TITLE: Mutually Orthogonal Latin Squares and $k$-nets

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2 Abstract

In this paper, we will be examining different sizes of sets of mutually orthogonal Latin squares of order 10 (MOLS(10)). We will do so by using different objects which are combinatorially equivalent to a set of MOLS(10), including $k$-nets of order 10 and decompositions of complete multipartite graphs into cliques whose parts are of size 10 (e.g. $K_{10,10,10,10}$). Specifically, we will be determining restrictions on the potential dimensions of a 6-net of order 10, which is combinatorially equivalent to four MOLS(10). Using these facts, we discuss the possibility of a computer search either finding or disproving the existence of a set of four MOLS(10).
3 Introduction

We begin by briefly discussing the history of Latin squares.

Definition 3.1. A Latin square of order $n$ is an $n \times n$ array of symbols from an $n$-set in which each row and column contains each symbol exactly once.

It is easy to show that Latin squares exist for all $n$. For example, the addition table in $\mathbb{Z}_n$ is a Latin square of order $n$.

Definition 3.2. Two Latin squares $L$ and $M$ are orthogonal if the $n^2$ ordered pairs $(L_{i,j}, M_{i,j})$ are distinct.

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 \\
2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 & 3
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 & 1 \\
4 & 0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 & 2
\end{bmatrix}
\]

Example 3.1. Two orthogonal Latin Squares of order 5.

By relabelling as necessary, we can ensure that $L$ and $M$ are composed of the numbers \{1, 2, \ldots, n\}. Then $L$ and $M$ being orthogonal means that for every ordered pair of elements $(x, y)$ from \{1, 2, \ldots, n\}, there is a unique cell $(i, j)$ such that $L_{i,j} = x$ and the $M_{i,j} = y$.

It is not surprising to find that, for certain orders, it is possible to find sets of more than two Latin squares such that the squares are all pairwise orthogonal. Such a set is called a set of mutually orthogonal Latin squares of order $n$ (abbreviated as $MOLS(n)$).

Latin squares have a long history in puzzle solving (logic puzzles often make use of Latin squares, and sudoku is a Latin square of order 9 with additional constraints). One of the earlier examples of such a puzzle was Euler’s 36 officers problem (1782). In it, Euler supposes that there are six regiments of six officers, each of different rank, and asks if there is a way to arrange the officers in a $6 \times 6$ array such that no row or column contains more than one of each rank or regiment. It is not hard to see that such a solution is essentially a pair of $MOLS(6)$ – one Latin square for rank and one Latin square for regiment, and the squares must be orthogonal as there is only one of each rank in each regiment. Euler correctly conjectured that there was no possible solution, although this was not proven until Tarry (1901) proved it exhaustively [2]. Stinson (1984) provided an alternative proof [3], using coding theory.
and the combinatorial equivalence of a pair of $MOLS(6)$ and another combinatorial object, called a transversal design. The thought process behind the work in this paper is largely similar to the one in his proof.

We thus know that there can be no pair of $MOLS(6)$. It is also relatively easy to determine that there can be at most $n - 1$ $MOLS(n)$ [1]. We thus arrive at the natural question: what is the maximum possible $MOLS(n)$ for a given $n$? This number is typically denoted $N(n)$.

In 1922, MacNeish showed that $N(n)$ is at least one less than the smallest prime power in the prime factorization of $n$ [4]. This means that $N(n) = n - 1$ when $n$ is a prime power. Thus for $2 \leq n \leq 9$, $N(n) = n - 1$ for all $n \neq 6$. We also know that $N(6) = 1$ from Tarry. This means that $n = 10$ is the first number for which $N(n)$ is unknown.

Stemming from the nonexistence of a pair of $MOLS(6)$, Euler conjectured that there would be no pair of $MOLS(n)$, for all $n$ congruent to 2 modulo 4. This was disproved by Bose and Shrikhande (1959) for $n = 22$ [5], and more generally by Bose, Shrikhande, and Parker (1960) for all such $n > 6$ [6]. Thus, $N(10) \geq 2$.

The current upper bound on $N(10)$ is thanks to Lam, Thiel and Swiercz (1989) [7]. They used a computer search to demonstrate the non-existence of a projective plane of order 10. The existence of a projective plane of order $n$ is combinatorially equivalent to $n - 1$ $MOLS(n)$, and so there are at most 8 $MOLS(10)$. Bruck (1963) showed that if $N(n) < n - 1$, then $N(n) < n - 1 - (2n)^{1/4}$ [8]. Hence, $N(10) < 10 - 1 - (20)^{1/4} \approx 6.89$, and so $N(10) \leq 6$.

The purpose of this paper is to examine ways to further restrict $N(10)$. This is done by compiling information from various sources, the majority of which comes from the PhD thesis of Dr. Leah Howard [1]. The material has been reordered and elucidated for the sake of accessibility, and select computations were double checked to ensure validity. Using this information, we look to draw conclusions on the feasibility of restrictions on $N(10)$.
4 Dimensions of $k$-Nets

Analyzing $N(n)$ directly through Latin squares is difficult. It is easier to start by looking at another combinatorial object, the $k$-net of order $n$.

**Definition 4.1.** A $k$-net of order $n$, denoted $N_k$, is an incidence structure of $n^2$ points and $nk$ lines, such that the following hold:

1. There are $n$ points on each line;
2. There are $k$ parallel classes with $n$ lines each; and
3. Every two non-parallel lines meet at exactly one point.
4. Lines being parallel forms an equivalence relation on the set of lines;

\[
\begin{array}{cccccccc}
00 & 01 & 02 & 03 & 04 \\
10 & 11 & 12 & 13 & 14 \\
20 & 21 & 22 & 23 & 24 \\
30 & 31 & 32 & 33 & 34 \\
40 & 41 & 42 & 43 & 44 \\
\end{array}
\]

**Example 4.1** A 3-net of order 5.

In the example above, the different parallel classes are represented by the first, second, and third digits of each point. The different lines within each parallel class are labeled 0, 1, 2, 3, and 4.

Note that each parallel class forms a partition of the points in the net, so that each point is thus on $k$ lines in the $k$-net, one in each parallel class.

It can be shown that a $k$-net of order $n$ is combinatorially equivalent to $k$-2 $MOLS(n)$ [1].

Then as $N(10) \leq 6$, there can be at most an 8-net of order 10. Since $N(10) \geq 2$, there is a 4-net of order 10. In this paper we will focus our efforts on the possible existence of a 6-net of order 10, which would be equivalent to four $MOLS(10)$.

**Definition 4.2.** A subnet $M$ of a net $N$ consists of the same points as $N$ and some of the parallel classes of $N$. 
Definition 4.3. Given a $k$-net $N_k$ of order $n$, label the points of $N_k$ as $p_1, p_2, \ldots, p_{n^2}$. The characteristic function of a line in $N_k$ is the vector in $F_2^{n^2}$ whose $i$th component is 1 if $p_i$ is on the line, and 0 otherwise. Let $C_2(N_k)$ be the vector space over $F_2$ generated by the characteristic functions of the lines in $N_k$. Let $D_2(N_k)$ be the vector space over $F_2$ generated by the differences of characteristic functions of lines $l - m$, where $l$ and $m$ are lines in the same parallel class.

$C_2(N_k)$ and $D_2(N_k)$ are examples of linear binary codes:

Definition 4.4. A linear $q$-ary code of length $n$ is a subspace of $F_q^n$. A codeword is a vector lying in the code.

