\frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta Q^*}}{2\beta}}\ , \tag{35}$$

which is an increasing, strictly concave, non-negative, real-valued function (ignoring paired non-positive real values when γ is even and paired non-real values when γ is odd) over all $Q^* \geq 0$. In the related special case of affine supply, $\gamma = 0$ so that $\varphi^{-1}(Q) = \alpha + \beta Q$ and

$$\psi^{-1}(Q^*) = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta Q^*}}{2\beta Q^*}\ , \tag{36}$$

which is a decreasing, strictly convex (and hence non-affine), non-negative, real-valued (inverse demand) function over all $Q^* > 0$.

The "augmented logarithmic" (supply or demand) case assumes that

$$\varphi^{-1}(Q) = \frac{\alpha}{Q} + \beta \ln Q\ , \tag{37}$$

where α and $\beta > 0$ are constants, which implies that

$$\xi(Q) = \alpha + \beta Q \ln Q \tag{38}$$

so that

$$\eta = \frac{\frac{\alpha}{\beta Q} + \ln Q}{1 - \frac{\alpha}{\beta Q}}\ , \tag{39}$$

$$\varepsilon = \frac{-\ln Q - 1}{1 - \frac{\alpha}{\beta Q}}\ , \tag{40}$$

and

$$[\exp(\ln Q)] \ln Q = \frac{Q^* - \alpha}{\beta} , \tag{41}$$

which implies in turn that

$$\ln Q = W\left(\tfrac{Q^* - \alpha}{\beta}\right) \tag{42}$$

by the definition of the Lambert W function.[9] Consequently,

$$\psi^{-1}(Q^*) = \frac{\exp\left[W\left(\tfrac{Q^* - \alpha}{\beta}\right)\right]}{Q^*} , \tag{43}$$

which is not necessarily a function and not necessarily everywhere decreasing over all $Q^* > 0$.

The preceding five cases would seem to constitute all those that are both analytic and have at least *sub*domains of positive prices (exchange rates) for which the associated supply and demand relations are, respectively, non-decreasing and non-increasing functions. The key to constructing them was to first specify a functional form for $\xi(\cdot)$ that has an analytic inverse $\xi^{-1}(\cdot)$. In doing so, it also has to be true that some parameterization of the form for $\xi(\cdot)$ has an associated form for $\varphi(\cdot)$ that is non-increasing or non-decreasing on some part of its domain and that the same parameterization of the form for $\xi^{-1}(\cdot)$ has an associated form for $\psi(\cdot)$ that is non-decreasing or non-increasing on some part of its domain.

Illustrations of the five cases are provided by the eight figures herein. Each figure includes (i) the graph of a specific parameterization of one of the five functional forms of $\varphi(\cdot)$ labelled "S" or "D" as appropriate with the vertical axis measured in units of foreign currency (¥) per unit of domestic currency ($) and the horizontal axis measured in z-illions[10] of units of domestic

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[9] $W(x)$ is defined as the solution to the equation $W(x) \exp W(x) = x$ and thereby "answers the question 'What power of Euler's number, multiplied by itself, produced the number x?'" (Lehtonen, 2016, p. 1111)

[10] That is, ones, tens, hundreds, thousands, tens of thousands, hundreds of thousands, millions, etc. corresponding to z

currency, (ii) the graphs of the associated $\xi(\cdot)$ and its mirror image in relation to the 45-degree line $\xi^{-1}(\cdot)$ with the horizontal axis measured in z-illions of units of domestic or foreign currency and the vertical axis measured in z-illions of units of foreign or domestic currency, (iii) the graph of the associated $\psi(\cdot)$ labelled "D*" or "S*" with the vertical axis measured in units of domestic currency per unit of foreign currency and the horizontal axis measured in z-illions of units of foreign currency, and (iv) a pair of dashed straight-line segments highlighting the one-for-one crossing points of $\varphi(\cdot)$ and $\psi(\cdot)$ and of $\xi(\cdot)$ and $\xi^{-1}(\cdot)$. In addition, Figure 1 includes a sequence of dotted-and-dashed straight-line segments that show how to read the constituent graphs from (a) $e = 2$ units of foreign currency per unit of domestic currency yielding $(3.9 - 2)/0.9 = 2.\bar{1}$ z-illions of units of domestic currency demanded by foreigners to (b) $2 \times 2.\bar{1} = 4.\bar{2}$ z-illions of units of foreign currency sold by foreigners in exchange for $2.\bar{1}$ z-illions of units of domestic currency to (c) $2.\bar{1}$ z-illions of units of domestic currency bought by foreigners in exchange for $4.\bar{2}$ z-illions of units of foreign currency to (d) $4.\bar{2}$ z-illions of units of foreign currency supplied by foreigners at $2.\bar{1}/4.\bar{2} = 0.5 = e^{-1}$ units of domestic currency per unit of foreign currency. Figure 5b is read in the same way; Figures 2a through 5a are read similarly, but in relation to the domestic supply of domestic currency S first and the domestic demand for foreign currency D* last (or in the reverse order starting with a value for e^{-1}).

