INTERNATIONAL ECONOMIC REVIEW
Vol. 44, No. 1, February 2003

A RESTRICTED-DOMAIN MULTILATERAL TEST APPROACH TO THE THEORY OF INTERNATIONAL COMPARISONS*

BY KEIR G. ARMSTRONG[1]

Carleton University, Canada

This article develops a novel multilateral test approach to the problem of international comparisons. Many of the associated tests are justified as "reasonable" using the fact that they are direct analogues to properties of the cost-of-living index. Further support is bestowed upon the new approach by showing that it is equivalent to an extended version of Diewert's (1986) multilateral test approach. Finally, a number of alternative multilateral comparison formulas are evaluated and the relative superiority of two of them is established.

1. INTRODUCTION

The economic approach to the theory of international comparisons developed in Pollak (1980, 1981), Diewert (1984), and Armstrong (2001) has a number of limitations. First, in deriving empirically useful results, it relies heavily on separability assumptions about the underlying aggregator functions that are unlikely to be correct. The most objectionable of these is the requirement that tastes or technologies be identical or, at the very least, be closely related across countries. Second, in some contexts, the key assumption that agents behave optimally in allocating their available resources may be inappropriate. Finally, implementation of the economic approach may require unobservable ex ante expectations about future prices to enable the calculation of rental prices of durable goods.

The test (or axiomatic) approach gets around these problems by focusing exclusively on axiomatic indexes; i.e., those based on ex post accounting data that are observable and treated as independent variables. Its ultimate objective is to specify a set of "reasonable" tests (or axioms or requirements) that is sufficient to determine a unique functional form for the index in question. Failing this, the specified tests can provide a basis for assessing the relative merits of alternative formulas motivated outside the test approach framework.

For the most part, the literature in this field is concerned with bilateral comparisons.[2] Working under the auspices of the United Nations International Comparison Project (ICP), Kravis et al. (1975, p. 54) were the first to develop a set of tests that is applicable in a multilateral context. The latest version of this set was

* Manuscript received October 2000; revised May 2001.

[1] The author wishes to thank Erwin Diewert, Ken White, and an anonymous referee for helpful comments on earlier versions of this article. Please address correspondence to: Keir G. Armstrong, Carleton University, Ottawa, ON KIS 5B6, Canada. Tel: 613 520-2600 (ext. 8696). E-mail: *karmstro@ccs.carleton.ca.*

[2] See, for example, Fisher (1927), Voeller (1981), and Eichhorn and Voeller (1983).

described by Gerardi (1982). Diewert (1986) proposed a more comprehensive system of multilateral tests and then used it to evaluate a number of different methods for making real output comparisons within a bloc of countries. Balk (1989) used Diewert's system to evaluate an additional output-comparison formula.

In the sections that follow, a new framework for making multilateral international comparisons is developed. The various tests that define this framework are set out in Section 2. Many of these tests can be justified as "reasonable" using the fact that they are direct analogues to properties of the cost-of-living index. Further support for the new approach is provided in Section 3 by showing that it is equivalent to an extended version of Diewert's (1986) multilateral test approach. Section 4 analyzes a number of alternative multilateral comparison formulas and Section 5 establishes the relative superiority of two of them. Further exploration of the relationships among these formulas is undertaken in Section 6, and the two that have the best axiomatic properties are shown to have justifications grounded in economic theory. Section 7 offers some concluding remarks.

2. DEFINITIONS

As in Armstrong (2001), the maintained domain of comparison involves a bloc of countries $\mathcal{N} := \{1, \ldots, n\}$ with $\mathbf{h} := (h_1, \ldots, h_n)^\top \in \mathbb{R}^n_{++}$ resident households, a set of consumer goods and services $\mathcal{M} := \{1, \ldots, m\}$ with country-specific national-currency-denominated prices

$$\mathbf{P} := (\mathbf{p}_1, \ldots, \mathbf{p}_n)^\top = \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix} \in \mathbb{R}^{nm}_{++}$$

and a vector of per household consumption bundles

$$\mathbf{X} := (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{pmatrix} \in \mathbb{R}^{nm}_{+}$$

Unlike that article, however, the underlying preferences that generate $\mathbf{X}$ are ignored. Further, the elements of $\mathbf{P}$, $\mathbf{X}$, and $\mathbf{h}$ are treated as independent variables.

From the viewpoint of the typical country-k household, the vectors $\mathbf{p}_i$, $\mathbf{p}_j$, and $\mathbf{x}_k$ $(i, j, k \in \mathcal{N})$ constitute the only available information that is relevant to the calculation of the purchasing power parity (PPP) between countries i and j. Prices outside i and j have no bearing on the cost of a commodity bundle in one of these countries relative to the cost of the same bundle in the other. Consumption bundles other than $\mathbf{x}_k$ are generated by preferences that may be very different from those of the typical country-k household. Thus, it would appear that the best way to make use of the available data in calculating PPPs that are specific to country k is by means of the fixed-weight index-number formula

$$(1) \qquad \theta(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_k) \equiv \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}$$

If the typical country-k household has preferences that admit very little substitution among the commodity types in $\mathcal{M}$, or if the various price vectors are not very different from one another, then this index will be approximately exact.

The most obvious way to think about PPPs that are relevant to the bloc as a whole is as an aggregate of the n country-specific PPPs. To reflect the democratic principle of "one person, one vote," the available data on numbers of households could be used to provide appropriate weights for the different countries in constructing such an aggregate. Following this logic, a bloc-specific PPP index for country i relative to country j is a function $\rho\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ defined over (i) the price vectors for the pair of countries being compared, (ii) all of the per household consumption bundles, and (iii) the vector of household numbers. Since there are $n - 2$ price vectors that are not arguments of this index (but could be, in principle), ρ is called a *restricted-domain* index. Examples of such indexes are presented in Section 4.

The first and second vectors of prices over which ρ is defined can be thought of as comparison and reference prices, respectively. Given that ρ is being viewed as an aggregate of country-specific PPPs and that, under the economic approach, the country-k PPP index is simply the (Konüs-type) cost-of-living index r_k, it seems reasonable to require that ρ depend on these prices in the same way that r_k does. Accordingly, the first four tests for ρ encompass the direct analogues to the "essential" properties of r_k: positivity (P1), nondecreasingness in the comparison prices (P2), positive linear homogeneity in the comparison prices (P3), and transitivity with respect to the reference and comparison prices (P4).[3]

Corresponding to P1 is the requirement that the value of ρ be a positive number. The motivation for this test comes from the fact that the PPP between any two countries is the number of currency units of the first country needed to buy a commodity bundle equivalent to one that can be bought with a single currency unit of the second country.

P. *Positivity*: For all $i, j \in \mathcal{N}$, $\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) > 0$.

The analogue to P2, called *positive monotonicity*, requires that an increase in one or more of the comparison prices cause the value of ρ to increase or remain the same.

M. *Positive Monotonicity*: For all $i, j \in \mathcal{N}$ and for all $\mathbf{p}'_i > \mathbf{p}_i$, $\rho(\mathbf{p}'_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \geq \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$.

[3] With the singular exception of Samuelson and Swamy's (1974, pp. 571–72) "dimensional invariance test" (see below), all other properties of r_k that appear in the literature are implied by one or more of P1–P4.

The P3-analogue, *linear homogeneity*, requires that a common proportional change in all comparison prices cause the same proportional change in the value of ρ.

H. *Linear Homogeneity*: For all $i, j \in \mathcal{N}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(\lambda\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$.

The test for ρ that corresponds to P4 is called *transitivity*. It requires that the PPP between two countries be equal to the product of the PPP between the first country and any third country and the PPP between the same third country and the second country.

T. *Transitivity*: For all $i, j \in \mathcal{N}$ and for all $t \in \mathcal{N}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Seven additional tests for ρ follow from the preceding four in the same way that the analogous properties of the cost-of-living index follow from P1–P4. The first of these additional tests, called *identity*, requires the value of ρ to be unity if the reference and comparison countries are one and the same.

I. *Identity*: For all $j \in \mathcal{N}$, $\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = 1$.

The second implied test for ρ, called *proportionality*, asserts that if the result of applying a common proportional change to a country's prices is compared with its original situation, the value of ρ is the factor of proportionality. Note that this requirement contains I as a special case.

PP. *Proportionality*: For all $j \in \mathcal{N}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(\lambda\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda$.

The third implied test, *country reversal*, asserts that if the reference and comparison countries are switched, the new value of ρ is the reciprocal of the old.

CR. *Country Reversal*: For all $i, j \in \mathcal{N}$,

$$\rho(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h}) = \frac{1}{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}$$

The fourth implied test, *negative monotonicity*, is the reference-price counterpart to M. It requires that an increase in one or more of the reference prices cause the value of ρ to decrease or remain the same.

NM. *Negative Monotonicity*: For all $i, j \in \mathcal{N}$ and for all $\mathbf{p}'_j > \mathbf{p}_j$, $\rho(\mathbf{p}_i, \mathbf{p}'_j, \mathbf{X}, \mathbf{h}) \leq \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$.

The fifth implied test, *homogeneity of degree minus one*, is the reference-price counterpart to H. It requires that a common proportional change in all reference prices cause the value of ρ to change by the reciprocal of the factor of proportionality.

HDM. *Homogeneity of Degree Minus One*: For all $i, j \in \mathcal{N}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(\mathbf{p}_i, \lambda\mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda^{-1}\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$.

The sixth implied test, *price dimensionality*, requires that a common proportional change in all reference and comparison prices have no effect on the value of ρ.

PD. *Price Dimensionality*: For all $i, j \in \mathcal{N}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(\lambda\mathbf{p}_i, \lambda\mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$.

The final implication of the four essential tests for ρ, the *mean value test*, asserts that the value of ρ lies between the smallest and the largest price relative $p_{i\ell}/p_{j\ell}$, $\ell \in \mathcal{M}$.

MV. *Mean Value Test*: For all $i, j \in \mathcal{N}$,

$$\min_{\ell \in \mathcal{M}} \left\{ \frac{p_{i\ell}}{p_{j\ell}} \right\} \leq \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \leq \max_{\ell \in \mathcal{M}} \left\{ \frac{p_{i\ell}}{p_{j\ell}} \right\}$$

THEOREM 1. *Suppose there exists a function* $\rho : \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P and T. Then ρ also satisfies (i) I; (ii) PP if H holds; (iii) CR; (iv) NM if M holds; (v) HDM if H holds; (vi) PD if H holds; (vii) MV if both H and M hold.*

The proof of this result, like all others in the article, can be found in the appendix.

The direct analogue to the invariance property of the cost-of-living index with respect to the dimensionality of each price and the position of each commodity in the "general list" is encompassed by a pair of tests. The first of these, called *commensurability*, requires that a change in the unit of measure of each commodity[4] have no effect on the value of ρ.

C. *Commensurability*: For all $i, j \in \mathcal{N}$ and for all $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_m)^\top \in \mathbb{R}^m_{++}$,

$$\rho\big(\hat{\boldsymbol{\lambda}}\mathbf{p}_i,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_j, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}\big) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

where $\hat{\boldsymbol{\lambda}}$ is the $m \times m$ diagonal matrix with $\hat{\lambda}_{\ell\ell} = \lambda_\ell$ for all $\ell \in \mathcal{M}$.

The second part of the invariance analogue is captured by *commodity symmetry*: a change in the ordering of the items in the general commodity list has no effect on the value of ρ.

CS. *Commodity Symmetry*: For all $i, j \in \mathcal{N}$ and for any permutation of the columns of the $m \times m$ identity matrix, denoted by $\tilde{\mathbf{I}}_m$,

$$\rho\big(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\big) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

[4] Such a change could include measuring the quantity of beer, say, in liters instead of gallons and the associated prices in currency units per liter instead of currency units per gallon.

The nature of the dependence of ρ on the matrix of per household quantities and the vector of household numbers cannot be established by analogy to properties of the cost-of-living index because neither set of variables is in the domain of this (latter) function. Consequently, from a theoretical economic standpoint, no pertinent test for ρ can be considered to be as desirable as those discussed above. From certain applied standpoints, however, this conclusion may not hold. Political or other noneconomic considerations could lead to the prioritization of a particular requirement for ρ that is not grounded in the economic approach.

One such requirement, *weight symmetry*, precludes the possibility that any country's total consumption bundle (or weight) plays a special role in the determination of ρ.

WS. *Weight Symmetry*: For all $i, j \in \mathcal{N}$ and for any permutation of the columns of the $n \times n$ identity matrix, denoted by $\tilde{\mathbf{I}}_n$,

$$\rho\left(\mathbf{p}_i, \mathbf{p}_j, \tilde{\mathbf{I}}_n^\top \mathbf{X}, \tilde{\mathbf{I}}_n^\top \mathbf{h}\right) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Another "ungrounded" test lives up to the name *population irrelevance* by granting equal treatment to every country, regardless of size.

PI. *Population Irrelevance*: For all $i, j \in \mathcal{N}$ and for all $\mathbf{h}' \in \mathbb{R}^n_{++}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}') = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

An obvious counterpart to the price dimensionality axiom discussed earlier, *quantity dimensionality* requires that a common proportional change in all per household quantities together with a possibly different proportional change in all household numbers have no effect on the value of ρ.

QD. *Quantity Dimensionality*: For all $i, j \in \mathcal{N}$ and for all $\beta, \gamma \in \mathbb{R}_{++}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h}) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

A stronger version of this requirement, *strong quantity dimensionality*, states that a common proportional change in the per household quantities of any country has no effect on the value of ρ.

SQD. *Strong Quantity Dimensionality*: For all $i, j \in \mathcal{N}$, for all $t \in \mathcal{N}$, and for all $\lambda \in \mathbb{R}_{++}$,

$$\rho\left(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\right) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

The importance of the distinction between total and per household quantities implicit in the definition of ρ is assessed by the *total quantities test*. It demands that a change in per household quantities and numbers of households such that all total quantities remain the same have no effect on the value of ρ.

TQ. *Total Quantities Test*: For all $i, j \in \mathcal{N}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

where $\hat{\mathbf{h}}$ is the $n \times n$ diagonal matrix with $\hat{h}_{kk} = h_k$ for all $k \in \mathcal{N}$ and $\mathbf{1}_n$ is the n-dimensional column vector of ones.

Bilateral versions of the following test have been proposed by several authors, beginning with Fisher (1911).

D. *Determinateness*: If any scalar argument in ρ tends to zero, then the value of ρ tends to a unique positive real number.

Opinions on the desirability of this requirement are usually expressed in a categorically unequivocal manner. At one extreme is Frisch (1930, p. 405) who "feel[s] a great repugnance against any index which does not satisfy the determinateness test." He justifies his position on practical grounds by adding that "the withdrawal or entry of any [new] commodity will often have to be performed as a limiting case when either the quantity . . . or the money value . . . decreases toward zero, respectively increases from zero." At the other extreme are Samuelson and Swamy (1974, p. 572) who consider the determinateness test to be "odd . . . and not at all . . . desirable . . . [because] it rules out the non-satiation assumptions often made in standard economic theory" thereby making it impossible for households to derive infinite utility when one or more prices vanish.

Next, three tests are considered that require the set of PPPs to change in a consistent manner as the size of the bloc changes; i.e., they require consistency in aggregation. First up is the *country partitioning test*. It says that if some country $t \in \mathcal{N}$ is partitioned into two new countries, each with the same per household consumption bundle $\mathbf{x}_t$, then none of the PPPs among the rest of the countries are affected. If, in addition, the two new countries have the same price vector $\mathbf{p}_t$, then each inherits the PPPs of the original country t.

CP. *Country Partitioning Test*: For all $t \in \mathcal{N}$ and for all $\lambda \in (0, 1)$,

$$\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)\, h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)$$
$$= \left\{ \begin{array}{lll} \rho_{ij} & \text{if} & i, j \in \mathcal{N}\backslash\{t\} \\ \rho_{it} & \text{if} & i \in \mathcal{N}\backslash\{t\},\, j \in \{t, n+1\} \\ \rho_{tj} & \text{if} & i \in \{t, n+1\},\, j \in \mathcal{N}\backslash\{t\} \\ \rho_{tt} & \text{if} & i, j \in \{t, n+1\} \end{array} \right\} \text{ and } \mathbf{p}_{n+1} = \mathbf{p}_t$$

where $\rho_{kl} := \rho(\mathbf{p}_k, \mathbf{p}_l, \mathbf{X}, \mathbf{h})$ for all $k, l \in \mathcal{N} \cup \{n+1\}$.

A stronger version of this requirement is the *strong country partitioning test*. It says that if some country $t \in \mathcal{N}$ is partitioned into two new countries, each with a per household consumption bundle that is possibly different from that of the other, then none of the PPPs among the rest of the countries are affected. If, in addition, the two new countries have the same price vector $\mathbf{p}_t$, then each inherits the PPPs of the original country t.

SCP. *Strong Country Partitioning Test*: For all $t \in \mathcal{N}$ and for all $(\mathbf{x}_t', \mathbf{x}_{n+1}, \lambda) \in \mathbb{R}_+^{2m} \times (0,1)$ such that $(1-\lambda)\mathbf{x}_t' + \lambda \mathbf{x}_{n+1} = \mathbf{x}_t$,

$$\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_t', \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)\, h_t, \ldots, h_n, \lambda h_t]^\top\big)$$
$$= \left\{ \begin{array}{lll} \rho_{ij} & \text{if} & i, j \in \mathcal{N}\backslash\{t\} \\ \rho_{it} & \text{if} & i \in \mathcal{N}\backslash\{t\},\, j \in \{t, n+1\} \\ \rho_{tj} & \text{if} & i \in \{t, n+1\},\, j \in \mathcal{N}\backslash\{t\} \\ \rho_{tt} & \text{if} & i, j \in \{t, n+1\} \end{array} \right\} \text{ and } \mathbf{p}_{n+1} = \mathbf{p}_t$$

where $\rho_{kl} := \rho(\mathbf{p}_k, \mathbf{p}_l, \mathbf{X}, \mathbf{h})$ for all $k, l \in \mathcal{N} \cup \{n+1\}$.

The third consistency-in-aggregation requirement, *tiny country irrelevance*, states that if the number of households in some country $t \in \mathcal{N}$ tends to zero, the PPPs among the remaining countries tend to those that would prevail if the bloc excluded country t altogether.

TCI. *Tiny Country Irrelevance*: For all $t \in \mathcal{N}$, for all $i, j \in \mathcal{N}\backslash\{t\}$, and for all $\lambda \in \mathbb{R}_{++}$,

$$\lim_{\lambda \to 0} \rho\big(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big) = \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t})$$

where $\mathbf{X}_{-t} := (\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n)^\top$ and $\mathbf{h}_{-t} := (h_1, \ldots, h_{t-1}, h_{t+1}, \ldots, h_n)^\top$.

The next axiom is called the *product test* because it asks that the product of the values of ρ and a bloc-specific per household consumption index $\tilde{\phi} : \mathbb{R}_{++}^{2m} \times \mathbb{R}_+^{2m+n(m+1)} \to \mathbb{R}$ be equal to the corresponding per household expenditure ratio.

PT. *Product Test*: For all $i, j \in \mathcal{N}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})\, \tilde{\phi}\, (\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \tag{2}$$

Note that once a functional form is established for ρ, $\tilde{\phi}$ can be defined implicitly by Equation (2). In this case, PT is a tautology.

The final axiom considered in this section is a strengthened version of PT. *Factor reversal* says that for any *bilateral* intrabloc price level comparison given by $\bar{\rho} : \mathbb{R}_{++}^{2m} \times \mathbb{R}_+^{2(m+1)} \to \mathbb{R}$, if the roles of prices and per household quantities are reversed, the result can be regarded as the corresponding per household consumption index.

FR. *Factor Reversal:* For any subbloc $\bar{\mathcal{N}} \subseteq \mathcal{N}$, $|\bar{\mathcal{N}}| = 2$, and for all $i, j \in \bar{\mathcal{N}}$,

$$\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)\bar{\rho}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i, \mathbf{p}_j, h_i, h_j) = \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j}$$

Note that FR is not a truly multilateral test since $\bar{\rho}$, unlike ρ, is not defined over all per household quantities and numbers of households. In bilateral contexts, the validity of this requirement has occasionally come into question during the past 80 or so years because of its lack of intuitive appeal. This is unfortunate because, as the following theorem demonstrates, FR is of critical importance in establishing the axiomatic characterization of bilateral PPP indexes.

THEOREM 2 (Funke and Voeller, 1978). *The bilateral PPP index* $\bar{\rho} : \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2(m+1)}_{+} \to \mathbb{R}$ *satisfies CR, FR, WS, and PI if and only if* $\bar{\rho}$ *is the country-j Fisher "ideal" PPP index; i.e.,*

$$\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j) = \left[\frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_i} \frac{\mathbf{p}_i^\top \mathbf{x}_j}{\mathbf{p}_j^\top \mathbf{x}_j}\right]^{\frac{1}{2}} \tag{3}$$

3. CONSUMPTION-SHARE EQUIVALENCE

The focus of this section is the translation of Diewert's (1986) multilateral test approach into the maintained domain of comparison. Following a detailed review and extension of the associated set of tests, a subset therefrom is shown to be equivalent to a subset of the restricted-domain tests developed in the preceding section. This result serves to enhance the validity and usefulness of both approaches.

In order to make it compatible with the test framework established above, Diewert's multilateral system of output indexes is treated as a system of bloc-specific (real) consumption indexes. Any such system is characterized by a function $\boldsymbol{\sigma}: \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^n$ defined over (i) all of the price vectors, (ii) all of the per household consumption bundles, and (iii) the vector of household numbers. The ith element ($i \in \mathcal{N}$) of the associated image vector $\boldsymbol{\sigma}(\mathbf{P}, \mathbf{X}, \mathbf{h}) := [\sigma_1(\mathbf{P}, \mathbf{X}, \mathbf{h}), \dots, \sigma_n(\mathbf{P}, \mathbf{X}, \mathbf{h})]^\top$ is to be interpreted as country i's share of total bloc consumption. Desirable properties for $\boldsymbol{\sigma}$, called share tests, are denoted by S1, S2, etc.