Definition 4.5. The dimension of a code $W$, denoted $\dim W$, is the dimension of $W$ as a vector space.

Definition 4.6. Given a linear binary code $W$ of length $n$, the dual code $W^\perp$ of $W$ is the orthogonal complement $\{x \in F_2^n | [x, w] = 0 \forall w \in W\}$, where $[x, w]$ is the dot product over $F_2^n$.

We can now begin analyzing $k$-nets of order $n$ through the use of $C_2(N_k)$ and $D_2(N_k)$.

Lemma 4.1. A 1-net of order $n$ has

$$\dim C_2(N_k) = n, \quad \dim D_2(N_k) = n - 1.$$ 

Proof. By definition, $C_2(N_1)$ is generated by the characteristic functions of $n$ parallel lines. These vectors must then be linearly independent, and so form a basis for $C_2(N_1)$, which thus has dimension $n$. To see the second equality, note first that $D_2(N_1) \subseteq C_2(N_1)$ (as $D_2(N_1)$ is constructed from elements of $C_2(N_1)$). Let $\{a_1, a_2, \ldots, a_n\}$ be the characteristic functions of the lines of $N_1$. Then $a_1 \in C_2(N_1)$ but $a_1 \notin D_2(N_1)$, and so $\dim D_2(N_1) < \dim C_2(N_1) = n$. However, the set $\{a_1 + a_2, a_1 + a_3, \ldots, a_1 + a_n\}$ is linearly independent, and all the elements are inside $D_2(N_1)$. Hence $\dim D_2(N_1) \geq n - 1$, and so $\dim D_2(N_1) = n - 1$. 

The next proposition requires the definition of a transversal of a net.

Definition 4.7. A transversal of a net of order $n$ is a set of $n$ points such that each line in the net has exactly one point in common with the transversal.

For example, in the 3-net of order 5 in Example 4.1, the set of points $\{000, 143, 231, 324, 412\}$ is a transversal.
Proposition 4.2. Let $N_k$ be a $k$-net of order $n$, where $n$ is even. If $k$ is even or $N_k$ has a transversal, then $\dim C_2(N_k) - \dim D_2(N_k) = k$.

Proof. First, let $m_i$ be a line in the $i$th parallel class. Then $C_2(N_k) = \langle m_1, m_2, \ldots, m_k, D_2(N_k) \rangle$, and so $\dim C_2(N_k) - \dim D_2(N_k) \leq k$. The result holds true for $k = 1$ by Lemma 4.1. For $k \geq 2$, it suffices to show that $\{m_1, m_2, \ldots, m_n\}$ are linearly independent over $D_2(N_k)$. Note that since $n$ is even, $[x, x] = 0 \forall x \in C_2(N_k)$. By definition of $N_k$, $[x, y] = 1 \forall x, y$ not in the same parallel class, and $[x, y] = 0 \forall x, y$ in the same parallel class. This means that $[x, y - z] = 0 \forall x, y, z \in C_2(N_k)$ where $y, z$ are in the same parallel class, and so $D_2(N_k) \subseteq C_2(N_k)$. Now suppose that $w = a_1m_1 + a_2m_2 + \ldots + a_km_k \in D_2(N_k) \subseteq C_2(N_k) \cap C_2(N_k)$. Let $l_j$ be a line in the $j$th parallel class. Then $0 \equiv [l_j, w] \equiv \sum_{i \neq j} a_i \pmod{2}$ (since $[l_j, m_j] = 0$). As this is true for each parallel class, $(k - 1)\sum_{i=1}^n a_i \equiv 0 \pmod{2}$.

If $k$ is even, $k - 1 \equiv 1 \pmod{2}$, and so $\sum_{i=1}^n a_i \equiv 0 \pmod{2}$. Since $\sum_{i \neq j} a_i \equiv 0 \pmod{2}$ $\forall 1 \leq j \leq k$, we have that $a_1 \equiv a_2 \equiv \ldots \equiv a_k \pmod{2}$, as required.

If instead $N_k$ has a transversal $t$, by definition, $[t, l] \equiv 1 \pmod{2}$ $\forall l \in C_2(N_k)$. Then $[t, l - m] \equiv 0 \pmod{2}$ $\forall l, m \in C_2(N_k)$, and so $t \in D_2(N_k)$. Hence $[t, w] \equiv 0 \pmod{2}$, and so $\sum_{i=1}^n a_i \equiv 0 \pmod{2}$. The argument then follows as above. \qed

This proposition allows us our first main result about $N_6$ of order 10.

Proposition 4.3. If a net $N_6$ of order 10 exists, $\dim C_2(N_6) \leq 53$.

Proof. Suppose $N_6$ is a net of order 10. Let $\dim C_2(N_6) = 53 + \alpha$, for some integer $\alpha$. Since 6 and 10 are even, by Prop. 4.2, $\dim D_2(N_6) = 47 + \alpha$. Since $C_2(N_6), D_2(N_6) \subseteq F_{100}^2$, by the rank-nullity theorem, $\dim D_2(N_6) = 53 - \alpha$. By the proof of Prop. 4.2, $D_2(N_6) \subseteq C_2(N_6)$, and so $C_2(N_6) \subseteq D_2(N_6)$ (by definition of orthogonal complements). We thus have that $53 + \alpha \leq 53 - \alpha$, and so $\alpha \leq 0$.

Thus, if it can be shown that any net $N_6$ of order 10 must contain an arrangement of lines such that the space spanned by those lines has dimension of at least 54, then $N_6$ cannot exist, and so $N(10) \leq 3$.

Proposition 4.3 is our first main result, and will motivate much of the work to come in this paper. There are, however, some weaker corollaries that are worth noting (these mostly follow from Prop. 4.2 and the proof of Prop 4.3):

Corollary 4.4. [1] If a collection of lines $N$ from a net $N_7$ of order 10 satisfies $\dim C_2(N) \geq 54$, there are at most five MOLS(10).
Corollary 4.5. [1] If a collection of lines $N$ from a net $N_8$ of order 10 satisfies $\dim C_2(N) \geq 55$, there are at most five $MOLS(10)$.

To go further, we need the use of relations over the net $N_k$.

Definition 4.8. A relation in a net is a subset of the lines of the net such that every point in the net lies on an even number of lines in the relation. This is equivalent to a linear dependency on the lines’ characteristic functions over $F_2$. Parallel classes are associated with the relation only if they provide a nonzero number of lines to the relation. A relation is trivial if it consists of an even number of entire parallel classes.

Definition 4.9. A configuration is a relation with the lines of one the parallel classes (which contributes a nonzero number of lines to the relation) deleted.

Definition 4.10. The weight of a parallel class with respect to a relation or configuration is the number of lines that class contributes to the relation. The type of a relation or configuration is a set of the non-zero weights of the parallel classes in that relation, arranged in ascending order.

For example, suppose that we had a net $N_6$ of order 10, and $N_6$ contained a relation consisting of 2 lines in each of three different parallel classes, 4 lines in another parallel class, and 6 lines in a fifth parallel class. Then the type of this relation would be $\{2, 2, 2, 4, 6\}$. Depending on which parallel class we removed from the relation, we could obtain configurations of type $\{2, 2, 2, 4\}$, type $\{2, 2, 2, 6\}$, or type $\{2, 2, 4, 6\}$.

