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TITLE: Mutually Orthogonal Latin Squares and k-nets

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My thanks to my all my friends from class. I may not be the most outgoing of people, but throughout school they accepted and encouraged me to be the best I could.

Contents

2 Abstract

In this paper, we will be examining different sizes of sets of mutually orthogonal Latin squares of order 10 ($MOLS(10)$). We will do so by using different objects which are combinatorially equivalent to a set of $MOLS(10)$, including k-nets of order 10 and decompositions of complete multipartite graphs into cliques whose parts are of size 10 (e.g. $K_{10,10,10,10}$). Specifically, we will be determining restrictions on the potential dimensions of a 6-net of order 10, which is combinatorially equivalent to four $MOLS(10)$. Using these facts, we discuss the possibility of a computer search either finding or disproving the existence of a set of four $MOLS(10)$.

3 Introduction

We begin by briefly discussing the history of Latin squares.

Definition 3.1. A **Latin square of order** n is an $n \times n$ array of symbols from an n-set in which each row and column contains each symbol exactly once.

It is easy to show that Latin squares exist for all n. For example, the addition table in $\mathbb{Z}_n$ is a Latin square of order n.

Definition 3.2. Two Latin squares L and M are **orthogonal** if the n^2 ordered pairs $(L_{i,j}, M_{i,j})$ are distinct.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix}$$

Example 3.1. Two orthogonal Latin Squares of order 5.

By relabelling as necessary, we can ensure that L and M are composed of the numbers {1, 2, . . . , n}. Then L and M being orthogonal means that for every ordered pair of elements (x, y) from {1, 2, . . . , n}, there is a unique cell (i, j) such that $L_{i,j} = x$ and the $M_{i,j} = y$.

It is not surprising to find that, for certain orders, it is possible to find sets of more than two Latin squares such that the squares are all pairwise orthogonal. Such a set is called a set of **mutually orthgonal Latin squares of order n** (abbreviated as $MOLS(n)$).

Latin squares have a long history in puzzle solving (logic puzzles often make use of Latin squares, and sudoku is a Latin square of order 9 with additional constraints). One of the earlier examples of such a puzzle was Euler's 36 officers problem (1782). In it, Euler supposes that there are six regiments of six officers, each of different rank, and asks if there is a way to arrange the officers in a 6×6 array such that no row or column contains more than one of each rank or regiment. It is not hard to see that such a solution is essentially a pair of $MOLS(6)$ – one Latin square for rank and one Latin square for regiment, and the squares must be orthogonal as there is only one of each rank in each regiment. Euler correctly conjectured that there was no possible solution, although this was not proven until Tarry (1901) proved it exhaustively [2]. Stinson (1984) provided an alternative proof [3], using coding theory and the combinatorial equivalence of a pair of $MOLS(6)$ and another combinatorial object, called a transversal design. The thought process behind the work in this paper is largely similar to the one in his proof.

We thus know that there can be no pair of $MOLS(6)$. It is also relatively easy to determine that there can be at most $n-1$ $MOLS(n)$ [1]. We thus arrive at the natural question: what is the maximum possible $MOLS(n)$ for a given n? This number is typically denoted $N(n)$.

In 1922, MacNeish showed that $N(n)$ is at least one less than the smallest prime power in the prime factorization of n [4]. This means that $N(n) = n - 1$ when n is a prime power. Thus for $2 \leq n \leq 9$, $N(n) = n - 1$ for all $n \neq 6$. We also know that $N(6) = 1$ from Tarry. This means that $n = 10$ is the first number for which $N(n)$ is unknown.

Stemming from the nonexistence of a pair of $MOLS(6)$, Euler conjectured that there would be no pair of $MOLS(n)$, for all n congruent to 2 modulo 4. This was disproved by Bose and Shrikhande (1959) for $n = 22$ [5], and more generally by Bose, Shrikhande, and Parker (1960) for all such $n > 6$ [6]. Thus, $N(10) \geq 2$.

The current upper bound on $N(10)$ is thanks to Lam, Thiel and Swiercz (1989) [7]. They used a computer search to demonstrate the non-existence of a projective plane of order 10. The existence of a projective plane of order n is combinatorially equivalent to $n-1$ $MOLS(n)$, and so there are at most 8 $MOLS(10)$. Bruck (1963) showed that if $N(n) < n-1$, then $N(n) < n - 1 - (2n)^{1/4}$ [8]. Hence, $N(10) < 10 - 1 - (20)^{1/4} \approx 6.89$, and so $N(10) \leq 6$.

The purpose of this paper is to examine ways to further restrict $N(10)$. This is done by compiling information from various sources, the majority of which comes from the PhD thesis of Dr. Leah Howard [1]. The material has been reordered and elucidated for the sake of accessibility, and select computations were double checked to ensure validity. Using this information, we look to draw conclusions on the feasibility of restrictions on $N(10)$.

4 Dimensions of k-Nets

Analyzing $N(n)$ directly through Latin squares is difficult. It is easier to start by looking at another combinatorial object, the k-net of order n.

Definition 4.1. A **k-net of order** n, denoted N_k, is an incidence structure of n^2 points and nk lines, such that the following hold:

1. There are n points on each line;
2. There are k parallel classes with n lines each; and
3. Every two non-parallel lines meet at exactly one point.
4. Lines being parallel forms an equivalence relation on the set of lines;

000	011	022	033	044
104	110	121	132	143
203	214	220	231	242
302	313	324	330	341
401	412	423	434	440

Example 4.1 A 3-net of order 5.