The first such property is the *fundamental share test*—so named because it is essential to the interpretation of $\boldsymbol{\sigma}$ as a system of consumption shares.

S1. *Fundamental Share Test*: $\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) > 0$ for all $i \in \mathcal{N}$ and $\sum \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) = 1$.

The next share test is called *weak proportionality*. It says that if all of the price vectors are proportional to one another, all of the per household quantity vectors are proportional to one another, and all of the household numbers are equal to one another, then country i's share of the total bloc consumption is equal to its (common) share in consumption of every item in the general commodity list.

S2. *Weak Proportionality*: For all $i \in \mathcal{N}$, for all $t \in \mathcal{N}$, for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$, for all $\gamma \in \mathbb{R}_{++}$, and for all $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_{++}$ such that $\sum \beta_k = 1$,

$$\sigma_i\big([\alpha_1\mathbf{p}_t, \ldots, \alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big) = \beta_i$$

A stronger version of this requirement is called *proportionality*. It says that if any country's per household quantity vector is multiplied by a positive scalar, then the ratio of the same country's consumption share to that of any other country is equal to the original (premultiplication) consumption-share ratio times the scalar; all other consumption-share ratios remain the same.

S3. *Proportionality*: For all $t \in \mathcal{N}$ and for all $\lambda \in \mathbb{R}_{++}$,

$$\sigma_i\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \lambda\mathbf{x}_t, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) = \begin{cases} \frac{\sigma_i(\mathbf{P},\mathbf{X},\mathbf{h})}{1+(\lambda-1)\sigma_t(\mathbf{P},\mathbf{X},\mathbf{h})} & \text{if} \quad i \in \mathcal{N}\backslash\{t\} \\ \frac{\lambda\sigma_t(\mathbf{P},\mathbf{X},\mathbf{h})}{1+(\lambda-1)\sigma_t(\mathbf{P},\mathbf{X},\mathbf{h})} & \text{if} \quad i = t \end{cases}$$

The fourth property, called the *monetary unit test*, states that multiplying each price vector, the matrix of per household quantities and the vector of household numbers by (possibly different) positive scalars has no effect on the consumption share of any country.

S4. *Monetary Unit Test*: For all $i \in \mathcal{N}$, for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$, and for all $(\beta, \gamma) \in \mathbb{R}^2_{++}$,

$$\sigma_i\big([\alpha_1\mathbf{p}_1, \ldots, \alpha_n\mathbf{p}_n]^\top, \beta\mathbf{X}, \gamma\mathbf{h}\big) = \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

The fifth share test, *commensurability*, requires the consumption shares to be invariant to changes in the units of measure of commodities.

S5. *Commensurability*: For all $i \in \mathcal{N}$ and for all $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_m)^\top \in \mathbb{R}^m_{++}$,

$$\sigma_i(\mathbf{P}\hat{\boldsymbol{\lambda}}, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}) = \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

where $\hat{\boldsymbol{\lambda}}$ is the $m \times m$ diagonal matrix with $\hat{\lambda}_{\ell\ell} = \lambda_\ell$ for all $\ell \in \mathcal{M}$.

The sixth test is called *country symmetry* because it requires that $\boldsymbol{\sigma}$ treat the prices and quantities of every country in the same manner.

S6. *Country Symmetry*: For any permutation of the columns of the $n \times n$ identity matrix, denoted by $\tilde{\mathbf{I}}_n$,

$$\boldsymbol{\sigma}\big(\tilde{\mathbf{I}}_n^\top\mathbf{P}, \tilde{\mathbf{I}}_n^\top\mathbf{X}, \tilde{\mathbf{I}}_n^\top\mathbf{h}\big) = \tilde{\mathbf{I}}_n^\top\boldsymbol{\sigma}(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

The preceding axiom makes the names of countries irrelevant to the determination of consumption shares. *Commodity symmetry* does the same for commodity names.

S7. *Commodity Symmetry*: For all $i \in \mathcal{N}$ and for any permutation of the columns of the $m \times m$ identity matrix, denoted by $\tilde{\mathbf{I}}_m$,

$$\sigma_i\left(\mathbf{P}\tilde{\mathbf{I}}_m^\top, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right) = \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

The following three tests for $\boldsymbol{\sigma}$ are consistency-in-aggregation requirements. The *country partitioning test* says that if some country $t \in \mathcal{N}$ is partitioned into two new countries, each with the same per household consumption bundle $\mathbf{x}_t$ and the same price vector $\mathbf{p}_t$, then none of the consumption shares among the rest of the countries are affected and the consumption-share ratio between the two new countries is equal to the corresponding ratio of household numbers.

S8. *Country Partitioning Test*: For all $t \in \mathcal{N}$ and for all $\lambda \in (0, 1)$,

$$\begin{aligned}&\bar{\sigma}_i\left([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)\\ &= \begin{cases} \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i \in \mathcal{N}\backslash\{t\} \\ (1-\lambda)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i = t \\ \lambda\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i = n+1 \end{cases}\end{aligned}$$

The second consistency-in-aggregation requirement for $\boldsymbol{\sigma}$, *tiny country irrelevance*, states that if the number of households in some country $t \in \mathcal{N}$ tends to zero, the consumption shares among the remaining countries tend to those that would prevail if the bloc excluded country t altogether.

S9. *Tiny Country Irrelevance*: For all $t \in \mathcal{N}$, for all $i \in \mathcal{N}\backslash\{t\}$, and for all $\lambda \in \mathbb{R}_{++}$,

$$\lim_{\lambda \to 0} \sigma_i\left(\mathbf{P}, \mathbf{X}, [h_1, \ldots, \lambda h_t, \ldots, h_n]^\top\right) = \bar{\sigma}_i(\mathbf{P}_{-t}, \mathbf{X}_{-t}, \mathbf{h}_{-t})$$

where $\mathbf{P}_{-t} := (\mathbf{p}_1, \ldots, \mathbf{p}_{t-1}, \mathbf{p}_{t+1}, \ldots, \mathbf{p}_n)^\top$, $\mathbf{X}_{-t} := (\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n)^\top$, and $\mathbf{h}_{-t} := (h_1, \ldots, h_{t-1}, h_{t+1}, \ldots, h_n)^\top$.

The last of the multilateral tests devised by Diewert (1986) is called *strong dependence on a bilateral formula*. Arguably the least compelling of the consistency-in-aggregation requirements, it asks that the consumption-share ratio between any two countries tend to the value given by some bilateral total-consumption index-number formula as the number of households in the rest of the bloc shrinks to zero.

S10. *Strong Dependence on a Bilateral Formula*: For all $j \in \mathcal{N}$, for all $i \in \mathcal{N}\backslash\{j\}$, and for all $\lambda \in \mathbb{R}_{++}$, there exists a function $\psi : \mathbb{R}_{++}^{2m} \times \mathbb{R}_{+}^{4m} \to \mathbb{R}$ such that

$$\begin{aligned}&\lim_{\lambda \to 0} \frac{\sigma_i\left(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1} \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_n]^\top\right)}{\sigma_j\left(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1} \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_n]^\top\right)}\\ &= \psi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)\end{aligned}$$

Each of the next five tests is original. The first, *monotonicity*, says that if one or more of the prices in some country are increased, ceteris paribus, then the percentage change in that country's expenditure deflator is at least as large as the percentage change in the expenditure deflator of any other country.

S11. *Monotonicity*: For all $i,\ j \in \mathcal{N}$ and for all $\mathbf{p}_i' > \mathbf{p}_i$,

$$\hat{\delta}_i \geq \hat{\delta}_j$$

where $\hat{\delta}_i$ and $\hat{\delta}_j$ are defined implicitly by

$$1 + \hat{\delta}_i = \frac{1 + \hat{s}_i}{1 + \hat{\sigma}_i}$$

and

$$1 + \hat{\delta}_j = \frac{1}{1 + \hat{\sigma}_j}$$

respectively, and

$$\hat{s}_i := \frac{\mathbf{p}_i'^{\top}\mathbf{x}_i}{\mathbf{p}_i^{\top}\mathbf{x}_i} - 1$$

$$\hat{\sigma}_i := \frac{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^{\top}, \mathbf{X}, \mathbf{h}\right)}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})} - 1$$

and

$$\hat{\sigma}_j := \frac{\sigma_j\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^{\top}, \mathbf{X}, \mathbf{h}\right)}{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})} - 1$$

THEOREM 3. *S1 and S11 implies* $\hat{\sigma}_i \leq \hat{s}_i$.

By rearranging the terms that result from substituting for $\hat{\sigma}_i$ and $\hat{s}_i$ using their respective definitions, the preceding inequality can be interpreted as meaning that an increase in one or more of the prices of country *i* causes its expenditure deflator to increase or remain the same:

$$\frac{h_i\mathbf{p}_i^{\top}\mathbf{x}_i}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})} \leq \frac{h_i\mathbf{p}_i'^{\top}\mathbf{x}_i}{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^{\top}, \mathbf{X}, \mathbf{h}\right)} \tag{4}$$

Since expenditure deflators are implicit PPP indexes, this requirement is clearly analogous to the (positive) monotonicity test for the explicit PPP index of Section 2.

The second new share test, *implicit identity*, asserts that if the prices of any two countries are equal to one another, the consumption-share ratio between them is equal to the corresponding total-expenditure ratio.

S12. *Implicit Identity*: For all $i,\ j \in \mathcal{N}$,

$$\frac{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_j, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)}{\sigma_j\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_j, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)} = \frac{h_i \mathbf{p}_j^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j}$$

As above, this requirement can be restated in terms of expenditure deflators: If one country's prices are the same as another's, then so are their expenditure deflators.

The third new requirement for $\boldsymbol{\sigma}$ is the *total quantities test*. It asks that a change in per household quantities and numbers of households such that all total quantities remain the same has no effect on the consumption share of any country.

S13. *Total Quantities Test*: For all $i \in \mathcal{N}$,

$$\sigma_i(\mathbf{P},\ \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n) = \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

where $\hat{\mathbf{h}}$ is the $n \times n$ diagonal matrix with $\hat{h}_{kk} = h_k$ for all $k \in \mathcal{N}$ and $\mathbf{1}_n$ is the n-dimensional column vector of ones.

A strengthened version of S8, the *strong country partitioning test* says that if some country $t \in \mathcal{N}$ is partitioned into two new countries, each with the same price vector $\mathbf{p}_t$ but possibly different per household consumption bundles, then none of the consumption shares among the rest of the countries are affected and the consumption-share ratio between the two new countries is equal to the corresponding total-expenditure ratio.

S14. *Strong Country Partitioning Test*: For all $t \in \mathcal{N}$ and for all $(\mathbf{x}_t', \mathbf{x}_{n+1}, \lambda) \in \mathbb{R}_+^{2m} \times (0, 1)$ such that $(1-\lambda)\mathbf{x}_t' + \lambda \mathbf{x}_{n+1} = \mathbf{x}_t$,

$$\begin{aligned}
&\bar{\sigma}_i\left(\left[\mathbf{P}^\top, \mathbf{p}_t\right]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}_t', \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, \left[h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t\right]^\top\right) \\
&\quad = \begin{cases} \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i \in \mathcal{N}\backslash\{t\} \\ (1-\lambda)\frac{\mathbf{p}_t^\top \mathbf{x}_t'}{\mathbf{p}_t^\top \mathbf{x}_t}\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i = t \\ \lambda \frac{\mathbf{p}_t^\top \mathbf{x}_{n+1}}{\mathbf{p}_t^\top \mathbf{x}_t}\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h}) & \text{if} \quad i = n+1 \end{cases}
\end{aligned}$$

The last axiom considered in this section is the *ratio test*. It provides a link between the two multilateral test approaches defined above by requiring that the ratio of any two countries' restricted-domain total consumption indexes[5] be equal to the corresponding consumption-share ratio.

RT. *Ratio Test*: For all $i, j \in \mathcal{N}$ and for all $k \in \mathcal{N}$,

$$\frac{h_i \tilde{\phi}(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k, \mathbf{X}, \mathbf{h})}{h_j \tilde{\phi}(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k, \mathbf{X}, \mathbf{h})} = \frac{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})} \tag{5}$$

Using this axiom together with three others, it is possible to derive the precise mathematical relationship between the consumption-share system $\boldsymbol{\sigma}$ and the restricted-domain PPP index ρ.

LEMMA 1. *Suppose there exists a function* $\rho : \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P, and a function* $\boldsymbol{\sigma} : \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^n$ *satisfying S1. Define the function* $\tilde{\phi}$: $\mathbb{R}^{2m}_{++} \times \mathbb{R}^{2m+n(m+1)}_{+} \to \mathbb{R}$ *implicitly by Equation (2) and suppose that* $(\tilde{\phi}, \boldsymbol{\sigma})$ *satisfies RT. Then*

$$\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1} \tag{6}$$

If, in addition, ρ *satisfies T, then*

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})} \tag{7}$$

Equation (6) enables the derivation of each of the nonfundamental share tests (S2–S14) from one or more of the tests for ρ.

LEMMA 2. *Suppose there exists a function* $\rho : \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P, and a function* $\boldsymbol{\sigma} : \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^n$ *satisfying S1. Define the function* $\tilde{\phi} : \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2m+n(m+1)}_{+} \to \mathbb{R}$ *implicitly by Equation (2) and suppose that* $(\tilde{\phi}, \boldsymbol{\sigma})$ *satisfies RT. Then* $\boldsymbol{\sigma}$ *satisfies (i) S2 if* ρ *satisfies H and HDM; (ii) S3 if* ρ *satisfies SQD and T; (iii) S4 if* ρ *satisfies H, HDM, and QD; (iv) S5 if* ρ *satisfies C; (v) S6 if* ρ *satisfies WS; (vi) S7 if* ρ *satisfies CS; (vii) S8 if* ρ *satisfies CP; (viii) S9 if* ρ *satisfies TCI; (ix) S10 if* ρ *satisfies TCI and T; (x) S11 if* ρ *satisfies M; (xi) S12 if* ρ *satisfies T; (xii) S13 if* ρ *satisfies TQ; (xiii) S14 if* ρ *satisfies SCP.*

The derivation of each of the tests for ρ—except P, T, PI, D, PT, and FR—from one or more of the share tests is enabled by Equation (7).

[5] Recall that indexes of this sort can be defined implicitly in terms of a restricted-domain PPP index by Equation (2).

LEMMA 3. *Suppose there exists a function* $\rho\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P and T, and a function* $\boldsymbol{\sigma}\colon \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^{n}$ *satisfying S1. Define the function* $\tilde{\phi}\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2m+n(m+1)}_{+} \to \mathbb{R}$ *implicitly by Equation (2) and suppose that* $(\tilde{\phi}, \boldsymbol{\sigma})$ *satisfies RT. Then* ρ *satisfies (i) M if* $\boldsymbol{\sigma}$ *satisfies S11; (ii) H if* $\boldsymbol{\sigma}$ *satisfies S4; (iii) C if* $\boldsymbol{\sigma}$ *satisfies S5; (iv) CS if* $\boldsymbol{\sigma}$ *satisfies S7; (v) WS if* $\boldsymbol{\sigma}$ *satisfies S6; (vi) QD if* $\boldsymbol{\sigma}$ *satisfies S4; (vii) SQD if* $\boldsymbol{\sigma}$ *satisfies S3; (viii) TQ if* $\boldsymbol{\sigma}$ *satisfies S13; (ix) CP if* $\boldsymbol{\sigma}$ *satisfies S8; (x) SCP if* $\boldsymbol{\sigma}$ *satisfies S14; (xi) TCI if* $\boldsymbol{\sigma}$ *satisfies S9.*

Under the hypothesis that $\boldsymbol{\sigma}$ together with $\tilde{\phi}$ defined implicitly in terms of ρ satisfies the ratio test, the next theorem establishes the equivalence of Diewert's (1986) multilateral test approach and that of Section 2 by combining the results presented in Lemmas 2 and 3.

THEOREM 4. *Suppose there exists a function* $\rho\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P and T, and a function* $\boldsymbol{\sigma}\colon \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^{n}$ *satisfying S1. Define the function* $\tilde{\phi}\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2m+n(m+1)}_{+} \to \mathbb{R}$ *implicitly by Equation (2) and suppose that* $(\tilde{\phi}, \boldsymbol{\sigma})$ *satisfies RT. Then*

(i) $\boldsymbol{\sigma}$ *satisfies S3 if and only if* ρ *satisfies SQD;*
(ii) $\boldsymbol{\sigma}$ *satisfies S4 if and only if* ρ *satisfies H and QD;*
(iii) $\boldsymbol{\sigma}$ *satisfies S5 if and only if* ρ *satisfies C;*
(iv) $\boldsymbol{\sigma}$ *satisfies S6 if and only if* ρ *satisfies WS;*
(v) $\boldsymbol{\sigma}$ *satisfies S7 if and only if* ρ *satisfies CS;*
(vi) $\boldsymbol{\sigma}$ *satisfies S8 if and only if* ρ *satisfies CP;*
(vii) $\boldsymbol{\sigma}$ *satisfies S9 and S10 if and only if* ρ *satisfies TCI;*
(viii) $\boldsymbol{\sigma}$ *satisfies S11 if and only if* ρ *satisfies M;*
(ix) $\boldsymbol{\sigma}$ *satisfies S12;*
(x) $\boldsymbol{\sigma}$ *satisfies S13 if and only if* ρ *satisfies TQ;*
(xi) $\boldsymbol{\sigma}$ *satisfies S14 if and only if* ρ *satisfies SCP.*

By stating that two independently developed test approaches imply one another, this theorem reinforces the "reasonableness" of both. It should be understood, however, that such equivalence holds only for a particular class of PPP indexes and a particular class of consumption-share systems. The next lemma shows that the transitivity axiom restricts the admissible ρ indexes to ratios of national price levels that are independent of foreign prices. The theorem that follows shows that national expenditures deflated by these price levels and then normalized to sum to unity comprise the class of admissible consumption shares. This restriction on $\boldsymbol{\sigma}$ is a direct consequence of the ratio test.

LEMMA 4 (Eichhorn, 1978, pp. 156–157). *The function* $\rho\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}_{++}$ *satisfies T if and only if, for some* $\delta\colon \mathbb{R}^{m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}_{++}$,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\delta(\mathbf{p}_i, \mathbf{X}, \mathbf{h})}{\delta(\mathbf{p}_j, \mathbf{X}, \mathbf{h})} \tag{8}$$

THEOREM 5. *Suppose there exists a function* $\rho\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ *satisfying P and T, and a function* $\boldsymbol{\sigma}\colon \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}^{n}$ *satisfying S1. Define the function* $\tilde{\phi}\colon \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2m+n(m+1)}_{+} \to \mathbb{R}$ *implicitly by Equation (2) and suppose that* $(\tilde{\phi}, \boldsymbol{\sigma})$ *satisfies RT. Then, for some* $\delta\colon \mathbb{R}^{m}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}_{++}$,

$$\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{\delta(\mathbf{p}_i, \mathbf{X}, \mathbf{h})} \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\delta(\mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1} \tag{9}$$

The practical value of consumption-share equivalence is that it enables the evaluation of indexes of the form (8) either directly via the axioms of Section 2 or indirectly via those of the present section. Consequently, any admissible restricted-domain PPP index can be compared with any consumption-share system under the share-test approach. Such comparisons are undertaken in Section 5.

4. SOME EXAMPLES

There are many different ways in which the available price and quantity data can be aggregated into a bloc-specific index of relative purchasing power. In this section, 12 such alternatives are presented and evaluated in the light of the foregoing pair of test approaches.

Patterned after the multiplicative democratic PPP index,[6] the *household democratic PPP index for country i relative to country j* is defined as the household-share-weighted geometric mean of the n country-specific PPPs given by (1):

$$\rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) := \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\bar{h}_k} \tag{10}$$

where $\bar{h}_k := h_k / \mathbf{1}_n^\top \mathbf{h}$ is the fraction of bloc households living in country k, $\mathbf{1}_n$ being the n-dimensional (column) vector of ones. By assigning each country-k PPP index a weight that is proportional to the number of households that it represents, ρ_{HD} affords equal treatment to all households in the bloc.

THEOREM 6. *The household democratic PPP index* ρ_{HD} *satisfies all of the restricted-domain tests except PI, TQ, SCP, and FR.*

COROLLARY 1. *The associated system of consumption shares,* $\boldsymbol{\sigma}_{HD}$, *defined by (6) with* $\rho := \rho_{HD}$, *satisfies all of the share tests except S13 and S14.*

A weaker democratic aggregation rule would treat countries as equals rather than households. Accordingly, define the *country democratic PPP index for country i relative to country j* as the unweighted geometric mean of the country-specific PPPs:

[6] See Section 3 of Armstrong (2001).

$$\rho_{CD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) := \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\frac{1}{n}} \tag{11}$$

THEOREM 7. *The country democratic PPP index ρ_{CD} satisfies all of the restricted-domain tests except CP, SCP, and TCI.*

COROLLARY 2. *The associated system of consumption shares, $\boldsymbol{\sigma}_{CD}$, defined by (6) with $\rho := \rho_{CD}$, satisfies all of the share tests except S8–S10 and S14.*

Although ρ_{CD} fails one fewer restricted-domain test than ρ_{HD}, the former's shortcomings can easily be seen to be much worse than the latter's. If, for example, the size of the bloc is likely to change over time, the benefit of satisfying PI, TQ, and FR will be more than offset by the cost of satisfying none of the consistency-in-aggregation requirements.