Proposition 4.6. Let $N_k$ have $j$th parallel class $\{l_{j1}, l_{j2}, \ldots, l_{jn}\}$, and suppose that $N_k$ contains a non-trivial relation $\sum_{i=1}^{n} a_i^{k}l_i = \sum_{i=1}^{n} a_i^{k-1}l_i - 1 + \ldots + \sum_{i=1}^{n} l_i$, where $a_i^j \in \{0, 1\}$. Let $a_j = |\{a_i^j \mid a_i^j = 1\}|$ (i.e. the weight of the $j$th parallel class). If $\exists j$ such that $a_j$ is odd, then $\dim C_2(N_k) - \dim D_2(N_k) < k$.

Proof. Relabel the parallel classes as necessary so that $l_{j1}$ has nonzero coefficient for each class in the relation with odd $a_j$. As the relation is over $F_2$, we can rewrite it as $\sum_{a_j \text{ odd}} l_{j1} = \sum a_i^{k}l_i + \sum a_i^{k-1}l_i - 1 + \ldots + \sum l_i$, where the summations on the right hand side are for $2 \leq i \leq n$ for classes that had odd weight and for $1 \leq i \leq n$ for classes that had even weight. Since each parallel class on the right hand side now contains an even number of lines, these lines can be paired up within the classes. As the relation is over $F_2$, this means that the right hand side is a sum of elements in $D_2(N_k)$, and so is an element in
$D_2(N_k)$. Thus, the left hand side must also be an element in $D_2(N_k)$. From the proof of 4.2, $C_2(N_k) = \langle l_1^1, l_1^2, \ldots, l_1^k, D_2(N_k) \rangle$, and so $\dim C_2(N_k) - \dim D_2(N_k) < k$ (As the $l_1^i$ are not linearly independent of $D_2(N_k)$).

These relations are important, as each relation in the net $N_6$ corresponds to a linear dependency in the row space of $C_2(N_6)$. Recall from Prop. 4.3 that (for order 10) we have $\dim C_2(N_6) \leq 53$. Since there are 60 lines in $N_6$ of order 10, this means that there are at least 7 relations in $N_6$. Analysing what these relations could look like will comprise the next couple of sections.
5 Determining Relations in $N_6$

In the last section, we determined that a net $N_6$ or order 10 would need to contain at least 7 relations. In this section, we look at what form these relations could take.

We begin with a couple of simple results which will drastically cut down the search space:

**Proposition 5.1.** Every parallel class has even weight in any non-empty relation in a net $N_6$ of order 10.

*Proof.* Since 6 and 10 are even, by Prop. 4.2, $\dim C_2(N_6) - \dim D_2(N_6) = 6$. Also, since non-parallel lines meet at exactly one point, the points on any line in the last parallel class forms a transversal for the subnet formed from the first $i$ parallel classes, $1 \leq i \leq 5$. Then by Prop. 4.2, $\dim C_2(N_i) - \dim D_2(N_i) = i, 1 \leq i \leq 5$. By Prop. 4.6, all the weights must be even.

Since a given relation contains a subset (possibly trivial) of the lines in each parallel class, we can consider the complement of this subset with respect to the parallel class. For example, if a relation contains four lines in a given parallel class, then the complement would be the six lines not included in the relation. We will describe the act of replacing the set of lines in a relation within a given parallel class with their complement with respect to that parallel class as **complementing the class**.

**Proposition 5.2.** All parallel classes, except the last one, can be assumed to have weight zero, two, or four in any non-empty relation in a net $N_6$ of order 10.

*Proof.* Suppose some parallel class other than the last one has weight six, eight, or ten. If we complement both this class and the last class, we obtain another relation (as we have essentially added the 1 vector twice, which equals 0 over $F_2$). This reduces the weight of the class in question to zero, two, or four.

With this, we can therefore restrict our search to relations which have zero, two, or four lines in each of the first five parallel classes.

Note that, of the at least seven distinct relations, five are easy to find: if the parallel classes of $N_6$ are $\{B_1, B_2, B_3, B_4, B_5, B_6\}$, then taking all of the lines from an even number of parallel classes forms a trivial relation. Such relations are spanned by the relations $\{B_1 \cup B_2, B_1 \cup B_3, B_1 \cup B_4, B_1 \cup B_5, B_1 \cup B_6\}$, meaning that such relations account for five of the at least seven distinct relations. We thus have at least two relations in $N_6$ which are non-trivial.

We begin by proving that any such non-trivial relation must contain lines from at least four parallel classes.
Proposition 5.3. There is no non-trivial relation of type \( \{a, b, c\} \) in a net \( N_6 \) of order 10.

Proof. Suppose that there is a non-trivial relation of type \( \{a, b, c\} \). Since the relation is non-trivial, at least one of \( a, b, \) or \( c \) is not 0 or 10. Without loss of generality, say that \( 0 < a < 10 \). Label the three parallel classes \( B_1, B_2, \) and \( B_3 \), where \( B_1 \) is the parallel class which has \( a \) lines in the relation.

By definition of the relation and since every point appears on exactly one line in each parallel class, every point in the relation must appear on exactly 2 lines. Since every line from \( B_1 \) intersects every line from \( B_2 \) or \( B_3 \) at exactly one point, \( b + c = 10 \) (we cannot have more than 10 lines between them, as any extra lines would have to intersect the line from \( B_1 \), which contradicts that the point appears on exactly 2 lines). Since some of the lines in \( B_1 \) are not in the relation, we get that \( b = c \) (all the lines from \( B_1 \) which are not in the relation from \( B_2 \) in the relation, and so those points must be matched by lines from \( B_3 \), and vice versa, because such points are on 2 lines). Thus \( b = c = 5 \), which contradicts Prop. 5.1. Hence, no such non-trivial relation can exist.

By a similar argument, it is clear that any configuration resulting from a relation must have at least ten lines between all parallel classes in the relation.

Now, to analyze these non-trivial relations, we will consider the weights of each point in the net. Recall from Def. 4.10 that the weight of a point in a relation (resp. configuration) is the number of lines in the relation (resp. configuration) which that point is on. Also, recall that each point in the net is on one line in each parallel class. This means that the point can have weight at most six. If we form a configuration from the relation, the point can have weight at most five.

For a given relation or configuration in a \( N_6 \) of order 10, let \( Z, S, D, T, Q, K, H \) represent the number of points of weight zero, one, two, three, four, five, and six respectively.

We begin by determining a relationship between \( S, T, \) and \( K \).

Lemma 5.4. Given a relation in a net \( N_6 \) of order 10, let \( s, t, \) and \( k \) represent the number of weight one, three, and five points respectively that meet a line from the last parallel class. Then \( s + t + k = 10 \) and \( s + 3t + 5k = l \), where \( l \) is the number of lines in the relation not in the last parallel class. Also, if there is a parallel class not included in the relation, then \( k = 0 \).

Proof. Since the line in the last parallel class contains ten points which must each appear an odd number of times elsewhere in the relation, \( s + t + k = 10 \). Since the line in the last parallel class must meet every other line in the relation which is not in the last class exactly
once, $s + 3t + 5k = l$. Since each point in a relation must appear on an even number of lines, if it does not appear on 6 lines, it must appear on at most 4, and so $k = 0$. \hfill \square

How this lemma relates $S$, $T$, and $K$ is thus dependent on the relation in question. Of note is that we must always have that $10 \mid S + T + K$.