In the example above, the different parallel classes are represented by the first, second, and third digits of each point. The different lines within each parallel class are labeled 0, 1, 2, 3, and 4.

Note that each parallel class forms a partition of the points in the net, so that each point is thus on k lines in the k-net, one in each parallel class.

It can be shown that a k-net of order n is combinatorially equivalent to k-2 $MOLS(n)$ [1].

Then as $N(10) \leq 6$, there can be at most an 8-net of order 10. Since $N(10) \geq 2$, there is a 4-net of order 10. In this paper we will focus our efforts on the possible existence of a 6-net of order 10, which would be equivalent to four $MOLS(10)$.

Definition 4.2. A **subnet** M of a net N consists of the same points as N and some of the parallel classes of N.

Definition 4.3. Given a k-net N_k of order n, label the points of N_k as $p_1, p_2, \ldots, p_{n^2}$. The **characteristic function** of a line in N_k is the vector in $F_2^{n^2}$ whose ith component is 1 if p_i is on the line, and 0 otherwise. Let $C_2(N_k)$ be the vector space over F_2 generated by the characteristic functions of the lines in N_k. Let $D_2(N_k)$ be the vector space over F_2 generated by the differences of characteristic functions of lines $l - m$, where l and m are lines in the same parallel class.

$C_2(N_k)$ and $D_2(N_k)$ are examples of **linear binary codes**:

Definition 4.4. A **linear q-ary code of length** n is a subspace of F_q^n. A **codeword** is a vector lying in the code.

Definition 4.5. The **dimension** of a code W, denoted $dim\ W$, is the dimension of W as a vector space.

Definition 4.6. Given a linear binary code W of length n, the **dual code** $W^\perp$ of W is the orthogonal complement $\{x \in F_2^n | \ [x, w] = 0 \ \forall \ w \in W\}$, where $[x, w]$ is the dot product over F_2^n.

We can now begin anaylzing k-nets of order n through the use of $C_2(N_k)$ and $D_2(N_k)$.

Lemma 4.1. *A 1-net of order n has*

$$dim\ C_2(N_k) = n,\ dim\ D_2(N_k) = n - 1.$$

Proof. By definition, $C_2(N_1)$ is generated by the characteristic functions of n parallel lines. These vectors must then be linearly independent, and so form a basis for $C_2(N_1)$, which thus has dimension n. To see the second equality, note first that $D_2(N_1) \subseteq C_2(N_1)$ (as $D_2(N_1)$ is constructed from elements of $C_2(N_1)$). Let $\{a_1, a_2, \ldots, a_n\}$ be the characteristic functions of the lines of N_1. Then $a_1 \in C_2(N_1)$ but $a_1 \notin D_2(N_1)$, and so dim $D_2(N_1) <$ dim $C_2(N_1) = n$. However, the set $\{a_1 + a_2, a_1 + a_3, \ldots, a_1 + a_n\}$ is linearly independent, and all the elements are inside $D_2(N_1)$. Hence dim $D_2(N_1) \geq n - 1$, and so dim $D_2(N_1) = n - 1$. □

The next proposition requires the definition of a **transversal** of a net.

Definition 4.7. A **transversal** of a net of order n is a set of n points such that each line in the net has exactly one point in common with the transversal.

For example, in the 3-net of order 5 in Example 4.1, the set of points $\{000, 143, 231, 324, 412\}$ is a transversal.

Proposition 4.2. *Let N_k be a k-net of order n, where n is even. If k is even or N_k has a transversal, then dim $C_2(N_k)-$dim $D_2(N_k) = k$.*

Proof. First, let m_i be a line in the ith parallel class. Then $C_2(N_k) = \langle m_1, m_2, \ldots, m_k, D_2(N_k)\rangle$, and so dim $C_2(N_k) -$ dim $D_2(N_k) \leq k$. The result holds true for $k = 1$ by Lemma 4.1. For $k \geq 2$, it suffices to show that $\{m_1, m_2, \ldots, m_n\}$ are linearly independent over $D_2(N_k)$. Note that since n is even, $[x, x] = 0\ \forall x \in C_2(N_k)$. By definition of N_k, $[x, y] = 1\ \forall x$, y not in the same parallel class, and $[x, y] = 0\ \forall x, y$ in the same parallel class. This means that $[x, y-z] = 0\ \forall x,\ y,\ z \in C_2(N_k)$ where y, z are in the same parallel class, and so $D_2(N_k) \subseteq C_2(N_k)^{\perp}$. Now suppose that $w = a_1m_1 + a_2m_2 + \ldots + a_km_k \in D_2(N_k) \subseteq C_2(N_k) \cap C_2(N_k)^{\perp}$. Let l_j be a line in the jth parallel class. Then $0 \equiv [l_j, w] \equiv \sum_{i\neq j} a_i \pmod 2$ (since $[l_j, m_j] = 0$). As this is true for each parallel class, $(k-1)\sum_{i=1}^{n} a_i \equiv 0 \pmod 2$.

If k is even, $k - 1 \equiv 1 \pmod 2$, and so $\sum_{i=1}^{n} a_i \equiv 0 \pmod 2$. Since $\sum_{i\neq j} a_i \equiv 0 \pmod 2$ $\forall 1 \leq j \leq k$, we have that $a_1 \equiv a_2 \equiv \ldots \equiv a_k \pmod 2$, as required.