The preceding PPP indexes can be regarded as examples of "external average" formulas. In each case, the per household country-k basket $\mathbf{x}_k$ is priced at both $\mathbf{p}_i$ and $\mathbf{p}_j$ for all $k \in \mathcal{N}$, and then an average over the resulting n relative costs is calculated. An alternative methodology along similar lines would be to compute an average over the country-k baskets *before* doing the costing at $\mathbf{p}_i$ and $\mathbf{p}_j$. Such an "internal average" formula was once used by the United Nations Economic Commission for Latin America (ECLA) for measuring relative purchasing powers among the countries of Central and South America. Specifically, the *ECLA* or *average basket PPP index for country i relative to country j* is defined as the ratio of the cost of the bloc per household consumption bundle in the two countries being compared:

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) := \frac{\mathbf{p}_i^\top (\mathbf{X}^\top \bar{\mathbf{h}})}{\mathbf{p}_j^\top (\mathbf{X}^\top \bar{\mathbf{h}})} \tag{12}$$

By substituting for $\bar{\mathbf{h}}$ using its definition and rearranging terms, (12) can be rewritten as the axiomatic analogue to the (Prais–Pollak) plutocratic PPP index:[7]

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\sum_k h_k \mathbf{p}_i^\top \mathbf{x}_k}{\sum_k h_k \mathbf{p}_j^\top \mathbf{x}_k} \tag{13}$$

THEOREM 8. *The average basket PPP index ρ_{AB} satisfies all of the restricted-domain tests except PI, SQD, and FR.*

COROLLARY 3. *The associated system of consumption shares, $\boldsymbol{\sigma}_{AB}$, defined by (6) with $\rho := \rho_{AB}$, satisfies all of the share tests except S3.*

Since TQ is arguably neither "desirable" nor "undesirable" as a requirement for ρ, comparison of Theorems 6 and 8 reveals that the relative merit of ρ_{AB} and

[7] See Section 3 of Armstrong (2001).

ρ_{HD} depends on the relative desirability of SCP and SQD. However, due to the fact that weaker versions of these tests hold for both indexes, any preference for one over the other is unlikely to be very intense.

Most of the multilateral PPP indexes considered in the literature are not independent of prices outside the countries being compared. In order to accommodate this fact, it is necessary to introduce a class of bloc-specific PPP indexes that is more general than that of Section 2. Accordingly, an *un*restricted-domain bloc-specific PPP index for country i relative to country j is a function $\rho^{ij}: \mathbb{R}^{nm}_{++} \times \mathbb{R}^{n(m+1)}_{+} \to \mathbb{R}$ with image $\rho^{ij}(\mathbf{P}, \mathbf{X}, \mathbf{h})$. For a given system of consumption shares $\boldsymbol{\sigma}$, ρ^{ij} is defined by the right-hand side of Equation (7). Clearly, such an index has a restricted domain if and only if there exists a function $\tilde{\phi}$ such that $(\tilde{\phi}, \boldsymbol{\sigma})$ satisfies RT.

Kravis (1984, p. 10) pointed out that early multilateral comparison methods were based on bilateral index-number formulas. The simplest and most popular of these methods involved the use of the Laspeyres formula in making binary comparisons between a preselected base country and each of the other countries in the bloc. The first use of this sort of "star system"[8] was by the British Board of Trade (1908–1911) in a series of inquiries into the costs of living of workers in the major industrial centres of the United Kingdom, Germany, France, Belgium, and the United States. In general, bilateral-formula-based multilateral comparison methods can depend on any index-number formula of the form $\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)$. Thus, for a given base country $k \in \mathcal{N}$, the *country-k star system of consumption shares* is defined by

$$\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) := \frac{h_i \phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_j h_j \phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)} \tag{14}$$

Recall that a bilateral PPP index for country i relative to country j is a function $\bar{\rho}: \mathbb{R}^{2m}_{++} \times \mathbb{R}^{2(m+1)}_{+} \to \mathbb{R}$ with image $\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)$. If $\bar{\rho}$ satisfies PI, then, using Equation (2) with $\rho := \bar{\rho}$ and $h_i = h_j = 1$, the associated consumption index is defined as

$$\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) := \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \bigg/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1) \tag{15}$$

THEOREM 9. *Suppose $\bar{\rho}$ satisfies P, M, H, PP, HDM, C, CS, PI, and SQD. Then the country-k star system $\boldsymbol{\sigma}_{k*}$ with ϕ defined by (15) satisfies all the share tests except S6, S9, S10, S12, and S14. Moreover, ρ^{ij}_{k*} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{k*}$ is not a restricted-domain PPP index.*

A second multilateral comparison method based on a bilateral formula is known by the initials of its three independent rediscoverers, Eltetö and Köves (1964) and

[8] Named for the fact that its graph, constructed by associating nodes with countries and edges with admissible binary comparisons, looks like a star.

Szulc (1964). The (*generalized*) *EKS system of consumption shares* is defined by[9]

$$\sigma_{EKS,i}(\mathbf{P},\mathbf{X},\mathbf{h}) := \frac{h_i \prod_k [\phi(\mathbf{p}_i,\mathbf{p}_k,\mathbf{x}_i,\mathbf{x}_k)]^{\frac{1}{n}}}{\sum_j h_j \prod_l [\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}} \tag{16}$$

THEOREM 10. *Suppose $\bar{\rho}$ satisfies P, M, H, PP, HDM, C, CS, PI, and SQD. Then the EKS system $\boldsymbol{\sigma}_{EKS}$ with ϕ defined by (15) satisfies all the share tests except S8–S10, S12, and S14. Moreover, ρ^{ij}_{EKS} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{EKS}$ is not a restricted-domain PPP index.*

A third bilateral-formula-based multilateral comparison method is due to Diewert (1986, p. 25). His *own-share system of consumption indexes* is defined by

$$\sigma_{OS,i}(\mathbf{P},\mathbf{X},\mathbf{h}) := \frac{h_i\left\{\sum_k h_k [\phi(\mathbf{p}_i,\mathbf{p}_k,\mathbf{x}_i,\mathbf{x}_k)]^{-1}\right\}^{-1}}{\sum_j h_j\left\{\sum_l h_l [\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{-1}\right\}^{-1}} \tag{17}$$

THEOREM 11. *Suppose $\bar{\rho}$ satisfies P, M, H, PP, HDM, C, CS, PI, and SQD. Then the own-share system $\boldsymbol{\sigma}_{OS}$ with ϕ defined by (15) satisfies all the share tests except S3, S12, and S14. Moreover, ρ^{ij}_{OS} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{OS}$ is not a restricted-domain PPP index.*

The next three multilateral methods are based on weighted averages of the country-k star systems. Respectively, the *democratic weights*, *plutocratic weights*, and *quantity weights consumption-share systems* are defined by

$$\sigma_{DW,i}(\mathbf{P},\mathbf{X},\mathbf{h}) := \sum_k \frac{1}{n}\sigma_{k*,i}(\mathbf{P},\mathbf{X},\mathbf{h}) \tag{18}$$

$$\sigma_{PW,i}(\hat{\gamma}\mathbf{P},\mathbf{X},\mathbf{h}) := \sum_k s_k(\hat{\gamma}\mathbf{P},\mathbf{X},\mathbf{h})\sigma_{k*,i}(\hat{\gamma}\mathbf{P},\mathbf{X},\mathbf{h}) \tag{19}$$

and

$$\sigma_{QW,i}(\mathbf{P},\mathbf{X},\mathbf{h}) := \sum_k \sigma_{OS,k}(\mathbf{P},\mathbf{X},\mathbf{h})\sigma_{k*,i}(\mathbf{P},\mathbf{X},\mathbf{h}) \tag{20}$$

where

$$s_k(\hat{\gamma}\mathbf{P},\mathbf{X},\mathbf{h}) := \frac{h_k(\gamma_k\mathbf{p}_k)^\top\mathbf{x}_k}{\sum h_l(\gamma_l\mathbf{p}_l)^\top\mathbf{x}_l} \tag{21}$$

[9] In the version of this index advanced by Eltetö and Köves (1964) and Szulc (1964), the Fisher "ideal" formula was used in place of ϕ. The more general version stated here is due to Gini (1931, p. 12).

is country k's share of (nominal) bloc expenditure, $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n)^\top$ is a vector of exchange rates, and $\hat{\boldsymbol{\gamma}}$ is the $n \times n$ diagonal matrix with $\hat{\gamma}_{kk} = \gamma_k$ for all $k \in \mathcal{N}$.[10]

THEOREM 12. *Suppose $\bar{\rho}$ satisfies P, M, H, PP, HDM, C, CS, PI, and SQD. Then (i) the democratic weights system $\boldsymbol{\sigma}_{DW}$ with $\sigma_{k*,i}$ defined by (14) and ϕ defined by (15) satisfies all the share tests except S3, S8–S10, S12, and S14; (ii) the plutocratic weights system $\boldsymbol{\sigma}_{PW}$ with $\sigma_{k*,i}$ defined by (14) and ϕ defined by (15) satisfies all the share tests except S3, S4, S12, and S14; and (iii) the quantity weights system $\boldsymbol{\sigma}_{QW}$ with $\sigma_{k*,i}$ defined by (14) and ϕ defined by (15) satisfies all the share tests except S3, S12, and S14. Moreover, ρ^{ij}_{DW} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{DW}$, ρ^{ij}_{PW} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{PW}$, and ρ^{ij}_{QW} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{QW}$ are not restricted-domain PPP indexes.*

Returning now to multilateral methods that are not based on a (general) bilateral formula, two additional procedures deserve consideration. The first is a proposal by Geary (1958) that was later amplified by Khamis (1970, 1972); the second is van Ijzeren's (1956) weighted balanced method.

The Geary–Khamis or GK consumption shares are found by solving the following system of equations:

$$\sigma_i = \sum_{\ell} \pi_\ell \, [h_i \, x_{i\ell}], \quad i = 1, \ldots, n \tag{22a}$$

$$\pi_\ell = \frac{\sum_i \omega_{i\ell}\sigma_i}{\sum_k h_k \, x_{k\ell}}, \quad \ell = 1, \ldots, m \tag{22b}$$

where $\omega_{i\ell} := p_{i\ell} x_{i\ell} / \mathbf{p}_i^\top \mathbf{x}_i$ is the ℓth country-i per household expenditure share. Equations (22b) define the "international price" of each commodity as the ratio of the per household expenditure-share-weighted sum of the n consumption shares to the total quantity consumed. Equations (22a) define the share of bloc consumption for each country as the cost of its national basket at international prices.

The $n + m$ equations (22) are not independent since each constituent set implies

$$\sum_{\ell} \pi_\ell \sum_i h_i \, x_{i\ell} = \sum \sigma_i \tag{23}$$

and, consequently, at least one nontrivial solution exists. Khamis (1970, Section 3) showed that, subject to any normalization on the σ_is, the system consisting of any $n + m - 1$ of the equations (22) has a unique positive solution.

[10] Since γ_k is the price of a unit of country k's currency in terms of some numéraire currency, $\hat{\boldsymbol{\gamma}}\mathbf{P}$ is the matrix of numéraire-denominated bloc commodity prices.

Under the normalization $\sum \sigma_i = 1$, this solution is denoted by $\boldsymbol{\sigma}_{GK}(\mathbf{P}, \mathbf{X}, \mathbf{h}) := [\sigma_{GK,1}(\mathbf{P}, \mathbf{X}, \mathbf{h}), \ldots, \sigma_{GK,n}(\mathbf{P}, \mathbf{X}, \mathbf{h})]^\top$.

THEOREM 13. *The Geary–Khamis system $\boldsymbol{\sigma}_{GK}$ satisfies all of the share tests except S3, S12, and S14. Moreover, ρ_{GK}^{ij} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{GK}$ is not a restricted-domain PPP index.*

The existence of international prices $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_m)^\top$ such that $\boldsymbol{\sigma} \equiv (\hat{\mathbf{h}}\mathbf{X})\boldsymbol{\pi}$ (Equations (22a)) has been considered by some to be a desirable property of multilateral comparison methods in its own right. This property, called *additivity*, was one of four that Kravis et al. (1975, p. 55) insisted that their multilateral indexes possess.[11] Their justification for this requirement is rather vague and expedient: "[Additivity is] important for a system of comparisons that can be used readily by the scholar and man of affairs who does not wish to make a detailed study of index-number problems before using the comparisons." Hill (1982, p. 48) justified additivity in a similar manner by stating that international organizations such as the European Union prefer aggregates valued at common prices $\boldsymbol{\pi}$ "so that the figures for different countries can actually be added together in a meaningful manner and not simply compared with each other"—i.e., the (unnormalized) σ_is at any level of aggregation are real quantities of the same dimensionality.

Such justifications are easily superseded, however, when additivity is considered in the light of the economic approach. In particular, Diewert (1999, pp. 48–50) used a simple indifference-curve diagram to demonstrate the general impossibility of "an additive multilateral method with good economic properties (i.e., a lack of substitution bias)" if $n \geq 3$ and $m \geq 2$. The bottom line in respect of additivity is that it is not a reasonable requirement of multilateral comparison formulas, which is precisely why it was ignored under the test approaches of Sections 2 and 3.

The consumption shares associated with van Ijzeren's weighted balanced method are found by solving the following system of equations:

$$\text{(24)} \qquad \sum_{k \neq i} a_k \frac{\mathbf{p}_i^\top (h_k \mathbf{x}_k)}{\mathbf{p}_i^\top (h_i \mathbf{x}_i)} \frac{\sigma_i}{\sigma_k} = \sum_{k \neq i} a_k \frac{\mathbf{p}_k^\top (h_i \mathbf{x}_i)}{\mathbf{p}_k^\top (h_k \mathbf{x}_k)} \frac{\sigma_k}{\sigma_i}, \; i = 1, \ldots, n$$

where a_k is the country-k "weighting coefficient." If $\xi_1 \equiv \mathbf{p}_1^\top (h_1 \mathbf{x}_1)/\sigma_1, \ldots, \xi_n \equiv \mathbf{p}_n^\top (h_n \mathbf{x}_n)/\sigma_n$ are called "equivalents," the left-hand side of (24) is the number of equivalents that would be required to buy, in country i, the quantities in the weighted national baskets that can be bought for one equivalent in countries $1, \ldots, i-1, i+1, \ldots, n$. The right-hand side is the number of equivalents that would be required to buy, in each of countries $1, \ldots, i-1, i+1, \ldots, n$, the weighted quantities purchased in country i for one equivalent. The balanced method asserts that, for $i = 1, \ldots, n$, these two quantities of money are equal.

Van Ijzeren (1956, pp. 25–27) showed that, subject to any normalization on the σ_is, the system consisting of any $n - 1$ of equations (24) has a

[11] The other three were WS, PT, and T.

unique positive solution. Under the normalization $\sum \sigma_i = 1$, this solution is denoted by $\boldsymbol{\sigma}_{VH}(\mathbf{P}, \mathbf{X}, \mathbf{h}) := [\sigma_{VH,1}(\mathbf{P}, \mathbf{X}, \mathbf{h}), \ldots, \sigma_{VH,n}(\mathbf{P}, \mathbf{X}, \mathbf{h})]^\top$ if $a_k := h_k$, and $\boldsymbol{\sigma}_{VQ}(\mathbf{P}, \mathbf{X}, \mathbf{h}) := [\sigma_{VQ,1}(\mathbf{P}, \mathbf{X}, \mathbf{h}), \ldots, \sigma_{VQ,n}(\mathbf{P}, \mathbf{X}, \mathbf{h})]^\top$ if $a_k := \sigma_k / h_k$. The former weighting scheme originates with van Ijzeren (1956, pp. 3–5); the latter with van Ijzeren (1983, p. 45).

THEOREM 14. *The household-weighted van Ijzeren system $\boldsymbol{\sigma}_{VH}$ satisfies all of the share tests except S12–S14; the quantity-weighted van Ijzeren system $\boldsymbol{\sigma}_{VQ}$ satisfies all of the share tests except S3, S8–S10, and S12–S14. Moreover, ρ^{ij}_{VH} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{VH}$ and ρ^{ij}_{VQ} defined by the right-hand side of (7) with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{VQ}$ are not restricted-domain PPP indexes.*

5. A DOMINANCE HIERARCHY

One multilateral comparison method is said to "dominate" another if, in addition to satisfying every potentially desirable share test satisfied by the second method, the first method satisfies at least one other such test. Using this criterion in conjunction with the three corollaries and the final six theorems of the preceding section, a merit-based hierarchy among the associated methods can be established.

Since the EKS system satisfies S3 in addition to satisfying every share test satisfied by the democratic weights system, the former method dominates the latter. Due to the fact that the total quantities test S13 is value-neutral, the democratic weights method neither dominates nor is dominated by van Ijzeren's quantity-weighted balanced method. Consequently, since neither of these methods satisfies S8–S10, both are dominated by the GK, own-share, and quantity weights methods in addition to being dominated by the EKS method. By virtue of satisfying S4, the GK, own-share, and quantity weights methods dominate the plutocratic weights method as well. In turn, these methods are dominated by the average basket method, which satisfies two further tests (S12 and S14).

By virtue of satisfying S12, the country democratic method dominates the EKS method. Since S13 is value-neutral, van Ijzeren's household-weighted balanced method dominates the EKS method (by S8–S10), the k-star method (by S6, S9, and S10), and the GK, own-share, and quantity weights methods (by S3). Similarly, the household democratic method dominates the country democratic method (by S8–S10) and the household-weighted balanced method (by S12). Thus, only the average basket and household democratic methods are undominated.

The hierarchy of multilateral comparison formulas is illustrated by Figure 1. Therein, the 12 methods under consideration are grouped in boxes according to the tests they satisfy: Methods satisfying the same tests are contained in the same box; methods satisfying different tests are contained in different boxes. These boxes are arranged so that the vertical distance between any pair of them is proportional to the difference in the number of tests satisfied by the methods inside. The higher up a given method is in the diagram, the more tests it satisfies. The dominance of one method over another is represented by a straight line connecting the boxes

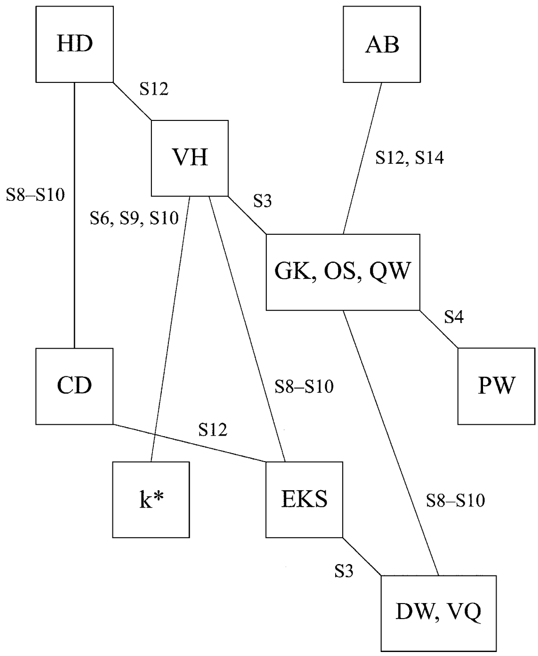

FIGURE 1

HIERARCHY OF MULTILATERAL COMPARISON FORMULAS

that hold them. Each of these lines is labeled with the names of the tests that are satisfied by the methods in the higher box but not by the methods in the lower one.

6. FURTHER IMPLICATIONS

The relationships between the three restricted-domain methods and the rest of the dominance hierarchy can be explored further using the results at the end of Section 3. In undertaking this exploration, the present section establishes that the two undominated methods are special cases of more general methods further down in the hierarchy that have been shown (elsewhere) to have exact index-number interpretations.

Recall that Lemma 4 asserts the existence of a price level function δ that satisfies (8) for a given restricted-domain PPP index ρ. Since (11) can be rewritten as

$$\rho_{CD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\prod_k \left[\mathbf{p}_i^\top \mathbf{x}_k\right]^{\frac{1}{n}}}{\prod_k \left[\mathbf{p}_j^\top \mathbf{x}_k\right]^{\frac{1}{n}}} \tag{25}$$

it is clear that

$$\delta_{CD}(\mathbf{p}_i, \mathbf{X}, \mathbf{h}) := \prod_k \left[\mathbf{p}_i^\top \mathbf{x}_k\right]^{\frac{1}{n}} \tag{26}$$

is the country democratic price level function. By Theorem 5, $\boldsymbol{\sigma}_{CD}$ and δ_{CD} satisfy (9). Substituting for δ_{CD} using (26) in this equation yields the functional form of the associated consumption shares:

$$\begin{aligned}\sigma_{CD,i}(\mathbf{P},\mathbf{X},\mathbf{h}) &= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{\prod_k [\mathbf{p}_i^\top \mathbf{x}_k]^{\frac{1}{n}}} \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\prod_l [\mathbf{p}_j^\top \mathbf{x}_l]^{\frac{1}{n}}} \right\}^{-1} \\ &= h_i \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_i^\top \mathbf{x}_k} \right]^{\frac{1}{n}} \left\{ \sum_j h_j \prod_l \left[\frac{\mathbf{p}_j^\top \mathbf{x}_j}{\mathbf{p}_j^\top \mathbf{x}_l} \right]^{\frac{1}{n}} \right\}^{-1} \\ &= \frac{h_i \prod_k [\theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)]^{\frac{1}{n}}}{\sum_j h_j \prod_l [\theta(\mathbf{x}_j, \mathbf{x}_l, \mathbf{p}_j)]^{\frac{1}{n}}} \qquad (27)\end{aligned}$$

where

$$\theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i) \equiv \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_i^\top \mathbf{x}_k} \qquad (28)$$

denotes the Paasche consumption index. Comparison of (27) with (16) reveals that the country democratic method is simply the EKS method with $\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k) := \theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)$. Since the Paasche consumption index is invariant with respect to the reference-country prices $\mathbf{p}_k$ whereas ϕ in general is not, this result is consistent with the fact that the country democratic method dominates the (generalized) EKS method.