Now suppose we have a configuration in $N_6$. By Prop. 5.2, we can remove the only parallel class that might have weight greater than four, and so we need only consider configurations with weights zero, two, or four. Let $p$ be the number of pairs of lines from different parallel classes which are each in the configuration, and let $c$ be the number of pairs of lines from different parallel classes which are not in the configuration. For example, a configuration of type $\{x, y, z\}$ would have $p = xy + xz + yz$ and $c = (10 - x)(10 - y) + (10 - x)(10 - z) + (10 - y)(10 - z)$. We then obtain the following equations:

$$Z + S + D + T + Q + K = 100$$
$$D + 3T + 6Q + 10K = p$$
$$\frac{l(l - 1)}{2}Z + \frac{(l - 1)(l - 2)}{2}S + \frac{(l - 2)(l - 3)}{2}D + \frac{(l - 3)(l - 4)}{2}T = c$$

where $l$ is the number of parallel classes involved in the configuration. These equations come from points in the net, pairs of lines inside the configuration, and pairs of lines outside the configuration, respectively.

We cannot have a $\{2, 2, 2, 2\}$ configuration, as it does not contain at least 10 lines. Hence, a $\{2, 2, 2, 4\}$ configuration is the smallest possible configuration in four classes in $N_6$. If such a configuration exists, it must also exist in a subnet $N_5$ of $N_6$.

Now consider a $\{2, 2, 4, 4\}$ configuration. By Lemma 5.4, we see that $s + t + k = 10$ and $s + 3t + 5k = l = 12$, so that $s = 9, t = 1$, and $k = 0$. Hence, $K = 0$ and $S = 9T$ (since the lemma holds for every line in the deleted class). Also, note that as there are two parallel classes with two lines each, we must have that $Q \leq 4$ (as there are only four points of intersection between the lines in the first two parallel classes).

This results in the system of equations:

$$Z + S + D + T + Q = 100$$
$$D + 3T + 6Q = 52$$
$$6Z + 3S + D = 292$$
\[ S = 9T \]

As \( Z, S, D, T, Q \) are all non-negative integers, the only two solutions are:

\[
(Z, S, D, T, Q) = (12, 72, 4, 8, 4)
\]

\[
(Z, S, D, T, Q) = (25, 36, 34, 4, 1)
\]

We can solve the system of equations to find any potential \((Z, S, D, T, Q, K)\) for a configuration of any given type. Before listing the results, however, we consider a way to pair up results, to reduce the total number of cases to consider.

Since configurations are formed by deleting a parallel class from a relation, if a relation has classes of both weight two and four (and no other weights), we can produce a different configuration depending on the weight of the class we delete. If a relation contains a class of weight six or eight (which, by our construction, must be the class deleted for the configuration), we can complement that class and another to produce a different relation, which can be considered analogous to the first relation. For example, classes of weight two and six can be complemented to classes of weight eight and four. This motivates the following definitions:

**Definition 5.1.** Any two configurations corresponding to the same relation are called **analogues**. Any configuration with itself as an analogue is called a **self-analogue**. Two configurations which come from relations which can be complemented to one another, as above, are called **complement analogues**.

It can be shown that any configuration with all classes the same weight has only itself as an analogue. Also, the solutions to the above system of equations shows that any relation with a class of weight eight must also contain a class of weight four, and any relation with a class of weight six must also contain a class of weight two. Thus, all configurations stemming from such relations must have at least one complement analogue. If a solution to the system of equations should have an analogue of some form which can’t exist, then the original solution is not a valid relation in \( N_6 \).

By pairing solutions to the systems of equations above according to what solutions are some form of analogue of one another, we obtain a classification of the various potential relations in the net \( N_6 \): (Please note that this list is a summary of the results from various non-trivial analysis and computations, which are omitted for the sake of brevity)[1]

1. Configuration of type \( \{4, 4, 4\} \) with \((Z, S, D, T) = (24, 36, 36, 4)\) is a self-analogue
2. Type \( \{2, 2, 2, 4\} \) with \((Z, S, D, T, Q) = (24, 60, 12, 0, 4)\) is a complement analogue of type \( \{2, 2, 4, 4\} \) with \((Z, S, D, T, Q) = (12, 72, 4, 8, 4)\).

3. Type \( \{2, 2, 4, 4\} \) with \((Z, S, D, T, Q) = (25, 36, 34, 4, 1)\) is an analogue of type \( \{2, 4, 4, 4\} \) with \((Z, S, D, T, Q) = (25, 16, 54, 4, 1)\).

4. Type \( \{2, 4, 4, 4\} \) with \((Z, S, D, T, Q) = (14, 48, 24, 12, 2)\) is a complement analogue of type \( \{4, 4, 4, 4\} \) with \((Z, S, D, T, Q) = (6, 56, 12, 24, 2)\).

5. Type \( \{4, 4, 4, 4\} \) with \((Z, S, D, T, Q) = (15, 28, 42, 12, 3)\) is a self-analogue.

6. Type \( \{2, 2, 2, 2, 2\} \) with \((Z, S, D, T, Q, K) = (25, 60, 10, 0, 5, 0)\) is a complement analogue of type \( \{2, 2, 2, 2, 4\} \) with \((Z, S, D, T, Q, K) = (13, 72, 2, 8, 5, 0)\).

7. Type \( \{2, 2, 2, 2, 2\} \) with \((Z, S, D, T, Q, K) = (40, 20, 40, 0, 0, 0)\) is a self-analogue.

8. Type \( \{2, 2, 2, 2, 4\} \) with \((Z, S, D, T, Q, K) = (26, 36, 32, 4, 2, 0)\) is an analogue of type \( \{2, 2, 2, 4, 4\} \) with \((Z, S, D, T, Q, K) = (26, 16, 52, 4, 2, 0)\).

9. Types \( \{2, 2, 2, 4, 4\} \) with \((Z, S, D, T, Q, K) = (15, 48, 22, 12, 3, 0), (14, 49, 24, 10, 2, 1), (13, 50, 26, 8, 1, 2), \) and \((12, 51, 28, 6, 0, 3)\) each have at least one complement analogue from types \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (7, 56, 10, 24, 3, 0), (6, 57, 12, 22, 2, 1), (5, 58, 14, 20, 1, 2), \) and \((4, 59, 16, 18, 0, 3)\).

10. Type \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (16, 28, 40, 12, 4, 0)\) is an analogue of type \( \{2, 4, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (16, 12, 56, 8, 8, 0)\).

11. Type \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (15, 29, 42, 10, 3, 1)\) is an analogue of type \( \{2, 4, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (15, 13, 58, 6, 7, 1)\).

12. Type \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (14, 30, 44, 8, 2, 2)\) is an analogue of type \( \{2, 4, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (14, 14, 60, 4, 6, 2)\).

13. Type \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (13, 31, 46, 6, 1, 3)\) is an analogue of type \( \{2, 4, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (13, 15, 62, 2, 5, 3)\).