If instead N_k has a transversal t, by definition, $[t, l] \equiv 1 \pmod 2\ \forall l \in C_2(N_k)$. Then $[t, l - m] \equiv 0 \pmod 2\ \forall l,\ m \in C_2(N_k)$, and so $t \in D_2(N_k)^{\perp}$. Hence $[t, w] \equiv 0 \pmod 2$, and so $\sum_{i=1}^{n} a_i \equiv 0 \pmod 2$. The argument then follows as above. □

This proposition allows us our first main result about N_6 of order 10.

Proposition 4.3. *If a net N_6 of order 10 exists, dim $C_2(N_6) \leq 53$.*

Proof. Suppose N_6 is a net of order 10. Let dim $C_2(N_6) = 53 + \alpha$, for some integer α. Since 6 and 10 are even, by Prop. 4.2, dim $D_2(N_6) = 47 + \alpha$. Since $C_2(N_6), D_2(N_6) \subseteq F_2^{100}$, by the rank-nullity theorem, dim $D_2(N_6)^{\perp} = 53 - \alpha$. By the proof of Prop. 4.2, $D_2(N_6) \subseteq C_2(N_6)^{\perp}$, and so $C_2(N_6) \subseteq D_2(N_6)^{\perp}$ (by definition of orthogonal complements). We thus have that $53 + \alpha \leq 53 - \alpha$, and so $\alpha \leq 0$. □

Thus, if it can be shown that any net N_6 of order 10 must contain an arrangement of lines such that the space spanned by those lines has dimension of at least 54, then N_6 cannot exist, and so $N(10) \leq 3$.

Proposition 4.3 is our first main result, and will motivate much of the work to come in this paper. There are, however, some weaker corollaries that are worth noting (these mostly follow from Prop. 4.2 and the proof of Prop 4.3):

Corollary 4.4. *[1] If a collection of lines N from a net N_7 of order 10 satisfies dim $C_2(N) \geq 54$, there are at most five $MOLS(10)$.*

Corollary 4.5. *[1] If a collection of lines N from a net N_8 of order 10 satisfies dim $C_2(N) \geq 55$, there are at most five $MOLS(10)$.*

To go further, we need the use of **relations** over the net N_k.

Definition 4.8. A **relation** in a net is a subset of the lines of the net such that every point in the net lies on an even number of lines in the relation. This is equivalent to a linear dependency on the lines' characteristic functions over F_2. Parallel classes are **associated** with the relation only if they provide a nonzero number of lines to the relation. A relation is **trivial** if it consists of an even number of entire parallel classes.

Definition 4.9. A **configuration** is a relation with the lines of one the parallel classes (which contributes a nonzero number of lines to the relation) deleted.

Definition 4.10. The **weight** of a parallel class with respect to a relation is the number of lines that class contributes to the relation. The **type** of a relation or configuration is a set of the non-zero weights of the parallel classes in that relation, arranged in ascending order. The **weight** of a point with respect to a relation or configuration is the number of lines in that relation or configuration the point lies on.

For example, suppose that we had a net N_6 of order 10, and N_6 contained a relation consisting of 2 lines in each of three different parallel classes, 4 lines in another parallel class, and 6 lines in a fifth parallel class. Then the type of this relation would be $\{2, 2, 2, 4, 6\}$. Depending on which parallel class we removed from the relation, we could obtain configurations of type $\{2, 2, 2, 4\}$, type $\{2, 2, 2, 6\}$, or type $\{2, 2, 4, 6\}$.

Proposition 4.6. *Let N_k have jth parallel class $\{l_1^j, l_2^j, \ldots, l_n^j\}$, and suppose that N_k contains a non-trivial relation $\sum_{i=1}^n a_i^k l_i^k + \sum_{i=1}^n a_i^{k-1} l_i^{k-1} + \ldots + \sum_{i=1}^n a_i^1 l_i^1 = 0$ over F_2, where $a_i^j \in \{0, 1\}$. Let $a^j = |\{a_i^j \mid a_i^j = 1\}|$ (i.e. the weight of the jth parallel class). If $\exists j$ such that a^j is odd, then dim $C_2(N_k) - dimD_2(N_k) < k$.*

Proof. Relabel the parallel classes as necessary so that l_1^j has nonzero coefficient for each class in the relation with odd a^j. As the relation is over F_2, we can rewrite it as $\sum_{a^j \text{ odd}} l_1^j = \sum a_i^k l_i^k + \sum a_i^{k-1} l_i^{k-1} + \ldots + \sum a_i^1 l_i^1$, where the summations on the right hand side are for $2 \leq i \leq n$ for classes that had odd weight and for $1 \leq i \leq n$ for classes that had even weight. Since each parallel class on the right hand side now contains an even number of lines, these lines can be paired up within the classes. As the relation is over F_2, this means that the right hand side is a sum of elements in $D_2(N_k)$, and so is an element in

$D_2(N_k)$. Thus, the left hand side must also be an element in $D_2(N_k)$. From the proof of 4.2, $C_2(N_k) = \langle l_1^1, l_1^2, \ldots, l_1^k, D_2(N_k)\rangle$, and so dim $C_2(N_k) - \text{dim}D_2(N_k) < k$ (As the l_1^i are not linearly independent of $D_2(N_k)$). $\square$

These relations are important, as each relation in the net N_6 corresponds to a linear dependency in the row space of $C_2(N_6)$. Recall from Prop. 4.3 that (for order 10) we have dim $C_2(N_6) \leq 53$. Since there are 60 lines in N_6 of order 10, this means that there are at least 7 relations in N_6. Analysing what these relations could look like will comprise the next couple of sections.