3. Conclusion

The "factors generally thought to influence exchange rates … include a country's inflation

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equal to zero, one, two, three, four, five, six, etc. Note that the etymology of the word "zillion" is "*Z* (perh[aps] = unknown quantity) + MILLION" according to the Oxford English Dictionary (and other expert sources).

rate, real economic growth rate, interest rates relative to the rest of the world, and private speculation. Theories of the exchange rate differ because of the assumptions they make about the importance of these factors" (Pearce, 1983, p. 21). As criticisms have been levelled against all of these theories mostly on the basis of what each leaves out of consideration of necessity in aid of maintaining tractability, it would seem that direct estimation of foreign-exchange supply and demand is the best approach to modelling exchange-rate determination empirically. To do so would require solving the associated identification problem, of course, which can be done in principle using supply- or demand-side instruments as explained by MacKay and Miller (2019). The product demand schedule specified therein (by equation 1) takes a semi-linear form analogous to $h(Q) = h(\varphi(e))$, where the function $h(\cdot)$ is such that the composition $h \circ \varphi$ is affine in e and hence amenable to linear regression analysis. The specification of the foreign-demand-for-domestic-currency function $\varphi(\cdot)$ would be informed by the results in Section 2 above and the (translation) part of $h \circ \varphi$ that is not dependent on e would be a linear function of the remaining relevant independent variables. A similar procedure could be used to construct a corresponding estimable supply equation. Further details on the construction and estimation of such a system are beyond the scope of this paper and await future research.

In addition to providing necessary structure for the possible estimation of the demand for and the supply of foreign exchange, the mathematical analysis herein constitutes a pedagogy for introducing and elaborating upon the peculiar geometry of such markets. An understanding of the basis for the two equivalent (economic) perspectives on a given market for foreign exchange as well as the relationship between the shape of supply or demand in one and the shape of demand or supply in the other would seem to be essential learning outcomes for any course on open-economy macroeconomics or international finance. Pace Machlup (1939) quoted in Section 1, the *depth* of understanding in relation to these learning outcomes would depend on the level of the course and its prerequisites in terms of prior mathematics training.

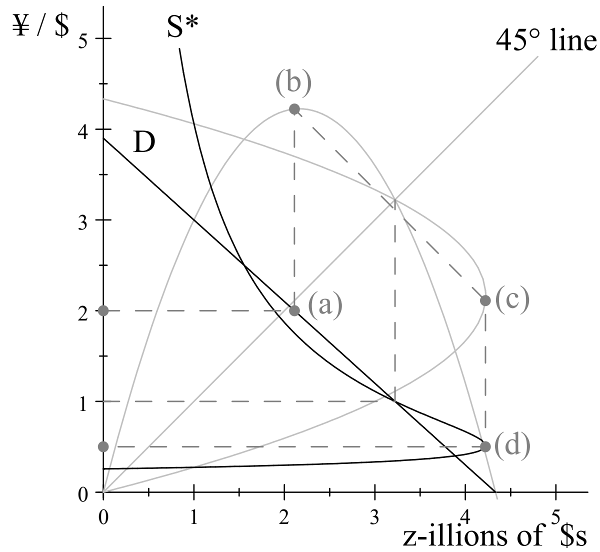

FIGURE 1. The affine demand case with $\alpha = 3.9$ and $\beta = 0.9$.