14. Type \( \{2, 2, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (12, 32, 48, 4, 0, 4)\) is an analogue of type \( \{2, 4, 4, 4, 4\} \) with \((Z, S, D, T, Q, K) = (12, 16, 64, 0, 4, 4)\).
15. Types \{2, 4, 4, 4, 4\} with \((Z, S, D, T, Q, K) = (9, 36, 26, 24, 5, 0), (8, 37, 28, 22, 4, 1), (7, 38, 30, 20, 3, 2), (6, 39, 32, 18, 2, 3), (5, 40, 34, 16, 1, 4), and (4, 41, 36, 14, 0, 5)\) all have at least one complement analogue from types \{4, 4, 4, 4, 4\} with \((Z, S, D, T, Q, K) = (5, 40, 10, 40, 5, 0), (4, 41, 12, 38, 4, 1), (3, 42, 14, 36, 3, 2), (2, 43, 16, 34, 2, 3), (1, 44, 18, 32, 1, 4), and (0, 45, 20, 30, 0, 5)\)

16. Types \{4, 4, 4, 4, 4\} with \((Z, S, D, T, Q, K) = (10, 20, 40, 10, 0), (9, 21, 42, 18, 9, 1), (8, 22, 44, 16, 8, 2), (7, 23, 46, 14, 7, 3), (6, 24, 48, 12, 6, 4), (5, 25, 50, 10, 5, 5), (4, 26, 52, 8, 4, 6), (3, 27, 54, 6, 3, 7), and (2, 28, 56, 4, 2, 8)\) are all self-analogues.

This gives us a list of all possible relations in a net \(N_6\) of order 10. We know that at least two of these must exist in any such \(N_6\). In the next section, we will use these relations to make further deductions about the dimension of \(C_2(N_6)\).
6 The Dimension’s Lower Bound

We now know the possible relations in a net \( N_6 \) of order 10. In this section we will make use of these relations to find a lower bound on \( \dim C_2(N_6) \), as well as some other implications of higher orders of \( \dim C_2(N_6) \).

First, recall that there are at least seven linearly independent relations in a net \( N_6 \) of order 10, five of which are the trivial relations. Now suppose that there are, in fact, at least ten linearly independent relations, that is, that \( \dim C_2(N_6) \leq 50 \). This means that there are five such non-trivial relations, \( S_1, S_2, S_3, S_4, \) and \( S_5 \).

Now let \( B_1 \) be an arbitrary parallel class in the net \( N_6 \). \( B_1 \) has some set (possibly empty) of lines in each relation, and some lines may be in multiple relations. Let \( T_i \) be the set of lines in \( B_1 \) which are in relation \( S_i \) and let \( s_i = |T_i|, 1 \leq i \leq 5 \). In order to properly assess the lines in \( B_1 \), we partition each \( T_i \) as follows:

\[
T_1 = A \cup A5 \cup B \cup B5 \cup C \cup C5 \cup D \cup D5 \cup E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5
\]
\[
T_2 = C \cup C5 \cup D \cup D5 \cup G \cup G5 \cup H \cup H5 \cup K \cup K5 \cup L \cup L5 \cup M \cup M5 \cup N \cup N5
\]
\[
T_3 = E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5 \cup I \cup I5 \cup J \cup J5 \cup K \cup K5 \cup L \cup L5
\]
\[
T_4 = A \cup A5 \cup C \cup C5 \cup E \cup E5 \cup G \cup G5 \cup I \cup I5 \cup K \cup K5 \cup M \cup M5 \cup O \cup O5
\]
\[
T_5 = A5 \cup B5 \cup C5 \cup D5 \cup E5 \cup F5 \cup G5 \cup H5 \cup I5 \cup J5 \cup K5 \cup L5 \cup M5 \cup N5 \cup O5 \cup X5
\]

(Note here that, for example, \( A \) and \( A5 \) are not related as subsets of \( B \) - we simply need to label 32 distinct regions, which is more than there are letters in the English alphabet.)

In this way, we partition \( B_1 \) as \( A \cup A5 \cup \ldots \cup X \cup X5 \), where each subset represents the lines that are in a specific combination of relations (for example, \( G = T_2 \cap T_3 \cap T_4 \), and so is the set of lines in \( B_1 \) and in \( S_2, S_3, \) and \( S_4 \)). Here, \( X \) is the set of lines that are in none of the relations.

As any linear combination of relations is itself a relation, we can form new relations from \( S_1, \ldots S_5 \) by taking combinations of these five relations. This results in \( \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 26 \) new relations. We can use these relations for a system of linear equations. For each subset, let the lower case variable be the number of lines contained in the upper case region. For example, \( a = |A| \) and \( o5 = |O5| \).

Then each new relation has a number of lines in \( B_1 \) equal to the sum of 16 of the variables, found by summing up the variables in the component relations over \( F_2 \). For example, the relation given by \( T_1 + T_2 \) has the following number of lines in \( B_1 \):

\[ a + b + c + f + k + l + m + n + a5 + b5 + e5 + f5 + k5 + l5 + m5 + n5 \]

By Prop 5.1, each of these new sums must be one of \( \{0, 2, 4, 6, 8, 10\} \). Also, since the 32 subsets partition \( B_1 \), we must have that \( a + a5 + \ldots + x + x5 = 10 \).
Proposition 6.1. Suppose \( N_6 \) is a net of order 10. If \( \dim C_2(N_6) \leq 50 \), then \( N_6 \) contains a relation not complementable to one of type \( \{4, 4, 4, 4, 4\} \).

Proof. Since \( \dim C_2(N_6) \leq 50 \), we can use the construction above. If one of the \( S_i \) is not complementable to type \( \{4, 4, 4, 4, 4\} \), we are done. If not, then there is a parallel class \( B_1 \) such that \( s_i = 6 \) \( \forall 1 \leq i \leq 5 \). Consider all systems of equations described above where \( s_i = 6 \), the 26 quantities for the other relations are in the set \( \{0, 2, 4, 6, 8, 10\} \), and \( a + a5 + \ldots + x + x5 = 10 \). For each such solution, of the 26 other relations, there is either one relation with 0 or 10 lines in \( B_1 \), or at least three relations with 2 or 8 lines in \( B_1 \). As the \( S_i \) are linearly independent, none of these new relations can be the zero relation, and so we must have a relation with at least one class of weight zero, two, eight, or ten. Such a relation cannot be complementable to one of type \( \{4, 4, 4, 4, 4\} \).

A similar argument shows that if there is an \( N_4 \) of order 10 has five or more linearly independent non-trivial relations, at least one relation must not be complementable to one of type \( \{4, 4, 4, 4, 4\} \). This produces the following result:

Proposition 6.2. Given a net \( N_4 \) of order 10, \( \dim C_2(N_4) \geq 33 \).

Proof. Suppose that \( N_4 \) contains at least five linearly independent non-trivial relations. Then by above, \( N_4 \) contains a non-trivial relation with a class of weight zero, two, eight, or ten. This contradicts the list of configurations from the last chapter, which shows that the only possible relation in \( N_4 \) is one complementable to one of type \( \{4, 4, 4, 4\} \). As such, \( N_4 \) contains at most four linearly independent non-trivial relations, as well as the three trivial relations generated by the parallel classes. As there are 40 lines in \( N_4 \), we thus have that \( \dim C_2(N_k) \geq 40 - 4 - 3 = 33 \).

We can now use a simple result to extend this result about \( N_4 \) to one about \( N_6 \).

Proposition 6.3. Let \( N_k \) be a net of order \( n \), where \( n \) is even. If \( k \) is even or \( N_k \) has a transversal, then \( \dim C_2(N_k) > \dim C_2(N_{k-1}) \).

Proof. By Prop. 5.1, all parallel classes have even weight in any relation in \( N_k \). Thus the characteristic function of any single line in the last parallel class of \( N_k \) must be linearly independent over \( C_2(N_{k-1}) \) (as otherwise we have a relation with a parallel class of weight one), and so \( \dim C_2(N_k) > \dim C_2(N_{k-1}) \).