5 Determining Relations in N_6

In the last section, we determined that a net N_6 or order 10 would need to contain at least 7 relations. In this section, we look at what form these relations could take.

We begin with a couple of simple results which will drastically cut down the search space:

Proposition 5.1. *Every parallel class has even weight in any non-empty relation in a net N_6 of order 10.*

Proof. Since 6 and 10 are even, by Prop. 4.2, dim $C_2(N_6) - \dim D_2(N_6) = 6$. Also, since non-parallel lines meet at exactly one point, the points on any line in the last parallel class forms a transversal for the subnet formed from the first i parallel classes, $1 \leq i \leq 5$. Then by Prop. 4.2, dim $C_2(N_i) - \dim D_2(N_i) = i, 1 \leq i \leq 5$. By Prop. 4.6, all the weights must be even. □

Since a given relation contains a subset (possibly trivial) of the lines in each parallel class, we can consider the complement of this subset with respect to the parallel class. For example, if a relation contains four lines in a given parallel class, then the complement would be the six lines not included in the relation. We will describe the act of replacing the set of lines in a relation within a given parallel class with their complement with respect to that parallel class as **complementing the class**.

Proposition 5.2. *All parallel classes, except the last one, can be assumed to have weight zero, two, or four in any non-empty relation in a net N_6 of order 10.*

Proof. Suppose some parallel class other than the last one has weight six, eight, or ten. If we complement both this class and the last class, we obtain another relation (as we have essentially added the 1 vector twice, which equals 0 over F_2). This reduces the weight of the class in question to zero, two, or four. □

With this, we can therefore restrict our search to relations which have zero, two, or four lines in each of the first five parallel classes.

Note that, of the at least seven distinct relations, five are easy to find: if the parallel classes of N_6 are $\{B_1, B_2, B_3, B_4, B_5, B_6\}$, then taking all of the lines from an even number of parallel classes forms a trivial relation. Such relations are spanned by the relations $\{B_1 \cup B_2, B_1 \cup B_3, B_1 \cup B_4, B_1 \cup B_5, B_1 \cup B_6\}$, meaning that such relations account for five of the at least seven distinct relations. We thus have at least two relations in N_6 which are non-trivial.

We begin by proving that any such non-trivial relation must contain lines from at least four parallel classes.

Proposition 5.3. *There is no non-trivial relation of type* $\{a, b, c\}$ *in a net* N_6 *of order 10.*

Proof. Suppose that there is a non-trivial relation of type $\{a, b, c\}$. Since the relation is non-trivial, at least one of a, b, or c is not 0 or 10. Without loss of generality, say that $0 < a < 10$. Label the three parallel classes B_1, B_2, and B_3, where B_1 is the parallel class which has a lines in the relation.

By definition of the relation and since every point appears on exactly one line in each parallel class, every point in the relation must appear on exactly 2 lines. Since every line from B_1 intersects every line from B_2 or B_3 at exactly one point, $b + c = 10$ (we cannot have more than 10 lines between them, as any extra lines would have to intersect the line from B_1, which contradicts that the point appears on exactly 2 lines). Since some of the lines in B_1 are not in the relation, we get that $b = c$ (all the lines from B_1 which are not in the relation from intersect the ones that are from B_2 in the relation, and so those points must be matched by lines from B_3, and vice versa, because such points are on 2 lines). Thus $b = c = 5$, which contradicts Prop. 5.1. Hence, no such non-trivial relation can exist. □

By a similar argument, it is clear that any configuration resulting from a relation must have at least ten lines between all parallel classes in the relation.

Now, to analyze these non-trivial relations, we will consider the weights of each point in the net. Recall from Def. 4.10 that the **weight** of a point in a relation (resp. configuration) is the number of lines in the relation (resp. configuration) which that point is on. Also, recall that each point in the net is on one line in each parallel class. This means that the point can have weight at most six. If we form a configuration from the relation, the point can have weight at most five.

For a given relation or configuration in a N_6 of order 10, let Z, S, D, T, Q, K, H represent the number of points of weight zero, one, two, three, four, five, and six respectively.

We begin by determining a relationship between S, T, and K.

Lemma 5.4. *Given a relation in a net* N_6 *of order 10, let* s, t, *and* k *represent the number of weight one, three, and five points respectively that meet a line from the last parallel class. Then* $s + t + k = 10$ *and* $s + 3t + 5k = l$, *where* l *is the number of lines in the relation not in the last parallel class. Also, if there is a parallel class not included in the relation, then* $k = 0$.

Proof. Since the line in the last parallel class contains ten points which must each appear an odd number of times elsewhere in the relation, $s + t + k = 10$. Since the line in the last parallel class must meet every other line in the relation which is not in the last class exactly once, $s+3t+5k=l$. Since each point in a relation must appear on an even number of lines, if it does not appear on 6 lines, it must appear on at most 4, and so $k=0$. □

How this lemma relates S, T, and K is thus dependent on the relation in question. Of note is that we must always have that $10 \mid S+T+K$.