Applying the same procedure to the household democratic PPP index defined by (10) yields

$$\delta_{HD}(\mathbf{p}_i, \mathbf{X}, \mathbf{h}) := \prod_k [\mathbf{p}_i^\top \mathbf{x}_k]^{\bar{h}_k} \qquad (29)$$

and

$$\sigma_{HD,i}(\mathbf{P},\mathbf{X},\mathbf{h}) = \frac{h_i \prod_k [\theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)]^{\bar{h}_k}}{\sum_j h_j \prod_l [\theta(\mathbf{x}_j, \mathbf{x}_l, \mathbf{p}_j)]^{\bar{h}_l}} \qquad (30)$$

Since $\boldsymbol{\sigma}_{HD}$ aggregates over households in the same way that $\boldsymbol{\sigma}_{CD}$ aggregates over countries, the household democratic method can be characterized as a household-based version of the EKS method with $\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k) := \theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)$.

By (13) and Lemma 4, the price-level function corresponding to the average basket PPP index is

$$\delta_{AB}(\mathbf{p}_i, \mathbf{X}, \mathbf{h}) := \sum_k h_k \, \mathbf{p}_i^\top \mathbf{x}_k \qquad (31)$$

Substituting for $\delta := \delta_{AB}$ in Equation (9) with $\sigma_i := \sigma_{AB,i}$ yields the functional form of the average basket consumption shares:

$$\sigma_{AB,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \frac{h_i\,\mathbf{p}_i^\top \mathbf{x}_i}{\sum_k h_k\,\mathbf{p}_i^\top \mathbf{x}_k}\left\{\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\sum_l h_l \mathbf{p}_j^\top \mathbf{x}_l}\right\}^{-1}$$

$$= \frac{h_i\left\{\sum_k h_k\left[\frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_i^\top \mathbf{x}_k}\right]^{-1}\right\}^{-1}}{\sum_j h_j\left\{\sum_l h_l\left[\frac{\mathbf{p}_j^\top \mathbf{x}_j}{\mathbf{p}_j^\top \mathbf{x}_l}\right]^{-1}\right\}^{-1}}$$

(32) $$= \frac{h_i\left\{\sum_k h_k[\theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)]^{-1}\right\}^{-1}}{\sum_j h_j\left\{\sum_l h_l[\theta(\mathbf{x}_j, \mathbf{x}_l, \mathbf{p}_j)]^{-1}\right\}^{-1}}$$

Comparison of (32) with (17) reveals that the average basket method is simply the own-share method with $\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k) := \theta(\mathbf{x}_i, \mathbf{x}_k, \mathbf{p}_i)$. Thus, as shown to be true of the EKS method, the own-share method can be made to satisfy additional tests by using the Paasche consumption index in place of one that is affected by the choice of reference-country prices.

Under the economic approach, Armstrong (2001, Section 6) showed that the EKS system is a direct approximation for the generalized mean-of-order-zero system of "true" consumption indexes when based on a bilateral axiomatic per household consumption index that is exact for a positively linearly homogeneous utility function. In other words, the use of the EKS system has a justification grounded in economic theory. Basing the EKS system on the Paasche consumption index moves it to a higher position in the (axiomatic) dominance hierarchy by enabling it to satisfy S12. Note that, by Diewert (1981, Theorems 19 and 20), the Paasche consumption index is exact for both the linear utility function $u(\mathbf{x}) := \mathbf{a}^\top \mathbf{x}$ and the Leontief utility function $u(\mathbf{x}) := \min_{\mathbf{x}} \{x_1/b_1, \ldots, x_m/b_m\}$, where $\mathbf{a} := (a_1, \ldots, a_m)^\top$ and $(b_1, \ldots, b_m)$ are vectors of positive constants. Shifting the basis of the EKS system from countries to households yields the further axiomatic improvement of consistency in aggregation (S8–S10), thereby moving it to the top of the dominance hierarchy.

Armstrong (2001, Section 6) provided an economic justification for the own-share system as well by showing it to be a direct approximation for the plutocratic system of "true" consumption indexes when based on a bilateral axiomatic per household consumption index that is exact for a positively linearly homogeneous utility function. Use of the Paasche consumption index moves the own-share system to the top of the dominance hierarchy by enabling it to satisfy S12 and S14.

It should be clear from the foregoing analysis that the principal benefit of restricted-domain consumption-share systems is that they—and they alone—satisfy S12. Recall that this property asserts the equality of two countries' expenditure deflators whenever they have the same prices. In a sense, then, S12 is a minimal statement of independence of irrelevant (i.e., third-country) prices.

From the perspective of the economic approach, the "cost" of the restricted-domain assumption is that the resulting consumption-share system is based on a bilateral index that is exact for the preference functions that exhibit either perfect substitutability or perfect complementarity. Since neither of these functional forms is flexible in the sense of providing a second-order (differential) approximation to an arbitrary twice-continuously differentiable function (at a point), it could be argued that restricted-domain systems are inferior to their unrestricted-domain counterparts when the latter are based on a superlative bilateral index.[12] But this argument presupposes identical preferences throughout the bloc, thereby allowing the price-consumption data of one country to provide information about the purchasing power of another. Given that preferences are, in general, fairly different across countries, it makes good economic sense to restrict the price domain a priori. Furthermore, since actual international comparisons of consumption are based on a relatively small number of broad categories of goods and services referred to as "basic headings,"[13] it is not unreasonable to assume a low degree of substitutability among them. Thus, a consumption-share system based on a bilateral index that is exact for a Leontief utility function (defined over m basic headings) may be eminently appropriate.

To recapitulate, the test approach of Section 3 enables the construction of a dominance hierarchy among multilateral comparison methods. And the exact approach allows the identification of two sequences of closely related methods that have justifications grounded in economic theory. Since both of these sequences terminate at the top of the dominance hierarchy, the corresponding methods—the household democratic and the average basket—can be said to be supported by both approaches.

In the limiting case of two countries ($\bar{\mathcal{N}} \equiv \{i, j\}$) with one household apiece ($h_i = h_j = 1$), 10 of the considered multilateral formulas (with ϕ set equal to the Fisher "ideal" index in the case of those based on a general bilateral formula) reduce to the Fisher "ideal" consumption index; i.e.,

$$\begin{aligned}\frac{\bar{\sigma}_i\left([\mathbf{p}_i, \mathbf{p}_j]^\top, [\mathbf{x}_i, \mathbf{x}_j]^\top, [1, 1]^\top\right)}{\bar{\sigma}_j\left([\mathbf{p}_i, \mathbf{p}_j]^\top, [\mathbf{x}_i, \mathbf{x}_j]^\top, [1, 1]^\top\right)} &= [\theta(\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i)\, \theta(\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_j)]^{\frac{1}{2}} \\ &=: \phi_F(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)\end{aligned} \tag{33}$$

The two exceptions are the average basket formula, which reduces to the Edgeworth–Marshall consumption index

$$\phi_{EM}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) := \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \frac{\mathbf{p}_j^\top (\mathbf{x}_i + \mathbf{x}_j)}{\mathbf{p}_i^\top (\mathbf{x}_i + \mathbf{x}_j)} \tag{34}$$

and the GK system, which reduces to the "bilateral GK" consumption index

[12] A bilateral index is said to be "superlative" if it is exact for a flexible functional form. See Diewert (1976, p. 117).

[13] In 1990, for example, "Final Consumption of Resident Households" consisted of 159 basic headings under the OECD's classification and 215 under Eurostat's.

$$\phi_{BGK}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) := \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \frac{\sum_\ell \frac{x_{i\ell} x_{j\ell}}{x_{i\ell} + x_{j\ell}} p_{j\ell}}{\sum_\ell \frac{x_{i\ell} x_{j\ell}}{x_{i\ell} + x_{j\ell}} p_{i\ell}} \tag{35}$$

The average basket exception is significant because, from the viewpoint of the bilateral test approach to index-number theory, the Fisher consumption index passes more tests than the Edgeworth–Marshall consumption index. Of the 20 bilateral tests listed in Diewert (1992, pp. 214–221), the former index fails none whereas the latter fails three—namely, homogeneity of degree zero in comparison-country quantities, homogeneity of degree zero in reference-country quantities, and "price weights symmetry." Thus, in the context of the relevant limits, the household democratic method can be said to dominate the average basket method.

7. CONCLUDING REMARKS

The novel feature of the test approach developed in Section 2 is the imposition of an economically sensible restriction on the price domain of admissible PPP indexes. Consequently, most of the multilateral comparison methods proposed in the literature are summarily ruled out. That this should be the case is reinforced by the fact that, under an extended version of Diewert's (1986) test approach, the best methods are those associated with a restricted-domain PPP index.

Kravis et al. (1975, p. 66) stated that "[e]conomic theory gives no explicit procedure for ... [determining PPPs] in the sense of providing a specific computing algorithm." The present article in conjunction with Armstrong (2001) demonstrates that this is not so. The latter article provides rigorous exact index-number interpretations for the two methods that are based on a general bilateral formula: the EKS method and the own-share method. Herein, the restricted-domain instances of these methods (derived by basing them on a *specific* bilateral formula) are shown to be precisely the ones that dominate all others under the test approach.

APPENDIX

PROOF OF THEOREM 1.

(i) By T, for any $i \in \mathcal{N}$, $\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$. Thus, by P, $\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = 1$.

(ii) By H and then I, $\rho(\lambda\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda$.

(iii) By P,

$$\begin{aligned}
\rho(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h}) &= \frac{\rho(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h})\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \\
&= \frac{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}, \text{ by T} \\
&= \frac{1}{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}, \text{ by I}
\end{aligned}$$

(iv) For any $\mathbf{p}'_j > \mathbf{p}_j$,

$$\begin{aligned}\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) &= \frac{1}{\rho(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h})}, \text{ by CR} \\ &\geq \frac{1}{\rho(\mathbf{p}'_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h})}, \text{ by M} \\ &= \rho(\mathbf{p}_i, \mathbf{p}'_j, \mathbf{X}, \mathbf{h}), \text{ by CR}\end{aligned}$$

(v) By CR,

$$\begin{aligned}\rho(\mathbf{p}_i, \lambda\mathbf{p}_j, \mathbf{X}, \mathbf{h}) &= \frac{1}{\rho(\lambda\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h})} \\ &= \frac{1}{\lambda\rho(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, \mathbf{h})}, \text{ by H} \\ &= \lambda^{-1}\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}), \text{ by CR}\end{aligned}$$

(vi) By H and HDM,

$$\rho(\lambda\mathbf{p}_i, \lambda\mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lambda^{-1}\lambda\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

(vii) By PP,

$$\begin{aligned}\alpha := \min_{\ell\in\mathcal{M}}\left\{\frac{p_{i\ell}}{p_{j\ell}}\right\} &= \rho\left(\alpha\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}\right) \\ &\leq \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}), \text{ by M since } \mathbf{p}_i \geq \alpha\mathbf{p}_j \\ \beta := \max_{\ell\in\mathcal{M}}\left\{\frac{p_{i\ell}}{p_{j\ell}}\right\} &= \rho\left(\beta\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}\right) \\ &\geq \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}), \text{ by M since } \mathbf{p}_i \leq \beta\mathbf{p}_j \quad \blacksquare\end{aligned}$$

PROOF OF THEOREM 2. Necessity:

$$\begin{aligned}&[\bar\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)]^2 \\ &= \frac{\mathbf{p}_i^\top\mathbf{x}_i}{\mathbf{p}_j^\top\mathbf{x}_j}\frac{\bar\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)}{\bar\rho(\mathbf{x}_i, \mathbf{x}_j, \mathbf{p}_i, \mathbf{p}_j, h_i, h_j)}, \text{ by FR} \\ &= \frac{\mathbf{p}_i^\top\mathbf{x}_i}{\mathbf{p}_j^\top\mathbf{x}_j}\bar\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)\bar\rho(\mathbf{x}_j, \mathbf{x}_i, \mathbf{p}_i, \mathbf{p}_j, h_i, h_j), \text{ by CR} \\ &= \frac{\mathbf{p}_i^\top\mathbf{x}_i}{\mathbf{p}_j^\top\mathbf{x}_j}\bar\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_j, \mathbf{x}_i, h_j, h_i)\bar\rho(\mathbf{x}_j, \mathbf{x}_i, \mathbf{p}_i, \mathbf{p}_j, h_i, h_j), \text{ by WS} \\ &= \frac{\mathbf{p}_i^\top\mathbf{x}_i}{\mathbf{p}_j^\top\mathbf{x}_j}\frac{\mathbf{p}_i^\top\mathbf{x}_j}{\mathbf{p}_j^\top\mathbf{x}_i}, \text{ by FR and PI}\end{aligned}$$

Sufficiency: Straightforward.

PROOF OF THEOREM 3. Suppose $\hat{\sigma}_i > \hat{s}_i$. Since $\mathbf{p}_i' > \mathbf{p}_i$ implies $\hat{s}_i \geq 0$, it must be the case that $\hat{\sigma}_i > 0$. By S11, $\hat{\sigma}_j \geq (\hat{\sigma}_i - \hat{s}_i)/(1+\hat{s}_i) > 0$ for all $j \in \mathcal{N}\backslash\{i\}$. By S1, $\sum_k \sigma_k(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n, \mathbf{X}, \mathbf{h}) = \sum \sigma_k(\mathbf{P}, \mathbf{X}, \mathbf{h}) = 1$. But $\hat{\sigma}_k > 0$ for all $k \in \mathcal{N}$ implies that $\sum_k \sigma_k(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n, \mathbf{X}, \mathbf{h}) > \sum \sigma_k(\mathbf{P}, \mathbf{X}, \mathbf{h})$. ■

PROOF OF LEMMA 1. By RT,

$$\begin{aligned}
\frac{\sigma_i(\mathbf{P},\mathbf{X},\mathbf{h})}{\sigma_j(\mathbf{P},\mathbf{X},\mathbf{h})} &= \frac{h_i\tilde{\phi}(\mathbf{p}_i,\mathbf{p}_j,\mathbf{x}_i,\mathbf{x}_j,\mathbf{X},\mathbf{h})}{h_j\tilde{\phi}(\mathbf{p}_j,\mathbf{p}_j,\mathbf{x}_j,\mathbf{x}_j,\mathbf{X},\mathbf{h})} \\
&= \frac{h_i\mathbf{p}_i^\top\mathbf{x}_i}{h_j\mathbf{p}_j^\top\mathbf{x}_j}\frac{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})}, \text{ by (2)} \qquad \text{(A.1)}\\
&= \left\{\frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}\right\}^{-1}, \text{ by T}\\
&= \left\{\frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})\right\}^{-1}\\
\Leftrightarrow \rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h}) &= \frac{h_i\mathbf{p}_i^\top\mathbf{x}_i}{h_j\mathbf{p}_j^\top\mathbf{x}_j}\frac{\sigma_j(\mathbf{P},\mathbf{X},\mathbf{h})}{\sigma_i(\mathbf{P},\mathbf{X},\mathbf{h})}
\end{aligned}$$

From (A.1),

$$\begin{aligned}
&\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}\sigma_i(\mathbf{P},\mathbf{X},\mathbf{h}) = \sum \sigma_j(\mathbf{P},\mathbf{X},\mathbf{h})\\
&\Leftrightarrow \sigma_i(\mathbf{P},\mathbf{X},\mathbf{h}) = \left\{\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}\right\}^{-1}, \text{ by S1}
\end{aligned}$$

■

PROOF OF LEMMA 2.

(i) By (6),

$$\begin{aligned}
&\sigma_i\big([\alpha_1\mathbf{p}_t,\ldots,\alpha_n\mathbf{p}_t]^\top,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)\\
&= \left\{\sum_j \frac{\gamma(\alpha_j\mathbf{p}_t)^\top(\beta_j\mathbf{x}_t)}{\gamma(\alpha_i\mathbf{p}_t)^\top(\beta_i\mathbf{x}_t)}\frac{\rho\big(\alpha_i\mathbf{p}_t,\alpha_j\mathbf{p}_t,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)}{\rho\big(\alpha_j\mathbf{p}_t,\alpha_j\mathbf{p}_t,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)}\right\}^{-1}\\
&= \left\{\sum_j \frac{\beta_j}{\beta_i}\frac{\rho\big(\mathbf{p}_t,\mathbf{p}_t,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)}{\rho\big(\mathbf{p}_t,\mathbf{p}_t,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)}\right\}^{-1}, \text{ by H and HDM}\\
&= \beta_i \text{ since } \sum \beta_j = 1
\end{aligned}$$

(ii) By (6), for any $i \in \mathcal{N}\backslash\{t\}$,

$$\sigma_i\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)$$

$$= \left\{\sum_{j\neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\rho\big(\mathbf{p}_j, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}\right.$$

$$\left. + \frac{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\big(\mathbf{p}_i, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\rho\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}\right\}^{-1}$$

$$= \left\{\sum_{j} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right.$$

$$\left. + (\lambda - 1)\frac{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_t, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}\right\}^{-1}, \text{ by SQD}$$

$$= \left\{[\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})]^{-1} + (\lambda - 1)\left[\frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}\right]^{-1}\right\}^{-1}$$

$$= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})\left\{1 + (\lambda - 1)\left[\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right]^{-1}\right\}^{-1}, \text{ by T}$$

$$= \frac{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

$$\sigma_t\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)$$

$$= \left\{\sum_{j\neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)} \frac{\rho\big(\mathbf{p}_t, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\rho\big(\mathbf{p}_j, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}\right.$$

$$\left. + \frac{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\rho\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}\right\}^{-1}$$

$$= \lambda\left\{\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} + (\lambda - 1)\right\}^{-1}, \text{ by SQD}$$

$$= \frac{\lambda\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

(iii) By (6),

$$\sigma_i\big([\alpha_1\mathbf{p}_1, \ldots, \alpha_n\mathbf{p}_n]^\top, \beta\mathbf{X}, \gamma\mathbf{h}\big)$$

$$= \left\{\sum_j \frac{\gamma h_j (\alpha_j\mathbf{p}_j)^\top (\beta\mathbf{x}_j)}{\gamma h_i (\alpha_i\mathbf{p}_i)^\top (\beta\mathbf{x}_i)} \frac{\rho(\alpha_i\mathbf{p}_i, \alpha_j\mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h})}{\rho(\alpha_j\mathbf{p}_j, \alpha_j\mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h})}\right\}^{-1}$$

$$= \left\{ \sum_j \frac{\alpha_j}{\alpha_i} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho(\alpha_i \mathbf{p}_i, \alpha_j \mathbf{p}_j, \beta \mathbf{X}, \gamma \mathbf{h})}{\rho(\alpha_j \mathbf{p}_j, \alpha_j \mathbf{p}_j, \beta \mathbf{X}, \gamma \mathbf{h})} \right\}^{-1}$$

$$= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}), \text{ by H, HDM and QD.}$$

(iv) By (6),

$$\sigma_i(\mathbf{P}\hat{\boldsymbol{\lambda}}, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h})$$

$$= \left\{ \sum_j \frac{h_j(\hat{\boldsymbol{\lambda}}\mathbf{p}_j)^\top (\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j)}{h_i(\hat{\boldsymbol{\lambda}}\mathbf{p}_i)^\top (\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i)} \frac{\rho(\hat{\boldsymbol{\lambda}}\mathbf{p}_i, \hat{\boldsymbol{\lambda}}\mathbf{p}_j, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h})}{\rho(\hat{\boldsymbol{\lambda}}\mathbf{p}_j, \hat{\boldsymbol{\lambda}}\mathbf{p}_j, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h})} \right\}^{-1}$$

$$= \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top (\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_j}{h_i \mathbf{p}_i^\top (\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1}, \text{ by C}$$

$$= \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top (\mathbf{I}_m)\mathbf{x}_j}{h_i \mathbf{p}_i^\top (\mathbf{I}_m)\mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1}$$

$$= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

(v) Consider the bijective mapping $\varphi : \mathcal{N} \to \mathcal{N}$ that satisfies

$$\text{col}_{\varphi(t)}\tilde{\mathbf{I}}_n = \text{col}_t\, \mathbf{I}_n \tag{A.2}$$

for any $t \in \mathcal{N}$. For all $k \in \mathcal{N}$, let

$$\left[\mathbf{p}'_{\varphi(k)}, \mathbf{x}'_{\varphi(k)}, h'_{\varphi(k)}\right] := [\mathbf{P}^\top, \mathbf{X}^\top, \mathbf{h}^\top]\text{col}_{\varphi(k)}\, \tilde{\mathbf{I}}_n \tag{A.3}$$

$$= [\mathbf{P}^\top, \mathbf{X}^\top, \mathbf{h}^\top]\text{col}_k\, \mathbf{I}_n, \text{ by (A.2)}$$

$$=: [\mathbf{p}_k, \mathbf{x}_k, h_k] \tag{A.4}$$

Now, by (6),

$$\sigma_{\varphi(i)}\left(\tilde{\mathbf{I}}_n^\top \mathbf{P}, \tilde{\mathbf{I}}_n^\top \mathbf{X}, \tilde{\mathbf{I}}_n^\top \mathbf{h}\right)$$

$$= \left\{ \sum_j \frac{h'_{\varphi(j)}\, \mathbf{p}'^\top_{\varphi(j)} \mathbf{x}'_{\varphi(j)}}{h'_{\varphi(i)}\, \mathbf{p}'^\top_{\varphi(i)} \mathbf{x}'_{\varphi(i)}} \frac{\rho\left(\mathbf{p}'_{\varphi(i)}, \mathbf{p}'_{\varphi(j)}, \tilde{\mathbf{I}}_n^\top \mathbf{X}, \tilde{\mathbf{I}}_n^\top \mathbf{h}\right)}{\rho\left(\mathbf{p}'_{\varphi(j)}, \mathbf{p}'_{\varphi(j)}, \tilde{\mathbf{I}}_n^\top \mathbf{X}, \tilde{\mathbf{I}}_n^\top \mathbf{h}\right)} \right\}^{-1}$$

$$= \left\{ \sum_j \frac{h_j\, \mathbf{p}_j^\top \mathbf{x}_j}{h_i\, \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1}, \text{ by (A.4) and WS}$$

$$= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

(vi) By (6),

$$\begin{aligned}
&\sigma_i\left(\mathbf{P}\tilde{\mathbf{I}}_m^\top, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right) \\
&= \left\{\sum_j \frac{h_j\left(\tilde{\mathbf{I}}_m\mathbf{p}_j\right)^\top\left(\tilde{\mathbf{I}}_m\mathbf{x}_j\right)}{h_i\left(\tilde{\mathbf{I}}_m\mathbf{p}_i\right)^\top\left(\tilde{\mathbf{I}}_m\mathbf{x}_i\right)} \frac{\rho\left(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right)}{\rho\left(\tilde{\mathbf{I}}_m\mathbf{p}_j, \tilde{\mathbf{I}}_m\mathbf{p}_j, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right)}\right\}^{-1} \\
&= \left\{\sum_j \frac{h_j\,\mathbf{p}_j^\top\left(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m\right)\mathbf{x}_j}{h_i\,\mathbf{p}_i^\top\left(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m\right)\mathbf{x}_i} \frac{\rho\left(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}\right)}{\rho\left(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h}\right)}\right\}^{-1}, \text{ by CS} \\
&= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) \text{ since } \tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m = \mathbf{I}_m
\end{aligned}$$