Corollary 6.4. If \( N_6 \) is a net of order 10, then \( \dim C_2(N_6) \geq 35 \).
Proof. Since \( N_6 \) exists, the subnet \( N_5 \) of \( N_6 \) must have a transversal (for example, the points of any line in the last parallel class of \( N_6 \) are such a transversal). Applying Prop. 6.3 with \( k = 6 \) and \( k = 5 \) along with Prop. 6.2 gives that \( \dim C_2(N_6) \geq \dim C_2(N_5) + 1 \geq (C_2(N_4) + 1) + 1 \geq 33 + 1 + 1 = 35. \)

With this we have our second main result of the paper. We now know that \( 35 \leq \dim C_2(N_6) \leq 53. \) While we cannot directly limit the range further, we can consider what happens if \( \dim C_2(N_k) \) is a specific number in the range:

**Corollary 6.5.** If \( N_6 \) is a net of order 10 such that \( \dim C_2(N_6) < 47 \), then there is a non-trivial relation in four or five classes in the net.

**Proof.** Since 6 and 10 are even, by Prop. 6.3, \( \dim C_2(N_5) \leq \dim C_2(N_6) - 1 < 47 - 1 = 46 = 50 - 4. \) Since there are 50 lines in \( N_5 \), this means that there is a nontrivial relation in the subnet \( N_5 \) of \( N_6 \), which must necessarily be a relation in four or five classes.

**Proposition 6.6.** If \( N_6 \) is a net of order 10 such that \( \dim C_2(N_6) = 53 \), then any net \( N_7 \) containing this net has no transversal.

**Proof.** Since 6 and 10 are even, by Prop. 4.2, \( \dim D_2(N_6) = \dim C_2(N_6) - 6 = 53 - 6 = 47. \) Since \( D_2(N_6) \subseteq F_2^{100} \), by the rank-nullity theorem, \( \dim D_2(N_6)^\perp = 100 - \dim D_2(N_6) = 100 - 47 = 53. \) By the proof of Prop. 4.3, \( C_2(N_6) \subseteq D_2(N_6)^\perp \), and so \( C_2(N_6) = D_2(N_6). \) By the proof of Prop. 4.2, any transversal \( t \) of \( N_6 \) has \( t \in D_2(N_6)^\perp = C_2(N_6). \) Then the characteristic function of any line in the last parallel class of any \( N_7 \) containing \( N_6 \) is in \( C_2(N_6) \), and so \( C_2(N_7) = C_2(N_6) \). Then by Prop. 4.2, \( N_7 \) cannot have a transversal.

Note that this means that this \( N_7 \) cannot be extended to an \( N_8 \). Hence if a net \( N_8 \) of order 10 exists, \( \dim C_2(N_6) \leq 52 \) \( \forall \) subnets \( N_6 \) of \( N_8 \).

We can also modify the argument used in Prop. 6.1 to the cases where there are three or four non-trivial relations in the net \( N_6 \) to produce the following result:

**Proposition 6.7.** [1] If \( N_6 \) is a net of order 10 such that \( \dim C_2(N_6) = 51 \) or 52, then \( N_6 \) contains a non-trivial relation not complementable to one of type \( \{2, 2, 2, 2, 2, 2\} \).

With this, we have found various restrictions we can place on the dimension of a potential \( N_6 \). In the next section, we will take the potential relations, listed in Section 5, and analyze them further.
7 Analyzing Structures of Relations

In Section 5, we discovered the list of possible configurations in a net $N_6$. While the list covered the weights of the points for that relation, it does not describe how these points are arranged among the parallel classes. In this section, we make use of decompositions of multipartite graphs into cliques in order to better understand the structure of these relations.

Definition 7.1. A graph $G = (V, E)$ is a set of vertices $V$ and a set of edges $E$, where each edge is an unordered pair of vertices in $V$. The order of a graph $G = (V, E)$ is $|V|$.

Definition 7.2. A graph $G = (V, E)$ is complete if $E$ consists of all possible pairs of distinct vertices of $V$. We use the notation $K_n$ for the complete graph of order $n$. We call a complete graph a clique when it appears as a subgraph of a larger graph. A complete multipartite graph with parts $P_1, P_2, \ldots, P_r$ is a graph $G = (V, E)$ such that $P_1, P_2, \ldots, P_r$ form a partition of $V$, and an edge $e = \{v_i, v_j\} \in E$ if and only if $v_i \in P_i$ and $v_j \in P_j$ for some $i \neq j$. We use the notation $K_{n_1, n_2, \ldots, n_r}$ for the complete multipartite graph with parts of size $|P_i| = n_i$.

Definition 7.3. A packing of graphs $G_1, G_2, \ldots, G_r$ into a graph $G$ is an assignment of a vertex in $V(G)$ to each vertex of $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_r)$ such that the edges of $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_r)$ are mapped into the edges of $E(G)$ injectively. A packing into a graph $G$ is said to be a decomposition of $G$ if it covers all the edges of $G$.

It can be shown that a relation or configuration in a net $N_k$ of type $\{n_1, n_2, \ldots, n_k\}$ is combinatorially equivalent to a decomposition of the complete multipartite graph $K_{n_1, n_2, \ldots, n_k}$ into cliques [1]. In this equivalence, the lines of the configuration or relation are vertices in the graph. Also, a point in the configuration or relation is a clique in the decomposition of the graph, and its weight is the order of the clique.

We can therefore see how the differently weighted points of a relation are arranged in the parallel classes in the net by seeing how the vertices of corresponding cliques are distributed among the corresponding parts of the complete multipartite graph.

Given a decomposition of a complete multipartite graph corresponding to a relation, let $Q_i, P_i$ be the number of $K_4$ and $K_2$ respectively incident with any vertex of a part $B$ where $|B| = i$. (Note that we only count the cliques coming from the decomposition) For this part $B$ of the graph, let $P$ be the number of edges of the graph incident with a vertex of that part, and let $C = 10i\alpha - P$, where $\alpha$ is the number of parts in the graph. In terms of the net, $P$ is the number of pairs of lines in the relation with one line in the parallel class...
corresponding to $B$. $C$ represents the number of pairs of lines where one line is in the relation and in the parallel class corresponding to $B$, while the other line is in a different parallel class and not in the relation. For example, for the complete multipartite graph $K_{2,4,4,4,6}$ corresponding to the relation of type $\{2,4,4,4,6\}$, we have that $P = 2(4 + 4 + 4 + 6) = 36$ and $C = 2(6 + 6 + 6 + 4) = 44$.

**Proposition 7.1.** Given a relation in a net $N_6$ of order 10 and its representation as a decomposition of a complete multipartite graph, for each $i$ the number of $K_i$ incident with a vertex of each part of the graph is only dependent on the size of that part.

**Proof.** We prove this using $P$ and $C$, as defined above. For a relation in four classes, the only possible relation (of type $\{4,4,4,4\}$) gives $3Q_4 + D_4 = P = 48$ and $2D_4 = C = 72$ (by using the meaning of $P$ and $C$ in terms of the net). Solving gives $(Q_4, D_4) = (4,36)$. For a relation in five classes, we have that $3Q_i + D_i = P$ and $Q_i + 3D_i = C$. Solving gives $(Q_i, D_i) = (\frac{3P-C}{8}, \frac{3C-P}{8})$. For a relation in six classes, we have that $5H + 3Q_i + D_i = P$ and $2Q_i + 4D_i = C$. Solving gives $(Q_i, D_i) = (\frac{4P-C}{10} - 2H, \frac{3C-2P}{10} + H)$. Since $P$ and $C$ depend on the relation in question and $i$, and $H$ depends only on the relation, the result follows.