Now suppose we have a configuration in N_6. By Prop. 5.2, we can remove the only parallel class that might have weight greater than four, and so we need only consider configurations with weights zero, two, or four. Let p be the number of pairs of lines from different parallel classes which are each in the configuration, and let c be the number of pairs of lines from different parallel classes which are **not** in the configuration. For example, a configuration of type $\{x,y,z\}$ would have $p=xy+xz+yz$ and $c=(10-x)(10-y)+(10-x)(10-z)+(10-y)(10-z)$. We then obtain the following equations:

$$Z+S+D+T+Q+K=100$$

$$D+3T+6Q+10K=p$$

$$\frac{l(l-1)}{2}Z+\frac{(l-1)(l-2)}{2}S+\frac{(l-2)(l-3)}{2}D+\frac{(l-3)(l-4)}{2}T=c$$

where l is the number of parallel classes involved in the configuration. These equations come from points in the net, pairs of lines inside the configuration, and pairs of lines outside the configuration, respectively.

We cannot have a $\{2,2,2,2\}$ configuration, as it does not contain at least 10 lines. Hence, a $\{2,2,2,4\}$ configuration is the smallest possible configuration in four classes in N_6. If such a configuration exists, it must also exist in a subnet N_5 of N_6.

Now consider a $\{2,2,4,4\}$ configuration. By Lemma 5.4, we see that $s+t+k=10$ and $s+3t+5k=l=12$, so that $s=9, t=1$, and $k=0$. Hence, $K=0$ and $S=9T$ (since the lemma holds for every line in the deleted class). Also, note that as there are two parallel classes with two lines each, we must have that $Q \leq 4$ (as there are only four points of intersection between the lines in the first two parallel classes).

This results in the system of equations:

$$Z+S+D+T+Q=100$$

$$D+3T+6Q=52$$

$$6Z+3S+D=292$$

$$S = 9T$$

As Z, S, D, T, Q are all non-negative integers, the only two solutions are:

$$(Z, S, D, T, Q) = (12, 72, 4, 8, 4)$$

$$(Z, S, D, T, Q) = (25, 36, 34, 4, 1)$$

We can solve the system of equations to find any potential (Z, S, D, T, Q, K) for a configuration of any given type. Before listing the results, however, we consider a way to pair up results, to reduce the total number of cases to consider.

Since configurations are formed by deleting a parallel class from a relation, if a relation has classes of both weight two and four (and no other weights), we can produce a different configuration depending on the weight of the class we delete. If a relation contains a class of weight six or eight (which, by our constuction, must be the class deleted for the configuration), we can complement that class and another to produce a different relation, which can be considered analogous to the first relation. For example, classes of weight two and six can be complemented to classes of weight eight and four. This motivates the following definitions:

Definition 5.1. Any two configurations corresponding to the same relation are called **analogues**. Any configuration with itself as an analogue is called a **self-analogue**. Two configurations which come from relations which can be complemented to one another, as above, are called **complement analogues**.

It can be shown that any configuration with all classes the same weight has only itself as an analogue. Also, the solutions to the above system of equations shows that any relation with a class of weight eight must also contain a class of weight four, and any relation with a class of weight six must also contain a class of weight two. Thus, all configurations stemming from such relations must have at least one complement analogue. If a solution to the system of equations should have an analogue of some form which can't exist, then the original solution is not a valid relation in N_6.

By pairing solutions to the systems of equations above according to what solutions are some form of analogue of one another, we obtain a classification of the various potential relations in the net N_6: (Please note that this list is a summary of the results from various non-trivial analysis and computations, which are ommited for the sake of brevity)[1]

1. Configuration of type $\{4, 4, 4\}$ with $(Z, S, D, T) = (24, 36, 36, 4)$ is a self-analogue

2. Type $\{2,2,2,4\}$ with $(Z,S,D,T,Q) = (24,60,12,0,4)$ is a complement analogue of type $(2,2,4,4)$ with $(Z,S,D,T,Q) = (12,72,4,8,4)$

3. Type $\{2,2,4,4\}$ with $(Z,S,D,T,Q) = (25,36,34,4,1)$ is an analogue of type $\{2,4,4,4\}$ with $(Z,S,D,T,Q) = (25,16,54,4,1)$

4. Type $\{2,4,4,4\}$ with $(Z,S,D,T,Q) = (14,48,24,12,2)$ is a complement analogue of type $\{4,4,4,4\}$ with $(Z,S,D,T,Q) = (6,56,12,24,2)$

5. Type $\{4,4,4,4\}$ with $(Z,S,D,T,Q) = (15,28,42,12,3)$ is a self-analogue

6. Type $\{2,2,2,2,2\}$ with $(Z,S,D,T,Q,K) = (25,60,10,0,5,0)$ is a complement analogue of type $\{2,2,2,2,4\}$ with $(Z,S,D,T,Q,K) = (13,72,2,8,5,0)$

7. Type $\{2,2,2,2,2\}$ with $(Z,S,D,T,Q,K) = (40,20,40,0,0,0)$ is a self-analogue

8. Type $\{2,2,2,2,4\}$ with $(Z,S,D,T,Q,K) = (26,36,32,4,2,0)$ is an analogue of type $\{2,2,2,4,4\}$ with $(Z,S,D,T,Q,K) = (26,16,52,4,2,0)$