(vii) By (6), for any $i \in \mathcal{N}\backslash\{t\}$,

$$\begin{aligned}
&\bar{\sigma}_i\left([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right) \\
&= \left\{\sum_{j\neq t} \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i} \right. \\
&\quad \times \frac{\bar{\rho}\left(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_j, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)} \\
&\quad + \frac{[(1-\lambda)+\lambda]h_t\mathbf{p}_t^\top\mathbf{x}_t}{h_i\mathbf{p}_i^\top\mathbf{x}_i} \\
&\quad \left. \times \frac{\bar{\rho}\left(\mathbf{p}_i, \mathbf{p}_t, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}\right\}^{-1} \\
&= \left\{\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i} \frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right\}^{-1}, \text{ by CP} \\
&= \sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})
\end{aligned}$$

$$\begin{aligned}
&\bar{\sigma}_t\left([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)\,h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right) \\
&= \left\{\sum_{j\neq t} \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{(1-\lambda)h_t\mathbf{p}_t^\top\mathbf{x}_t} \right. \\
&\quad \times \frac{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_j, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_j, \mathbf{p}_j, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)} \\
&\quad + \frac{[(1-\lambda)+\lambda]h_t\,\mathbf{p}_t^\top\mathbf{x}_t}{(1-\lambda)h_t\,\mathbf{p}_t^\top\mathbf{x}_t}
\end{aligned}$$

$$\left. \times \frac{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_t, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)\, h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_t, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)\, h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)} \right\}^{-1}$$

$$= \left\{ (1-\lambda)^{-1} \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1}, \text{ by CP}$$

$$= (1-\lambda)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

$$\bar{\sigma}_{n+1}\left(\left[\mathbf{P}^\top, \mathbf{p}_t\right]^\top, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)$$

$$= \left\{ \sum_{j \neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_t} \right.$$

$$\times \frac{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_j, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}$$

$$+ \frac{[(1-\lambda)+\lambda]h_t\, \mathbf{p}_t^\top \mathbf{x}_t}{\lambda h_t\, \mathbf{p}_t^\top \mathbf{x}_t}$$

$$\left. \times \frac{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_t, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)}{\bar{\rho}\left(\mathbf{p}_t, \mathbf{p}_t, \left[\mathbf{X}^\top, \mathbf{x}_t\right]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\right)} \right\}^{-1}$$

$$= \left\{ \lambda^{-1} \sum_j \frac{h_j\, \mathbf{p}_j^\top \mathbf{x}_j}{h_t\, \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1}, \text{ by CP}$$

$$= \lambda \sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

(viii) By (6),

$$\lim_{\lambda \to 0} \sigma_i\left(\mathbf{P}, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)$$

$$= \lim_{\lambda \to 0} \left\{ \sum_{j \neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\left(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)}{\rho\left(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)} \right.$$

$$\left. + \frac{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_t}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\left(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)}{\rho\left(\mathbf{p}_t, \mathbf{p}_t, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)} \right\}^{-1}$$

$$= \left\{ \sum_{j \neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\lim_{\lambda \to 0} \rho\left(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)}{\lim_{\lambda \to 0} \rho\left(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)} \right.$$

$$\left. + \frac{h_t \mathbf{p}_t^\top \mathbf{x}_t}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \lim_{\lambda \to 0} \lambda \frac{\lim_{\lambda \to 0} \rho\left(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)}{\lim_{\lambda \to 0} \rho\left(\mathbf{p}_t, \mathbf{p}_t, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)} \right\}^{-1}$$

$$= \left\{ \sum_{j \neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t})}{\bar{\rho}(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t})} \right\}^{-1}, \text{ by TCI}$$

$$= \bar{\sigma}_i(\mathbf{P}_{-t}, \mathbf{X}_{-t}, \mathbf{h}_{-t})$$

(ix) By (6),

$$\lim_{\lambda \to 0} \frac{\sigma_i(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1}, \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_n]^\top)}{\sigma_j(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1}, \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_n]^\top)}$$

$$= \lim_{\lambda \to 0} \frac{\sum_{k \neq i,j} \frac{\lambda h_k \mathbf{p}_k^\top \mathbf{x}_k}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\rho\left(\mathbf{p}_j, \mathbf{p}_k, \mathbf{X}, [\ldots]^\top\right)}{\rho\left(\mathbf{p}_k, \mathbf{p}_k, \mathbf{X}, [\ldots]^\top\right)} + 1 + \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\rho\left(\mathbf{p}_j, \mathbf{p}_i, \mathbf{X}, [\ldots]^\top\right)}{\rho\left(\mathbf{p}_i, \mathbf{p}_i, \mathbf{X}, [\ldots]^\top\right)}}{\sum_{k \neq i,j} \frac{\lambda h_k \mathbf{p}_k^\top \mathbf{x}_k}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\left(\mathbf{p}_i, \mathbf{p}_k, \mathbf{X}, [\ldots]^\top\right)}{\rho\left(\mathbf{p}_k, \mathbf{p}_k, \mathbf{X}, [\ldots]^\top\right)} + 1 + \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho\left(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [\ldots]^\top\right)}{\rho\left(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, [\ldots]^\top\right)}}$$

$$= \frac{1 + \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\bar{\rho}(\mathbf{p}_j, \mathbf{p}_i, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)}{\bar{\rho}(\mathbf{p}_i, \mathbf{p}_i, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)}}{1 + \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)}{\bar{\rho}(\mathbf{p}_j, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j)}}, \text{ by TCI}$$

$$= \frac{\mathbf{p}_i^\top (h_i \mathbf{x}_i)}{\mathbf{p}_j^\top (h_j \mathbf{x}_j)} \Big/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j), \text{ by T}$$

$$= \bar{\bar{\phi}}(\mathbf{p}_i, \mathbf{p}_j, h_i \mathbf{x}_i, h_j \mathbf{x}_j, \mathbf{x}_i, \mathbf{x}_j, h_i, h_j), \text{ by (2)}$$

(x) For any $\mathbf{p}_i' > \mathbf{p}_i$,

$$\frac{\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \leq \frac{\rho(\mathbf{p}_i', \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}, \text{ by P and M}$$

$$\Leftrightarrow \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

$$\leq \frac{h_i \mathbf{p}_i'^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)}{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)}, \text{ by (A.1)}$$

$$\Leftrightarrow \frac{\sigma_j([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h})}{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

$$\geq \left[\frac{\mathbf{p}_i'^\top \mathbf{x}_i}{\mathbf{p}_i^\top \mathbf{x}_i}\right]^{-1} \frac{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

(xi) If $\mathbf{p}_i = \mathbf{p}_j$ then

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = 1, \text{ by I } (\Leftarrow \text{ P and T})$$

$$\Leftrightarrow \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_j, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)}{\sigma_i\left([\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_j, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\right)} = 1, \text{ by (7)}$$

(xii) By (6),

$$\sigma_i(\mathbf{P},\ \hat{\mathbf{h}}\mathbf{X},\mathbf{1}_n) = \left\{\sum_j \frac{\mathbf{p}_j^\top(h_j\mathbf{x}_j)}{\mathbf{p}_i^\top(h_i\mathbf{x}_i)}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\hat{\mathbf{h}}\mathbf{X},\mathbf{1}_n)}{\rho(\mathbf{p}_j,\mathbf{p}_j,\hat{\mathbf{h}}\mathbf{X},\mathbf{1}_n)}\right\}^{-1}$$

$$= \left\{\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}\right\}^{-1}, \text{ by TQ}$$

$$= \sigma_i(\mathbf{P},\mathbf{X},\mathbf{h})$$

(xiii) By (6), for any $i \in \mathcal{N}\backslash\{t\}$,

$$\bar{\sigma}_i([\mathbf{P}^\top,\mathbf{p}_t]^\top,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)$$

$$= \left\{\sum_{j\neq t}\frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\right.$$

$$\times\frac{\bar{\rho}(\mathbf{p}_i,\mathbf{p}_j,[\mathbf{x}_1,\dots,\mathbf{x}_t',\ \dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}{\bar{\rho}(\mathbf{p}_j,\mathbf{p}_j,[\mathbf{x}_1,\dots,\mathbf{x}_t',\ \dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}$$

$$+\frac{h_t\mathbf{p}_t^\top[(1-\lambda)\mathbf{x}_t'+\lambda\mathbf{x}_{n+1}]}{h_i\mathbf{p}_i^\top\mathbf{x}_i}$$

$$\left.\times\frac{\bar{\rho}(\mathbf{p}_i,\mathbf{p}_t,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}{\bar{\rho}(\mathbf{p}_t,\mathbf{p}_t,[\mathbf{x}_1,\dots,\mathbf{x}_t',\ \dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}\right\}^{-1}$$

$$= \left\{\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i}\frac{\rho(\mathbf{p}_i,\mathbf{p}_j,\mathbf{X},\mathbf{h})}{\rho(\mathbf{p}_j,\mathbf{p}_j,\mathbf{X},\mathbf{h})}\right\}^{-1},$$

by SCP and since $(1-\lambda)\mathbf{x}_t'+\lambda\mathbf{x}_{n+1}=\mathbf{x}_t$

$$= \sigma_i(\mathbf{P},\mathbf{X},\mathbf{h})$$

$$\bar{\sigma}_t([\mathbf{P}^\top,\mathbf{p}_t]^\top,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)$$

$$= \left\{\sum_{j\neq t}\frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{(1-\lambda)h_t\mathbf{p}_t^\top\mathbf{x}_t'}\right.$$

$$\times\frac{\bar{\rho}(\mathbf{p}_t,\mathbf{p}_j,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}{\bar{\rho}(\mathbf{p}_j,\mathbf{p}_j,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}$$

$$+\frac{h_t\mathbf{p}_t^\top[(1-\lambda)\mathbf{x}_t'+\lambda\mathbf{x}_{n+1}]}{(1-\lambda)h_t\mathbf{p}_t^\top\mathbf{x}_t'}$$

$$\left.\times\frac{\bar{\rho}(\mathbf{p}_t,\mathbf{p}_t,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}{\bar{\rho}(\mathbf{p}_t,\mathbf{p}_t,[\mathbf{x}_1,\dots,\mathbf{x}_t',\dots,\mathbf{x}_n,\mathbf{x}_{n+1}]^\top,[h_1,\dots,(1-\lambda)h_t,\dots,h_n,\lambda h_t]^\top)}\right\}^{-1}$$

$$= \left\{ \left[(1-\lambda) \frac{\mathbf{p}_t^\top \mathbf{x}_t'}{\mathbf{p}_t^\top \mathbf{x}_t} \right]^{-1} \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1},$$

by SCP and since $(1-\lambda)\mathbf{x}_t' + \lambda \mathbf{x}_{n+1} = \mathbf{x}_t$

$$= (1-\lambda) \frac{\mathbf{p}_t^\top \mathbf{x}_t'}{\mathbf{p}_t^\top \mathbf{x}_t} \sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

$$\bar{\sigma}_{n+1}\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \dots, \mathbf{x}_t', \dots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top\big)$$

$$= \left\{ \sum_{j \neq t} \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_{n+1}} \right.$$

$$\times \frac{\bar{\rho}\big(\mathbf{p}_t, \mathbf{p}_j, [\mathbf{x}_1, \dots, \mathbf{x}_t', \dots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top\big)}{\bar{\rho}\big(\mathbf{p}_j, \mathbf{p}_j, [\mathbf{x}_1, \dots, \mathbf{x}_t', \dots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top\big)}$$

$$+ \frac{h_t \mathbf{p}_t^\top [(1-\lambda)\mathbf{x}_t' + \lambda \mathbf{x}_{n+1}]}{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_{n+1}}$$

$$\left. \times \frac{\bar{\rho}\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \dots, \mathbf{x}_t', \dots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top\big)}{\bar{\rho}\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \dots, \mathbf{x}_t', \dots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top\big)} \right\}^{-1}$$

$$= \left\{ \left[\lambda \frac{\mathbf{p}_t^\top \mathbf{x}_{n+1}}{\mathbf{p}_t^\top \mathbf{x}_t} \right]^{-1} \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1},$$

by SCP and since $(1-\lambda)\mathbf{x}_t' + \lambda \mathbf{x}_{n+1} = \mathbf{x}_t$

$$= \lambda \frac{\mathbf{p}_t^\top \mathbf{x}_{n+1}}{\mathbf{p}_t^\top \mathbf{x}_t} \sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h}) \qquad \blacksquare$$

PROOF OF LEMMA 3.

(i) By (7), for any $\mathbf{p}_i' > \mathbf{p}_i$,

$$\rho(\mathbf{p}_i', \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{h_i \mathbf{p}_i'^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big([\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \dots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\big)}{\sigma_i\big([\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \dots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\big)}$$

$$\geq \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S11}$$

$$= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

(ii) By (7),

$$\rho(\lambda \mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{h_i (\lambda \mathbf{p}_i)^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big([\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \lambda \mathbf{p}_i, \mathbf{p}_{i+1}, \dots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\big)}{\sigma_i\big([\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \lambda \mathbf{p}_i, \mathbf{p}_{i+1}, \dots, \mathbf{p}_n]^\top, \mathbf{X}, \mathbf{h}\big)}$$

$$= \lambda \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S4}$$

$$= \lambda \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

(iii) By (7),

$$\begin{aligned}
\rho\big(\hat{\boldsymbol{\lambda}}\mathbf{p}_i, \hat{\boldsymbol{\lambda}}\mathbf{p}_j, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}\big) &= \frac{h_i(\hat{\boldsymbol{\lambda}}\mathbf{p}_i)^\top(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i)}{h_j(\hat{\boldsymbol{\lambda}}\mathbf{p}_j)^\top(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j)} \frac{\sigma_j(\mathbf{P}\hat{\boldsymbol{\lambda}}, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h})}{\sigma_i(\mathbf{P}\hat{\boldsymbol{\lambda}}, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h})} \\
&= \frac{h_i \mathbf{p}_i^\top(\hat{\boldsymbol{\lambda}}^\top \hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_i}{h_j \mathbf{p}_j^\top(\hat{\boldsymbol{\lambda}}^\top \hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S5} \\
&= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \hat{\boldsymbol{\lambda}}^\top \hat{\boldsymbol{\lambda}}^{-1} = \mathbf{I}_m
\end{aligned}$$

(iv) By (7),

$$\begin{aligned}
\rho\left(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right) &= \frac{h_i(\tilde{\mathbf{I}}_m\mathbf{p}_i)^\top(\tilde{\mathbf{I}}_m\mathbf{x}_i)}{h_j(\tilde{\mathbf{I}}_m\mathbf{p}_j)^\top(\tilde{\mathbf{I}}_m\mathbf{x}_j)} \frac{\sigma_j\left(\mathbf{P}\tilde{\mathbf{I}}_m^\top, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right)}{\sigma_i\left(\mathbf{P}\tilde{\mathbf{I}}_m^\top, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\right)} \\
&= \frac{h_i \mathbf{p}_i^\top\left(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m\right)\mathbf{x}_i}{h_j \mathbf{p}_j^\top\left(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m\right)\mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S7} \\
&= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m = \mathbf{I}_m
\end{aligned}$$

(v) Consider the bijective mapping $\varphi : \mathcal{N} \to \mathcal{N}$ that satisfies (A.2) for any $t \in \mathcal{N}$ and let (A.3) hold for all $k \in \mathcal{N}$. Now,

$$\begin{aligned}
\rho\left(\mathbf{p}_i, \mathbf{p}_j, \tilde{\mathbf{I}}_n^\top\mathbf{X}, \tilde{\mathbf{I}}_n^\top\mathbf{h}\right) &= \rho\big(\mathbf{p}'_{\varphi(i)}, \mathbf{p}'_{\varphi(j)}, \tilde{\mathbf{I}}_n^\top\mathbf{X}, \tilde{\mathbf{I}}_n^\top\mathbf{h}\big), \text{ by (A.4)} \\
&= \frac{h'_{\varphi(i)}\mathbf{p}'^\top_{\varphi(i)}\mathbf{x}'_{\varphi(i)}}{h'_{\varphi(j)}\mathbf{p}'^\top_{\varphi(j)}\mathbf{x}'_{\varphi(j)}} \frac{\sigma_{\varphi(j)}\big(\tilde{\mathbf{I}}_n^\top\mathbf{P}, \tilde{\mathbf{I}}_n^\top\mathbf{X}, \tilde{\mathbf{I}}_n^\top\mathbf{h}\big)}{\sigma_{\varphi(i)}\big(\tilde{\mathbf{I}}_n^\top\mathbf{P}, \tilde{\mathbf{I}}_n^\top\mathbf{X}, \tilde{\mathbf{I}}_n^\top\mathbf{h}\big)}, \text{ by (7)} \\
&= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by (A.4) and S6} \\
&= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

(vi) By (7),

$$\begin{aligned}
\rho(\mathbf{p}_i, \mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h}) &= \frac{(\gamma h_i)\mathbf{p}_i^\top(\beta\mathbf{x}_i)}{(\gamma h_j)\mathbf{p}_j^\top(\beta\mathbf{x}_j)} \frac{\sigma_j(\mathbf{P}, \beta\mathbf{X}, \gamma\mathbf{h})}{\sigma_i(\mathbf{P}, \beta\mathbf{X}, \gamma\mathbf{h})} \\
&= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S4} \\
&= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

(vii) By (7), for any $i, j \in \mathcal{N}\backslash\{t\}$,

$$\begin{aligned}
&\rho\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\sigma_i\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \lambda\mathbf{x}_t, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)} \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})} \frac{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S3} \\
&\quad = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

$$\begin{aligned}
&\rho\big(\mathbf{p}_i, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)} \frac{\sigma_t\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\sigma_i\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)} \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{\lambda\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})} \frac{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\lambda\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S3} \\
&\quad = \rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})
\end{aligned}$$

$$\begin{aligned}
&\rho\big(\mathbf{p}_t, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) \\
&\quad = \frac{h_t \mathbf{p}_t^\top (\lambda\mathbf{x}_t)}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)}{\sigma_t\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)} \\
&\quad = \frac{h_t \mathbf{p}_t^\top \mathbf{x}_t}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\lambda\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})} \frac{1 + (\lambda - 1)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\lambda\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S3} \\
&\quad = \rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

$$\begin{aligned}
\rho\big(\mathbf{p}_t, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) &= 1, \text{ by I } (\Leftarrow \text{ P and T}) \\
&= \rho(\mathbf{p}_t, \mathbf{p}_t, \mathbf{X}, \mathbf{h}), \text{ by I}
\end{aligned}$$

(viii) By (7),

$$\begin{aligned}
\rho(\mathbf{p}_i, \mathbf{p}_j, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n) &= \frac{\mathbf{p}_i^\top (h_i \mathbf{x}_i)}{\mathbf{p}_j^\top (h_j \mathbf{x}_j)} \frac{\sigma_j(\mathbf{P}, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n)}{\sigma_i(\mathbf{P}, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n)} \\
&= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S13} \\
&= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

(ix) By (7), for any $i,\ j \in \mathcal{N}\backslash\{t\}$,

$$\begin{aligned}
&\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big) \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \\
&\qquad \times \frac{\bar{\sigma}_j\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_i\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)} \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S8} \\
&\quad = \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

$$\begin{aligned}
&\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_t, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big) \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{(1-\lambda) h_t \mathbf{p}_t^\top \mathbf{x}_t} \\
&\qquad \times \frac{\bar{\sigma}_t\big(\big[\mathbf{P}^\top, \mathbf{p}_t\big]^\top, \big[\mathbf{X}^\top, \mathbf{x}_t\big]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_i\big(\big[\mathbf{P}^\top, \mathbf{p}_t\big]^\top, \big[\mathbf{X}^\top, \mathbf{x}_t\big]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)} \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{(1-\lambda) h_t \mathbf{p}_t^\top \mathbf{x}_t} \frac{(1-\lambda)\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S8} \\
&\quad = \rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})
\end{aligned}$$

and if $\mathbf{p}_{n+1} = \mathbf{p}_t$, then

$$\begin{aligned}
&\bar{\rho}\big(\mathbf{p}_{n+1}, \mathbf{p}_j, \big[\mathbf{X}^\top, \mathbf{x}_t\big]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big) \\
&\quad = \frac{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_t}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \\
&\qquad \times \frac{\bar{\sigma}_j\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_{n+1}\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \ldots, h_{t-1}, (1-\lambda)h_t, h_{t+1}, \ldots, h_n, \lambda h_t]^\top\big)} \\
&\quad = \frac{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_t}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\lambda \sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S8} \\
&\quad = \rho(\mathbf{p}_j, \mathbf{p}_t, \mathbf{X}, \mathbf{h})
\end{aligned}$$