For the next proposition we introduce $Q_{iL}$ and $D_{iL}$, which are the number of $K_4$ and $K_2$ respectively which are incident with a given vertex in a part of size $L$.

**Proposition 7.2.** Given a relation in a net $N_6$ of order 10 and its representation as a decomposition of a complete multipartite graph, the values $Q_{iL}$ and $D_{iL}$ depend only on the relation, the cardinality $i$ of the part, and the number of $K_6$ incident with the vertex.

**Proof.** In relations in four or five classes, there are no points of weight six. We then obtain the equations $3Q_{iL} + D_{iL} = \frac{P}{i}$ (from the number of edges adjacent to that vertex) and $Q_{iL} + D_{iL} = 10$ (as the line in the net has ten points, the vertex in the graph is on ten cliques). Since $P + C = 10i\alpha$, where $\alpha$ is the number of classes in the relation, we get that $Q_{iL} + D_{iL} = \frac{P+C}{i\alpha}$. Then by Prop. 7.1, we can see that $(Q_i, D_i) = (iQ_{iL}, iD_{iL})$, and so $(Q_{iL}, D_{iL}) = (Q_i, D_i)$. For a relation in six classes, consider a specific vertex in the graph, and let $h$ be the number of $K_6$ incident with that vertex. We then get that $5h + 3Q_{iL} + D_{iL} = \frac{P}{i}$ and $h + Q_{iL} + D_{iL} = 10$. As these equations are linearly independent, there is a unique solution, and the result follows.

Using this construction, we can find the values of $Q_i$, $D_i$, and (if relevant) $Q_{iL}$ and $D_{iL}$ for each of the relations found in Section 5. For instance, the relation listed as (6) of type
\{2,2,2,2,6\} has \((Q_6,D_6) = (0,60)\), \((Q_2,D_2) = (4,16)\), and its complement analogue of type \{2,2,2,2,4,8\} has \((Q_8,D_8) = (8,72)\), \((Q_4,D_4) = (12,28)\), and \((Q_2,D_2) = (8,12)\).

Before listing our results for this section, we first introduce an additional set of equations which can further determine the structure of the relations.

Given a relation over six classes, let \(P_1, P_2, P_3, P_4, P_5, P_6\) be the six parts in the corresponding decomposition of a complete multipartite graph (note that the labels need not be in any particular order). Let \(a\) be the number of edges between two of \(P_1, P_2, P_3,\) and \(P_4,\) and let \(b\) be the number of edges between \(P_5\) and one of \(P_1, P_2, P_3,\) and \(P_4\). Now consider the configuration produced when \(P_6\) is deleted. Let \(\alpha\) and \(\beta\) be the number of \(K_4\) and \(K_3\), respectively, in the configuration and incident with \(P_5\). From how we defined \(a\) and \(b\), we get that

\[3\alpha + 6(Q - \alpha) + \beta + 3(T - \beta) + 6K \leq a\]

\[3\alpha + 2\beta + 4K \leq b\]

Together, these give that \(6Q + 3T + 6K - a \leq 3\alpha + 2\beta \leq b - 4K\) for \(\alpha \leq Q\) and \(\beta \leq T\) (which must hold by how we defined \(\alpha\) and \(\beta\)). Since the \(K_3\) in the configuration must be \(K_4\) in the original relation, we have that \(\beta + K \leq |P_5| \times |P_6|\) (since the left hand side counts all cliques adjacent to both \(P_5\) and \(P_6\) in the original decomposition, and in a decomposition each edge can only belong to one clique). As the configuration has \(\alpha\ K_4\) incident with \(P_4\), it must have \(Q - \alpha\ K_4\) incident with all of \(P_1, P_2, P_3\) and \(P_4\). Because any \(K_4\) in the original relation adjacent to \(P_5\) is either a \(K_4\) or a \(K_3\) adjacent to \(P_5\) in the configuration, we have that \(\alpha + \beta = Q_{|P_5|}\). Finally, since \(Q - \alpha\) is the number of \(K_4\) adjacent to neither \(P_5\) nor \(P_6\), we must have that \(Q - \alpha\) is no greater than \(H\) in the relation formed by complementing \(P_5\) and \(P_6\).

With this additional structure in our relations, it now becomes possible to rule out one of the possible relations from Section 5. From that list, there must be five \(K_4\) and eight \(K_3\) in the corresponding decomposition of the complete multipartite graph \(K_{2,2,2,2,4}\).

**Proposition 7.3.** There is no packing of five copies of \(K_4\) and eight copies of \(K_3\) into the complete multipartite graph \(K_{2,2,2,2,4}\) which corresponds to a configuration of type \(\{2,2,2,2,4\}\).

**Proof.** Label the parts of the graph \(P_1, P_2, P_3, P_4,\) and \(P_5\), where \(P_5\) is the part of cardinality four. From before, we have that \(Q_4 = 12\) in the original relation, and so if such a packing exists, then \(\alpha + \beta = 12\). Also, \(a = (2)(2)(\binom{4}{2}) = 24\) and \(b = 4(2 + 2 + 2 + 2) = 32\). This gives us that \(6(5) + 3(8) + 6(0) - 24 \leq 3\alpha + 2\beta \leq 32 - 4(0)\), and so \(30 \leq 3\alpha + 2\beta \leq 32\). Since
\(\alpha \leq Q = 5\) and \(\beta \leq T = 8\), for the lower bound to hold, we must have that \(\alpha = 5\) and \(\beta = 8\). But then \(13 = 5 + 8 = \alpha + \beta = 12\), a contradiction.

**Corollary 7.4.** There can be no relation of type \(\{2, 2, 2, 4, 8\}\) in a net \(N_6\) of order 10.

**Proof.** If such a relation exists, then by removing the parallel class of weight eight, we have that a configuration of type \(\{2, 2, 2, 4\}\) exists in the subnet \(N_5\). Then by combinatorial equivalence, we must have the corresponding decomposition of the complete multipartite graph \(K_{2,2,2,4}\) which includes five copies of \(K_4\) and eight copies of \(K_3\), contradicting Prop. 7.3.

With these sets of equations, we can now extend the list from Section 5 by elaborating on the structures of the relations. Note that we can also determine what a line in a parallel class not in the relation can look like, as each such line must intersect all lines in the relation at one point, and there are 10 points on the line. This gives that \(Q + D + Z = 10\) and \(4Q + 2D = l\), where \(l\) is the number of lines in the relation.

We begin with the relations in four or five classes, as in these \(k = 0\), which simplifies the systems of equations. In the following lists, when describing the various \(Q\), a number indicates that the point is incident with that parallel class, while a * indicates that the point is not incident with that parallel class. (This list is again a summary of the results from various non-trivial analysis and computations, which are omitted for the sake of brevity)[1]

1. Relation of type \(\{4, 4, 4, 4\}\). Four \(Q\) of type 4444, the remaining pairs are covered by \(D\). Each line in the fifth or sixth class meets one of: \((4Q, 6Z), (3Q, 2D, 5Z), (2Q, 4D, 4Z), (1Q, 6D, 3Z), (8D, 2Z)\).