9. Types $\{2,2,2,4,4\}$ with $(Z,S,D,T,Q,K) = (15,48,22,12,3,0), (14,49,24,10,2,1), (13,50,26,8,1,2)$, and $(12,51,28,6,0,3)$ each have at least one complement analogue from types $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (7,56,10,24,3,0)$, $(6,57,12,22,2,1)$, $(5,58,14,20,1,2)$, and $(4,59,16,18,0,3)$

10. Type $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (16,28,40,12,4,0)$ is an analogue of type $\{2,4,4,4,4\}$ with $(Z,S,D,T,Q,K) = (16,12,56,8,8,0)$

11. Type $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (15,29,42,10,3,1)$ is an analogue of type $\{2,4,4,4,4\}$ with $(Z,S,D,T,Q,K) = (15,13,58,6,7,1)$

12. Type $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (14,30,44,8,2,2)$ is an analogue of type $\{2,4,4,4,4\}$ with $(Z,S,D,T,Q,K) = (14,14,60,4,6,2)$

13. Type $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (13,31,46,6,1,3)$ is an analogue of type $\{2,4,4,4,4\}$ with $(Z,S,D,T,Q,K) = (13,15,62,2,5,3)$

14. Type $\{2,2,4,4,4\}$ with $(Z,S,D,T,Q,K) = (12,32,48,4,0,4)$ is an analogue of type $\{2,4,4,4,4\}$ with $(Z,S,D,T,Q,K) = (12,16,64,0,4,4)$

15. Types $\{2, 4, 4, 4, 4\}$ with $(Z, S, D, T, Q, K) = (9, 36, 26, 24, 5, 0), (8, 37, 28, 22, 4, 1), (7, 38, 30, 20, 3, 2),$ (and $(4, 41, 36, 14, 0, 5)$ all have at least one complement analogue from types $\{4, 4, 4, 4, 4\}$ with $(Z, S, D, T, Q, K) = (5, 40, 10, 40, 5, 0), (4, 41, 12, 38, 4, 1), (3, 42, 14, 36, 3, 2), (2, 43, 16, 34, 2, 3),$ $(1, 44, 18, 32, 1, 4)$, and $(0, 45, 20, 30, 0, 5)$

16. Types $\{4, 4, 4, 4, 4\}$ with $(Z, S, D, T, Q, K) = (10, 20, 40, 20, 10, 0), (9, 21, 42, 18, 9, 1),$ $(8, 22, 44, 16, 8, 2), (7, 23, 46, 14, 7, 3), (6, 24, 48, 12, 6, 4), (5, 25, 50, 10, 5, 5), (4, 26, 52, 8, 4, 6),$ $(3, 27, 54, 6, 3, 7)$, and $(2, 28, 56, 4, 2, 8)$ are all self-analogues

This gives us a list of all possible relations in a net N_6 of order 10. We know that at least two of these must exist in any such N_6. In the next section, we will use these relations to make further deductions about the dimension of $C_2(N_6)$.

6 The Dimension's Lower Bound

We now know the possible relations in a net N_6 of order 10. In this section we will make use of these relations to find a lower bound on dim $C_2(N_6)$, as well as some other implications of higher orders of dim $C_2(N_6)$.

First, recall that there are at least seven linearly independent relations in a net N_6 of order 10, five of which are the trivial relations. Now suppose that there are, in fact, at least ten linearly independent relations, that is, that dim $C_2(N_6) \leq 50$. This means that there are five such non-trivial relations, S_1, S_2, S_3, S_4, and S_5.

Now let B_1 be an arbitrary parallel class in the net N_6. B_1 has some set (possibly empty) of lines in each relation, and some lines may be in multiple relations. Let T_i be the set of lines in B_1 which are in relation S_i and let $s_i = |T_i|, 1 \leq i \leq 5$. In order to properly assess the lines in B_1, we partition each T_i as follows:

$T_1 = A \cup A5 \cup B \cup B5 \cup C \cup C5 \cup D \cup D5 \cup E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5$

$T_2 = C \cup C5 \cup D \cup D5 \cup G \cup G5 \cup H \cup H5 \cup K \cup K5 \cup L \cup L5 \cup M \cup M5 \cup N \cup N5$

$T_3 = E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5 \cup I \cup I5 \cup J \cup J5 \cup K \cup K5 \cup L \cup L5$

$T_4 = A \cup A5 \cup C \cup C5 \cup E \cup E5 \cup G \cup G5 \cup I \cup I5 \cup K \cup K5 \cup M \cup M5 \cup O \cup O5$

$T_5 = A5 \cup B5 \cup C5 \cup D5 \cup E5 \cup F5 \cup G5 \cup H5 \cup I5 \cup J5 \cup K5 \cup L5 \cup M5 \cup N5 \cup O5 \cup X5$

(Note here that, for example, A and $A5$ are not related as subsets of B - we simply need to label 32 distinct regions, which is more than there are letters in the English alphabet.)

In this way, we partition B_1 as $A \cup A5 \cup \ldots \cup X \cup X5$, where each subset represents the lines that are in a specific combination of relations (for example, $G = T_2 \cap T_3 \cap T_4$, and so is the set of lines in B_1 and in S_2, S_3, and S_4). Here, X is the set of lines that are in none of the relations.