(x) By (7), for any $i,\ j \in \mathcal{N}\backslash\{t\}$,

$$\begin{aligned}
&\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_t', \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big) \\
&\quad = \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j}
\end{aligned}$$

$$\times \frac{\bar{\sigma}_j\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_i\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}$$

$$= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big(\mathbf{P}, \mathbf{X}, \mathbf{h}\big)}{\sigma_i\big(\mathbf{P}, \mathbf{X}, \mathbf{h}\big)}, \text{ by S14}$$

$$= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

$$\bar{\rho}\big(\mathbf{p}_i, \mathbf{p}_t, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)$$

$$= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{(1-\lambda)h_t \mathbf{p}_t^\top \mathbf{x}_t}$$

$$\times \frac{\bar{\sigma}_t\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_i\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}$$

$$= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{(1-\lambda)h_t \mathbf{p}_t^\top \mathbf{x}'_t} \frac{(1-\lambda)\mathbf{p}_t^\top \mathbf{x}'_t}{\mathbf{p}_t^\top \mathbf{x}_t} \frac{\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S14}$$

$$= \rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})$$

and if $\mathbf{p}_{n+1} = \mathbf{p}_t$, then

$$\bar{\rho}\big(\mathbf{p}_{n+1}, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)$$

$$= \frac{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_{n+1}}{h_j \mathbf{p}_j^\top \mathbf{x}_j}$$

$$\times \frac{\bar{\sigma}_j\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}{\bar{\sigma}_{n+1}\big([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{x}_1, \ldots, \mathbf{x}'_t, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)}$$

$$= \frac{\lambda h_t \mathbf{p}_t^\top \mathbf{x}_{n+1}}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\mathbf{p}_t^\top \mathbf{x}_t}{\lambda \mathbf{p}_t^\top \mathbf{x}_{n+1}} \frac{\sigma_j(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_t(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by S14}$$

$$= \rho(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

(xi) By (7),

$$\lim_{\lambda \to 0} \rho\big(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big)$$

$$= \lim_{\lambda \to 0} \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\sigma_j\big(\mathbf{P}, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big)}{\sigma_i\big(\mathbf{P}, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big)}$$

$$= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{h_j \mathbf{p}_j^\top \mathbf{x}_j} \frac{\bar{\sigma}_j(\mathbf{P}_{-t}, \mathbf{X}_{-t}, \mathbf{h}_{-t})}{\bar{\sigma}_i(\mathbf{P}_{-t}, \mathbf{X}_{-t}, \mathbf{h}_{-t})}, \text{ by S9}$$

$$= \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t}) \qquad \blacksquare$$

PROOF OF THEOREM 4. Part (i) follows from Lemmas 2(ii) and 3(vii). Part (ii) follows from Lemma 2(iii), Theorem 1(v), and Lemma 3, parts (ii) and (vi). Part (iii) follows from Lemmas 2(iv) and 3(iii). Part (iv) follows from Lemmas 2(v) and 3(v). Part (v) follows from Lemmas 2(vi) and 3(iv). Part (vi) follows from Lemmas 2(vii) and 3(ix). Part (vii) follows from Lemmas 2 (viii), 2(ix), and 3(xi). Part (viii) follows from Lemmas 2(x) and 3(i). Part (ix) follows from Lemma 2(xi). Part (x) follows from Lemmas 2(xii) and 3(viii). Part (xi) follows from Lemmas 2(xiii) and 3(x). ■

PROOF OF LEMMA 4. Necessity: By T,

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\rho(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{p}_t, \mathbf{X}, \mathbf{h})}$$

Since the left-hand side and, consequently, the right-hand side of this equation is independent of $\mathbf{p}_t$, it can be rewritten as

$$\rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\rho(\mathbf{p}_i, \mathbf{1}_m, \mathbf{X}, \mathbf{h})}{\rho(\mathbf{p}_j, \mathbf{1}_m, \mathbf{X}, \mathbf{h})} =: \frac{\delta(\mathbf{p}_i, \mathbf{X}, \mathbf{h})}{\delta(\mathbf{p}_j, \mathbf{X}, \mathbf{h})}$$

where $\mathbf{1}_m$ is the m-dimensional column vector of ones.

Sufficiency: Straightforward.

PROOF OF THEOREM 5. By (6) and I ($\Leftarrow$ P and T),

$$\begin{aligned}\sigma_i(\mathbf{P}, \mathbf{X}, \mathbf{h}) &= \left\{\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \rho(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})\right\}^{-1} \\ &= \left\{\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\delta(\mathbf{p}_i, \mathbf{X}, \mathbf{h})}{\delta(\mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right\}^{-1}, \text{ by Lemma 4} \\ &= \frac{h_i \mathbf{p}_i^\top \mathbf{x}_i}{\delta(\mathbf{p}_i, \mathbf{X}, \mathbf{h})} \left\{\sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{\delta(\mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right\}^{-1}\end{aligned}$$

■

PROOF OF THEOREM 6. Positivity:

$$\rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{\bar{h}_k} > 0$$

Positive Monotonicity: For any $\mathbf{p}_i' > \mathbf{p}_i$,

$$\begin{aligned}\rho_{HD}(\mathbf{p}_i', \mathbf{p}_j, \mathbf{X}, \mathbf{h}) &= \prod_k \left[\frac{\mathbf{p}_i'^{\top}\mathbf{x}_k}{\mathbf{p}_j^{\top}\mathbf{x}_k}\right]^{\bar{h}_k} \\ &\geq \prod_k \left[\frac{\mathbf{p}_i^{\top}\mathbf{x}_k}{\mathbf{p}_j^{\top}\mathbf{x}_k}\right]^{\bar{h}_k} \\ &= \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})\end{aligned}$$

Linear Homogeneity: For any $\lambda \in \mathbb{R}_{++}$,

$$\rho_{HD}(\lambda\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_k \left[\frac{(\lambda\mathbf{p}_i)^{\top}\mathbf{x}_k}{\mathbf{p}_j^{\top}\mathbf{x}_k}\right]^{\bar{h}_k} = \lambda\rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Transitivity: For any $t \in \mathcal{N}$,

$$\rho_{HD}(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h})\rho_{HD}(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_k \left[\frac{\mathbf{p}_i^{\top}\mathbf{x}_k}{\mathbf{p}_t^{\top}\mathbf{x}_k}\frac{\mathbf{p}_t^{\top}\mathbf{x}_k}{\mathbf{p}_j^{\top}\mathbf{x}_k}\right]^{\bar{h}_k} = \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Commensurability: For any $\boldsymbol{\lambda} \in \mathbb{R}^m_{++}$,

$$\begin{aligned}\rho_{HD}\left(\hat{\boldsymbol{\lambda}}\mathbf{p}_i,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_j, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}\right) &= \prod_k \left[\frac{(\hat{\boldsymbol{\lambda}}\mathbf{p}_i)^{\top}(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_k)}{(\hat{\boldsymbol{\lambda}}\mathbf{p}_j)^{\top}(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_k)}\right]^{\bar{h}_k} \\ &= \prod_k \left[\frac{\mathbf{p}_i^{\top}(\hat{\boldsymbol{\lambda}}^{\top}\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_k}{\mathbf{p}_j^{\top}(\hat{\boldsymbol{\lambda}}^{\top}\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_k}\right]^{\bar{h}_k} \\ &= \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \hat{\boldsymbol{\lambda}}^{\top}\hat{\boldsymbol{\lambda}}^{-1} = \mathbf{I}_m\end{aligned}$$

Commodity Symmetry: For any permutation matrix $\tilde{\mathbf{I}}_m$,

$$\begin{aligned}\rho_{HD}\left(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \mathbf{X}\tilde{\mathbf{I}}_m^{\top}, \mathbf{h}\right) &= \prod_k \left[\frac{(\tilde{\mathbf{I}}_m\mathbf{p}_i)^{\top}(\tilde{\mathbf{I}}_m\mathbf{x}_k)}{(\tilde{\mathbf{I}}_m\mathbf{p}_j)^{\top}(\tilde{\mathbf{I}}_m\mathbf{x}_k)}\right]^{\bar{h}_k} \\ &= \prod_k \left[\frac{\mathbf{p}_i^{\top}(\tilde{\mathbf{I}}_m^{\top}\tilde{\mathbf{I}}_m)\mathbf{x}_k}{\mathbf{p}_j^{\top}(\tilde{\mathbf{I}}_m^{\top}\tilde{\mathbf{I}}_m)\mathbf{x}_k}\right]^{\bar{h}_k} \\ &= \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \tilde{\mathbf{I}}_m^{\top}\tilde{\mathbf{I}}_m = \mathbf{I}_m\end{aligned}$$

Weight Symmetry: Consider the bijective mapping $\varphi : \mathcal{N} \times \mathcal{N}$ that satisfies (A.2) for any $t \in \mathcal{N}$ and let (A.3) hold for all $k \in \mathcal{N}$. Now,

$$\rho_{HD}\big(\mathbf{p}_i, \mathbf{p}_j, \tilde{\mathbf{I}}_n^\top \mathbf{X}, \tilde{\mathbf{I}}_n^\top \mathbf{h}\big) := \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_{\varphi(k)}}{\mathbf{p}_j^\top \mathbf{x}_{\varphi(k)}}\right]^{\bar{h}_{\varphi(k)}} = \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Quantity Dimensionality: For any $\beta, \gamma \in \mathbb{R}_{++}$,

$$\rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h}) = \prod_k \left[\frac{\mathbf{p}_i^\top(\beta\mathbf{x}_k)}{\mathbf{p}_j^\top(\beta\mathbf{x}_k)}\right]^{\frac{\gamma h_k}{\sum \gamma h_t}} = \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Strong Quantity Dimensionality: For any $t \in \mathcal{N}$ and for any $\lambda \in \mathbb{R}_{++}$,

$$\begin{aligned}
&\rho_{HD}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big)\\
&= \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{\bar{h}_k} \left[\frac{\mathbf{p}_i^\top(\lambda\mathbf{x}_t)}{\mathbf{p}_j^\top(\lambda\mathbf{x}_t)}\right]^{\bar{h}_t}\\
&= \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})
\end{aligned}$$

Determinateness: For any $(\ell, t) \in \mathcal{M} \times \mathcal{N}$,

$$\lim_{p_{i\ell} \to 0} \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_k \left[\frac{\sum_{l \neq \ell} p_{il} x_{il}}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{\bar{h}_k} > 0$$

$$\lim_{p_{j\ell} \to 0} \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\sum_{l \neq \ell} p_{jl} x_{jl}}\right]^{\bar{h}_k} > 0$$

$$\lim_{x_{t\ell} \to 0} \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{\bar{h}_k} \left[\frac{\sum_{l \neq \ell} p_{il} x_{tl}}{\sum_{l \neq \ell} p_{jl} x_{tl}}\right]^{\bar{h}_t} > 0$$

$$\lim_{h_t \to 0} \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{h_k / \sum_{r \neq t} h_r} > 0$$

Country Partitioning Test: For any $t \in \mathcal{N}$ and for any $\lambda \in (0, 1)$,

$$\begin{aligned}
&\bar{\rho}_{HD}\big(\mathbf{p}_i, \mathbf{p}_j, \big[\mathbf{X}^\top, \mathbf{x}_t\big]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)\\
&= \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k}\right]^{\bar{h}_k} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_j^\top \mathbf{x}_t}\right]^{(1-\lambda)\bar{h}_t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_j^\top \mathbf{x}_t}\right]^{\lambda\bar{h}_t}
\end{aligned}$$

$$= \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\bar{h}_k} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_j^\top \mathbf{x}_t} \right]^{\bar{h}_t}$$

$$= \rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Tiny Country Irrelevance: For any $t \in \mathcal{N}$,

$$\lim_{\lambda \to 0} \rho_{HD}\left(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\right)$$

$$= \lim_{\lambda \to 0} \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\frac{h_k}{\sum_{r \neq t} h_r + \lambda h_t}} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_j^\top \mathbf{x}_t} \right]^{\frac{\lambda h_t}{\sum_{r \neq t} h_r + \lambda h_t}}$$

$$= \prod_{k \neq t} \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\frac{h_k}{\sum_{r \neq t} h_r}}$$

$$= \bar{\rho}_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t}) \qquad \blacksquare$$

PROOF OF COROLLARY 1. Since ρ_{HD} satisfies P,

$$\sigma_{HD,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\rho_{HD}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho_{HD}(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})} \right\}^{-1} > 0$$

Now,

$$\sum_i \sigma_{HD,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \sum_i \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \prod_k \left[\frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_j^\top \mathbf{x}_k} \right]^{\bar{h}_k} \right\}^{-1}$$

$$= \frac{\sum_i h_i \mathbf{p}_i^\top \mathbf{x}_i \prod_k \left[\mathbf{p}_i^\top \mathbf{x}_k\right]^{-\bar{h}_k}}{\sum_j h_j \mathbf{p}_j^\top \mathbf{x}_j \prod_k \left[\mathbf{p}_j^\top \mathbf{x}_k\right]^{-\bar{h}_k}}$$

$$= 1$$

Thus $\boldsymbol{\sigma}_{HD}$ satisfies S1. By Lemma 2, $\boldsymbol{\sigma}_{HD}$ satisfies S2–S12. ■

PROOF OF THEOREM 7. Straightforward.

PROOF OF COROLLARY 2. Straightforward.

PROOF OF THEOREM 8. Positivity:

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}_i^\top \sum h_k \mathbf{x}_k}{\mathbf{p}_j^\top \sum h_k \mathbf{x}_k} > 0$$

Positive Monotonicity: For any $\mathbf{p}'_i > \mathbf{p}_i$,

$$\rho_{AB}(\mathbf{p}'_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}'^{\top}_i \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} \sum h_k \mathbf{x}_k} \geq \frac{\mathbf{p}_i^{\top} \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} \sum h_k \mathbf{x}_k} = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Linear Homogeneity: For any $\lambda \in \mathbb{R}_{++}$,

$$\rho_{AB}(\lambda \mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{(\lambda \mathbf{p}_i)^{\top} \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} \sum h_k \mathbf{x}_k} = \lambda \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Transitivity: For any $t \in \mathcal{N}$,

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_t, \mathbf{X}, \mathbf{h}) \rho_{AB}(\mathbf{p}_t, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}_i^{\top} \sum h_k \mathbf{x}_k}{\mathbf{p}_t^{\top} \sum h_k \mathbf{x}_k} \frac{\mathbf{p}_t^{\top} \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} \sum h_k \mathbf{x}_k} = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Commensurability: For any $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_m)^{\top} \in \mathbb{R}^m_{++}$,

$$\begin{aligned}
\rho_{AB}\left(\hat{\boldsymbol{\lambda}} \mathbf{p}_i, \hat{\boldsymbol{\lambda}} \mathbf{p}_j, \mathbf{X} \hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}\right) &= \frac{(\hat{\boldsymbol{\lambda}} \mathbf{p}_i)^{\top} \sum h_k \left(\hat{\boldsymbol{\lambda}}^{-1} \mathbf{x}_k\right)}{(\hat{\boldsymbol{\lambda}} \mathbf{p}_j)^{\top} \sum h_k \left(\hat{\boldsymbol{\lambda}}^{-1} \mathbf{x}_k\right)} \\
&= \frac{\mathbf{p}_i^{\top} (\hat{\boldsymbol{\lambda}}^{\top} \hat{\boldsymbol{\lambda}}^{-1}) \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} (\hat{\boldsymbol{\lambda}}^{\top} \hat{\boldsymbol{\lambda}}^{-1}) \sum h_k \mathbf{x}_k} \\
&= \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \hat{\boldsymbol{\lambda}}^{\top} \hat{\boldsymbol{\lambda}}^{-1} = \mathbf{I}_m
\end{aligned}$$

Commodity Symmetry: For any permutation matrix $\tilde{\mathbf{I}}_m$,

$$\begin{aligned}
\rho_{AB}(\tilde{\mathbf{I}}_m \mathbf{p}_i, \tilde{\mathbf{I}}_m \mathbf{p}_j, \mathbf{X} \tilde{\mathbf{I}}_m^{\top}, \mathbf{h}) &= \frac{(\tilde{\mathbf{I}}_m \mathbf{p}_i)^{\top} \sum h_k (\tilde{\mathbf{I}}_m \mathbf{x}_k)}{(\tilde{\mathbf{I}}_m \mathbf{p}_j)^{\top} \sum h_k (\tilde{\mathbf{I}}_m \mathbf{x}_k)} \\
&= \frac{\mathbf{p}_i^{\top} (\tilde{\mathbf{I}}_m^{\top} \tilde{\mathbf{I}}_m) \sum h_k \mathbf{x}_k}{\mathbf{p}_j^{\top} (\tilde{\mathbf{I}}_m^{\top} \tilde{\mathbf{I}}_m) \sum h_k \mathbf{x}_k} \\
&= \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } \tilde{\mathbf{I}}_m^{\top} \tilde{\mathbf{I}}_m = \mathbf{I}_m
\end{aligned}$$

Weight Symmetry: Consider the bijective mapping $\varphi : \mathcal{N} \times \mathcal{N}$ that satisfies (A.2) for any $t \in \mathcal{N}$ and let (A.3) hold for all $k \in \mathcal{N}$. Now,

$$\rho_{AB}\left(\mathbf{p}_i, \mathbf{p}_j, \tilde{\mathbf{I}}_n^{\top} \mathbf{X}, \tilde{\mathbf{I}}_n^{\top} \mathbf{h}\right) = \frac{\mathbf{p}_i^{\top} \sum h_{\varphi(k)} \mathbf{x}_{\varphi(k)}}{\mathbf{p}_j^{\top} \sum h_{\varphi(k)} \mathbf{x}_{\varphi(k)}} = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Quantity Dimensionality: For any $(\beta, \gamma) \in \mathbb{R}^2_{++}$,

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \beta\mathbf{X}, \gamma\mathbf{h}) = \frac{\mathbf{p}_i^\top \sum(\gamma h_k)(\beta\mathbf{x}_k)}{\mathbf{p}_j^\top \sum(\gamma h_k)(\beta\mathbf{x}_k)} = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Total Quantities Test:

$$\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n) = \frac{\mathbf{p}_i^\top \sum(1)(h_k\mathbf{x}_k)}{\mathbf{p}_j^\top \sum(1)(h_k\mathbf{x}_k)} = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})$$

Determinateness: For any $(\ell, t) \in \mathcal{M} \times \mathcal{N}$,

$$\lim_{p_{i\ell}\to 0} \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\sum_{l\neq\ell} p_{il} \sum_k h_k x_{kl}}{\mathbf{p}_j^\top \sum h_k\mathbf{x}_k} > 0$$

$$\lim_{p_{j\ell}\to 0} \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}_i^\top \sum h_k\mathbf{x}_k}{\sum_{l\neq\ell} p_{jl} \sum_k h_k x_{kl}} > 0$$

$$\lim_{x_{t\ell}\to 0} \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \lim_{h_t\to 0} \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) = \frac{\mathbf{p}_i^\top \sum_{k\neq t} h_k\mathbf{x}_k}{\mathbf{p}_j^\top \sum_{k\neq t} h_k\mathbf{x}_k} > 0$$

Strong Country Partitioning Test: For any $(t, \lambda) \in \mathcal{N} \times (0, 1)$,

$$\begin{aligned}
&\bar{\rho}_{AB}\big(\mathbf{p}_i, \mathbf{p}_j, [\mathbf{x}_1, \ldots, \mathbf{x}_t', \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big) \\
&\quad = \frac{\mathbf{p}_i^\top\big[\sum_{k\neq t} h_k\mathbf{x}_k + (1-\lambda)h_t\mathbf{x}_t' + \lambda h_t\mathbf{x}_{n+1}\big]}{\mathbf{p}_j^\top\big[\sum_{k\neq t} h_k\mathbf{x}_k + (1-\lambda)h_t\mathbf{x}_t' + \lambda h_t\mathbf{x}_{n+1}\big]} \\
&\quad = \rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h}) \text{ since } (1-\lambda)\mathbf{x}_t' + \lambda\mathbf{x}_{n+1} = \mathbf{x}_t
\end{aligned}$$

Tiny Country Irrelevance: For any $t \in \mathcal{N}$,

$$\begin{aligned}
&\lim_{\lambda\to 0} \rho_{AB}\big(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big) \\
&\quad = \lim_{\lambda\to 0} \frac{\mathbf{p}_i^\top\big[\sum_{k\neq t} h_k\mathbf{x}_k + \lambda h_t\mathbf{x}_t\big]}{\mathbf{p}_j^\top\big[\sum_{k\neq t} h_k\mathbf{x}_k + \lambda h_t\mathbf{x}_t\big]} \\
&\quad = \bar{\rho}_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}_{-t}, \mathbf{h}_{-t})
\end{aligned}$$