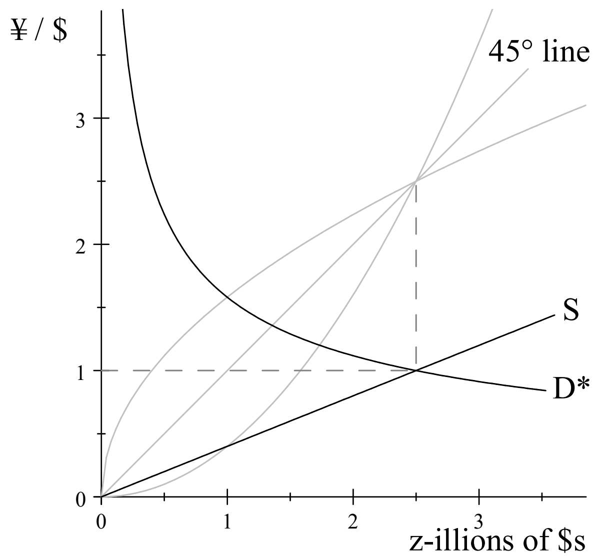

FIGURE 2A. The “augmented power” (linear supply) case with $\alpha = 0$, $\beta = 0.4$, and $\gamma = 1$.

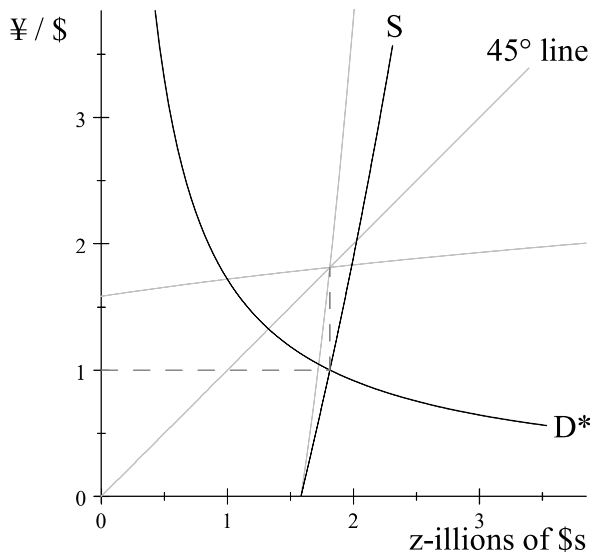

FIGURE 2B. The "augmented power" (nonlinear supply) case with $\alpha = -3$, $\beta = 0.6$, and $\gamma = 2.5$.

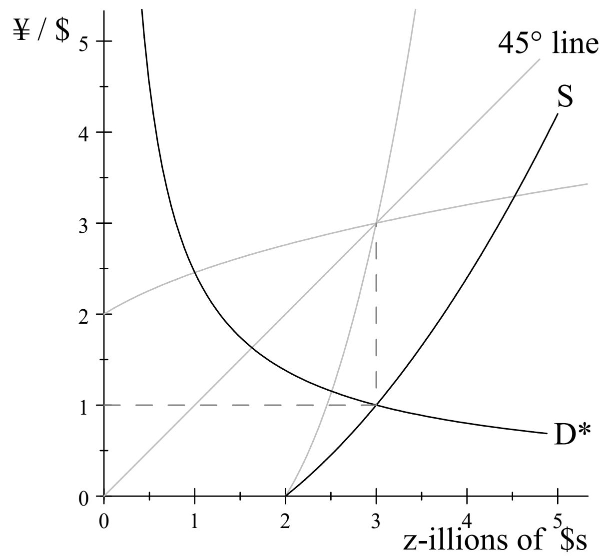

FIGURE 3. The "depressed quadratic" supply case with $\alpha = -0.8$ and $\beta = 0.2$.

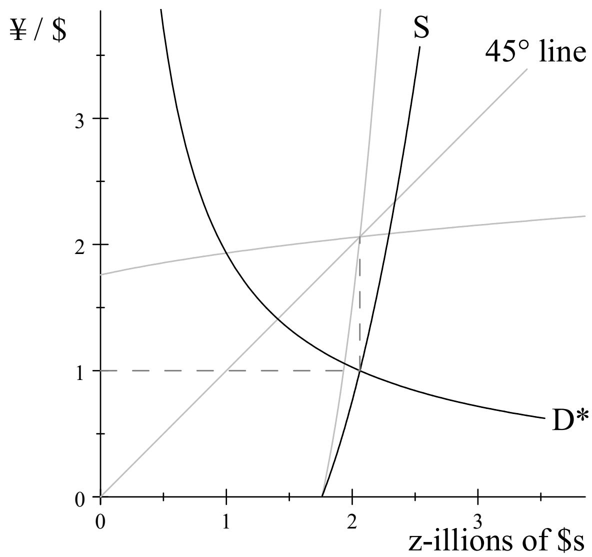

FIGURE 4A. The "generalized depressed cubic" supply case with $\alpha = -1.3$, $\beta = 0.42$, and $\gamma = 1$.