As any linear combination of relations is itself a relation, we can form new relations from $S_1, \ldots S_5$ by taking combinations of these five relations. This results in $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} =$ 26 new relations. We can use these relations for a system of linear equations. For each subset, let the lower case variable be the number of lines contained in the upper case region. For example, $a = |A|$ and $o5 = |O5|$.

Then each new relation has a number of lines in B_1 equal to the sum of 16 of the variables, found by summing up the variables in the component relations over F_2. For example, the relation given by $T_1 + T_2$ has he following number of lines in B_1:

$a + b + e + f + k + l + m + n + a5 + b5 + e5 + f5 + k5 + l5 + m5 + n5$

By Prop 5.1, each of these new sums must be one of $\{0, 2, 4, 6, 8, 10\}$. Also, since the 32 subsets partition B_1, we must have that $a + a5 + \ldots + x + x5 = 10$.

Proposition 6.1. *Suppose N_6 is a net of order 10. If dim $C_2(N_6) \leq 50$, then N_6 contains a relation not complementable to one of type* $\{4, 4, 4, 4, 4, 4\}$

Proof. Since dim $C_2(N_6) \leq 50$, we can use the construction above. If one of the S_i is not complementable to type $\{4, 4, 4, 4, 4, 4\}$, we are done. If not, then there is a parallel class B_1 such that $s_i = 6 \; \forall \; 1 \leq i \leq 5$. Consider all systems of equations described above where $s_i = 6$, the 26 quantities for the other relations are in the set $\{0, 2, 4, 6, 8, 10\}$, and $a + a5 + \ldots + x + x5 = 10$. For each such solution, of the 26 other relations, there is either one relation with 0 or 10 lines in B_1, or at least three relations with 2 or 8 lines in B_1. As the S_i are linearly independent, none of these new relations can be the zero relation, and so we must have a relation with at least one class of weight zero, two, eight, or ten. Such a relation cannot be comlpementable to one of type $\{4, 4, 4, 4, 4, 4\}$. □

A similar argument shows that if there is an N_4 of order 10 has five or more linearly independent non-trivial relations, at least one relation must not be complementable to one of type $\{4, 4, 4, 4\}$. This produces the following result:

Proposition 6.2. *Given a net N_4 of order 10, dim $C_2(N_4) \geq 33$.*

Proof. Suppose that N_4 contains at least five linearly independent non-trivial relations. Then by above, N_4 contains a non-trivial relation with a class of weight zero, two, eight, or ten. This contradicts the list of configurations from the last chapter, which shows that the only possible relation in N_4 is one complementable to one of type $\{4, 4, 4, 4\}$. As such, N_4 contains at most four linearly independent non-trivial relations, as well as the three trivial relations generated by the parallel classes. As there are 40 lines in N_4, we thus have that dim $C_2(N_k) \geq 40 - 4 - 3 = 33$. □

We can now use a simple result to extend this result about N_4 to one about N_6.

Proposition 6.3. *Let N_k be a net of order n, where n is even. If k is even or N_k has a transversal, then dim $C_2(N_k) >$ dim $C_2(N_{k-1})$.*

Proof. By Prop. 5.1, all parallel classes have even weight in any relation in N_k. Thus the characteristic function of any single line in the last parallel class of N_k must be linearly independent over $C_2(N_{k-1})$ (as otherwise we have a relation with a parallel class of weight one), and so dim $C_2(N_k) >$ dim $C_2(N_{k-1})$. □

Corollary 6.4. *If N_6 is a net of order 10, then dim $C_2(N_6) \geq 35$.*

Proof. Since N_6 exists, the subnet N_5 of N_6 must have a transversal (for example, the points of any line in the last parallel class of N_6 are such a transversal). Applying Prop. 6.3 with $k = 6$ and $k = 5$ along with Prop. 6.2 gives that dim $C_2(N_6) \geq$ dim $C_2(N_5) + 1 \geq$ dim $(C_2(N_4) + 1) + 1 \geq 33 + 1 + 1 = 35$. □

With this we have our second main result of the paper. We now know that $35 \leq$ dim $C_2(N_6) \leq 53$. While we cannot directly limit the range further, we can consider what happens if dim $C_2(N_k)$ is a specific number in the range:

Corollary 6.5. *If N_6 is a net of order 10 such that dim $C_2(N_6) < 47$, then there is a non-trivial relation in four or five classes in the net.*

Proof. Since 6 and 10 are even, by Prop. 6.3, dim $C_2(N_5) \leq$ dim $C_2(N_6) - 1 < 47 - 1 = 46 = 50 - 4$. Since there are 50 lines in N_5, this means that there is a nontrivial relation in the subnet N_5 of N_6, which must necesarrily be a relation in four or five classes. □

Proposition 6.6. *If N_6 is a net of order 10 such that dim $C_2(N_6) = 53$, then any net N_7 containing this net has no transversal.*

Proof. Since 6 and 10 are even, by Prop. 4.2, dim $D_2(N_6) =$ dim $C_2(N_6) - 6 = 53 - 6 = 47$. Since $D_2(N_6) \subseteq F_2^{100}$, by the rank-nullity theorem, dim $D_2(N_6)^{\perp} = 100 -$ dim $D_2(N_6) = 100 - 47 = 53$. By the proof of Prop. 4.3, $C_2(N_6) \subseteq D_2(N_6)^{\perp}$, and so $C_2(N_6) = D_2(N_6)$. By the proof of Prop. 4.2, any transversal t of N_6 has $t \in D_2(N_6)^{\perp} = C_2(N_6)$. Then the characteristic function of any line in the last parallel class of any N_7 containing N_6 is in $C_2(N_6)$, and so $C_2(N_7) = C_2(N_6)$. Then by Prop. 4.2, N_7 cannot have a transversal. □

Note that this means that this N_7 cannot be extended to an N_8. Hence if a net N_8 of order 10 exists, dim $C_2(N_6) \leq 52 \; \forall$ subnets N_6 of N_8.