■

PROOF OF COROLLARY 3. Since ρ_{AB} satisfies P,

$$\sigma_{AB,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) = \left\{\sum_j \frac{h_j\mathbf{p}_j^\top\mathbf{x}_j}{h_i\mathbf{p}_i^\top\mathbf{x}_i} \frac{\rho_{AB}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}{\rho_{AB}(\mathbf{p}_j, \mathbf{p}_j, \mathbf{X}, \mathbf{h})}\right\}^{-1} > 0$$

Now,

$$\begin{aligned}\sum \sigma_{AB,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) &= \sum_i \left\{ \sum_j \frac{h_j \mathbf{p}_j^\top \mathbf{x}_j}{h_i \mathbf{p}_i^\top \mathbf{x}_i} \frac{\mathbf{p}_i^\top \sum h_k \mathbf{x}_k}{\mathbf{p}_j^\top \sum h_k \mathbf{x}_k} \right\}^{-1} \\ &= \frac{\sum_i h_i \mathbf{p}_i^\top \mathbf{x}_i / \mathbf{p}_i^\top \sum h_k \mathbf{x}_k}{\sum_j h_j \mathbf{p}_j^\top \mathbf{x}_j / \mathbf{p}_j^\top \sum h_k \mathbf{x}_k} \\ &= 1\end{aligned}$$

Thus $\boldsymbol{\sigma}_{AB}$ satisfies S1. By Lemma 2, $\boldsymbol{\sigma}_{AB}$ satisfies S2 and S4–S14. ■

Proof of Theorem 9. The bilateral consumption index ϕ has the following six properties.[14]

Q1. Positivity:

$$\phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) = \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \Big/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1) > 0, \text{ by P}$$

Q2. Identity:

$$\phi(\lambda \mathbf{p}_j, \mathbf{p}_j, \mathbf{x}_j, \mathbf{x}_j) = \frac{(\lambda \mathbf{p}_j)^\top \mathbf{x}_j}{\mathbf{p}_j^\top \mathbf{x}_j} \Big/ \bar{\rho}(\lambda \mathbf{p}_j, \mathbf{p}_j, \mathbf{x}_j, \mathbf{x}_j, 1, 1) = 1, \text{ by PP}$$

Q3. Proportionality:

$$\begin{aligned}\phi(\mathbf{p}_i, \mathbf{p}_j, \beta_i \mathbf{x}_i, \beta_j \mathbf{x}_j) &= \frac{\mathbf{p}_i^\top (\beta_i \mathbf{x}_i)}{\mathbf{p}_j^\top (\beta_j \mathbf{x}_j)} \Big/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \beta_i \mathbf{x}_i, \beta_j \mathbf{x}_j, 1, 1) \\ &= \frac{\beta_i}{\beta_j} \frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \Big/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1), \text{ by SQD} \\ &= \frac{\beta_i}{\beta_j} \phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)\end{aligned}$$

Q4. Strong Monetary Unit Test:

$$\begin{aligned}&\phi(\alpha_i \mathbf{p}_i, \alpha_j \mathbf{p}_j, \beta \mathbf{x}_i, \beta \mathbf{x}_j) \\ &\quad = \frac{(\alpha_i \mathbf{p}_i)^\top (\beta \mathbf{x}_i)}{(\alpha_j \mathbf{p}_j)^\top (\beta \mathbf{x}_j)} \Big/ \bar{\rho}(\alpha_i \mathbf{p}_i, \alpha_j \mathbf{p}_j, \beta \mathbf{x}_i, \beta \mathbf{x}_j, 1, 1)\end{aligned}$$

[14] With the exception of Q3, these properties correspond to Diewert's (1986) "essential" bilateral tests BT1, BT2, BT4, BT5, and BT7. Since Q3 is implied by BT3 and BT6, Theorems 9, 10, 11, and 12 constitute a stronger version of Diewert (1986, Proposition 8).

$$= \frac{\alpha_i}{\alpha_j}\frac{\mathbf{p}_i^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \Bigg/ \left\{\frac{\alpha_i}{\alpha_j}\bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1)\right\}, \text{ by H, HDM and SQD}$$
$$= \phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j)$$

Q5. Commensurability:

$$\phi\big(\hat{\boldsymbol{\lambda}}\mathbf{p}_i,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_j,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j\big)$$
$$= \frac{(\hat{\boldsymbol{\lambda}}\mathbf{p}_i)^\top(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i)}{(\hat{\boldsymbol{\lambda}}\mathbf{p}_j)^\top(\hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j)} \Bigg/ \bar{\rho}\big(\hat{\boldsymbol{\lambda}}\mathbf{p}_i,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_j,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j, 1, 1\big)$$
$$= \frac{\mathbf{p}_i^\top(\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_i}{\mathbf{p}_j^\top(\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}}^{-1})\mathbf{x}_j} \Bigg/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1), \text{ by C}$$
$$= \phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) \text{ since } \hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}}^{-1} = \mathbf{I}_m$$

Q6. Commodity Symmetry:

$$\phi(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \tilde{\mathbf{I}}_m\mathbf{x}_i, \tilde{\mathbf{I}}_m\mathbf{x}_j)$$
$$= \frac{(\tilde{\mathbf{I}}_m\mathbf{p}_i)^\top(\tilde{\mathbf{I}}_m\mathbf{x}_i)}{(\tilde{\mathbf{I}}_m\mathbf{p}_j)^\top(\tilde{\mathbf{I}}_m\mathbf{x}_j)} \Bigg/ \bar{\rho}(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_j, \tilde{\mathbf{I}}_m\mathbf{x}_i, \tilde{\mathbf{I}}_m\mathbf{x}_j, 1, 1)$$
$$= \frac{\mathbf{p}_i^\top(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m)\mathbf{x}_i}{\mathbf{p}_j^\top(\tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m)\mathbf{x}_j} \Bigg/ \bar{\rho}(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j, 1, 1), \text{ by CS}$$
$$= \phi(\mathbf{p}_i, \mathbf{p}_j, \mathbf{x}_i, \mathbf{x}_j) \text{ since } \tilde{\mathbf{I}}_m^\top\tilde{\mathbf{I}}_m = \mathbf{I}_m$$

S1 holds by Q1 and since $\sum_i \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}) = 1$. S2:

$$\sigma_{k*,i}\big([\alpha_1\mathbf{p}_t, \ldots, \alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \frac{\gamma\phi(\alpha_i\mathbf{p}_t, \alpha_k\mathbf{p}_t, \beta_i\mathbf{x}_t, \beta_k\mathbf{x}_t)}{\sum_j \gamma\phi(\alpha_j\mathbf{p}_t, \alpha_k\mathbf{p}_t, \beta_j\mathbf{x}_t, \beta_k\mathbf{x}_t)}$$
$$= \frac{\frac{\beta_i}{\beta_k}\phi(\mathbf{p}_t, \mathbf{p}_t, \mathbf{x}_t, \mathbf{x}_t)}{\sum_j \frac{\beta_j}{\beta_k}\phi(\mathbf{p}_t, \mathbf{p}_t, \mathbf{x}_t, \mathbf{x}_t)}, \text{ by Q3 and Q4}$$
$$= \beta_i \text{ since } \sum \beta_j = 1$$

S3: For $i \neq t \neq k$,

$$
\begin{aligned}
&\sigma_{k*,i}\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) \\
&\quad = \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{j\neq t} h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \lambda\mathbf{x}_t, \mathbf{x}_k)} \\
&\quad = \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{j\neq t} h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + \lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)}, \text{ by Q3} \\
&\quad = \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{j} h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + (\lambda-1) h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)} \\
&\quad = \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)\big/\sum_j h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)}{1 + (\lambda-1) h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)\big/\sum_j h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)} \\
&\quad = \frac{\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda-1)\sigma_{k*,t}(\mathbf{P}, \mathbf{X}, \mathbf{h})}
\end{aligned}
$$

For $t \neq k$,

$$
\begin{aligned}
&\sigma_{k*,t}\big(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\big) \\
&\quad = \frac{h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \lambda\mathbf{x}_t, \mathbf{x}_k)}{\sum_{j\neq t} h_l\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \lambda\mathbf{x}_t, \mathbf{x}_k)} \\
&\quad = \frac{\lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)}{\sum_{j\neq t} h_l\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + \lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)}, \text{ by Q3} \\
&\quad = \frac{\lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)\big/\sum_j h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)}{1 + (\lambda-1) h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)\big/\sum_j h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)} \\
&\quad = \frac{\sigma_{k*,t}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda-1)\sigma_{k*,t}(\mathbf{P}, \mathbf{X}, \mathbf{h})}
\end{aligned}
$$

S4:

$$
\begin{aligned}
\sigma_{k*,i}\big([\alpha_1\mathbf{p}_1, \ldots, \alpha_n\mathbf{p}_n]^\top, \beta\mathbf{X}, \gamma\mathbf{h}\big) &= \frac{(\gamma h_i)\phi(\alpha_i\mathbf{p}_i, \alpha_k\mathbf{p}_k, \beta\mathbf{x}_i, \beta\mathbf{x}_k)}{\sum_j (\gamma h_j)\phi(\alpha_j\mathbf{p}_j, \alpha_k\mathbf{p}_k, \beta\mathbf{x}_j, \beta\mathbf{x}_k)} \\
&= \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_j h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)}, \text{ by Q4} \\
&= \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})
\end{aligned}
$$

S5:

$$
\begin{aligned}
\sigma_{k*,i}(\mathbf{P}\hat{\boldsymbol{\lambda}}, \mathbf{X}\hat{\boldsymbol{\lambda}}^{-1}, \mathbf{h}) &= \frac{h_i\phi(\hat{\boldsymbol{\lambda}}\mathbf{p}_i,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_k,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_i,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_k)}{\sum_j h_j\phi(\hat{\boldsymbol{\lambda}}\mathbf{p}_j,\ \hat{\boldsymbol{\lambda}}\mathbf{p}_k, \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_j,\ \hat{\boldsymbol{\lambda}}^{-1}\mathbf{x}_k)} \\
&= \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}), \text{ by Q5}
\end{aligned}
$$

S7:

$$\sigma_{k*,i}\big(\mathbf{P}\tilde{\mathbf{I}}_m^\top, \mathbf{X}\tilde{\mathbf{I}}_m^\top, \mathbf{h}\big) = \frac{h_i\phi(\tilde{\mathbf{I}}_m\mathbf{p}_i, \tilde{\mathbf{I}}_m\mathbf{p}_k, \tilde{\mathbf{I}}_m\mathbf{x}_i, \tilde{\mathbf{I}}_m\mathbf{x}_k)}{\sum_j h_j\phi(\tilde{\mathbf{I}}_m\mathbf{p}_j, \tilde{\mathbf{I}}_m\mathbf{p}_k, \tilde{\mathbf{I}}_m\mathbf{x}_j, \tilde{\mathbf{I}}_m\mathbf{x}_k)}$$
$$= \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h}), \text{ by Q6}$$

S8: For $k, i \in \mathcal{N}\backslash\{t\}$,

$$\bar{\sigma}_{k*,i}\big(\big[\mathbf{P}^\top, \mathbf{p}_t\big]^\top, \big[\mathbf{X}^\top, \mathbf{x}_t\big]^\top, [h_1, \ldots, (1-\lambda)h_t, \ldots, h_n, \lambda h_t]^\top\big)$$
$$= \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{j\neq t} h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + (1-\lambda)h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k) + \lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)}$$
$$= \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

S9: For $k \neq t \neq i$,

$$\lim_{\lambda\to 0} \sigma_{k*,i}\big(\mathbf{P}, \mathbf{X}, [h_1, \ldots, h_{t-1}, \lambda h_t, h_{t+1}, \ldots, h_n]^\top\big)$$
$$= \lim_{\lambda\to 0} \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{j\neq t} h_l\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k) + \lambda h_t\phi(\mathbf{p}_t, \mathbf{p}_k, \mathbf{x}_t, \mathbf{x}_k)}$$
$$= \bar{\sigma}_{k*,i}(\mathbf{P}_{-t}, \mathbf{X}_{-t}, \mathbf{h}_{-t})$$

S11: For $j \neq i \neq k$,

$$\frac{\sigma_{k*,j}(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n, \mathbf{X}, \mathbf{h})}{\sigma_{k*,j}(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$
$$= \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_{l\neq i} h_l\phi(\mathbf{p}_l, \mathbf{p}_k, \mathbf{x}_l, \mathbf{x}_k) + h_i\phi(\mathbf{p}_i', \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)} \frac{\sum_l h_l\phi(\mathbf{p}_l, \mathbf{p}_k, \mathbf{x}_l, \mathbf{x}_k)}{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}$$
$$= \frac{\mathbf{p}_i^\top\mathbf{x}_i}{\mathbf{p}_i'^\top\mathbf{x}_i} \frac{h_i\mathbf{p}_i'^\top\mathbf{x}_i/\mathbf{p}_k^\top\mathbf{x}_k/\bar{\rho}(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k, 1, 1)/\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sum_{l\neq i} h_l\phi(\mathbf{p}_l, \mathbf{p}_k, \mathbf{x}_l, \mathbf{x}_k) + h_i\phi(\mathbf{p}_i', \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}, \text{ by (15)}$$
$$\geq \left[\frac{\mathbf{p}_i'^\top\mathbf{x}_i}{\mathbf{p}_i^\top\mathbf{x}_i}\right]^{-1} \frac{h_i\mathbf{p}_i'^\top\mathbf{x}_i/\mathbf{p}_k^\top\mathbf{x}_k/\bar{\rho}(\mathbf{p}_i', \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k, 1, 1)/\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sum_{l\neq i} h_l\phi(\mathbf{p}_l, \mathbf{p}_k, \mathbf{x}_l, \mathbf{x}_k) + h_i\phi(\mathbf{p}_i', \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}, \text{ by M}$$
$$= \left[\frac{\mathbf{p}_i'^\top\mathbf{x}_i}{\mathbf{p}_i^\top\mathbf{x}_i}\right]^{-1} \frac{\sigma_{k*,i}(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_i', \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n, \mathbf{X}, \mathbf{h})}{\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}, \text{ by (15) and (14)}$$

S13:

$$\sigma_{k*,i}(\mathbf{P}, \hat{\mathbf{h}}\mathbf{X}, \mathbf{1}_n) = \frac{\phi(\mathbf{p}_i, \mathbf{p}_k, h_i\mathbf{x}_i, h_k\mathbf{x}_k)}{\sum_j \phi(\mathbf{p}_j, \mathbf{p}_k, h_j\mathbf{x}_j, h_k\mathbf{x}_k)}$$

$$= \frac{\frac{h_i}{h_k}\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{\sum_j \frac{h_j}{h_k}\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)}, \text{ by Q3}$$
$$= \sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})$$

Since

$$\frac{\sigma_{k*,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{\sigma_{k*,j}(\mathbf{P}, \mathbf{X}, \mathbf{h})} := \frac{h_i\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)}{h_j\phi(\mathbf{p}_j, \mathbf{p}_k, \mathbf{x}_j, \mathbf{x}_k)}$$

depends on prices other than $\mathbf{p}_i$ and $\mathbf{p}_j$, there does not exist a restricted-domain consumption index satisfying RT with $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{k*}$. Therefore, ρ^{ij}_{k*} is not a restricted-domain PPP index. ■

PROOF OF THEOREM 10. S2:

$$\sigma_{EKS,i}\left([\alpha_1\mathbf{p}_t, \ldots, \alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\right)$$
$$= \frac{(\gamma)\prod_k [\phi(\alpha_i\mathbf{p}_t, \alpha_k\mathbf{p}_t, \beta_i\mathbf{x}_t, \beta_k\mathbf{x}_t)]^{\frac{1}{n}}}{\sum_j(\gamma)\prod_l [\phi(\alpha_j\mathbf{p}_t, \alpha_l\mathbf{p}_t, \beta_j\mathbf{x}_t, \beta_l\mathbf{x}_t)]^{\frac{1}{n}}}$$
$$= \frac{\prod_k \left[\frac{\beta_i}{\beta_k}\phi(\mathbf{p}_t, \mathbf{p}_t, \mathbf{x}_t, \mathbf{x}_t)\right]^{\frac{1}{n}}}{\sum_j \prod_l \left[\frac{\beta_j}{\beta_l}\phi(\mathbf{p}_t, \mathbf{p}_t, \mathbf{x}_t, \mathbf{x}_t)\right]^{\frac{1}{n}}}, \text{ by Q3 and Q4}$$
$$= \frac{\beta_i \prod_k [\beta_k]^{-\frac{1}{n}}}{\sum_j \beta_j \prod_l [\beta_l]^{-\frac{1}{n}}}$$
$$= \beta_i \text{ since } \sum \beta_j = 1$$

S3: For $i \neq t$,

$$\sigma_{EKS,i}\left(\mathbf{P}, [\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \lambda\mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_n]^\top, \mathbf{h}\right)$$
$$= \frac{h_i [\phi(\mathbf{p}_i, \mathbf{p}_t, \mathbf{x}_i, \lambda\mathbf{x}_t)]^{\frac{1}{n}} \prod_{k\neq t} [\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)]^{\frac{1}{n}}}{\sum_{j\neq t} h_j [\phi(\mathbf{p}_j, \mathbf{p}_t, \mathbf{x}_j, \lambda\mathbf{x}_t)]^{\frac{1}{n}} \prod_{l\neq t} [\phi(\mathbf{p}_j, \mathbf{p}_l, \mathbf{x}_j, \mathbf{x}_l)]^{\frac{1}{n}} + h_t [\phi(\mathbf{p}_t, \mathbf{p}_t, \lambda\mathbf{x}_t, \lambda\mathbf{x}_t)]^{\frac{1}{n}} \prod_{l\neq t} [\phi(\mathbf{p}_t, \mathbf{p}_l, \lambda\mathbf{x}_t, \mathbf{x}_l)]^{\frac{1}{n}}}$$
$$= \frac{\lambda^{-\frac{1}{n}} h_i \prod_k [\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)]^{\frac{1}{n}}}{\lambda^{-\frac{1}{n}} \sum_{j\neq t} h_j \prod_l [\phi(\mathbf{p}_j, \mathbf{p}_l, \mathbf{x}_j, \mathbf{x}_l)]^{\frac{1}{n}} + \lambda^{1-\frac{1}{n}} h_t \prod_l [\phi(\mathbf{p}_t, \mathbf{p}_l, \mathbf{x}_t, \mathbf{x}_l)]^{\frac{1}{n}}}, \text{ by Q3}$$
$$= \frac{h_i \prod_k [\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)]^{\frac{1}{n}}}{\sum_j h_j \prod_l [\phi(\mathbf{p}_j, \mathbf{p}_l, \mathbf{x}_j, \mathbf{x}_l)]^{\frac{1}{n}} + (\lambda - 1)h_t \prod_l [\phi(\mathbf{p}_t, \mathbf{p}_l, \mathbf{x}_t, \mathbf{x}_l)]^{\frac{1}{n}}}$$
$$= \frac{h_i \prod_k [\phi(\mathbf{p}_i, \mathbf{p}_k, \mathbf{x}_i, \mathbf{x}_k)]^{\frac{1}{n}} \Big/ \sum_j h_j \prod_l [\phi(\mathbf{p}_j, \mathbf{p}_l, \mathbf{x}_j, \mathbf{x}_l)]^{\frac{1}{n}}}{1 + (\lambda - 1)h_t \prod_l [\phi(\mathbf{p}_t, \mathbf{p}_l, \mathbf{x}_t, \mathbf{x}_l)]^{\frac{1}{n}} \Big/ \sum_j h_j \prod_l [\phi(\mathbf{p}_j, \mathbf{p}_l, \mathbf{x}_j, \mathbf{x}_l)]^{\frac{1}{n}}}$$
$$= \frac{\sigma_{EKS,i}(\mathbf{P}, \mathbf{X}, \mathbf{h})}{1 + (\lambda - 1)\sigma_{EKS,t}(\mathbf{P}, \mathbf{X}, \mathbf{h})}$$

$$\sigma_{EKS,t}\Big(\mathbf{P},[\mathbf{x}_1,\ldots,\mathbf{x}_{t-1},\lambda\mathbf{x}_t,\mathbf{x}_{t+1},\ldots,\mathbf{x}_n]^\top,\mathbf{h}\Big)$$

$$= \frac{h_t\,[\phi(\mathbf{p}_t,\mathbf{p}_t,\lambda\mathbf{x}_t,\lambda\mathbf{x}_t)]^{\frac{1}{n}}\prod_{k\neq t}[\phi(\mathbf{p}_t,\mathbf{p}_k,\lambda\mathbf{x}_t,\mathbf{x}_k)]^{\frac{1}{n}}}{\sum_{j\neq t}h_j\,[\phi(\mathbf{p}_j,\mathbf{p}_t,\mathbf{x}_j,\lambda\mathbf{x}_t)]^{\frac{1}{n}}\prod_{l\neq t}[\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}+h_t\,[\phi(\mathbf{p}_t,\mathbf{p}_t,\lambda\mathbf{x}_t,\lambda\mathbf{x}_t)]^{\frac{1}{n}}\prod_{l\neq t}[\phi(\mathbf{p}_t,\mathbf{p}_l,\lambda\mathbf{x}_t,\mathbf{x}_l)]^{\frac{1}{n}}}$$

$$= \frac{\lambda^{-\frac{1}{n}}h_t\prod_k[\phi(\mathbf{p}_t,\mathbf{p}_k,\mathbf{x}_t,\mathbf{x}_k)]^{\frac{1}{n}}}{\lambda^{-\frac{1}{n}}\sum_{j\neq t}h_j\prod_l[\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}+\lambda^{1-\frac{1}{n}}h_t\prod_l[\phi(\mathbf{p}_t,\mathbf{p}_l,\mathbf{x}_t,\mathbf{x}_l)]^{\frac{1}{n}}},\ \text{by Q3}$$

$$= \frac{\lambda h_t\prod_k[\phi(\mathbf{p}_t,\mathbf{p}_k,\mathbf{x}_t,\mathbf{x}_k)]^{\frac{1}{n}}}{\sum_j h_j\prod_l[\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}+(\lambda-1)h_t\prod_l[\phi(\mathbf{p}_t,\mathbf{p}_l,\mathbf{x}_t,\mathbf{x}_l)]^{\frac{1}{n}}}$$

$$= \frac{\lambda h_t\prod_k[\phi(\mathbf{p}_t,\mathbf{p}_k,\mathbf{x}_t,\mathbf{x}_k)]^{\frac{1}{n}}\Big/\sum_j h_j\prod_l[\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}}{1+(\lambda-1)h_t\prod_l[\phi(\mathbf{p}_t,\mathbf{p}_l,\mathbf{x}_t,\mathbf{x}_l)]^{\frac{1}{n}}\Big/\sum_j h_j\prod_l[\phi(\mathbf{p}_j,\mathbf{p}_l,\mathbf{x}_j,\mathbf{x}_l)]^{\frac{1}{n}}}$$

$$= \frac{\lambda\sigma_{EKS,t}(\mathbf{P},\mathbf{X},\mathbf{h})}{1+(\lambda-1)\sigma_{EKS,t}(\mathbf{P},\mathbf{X},\mathbf{h})}$$

The remaining parts are straightforward.