2. Relation of type \(\{2, 2, 2, 4, 6\}\). Four \(Q\) of type 2224*, the remaining pairs are covered by \(D\). Each line in the sixth class meets one of: \((1Q, 6D, 3Z), (8D, 2Z)\). Complement analogue of type \(\{2, 2, 4, 4, 8\}\). Twelve \(Q\), four each of types 2244*, 2448, 2*448. Each line in the sixth class meets one of: \((2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)\).

3. Relation of type \(\{2, 2, 4, 4, 4\}\). Five \(Q\) of types 2244*, 2444, 2*444, *2444, the remaining pairs are covered by \(D\). Each line in the sixth class meets one of: \((3Q, 2D, 5Z), (2Q, 4D, 4Z), (1Q, 6D, 3Z), (8Q, 2Z)\).

4. Relation of type \(\{2, 4, 4, 4, 6\}\). Fourteen \(Q\), two each of types 2444*, 24*46, 244*6, and six of type *4446, the remaining pairs are covered by \(D\). Each line in the sixth class
meets one of: \((4Q, 2D, 4Z), (3Q, 4D, 3Z), (2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)\). Complement analogue of type \(\{4, 4, 4, 4, 8\}\). 26 \(Q\) with two of type \(4444^*\) and six each of types \(*4448, 4*448, 44*48, 444*8,\) the remaining pairs are covered by \(D\). Each line in the sixth class meets one of: \((4Q, 4D, 2Z), (3Q, 6D, Z), (2Q, 8D)\).

5. Relation of type \(\{4, 4, 4, 4, 4\}\). Fifteen \(Q\), three each of types \(*4444, 4*444, 44*44, 444*4, 4444^*,\) the remaining pairs are covered by \(D\). Each line in the sixth class meets one of: \((5Q, 5Z), (4Q, 2D, 4Z), (3Q, 4D, 3Z), (2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)\).

6. Relation ruled out above

7. Relation of type \(\{2, 2, 2, 2, 2, 2\}\). All possible pairs covered by \(D\).

For relations in six classes, the existence of points of weight six means that we now have a range of possibilities for structures. It is also far more difficult to list the arrangement of the \(Q\). Let \(a, b, \ldots, o\) represent the number of \(Q\) of the following forms:

\[
a : 1111** \quad b : 111*1* \quad c : 111**1 \quad d : 11*11* \quad e : 11*1*1 \\
f : 11**11 \quad g : 1*111* \quad h : 1*11*1 \quad i : 1*11*11 \quad j : 1**111 \\
k : 1*1111 \quad l : 1*1111 \quad m : 1**111 \quad n : 1**111 \quad o : **1111
\]

Then \(a + b + \ldots + o = Q\). Also, each type of \(Q\) has an upper bound by complementing the two nonadjacent classes. For example, a point of weight four of type \(a\) becomes a point of weight six in the relation formed by complementing the last two classes, and so \(a \leq H\) where \(H\) is taken from the new relation. Finally, we will require that \(j \geq n \geq o\) and \(b \geq c\) (where possible), in order to try to minimize relations which are simply permutations of the parallel classes of other relations.

Since the number of possibilities is too great to list outright, we will simply provide the number of possibilities for each relation. (This list is again a summary of the results of various non-trivial analysis and computations, which are omitted for the sake of brevity)[1]

1. Relation of type \(\{2, 2, 2, 2, 4, 4\}\) has analogues of types \(\{2, 2, 2, 2, 6, 6\}, \{2, 2, 2, 4, 6, 8\},\) and \(2, 2, 4, 4, 8, 8\) each having \(0 \leq H \leq 2\). Solving for \(a, b, \ldots, o\) gives 46 solutions.

2. Relations of type \(\{2, 2, 2, 4, 4, 6\}\) have analogues of types \(\{2, 4, 4, 6, 8, 8\}, \{2, 2, 2, 4, 6, 8\}, \{2, 2, 4, 4, 4, 8\}, \{2, 2, 2, 6, 6, 6\},\) and \(\{2, 2, 2, 4, 4, 6\}\), each having \(0 \leq H \leq 3\). Solving for \(a, b, \ldots, o\) gives 266 solutions.

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3. Relations of type \{2, 2, 4, 4, 4\} have analogues of types \{4, 4, 4, 4, 8, 8\}, \{2, 4, 4, 4, 6, 8\}, and \{2, 2, 4, 4, 6, 6\}, each having \(0 \leq H \leq 4\). Solving for \(a, b, \ldots, o\) gives 1546 solutions.

4. Relations of type \{2, 4, 4, 4, 4, 6\} have analogues of types \{4, 4, 4, 6, 6, 8\}, \{4, 4, 4, 4, 4, 8\}, \{2, 4, 4, 6, 6, 6\}, and \{2, 4, 4, 4, 4, 6\}, each having \(0 \leq H \leq 5\). Solving for \(a, b, \ldots, o\) gives 5658 solutions.

5. Relations of type \{4, 4, 4, 4, 4\} have analogues of type \{4, 4, 4, 4, 6, 6\} with \(0 \leq H \leq 6\). Solving for \(a, b, \ldots, o\) gives 470821 solutions.

This gives us a total of \(6 + 46 + 266 + 1546 + 5658 + 470821 = 478337\) different possible structures for a relation in a net \(N_6\) of order 10, of which only 7522 are not of type \{4, 4, 4, 4, 4\}. 
8 Conclusion

In this paper, we have been considering the possible existence of a net $N_6$ of order 10, which would imply the existence of four $MOLS(10)$. We have determined that should such a net exist, we would have that $35 \leq \dim C_2(N_6) \leq 53$, and that there would be at least two relations in the net, which have 478337 possible structures.

It is clear that these restrictions alone are insufficient to determine whether or not a net $N_6$ of order 10 exists. They do, however, suggest the possibility of using a computer search to answer this question conclusively.

Such a computer search would involve taking each potential structure for a relation $R$ in a net $N_6$, determining the corresponding matrix $C_2(R)$, and then seeing if $C_2(R)$ could be extended to $C_2(N_6)$ for a net $N_6$. Such a search would resemble the one used by Lam, Thiel and Swiercz to show that there is no projective plane of order 10.

Is such a search currently feasible? The search by Lam, Thiel and Swiercz for weight 19 codewords involved taking 45 initial cases of a $6 \times 19$ matrix, expanding it to a $111 \times 19$ matrix, and then attempting to expand the number of columns [7]. These 45 initial cases resulted in 639624 possible $111 \times 19$ matrices, which is similar in scope to the number of possible structures of relations we obtained in the last section. The subsequent expansion of the matrix is also similar in scope. As such, if it can be shown that there are few possibilities for $C_2(R)$ for a given structure of relation $R$ (up to permutation of row and column vectors), then such a search is entirely possible (especially considering the increase in computing power since 1989, when Lam, Thiel and Swiercz conducted their search). For example, there is only one possible $C_2(R)$ (up to permutation of row and column vectors) for a relation $R$ of type \{2, 2, 2, 2, 2, 2\}.

If there are too many possibilities for $C_2(R)$ for a given structure of relation $R$, it may still be useful to conduct a search for all structures of relation except those of type \{4, 4, 4, 4, 4\}. This limits us to 7522 possible structures, which should make for a feasible search unless the number of possible matrices is enormous for each structure. Assuming no net $N_6$ was found, this would then allow us to use the results of Section 6 to drastically limit the dimension of a potential net $N_6$ of order 10.

In either case, a computer search which would, at worst, drastically limit the scope of search for a net $N_6$ of order 10 does seem feasible, and worthy of consideration.
Bibliography