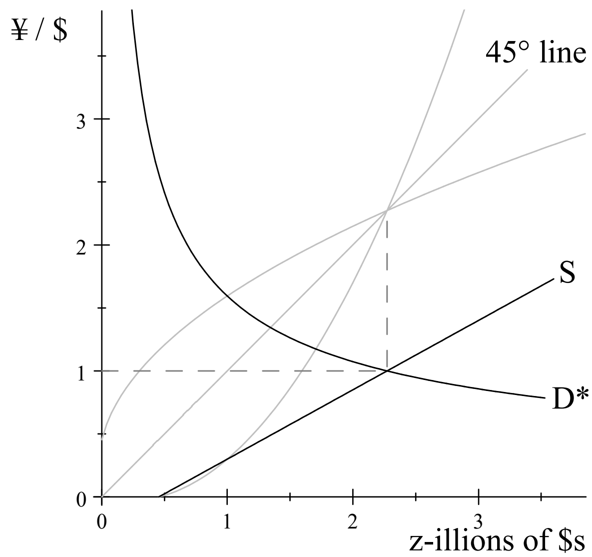

FIGURE 4B. The “generalized depressed cubic” (affine) supply case with $\alpha = -0.25$, $\beta = 0.55$, and $\gamma = 0$.

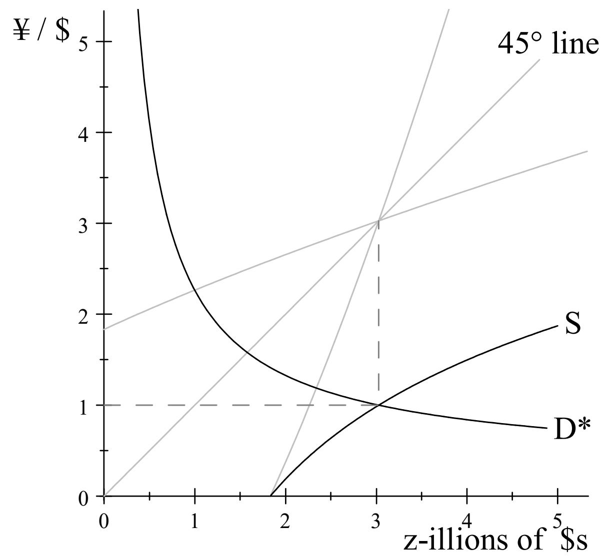

FIGURE 5A. The "augmented logarithmic" (supply) case with $\alpha = -1.5$ and $\beta = 1.35$.

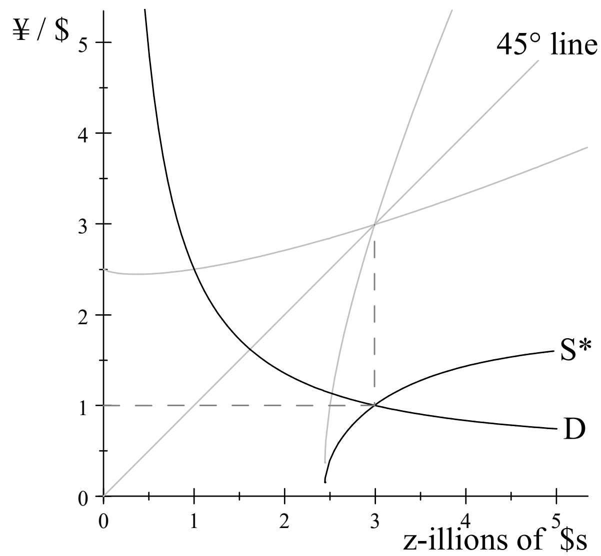

FIGURE 5B. The “augmented logarithmic” (demand) case with $\alpha = 2.5$ and $\beta = 0.15$.

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