Proof of Theorem 11. S2:

$$\sigma_{OS,i}\big([\alpha_1\mathbf{p}_t,\ldots,\alpha_n\mathbf{p}_t]^\top,[\beta_1\mathbf{x}_t,\ldots,\beta_n\mathbf{x}_t]^\top,[\gamma,\ldots,\gamma]^\top\big)$$

$$= \frac{(\gamma)\left\{\sum_k(\gamma)\,[\phi(\alpha_i\mathbf{p}_t,\alpha_k\mathbf{p}_t,\beta_i\mathbf{x}_t,\beta_k\mathbf{x}_t)]^{-1}\right\}^{-1}}{\sum_j(\gamma)\left\{\sum_l(\gamma)\,[\phi(\alpha_j\mathbf{p}_t,\alpha_l\mathbf{p}_t,\beta_j\mathbf{x}_t,\beta_l\mathbf{x}_t)]^{-1}\right\}^{-1}}$$

$$= \frac{\left\{\sum_k\left[\frac{\beta_i}{\beta_k}\phi(\mathbf{p}_t,\mathbf{p}_t,\mathbf{x}_t,\mathbf{x}_t)\right]^{-1}\right\}^{-1}}{\sum_j\left\{\sum_l\left[\frac{\beta_j}{\beta_l}\phi(\mathbf{p}_t,\mathbf{p}_t,\mathbf{x}_t,\mathbf{x}_t)\right]^{-1}\right\}^{-1}},\ \text{by Q3 and Q4}$$

$$= \frac{\beta_i\left\{\sum_k\beta_k\right\}^{-1}}{\sum_j\beta_j\left\{\sum_l\beta_l\right\}^{-1}}$$

$$= \beta_i \text{ since } \sum\beta_j = 1$$

S10:

$$\lim_{\lambda\to 0}\frac{\sigma_{OS,i}\big(\mathbf{P},\mathbf{X},[\lambda h_1,\ldots,\lambda h_{i-1},h_i,\lambda h_{i+1},\ldots,\lambda h_{j-1},h_j,\lambda h_{j+1},\ldots,\lambda h_n]^\top\big)}{\sigma_{OS,j}\big(\mathbf{P},\mathbf{X},[\lambda h_1,\ldots,\lambda h_{i-1},h_i,\lambda h_{i+1},\ldots,\lambda h_{j-1},h_j,\lambda h_{j+1},\ldots,\lambda h_n]^\top\big)}$$

$$= \frac{h_i\left\{h_i\,[\phi(\mathbf{p}_i,\mathbf{p}_i,\mathbf{x}_i,\mathbf{x}_i)]^{-1}+h_j\,[\phi(\mathbf{p}_i,\mathbf{p}_j,\mathbf{x}_i,\mathbf{x}_j)]^{-1}\right\}^{-1}}{h_j\left\{h_i\,[\phi(\mathbf{p}_j,\mathbf{p}_i,\mathbf{x}_j,\mathbf{x}_i)]^{-1}+h_j\,[\phi(\mathbf{p}_j,\mathbf{p}_j,\mathbf{x}_j,\mathbf{x}_j)]^{-1}\right\}^{-1}}$$

$$= \frac{1 + [\phi(\mathbf{p}_j, \mathbf{p}_i, h_j\mathbf{x}_j, h_i\mathbf{x}_i)]^{-1}}{1 + [\phi(\mathbf{p}_i, \mathbf{p}_j, h_i\mathbf{x}_i, h_j\mathbf{x}_j)]^{-1}}, \text{ by Q2 and Q3}$$
$$=: \psi(\mathbf{p}_i, \mathbf{p}_j, h_i\mathbf{x}_i, h_j\mathbf{x}_j, 1, 1)$$

The remaining parts are straightforward.

PROOF OF THEOREM 12. S2:

$$\sigma_{DW,i}\big([\alpha_1\mathbf{p}_t, \ldots, \alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \sum_k \frac{1}{n}\sigma_{k*,i}\big([\alpha_1\mathbf{p}_t, \ldots, \alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \beta_i, \text{ by Theorem 9}$$

$$\sigma_{PW,i}\big([\gamma_1\alpha_1\mathbf{p}_t, \ldots, \gamma_n\alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \sum_k \frac{(\gamma)(\gamma_k\alpha_k\mathbf{p}_t)^\top(\beta_k\mathbf{x}_t)}{\sum_l(\gamma)(\gamma_l\alpha_l\mathbf{p}_t)^\top(\beta_l\mathbf{x}_t)}$$
$$\times\, \sigma_{k*,i}\big([\gamma_1\alpha_1\mathbf{p}_t, \ldots, \gamma_n\alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \sum_k \frac{\gamma_k\alpha_k\beta_k}{\sum_l \gamma_l\alpha_l\beta_l}\beta_i, \text{ by Theorem 9}$$
$$= \beta_i$$

$$\sigma_{QW,i}\big([\gamma_1\alpha_1\mathbf{p}_t, \ldots, \gamma_n\alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \sum_k \sigma_{OS,k}\big([\gamma_1\alpha_1\mathbf{p}_t, \ldots, \gamma_n\alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$\times\, \sigma_{k*,i}\big([\gamma_1\alpha_1\mathbf{p}_t, \ldots, \gamma_n\alpha_n\mathbf{p}_t]^\top, [\beta_1\mathbf{x}_t, \ldots, \beta_n\mathbf{x}_t]^\top, [\gamma, \ldots, \gamma]^\top\big)$$
$$= \sum_k \beta_k\beta_i, \text{ by Theorems 11 and 9}$$
$$= \beta_i \text{ since } \sum \beta_k = 1$$

The remaining parts are straightforward.

PROOF OF THEOREM 13. Since $\mathbf{X} \in \mathbb{R}_+^{nm}$, $\boldsymbol{\sigma}_{GK}$ satisfies S1, S2 and S4–S9 by Diewert (1986, p. 50). For all $k \in \mathcal{N}$, let $\sigma_{\lambda,k} := \sigma_{GK,k}(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1}, \ldots, \lambda h_n]^\top)$. To show that $\boldsymbol{\sigma}_{GK}$ satisfies S10, substitute for π_ℓ in (22a) using (22b), replace h_k by λh_k for all $k \in \mathcal{N}\backslash\{i, j\}$, and take the limit as $\lambda \to 0$:

$$\sum_\ell \left[\sum_{k \neq i,j} \lim_{\lambda\to 0} \lambda\frac{h_k x_{k\ell}}{h_i x_{i\ell}} + 1 + \frac{h_j x_{j\ell}}{h_i x_{i\ell}}\right]^{-1} \sum_t \frac{p_{t\ell}x_{t\ell}}{\mathbf{p}_t^\top\mathbf{x}_t} \lim_{\lambda\to 0}\frac{\sigma_{\lambda,t}}{\sigma_{\lambda,i}} = 1$$

$$\Leftrightarrow \sum_\ell \frac{h_i x_{i\ell}}{h_i x_{i\ell} + h_j x_{j\ell}} \frac{p_{j\ell} x_{j\ell}}{\mathbf{p}_j^\top \mathbf{x}_j} \lim_{\lambda\to 0} \frac{\sigma_{\lambda,j}}{\sigma_{\lambda,i}} + \sum_\ell \frac{h_i x_{i\ell}}{h_i x_{i\ell} + h_j x_{j\ell}} \frac{p_{i\ell} x_{i\ell}}{\mathbf{p}_i^\top \mathbf{x}_i} = \sum_\ell \frac{p_{i\ell} x_{i\ell}}{\mathbf{p}_i^\top \mathbf{x}_i}$$

$$\Leftrightarrow \lim_{\lambda\to 0} \frac{\sigma_{\lambda,i}}{\sigma_{\lambda,j}} = \frac{\mathbf{p}_i^\top (h_i \mathbf{x}_i)}{\mathbf{p}_j^\top (h_j \mathbf{x}_j)} \frac{\sum_\ell p_{j\ell} \left[\frac{(h_i x_{i\ell})(h_j x_{j\ell})}{h_i x_{i\ell} + h_j x_{j\ell}}\right]}{\sum_\ell p_{i\ell} \left[\frac{(h_i x_{i\ell})(h_j x_{j\ell})}{h_i x_{i\ell} + h_j x_{j\ell}}\right]}$$

The remaining parts are straightforward.

PROOF OF THEOREM 14. S3: For all $k \in \mathcal{N}$, let $\bar{\sigma}_k := \sigma_{VH,k}(\mathbf{P}, [\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \lambda \mathbf{x}_t, \mathbf{x}_{t+1}, \dots, \mathbf{x}_n]^\top, \mathbf{h})$ and let $\sigma_k := \sigma_{VH,k}(\mathbf{P}, \mathbf{X}, \mathbf{h})$. For $i \neq t$,

$$\sum_{k\neq t} h_k^2 \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} + h_t^2 \frac{\mathbf{p}_i^\top (\lambda \mathbf{x}_t)}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_t} = \sum_{k\neq t} h_i^2 \frac{\mathbf{p}_k^\top \mathbf{x}_i}{\mathbf{p}_k^\top \mathbf{x}_k} \frac{\bar{\sigma}_k}{\bar{\sigma}_i} + h_i^2 \frac{\mathbf{p}_t^\top \mathbf{x}_i}{\mathbf{p}_t^\top (\lambda \mathbf{x}_t)} \frac{\bar{\sigma}_t}{\bar{\sigma}_i}$$

$$\leftrightarrow \sum_k h_k^2 \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\sigma_i}{\sigma_k} = \sum_k h_i^2 \frac{\mathbf{p}_k^\top \mathbf{x}_i}{\mathbf{p}_k^\top \mathbf{x}_k} \frac{\sigma_k}{\sigma_i}$$

$$\Leftrightarrow \frac{\bar{\sigma}_t}{\bar{\sigma}_i} = \lambda \frac{\sigma_t}{\sigma_i} \text{ and } \frac{\bar{\sigma}_k}{\bar{\sigma}_i} = \frac{\sigma_k}{\sigma_i} \text{ for all } k \in \mathcal{N}\backslash\{t\}$$

S8: For all $k \in \mathcal{N}$, let $\bar{\sigma}_k := \sigma_{VH,k}([\mathbf{P}^\top, \mathbf{p}_t]^\top, [\mathbf{X}^\top, \mathbf{x}_t]^\top, [h_1, \dots, (1-\lambda)h_t, \dots, h_n, \lambda h_t]^\top)$ and let $\sigma_k := \sigma_{VH,k}(\mathbf{P}, \mathbf{X}, \mathbf{h})$. For $i \neq t$,

$$\sum_{k\neq t} h_k^2 \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} + [(1-\lambda)h_t]^2 \frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_t} + (\lambda h_t)^2 \frac{\mathbf{p}_i^\top \mathbf{x}_t}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_{n+1}}$$

$$= \sum_{k\neq t} h_i^2 \frac{\mathbf{p}_k^\top \mathbf{x}_i}{\mathbf{p}_k^\top \mathbf{x}_k} \frac{\bar{\sigma}_k}{\bar{\sigma}_i} + h_i^2 \frac{\mathbf{p}_t^\top \mathbf{x}_i}{\mathbf{p}_t^\top \mathbf{x}_t} \frac{\bar{\sigma}_t}{\bar{\sigma}_i} + h_i^2 \frac{\mathbf{p}_t^\top \mathbf{x}_i}{\mathbf{p}_t^\top \mathbf{x}_t} \frac{\bar{\sigma}_{n+1}}{\bar{\sigma}_i}$$

$$\leftrightarrow \sum_k h_k^2 \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_i^\top \mathbf{x}_i} \frac{\sigma_i}{\sigma_k} = \sum_k h_i^2 \frac{\mathbf{p}_k^\top \mathbf{x}_i}{\mathbf{p}_k^\top \mathbf{x}_k} \frac{\sigma_k}{\sigma_i}$$

$$\Leftrightarrow (1-\lambda) \frac{\bar{\sigma}_i}{\bar{\sigma}_t} = \lambda \frac{\bar{\sigma}_i}{\bar{\sigma}_{n+1}} = \frac{\sigma_i}{\sigma_t} \text{ and } \frac{\bar{\sigma}_i}{\bar{\sigma}_k} = \frac{\sigma_i}{\sigma_k} \text{ for all } k \in \mathcal{N}\backslash\{t\}$$

Thus,

$$\sum_{k=1}^{n+1} \frac{\bar{\sigma}_k}{\bar{\sigma}_i} = \sum_{k\in\mathcal{N}\backslash\{t\}} \frac{\sigma_k}{\sigma_i} + (1-\lambda)\frac{\sigma_t}{\sigma_i} + \lambda \frac{\sigma_t}{\sigma_i}$$

$$\Leftrightarrow \bar{\sigma}_i = \sigma_i \text{ since } \sum_{k=1}^{n+1} \bar{\sigma}_k = \sum_{k=1}^{n} \sigma_k = 1$$

$$\Rightarrow \bar{\sigma}_t = (1-\lambda)\sigma_t \text{ and } \bar{\sigma}_{n+1} = \lambda \sigma_t$$

S10: For all $k \in \mathcal{N}$, let $\sigma_{\lambda,k} := \sigma_{VH,k}(\mathbf{P}, \mathbf{X}, [\lambda h_1, \ldots, \lambda h_{j-1}, h_j, \lambda h_{j+1}, \ldots, \lambda h_{i-1}, h_i, \lambda h_{i+1}, \ldots, \lambda h_n]^\top)$. Now,

$$\sum_{k \neq i,j} \lim_{\lambda \to 0} (\lambda h_k)^2 \frac{\mathbf{p}_i^\top \mathbf{x}_k}{\mathbf{p}_i^\top \mathbf{x}_i} \lim_{\lambda \to 0} \left(\frac{\sigma_{\lambda,i}}{\sigma_{\lambda,k}} \right) + h_j^2 \frac{\mathbf{p}_i^\top \mathbf{x}_j}{\mathbf{p}_i^\top \mathbf{x}_i} \lim_{\lambda \to 0} \left(\frac{\sigma_{\lambda,i}}{\sigma_{\lambda,j}} \right)$$

$$= \sum_{k \neq i,j} h_i^2 \frac{\mathbf{p}_k^\top \mathbf{x}_i}{\mathbf{p}_k^\top \mathbf{x}_k} \lim_{\lambda \to 0} \left(\frac{\sigma_{\lambda,k}}{\sigma_{\lambda,i}} \right) + h_i^2 \frac{\mathbf{p}_j^\top \mathbf{x}_i}{\mathbf{p}_j^\top \mathbf{x}_j} \lim_{\lambda \to 0} \left(\frac{\sigma_{\lambda,j}}{\sigma_{\lambda,i}} \right)$$

$$\Leftrightarrow \lim_{\lambda \to 0} \left(\frac{\sigma_{\lambda,i}}{\sigma_{\lambda,j}} \right) = \left[\frac{\mathbf{p}_i^\top (h_i \mathbf{x}_i)}{\mathbf{p}_i^\top (h_j \mathbf{x}_j)} \frac{\mathbf{p}_j^\top (h_i \mathbf{x}_i)}{\mathbf{p}_j^\top (h_j \mathbf{x}_j)} \right]^{\frac{1}{2}}$$

The remaining parts are straightforward.

REFERENCES

Armstrong, K. G., "Microeconomic Foundations for the Theory of International Comparisons," *Journal of Economic Theory* 101 (2001), 585–605.

Balk, B. M., "On van Ijzeren's Approach to International Comparisons and its Properties," *Statistical Papers* 30 (1989), 295–315.

Board of Trade, *Report on an Enquiry by the Board of Trade into Working Class Rents, Housing and Retail Prices together with the Rates of Wages in Certain Occupations in the Principal Industrial Towns of the United Kingdom, Germany, France, Belgium and the United States*, Cd. 3864, 4032, 4512, 5065 and 5609 (London: Printed for H.M. Stationary Office by Darling & Son, 1908–1911).

Diewert, W. E., "Exact and Superlative Index Numbers," *Journal of Econometrics* 4 (1976), 115–45.

———, "The Economic Theory of Index Numbers: A Survey," in A. Deaton, ed., *Essays in the Theory and Measurement of Consumer Behaviour in Honour of Sir Richard Stone* (Cambridge: Cambridge University Press, 1981), 163–208.

——— "Group Cost-of-Living Indexes: Approximations and Axiomatics," *Methods of Operations Research* 48 (1984), 23–45.

———, "Microeconomic Approaches to the Theory of International Comparisons," Discussion paper no. 86-31, Department of Economics, University of British Columbia, August 1986.

———, "Fisher Ideal Output, Input and Productivity Indexes Revisited," *Journal of Productivity Analysis* 3 (1992), 211–48.

———, "Axiomatic and Economic Approaches to International Comparisons," in A. Heston and R. E. Lipsey, eds., *International and Interarea Comparisons of Income, Output, and Prices*, *NBER Studies in Income and Wealth*, Vol. 61 (Chicago: University of Chicago Press, 1999), 13–87.

Eichhorn, W., *Functional Equations in Economics* (Reading, MA: Addison-Wesley Publishing Company, 1978).

Eichhorn, W., and J. Voeller, "Axiomatic Foundation of Price Indexes and Purchasing Power Parities," in W. E. Diewert and C. Montmarquette, eds., *Price Level Measurement: Proceedings from a Conference Sponsored by Statistics Canada* (Ottawa: Minister of Supply and Services, 1983), 411–50.

Eltetö, O., and P. Köves, "On a Problem of Index Number Computation Relating to International Comparisons," *Statisztikai Szemle* 42 (1964), 507–18.

Fisher, I., *The Purchasing Power of Money* (New York: Macmillan, 1911).

———, *The Making of Index Numbers: A Study of Their Varieties, Tests, and Reliability*, 3rd edition (New York: Augustus M. Kelley, 1967 reprint, 1927).

FRISCH, R., "Necessary and Sufficient Conditions Regarding the Form of an Index Number which Shall Meet Certain of Fisher's Tests," *Journal of the American Statistical Association* 25 (1930), 397–406.

FUNKE, H., AND J. VOELLER, "A Note on the Characterization of Fisher's Ideal Index," in W. Eichhorn et al., eds., *Theory and Applications of Economic Indices: Proceedings of an International Symposium held at the University of Karlsruhe* (Würzburg: Physica-Verlag, 1978), 177–81.

GEARY, R. C., "A Note on the Comparison of Exchange Rates and Purchasing Power between Countries," *Journal of the Royal Statistical Society A* 121 (1958), 97–99.

GERARDI, D., "Selected Problems of Inter-Country Comparisons on the Basis of the Experience of the EEC," *Review of Income and Wealth* 28 (1982), 381–405.

GINI, C., "On the Circular Test of Index Numbers," *Metron* 9 (1931), 3–24.

HILL, P., *Multilateral Measurements of Purchasing Power and Real GDP* (Luxembourg: Eurostat, 1982).

KHAMIS, S. H., "Properties and Conditions for the Existence of a New Type of Index Number," *Sankhya B* 32 (1970), 81–98.

———, "A New System of Index Numbers for National and International Purposes," *Journal of the Royal Statistical Society A* 135 (1972), 96–121.

KRAVIS, I. B., "Comparative Studies of National Incomes and Prices," *Journal of Economic Literature* 22 (1984), 1–39.

KRAVIS, I. B., Z. KENESSEY, A. HESTON, AND R. SUMMERS, *A System of International Comparisons of Gross Product and Purchasing Power* (Baltimore: Johns Hopkins University Press, 1975).

POLLAK, R. A., "Group Cost-of-Living Indexes," *American Economic Review* 70 (1980), 273–78.

———, "The Social Cost-of-Living Index," *Journal of Public Economics* 15 (1981), 311–36.

SAMUELSON, P. A., AND S. SWAMY, "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis," *American Economic Review* 64 (1974), 566–93.

SZULC, B., "Indices for Multiregional Comparisons," *Przeqlad Statystyczny* 3 (1964), 239–54.

VAN IJZEREN (YZEREN), J., "Three Methods of Comparing the Purchasing Power of Currencies," *Statistical Studies* 7 (The Hague: Centraal Bureau voor de Statiskiek, 1956).

———, "Index Numbers for Binary and Multilateral Comparison," *Statistical Studies* 34 (The Hague: Centraal Bureau voor de Statiskiek, 1983).

VOELLER, J., *Purchasing Power Parities for International Comparisons*, *Mathematical Systems in Economics*, No. 72 (Cambridge, MA: Oelgeschlager, Gunn & Hain, 1981).