We can also modify the argument used in Prop. 6.1 to the cases where there are three or four non-trivial relations in the net N_6 to produce the following result:

Proposition 6.7. *[1] If N_6 is a net of order 10 such that dim $C_2(N_6) = 51$ or 52, then N_6 contains a non-trivial relation not complementable to one of type $\{2, 2, 2, 2, 2, 2\}$.*

With this, we have found various restrictions we can place on the dimension of a potential N_6. In the next section, we will take the potential relations, listed in Section 5, and analyze them further.

7 Analyzing Structures of Relations

In Section 5, we discovered the list of possible configurations in a net N_6. While the list covered the weights of the points for that relation, it does not describe how these points are arranged among the parallel classes. In this section, we make use of decompositions of multipartite graphs into cliques in order to better understand the structure of these relations.

Definition 7.1. A **graph** $G = (V, E)$ is a set of vertices V and a set of edges E, where each edge is an unordered pair of vertices in V. The **order** of a graph $G = (V, E)$ is $|V|$.

Definition 7.2. A graph $G = (V, E)$ is **complete** if E consists of all possible pairs of distinct vertices of V. We use the notation K_n for the complete graph of order n. We call a complete graph a **clique** when it appears as a subgraph of a larger graph. A **complete multipartite graph** with parts $P_1, P_2, \ldots, P_r$ is a graph $G = (V, E)$ such that $P_1, P_2, \ldots, P_r$ form a partition of V, and an edge $e = \{v_i, v_j\} \in E$ if and only if $v_i \in P_i$ and $v_j \in P_j$ for some $i \neq j$. We use the notation $K_{n_1,n_2,\ldots,n_r}$ for the complete multipartite graph with parts of size $|P_i| = n_i$.

Definition 7.3. A **packing** of graphs $G_1, G_2, \ldots, G_r$ into a graph G is an assignment of a vertex in $V(G)$ to each vertex of $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_r)$ such that the edges of $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_r)$ are mapped into the edges of $E(G)$ injectively. A packing into a graph G is said to be a **decomposition** of G if it covers all the edges of G.

It can be shown that a relation or configuration in a net N_k of type $\{n_1, n_2, \ldots, n_k\}$ is combinatorially equivalent to a decomposition of the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ into cliques [1]. In this equivalence, the lines of the configuration or relation are vertices in the graph. Also, a point in the configuration or relation is a clique in the decomposition of the graph, and its weight is the order of the clique.

We can therefore see how the differently weighted points of a relation are arranged in the parallel classes in the net by seeing how the vertices of corresponding cliques are distributed among the corresponding parts of the complete multipartite graph.

Given a decomposition of a complete multipartite graph corresponding to a relation, let Q_i, P_i be the number of K_4 and K_2 respectively incident with any vertex of a part B where $|B| = i$. (Note that we only count the cliques coming from the decomposition) For this part B of the graph, let P be the number of edges of the graph incident with a vertex of that part, and let $C = 10i\alpha - P$, where α is the number of parts in the graph. In terms of the net, P is the number of pairs of lines in the relation with one line in the parallel class corresponding to B. C represents the number of pairs of lines where one line is in the relation and in the parallel class corresponding to B, while the other line is in a different parallel class and not in the relation. For example, for the complete multipartite graph $K_{2,4,4,4,6}$ corresponding to the relation of type $\{2,4,4,4,6\}$, we have that $P = 2(4+4+4+6) = 36$ and $C = 2(6+6+6+4) = 44$.

Proposition 7.1. *Given a relation in a net N_6 of order 10 and its representation as a decomposition of a complete multipartite graph, for each i the number of K_i incident with a vertex of each part of the graph is only dependent on the size of that part.*

Proof. We prove this using P and C, as defined above. For a relation in four classes, the only possible relation (of type $\{4,4,4,4\}$) gives $3Q_4 + D_4 = P = 48$ and $2D_4 = C = 72$ (by using the meaning of P and C in terms of the net). Solving gives $(Q_4, D_4) = (4, 36)$. For a relation in five classes, we have that $3Q_i + D_i = P$ and $Q_i + 3D_i = C$. Solving gives $(Q_i, D_i) = (\frac{3P-C}{8}, \frac{3C-P}{8})$. For a relation in six classes, we have that $5H + 3Q_i + D_i = P$ and $2Q_i + 4D_i = C$. Solving gives $(Q_i, D_i) = (\frac{4P-C}{10} - 2H, \frac{3C-2P}{10} + H)$. Since P and C depend on the relation in question and i, and H depends only on the relation, the result follows. □

For the next proposition we introduce Q_{iL} and D_{iL}, which are the number of K_4 and K_2 respectively which are incident with a given vertex in a part of size L.

Proposition 7.2. *Given a relation in a net N_6 of order 10 and its representation as a decomposition of a complete multipartite graph, the values Q_{iL} and D_{iL} depend only on the relation, the cardinality i of the part, and the number of K_6 incident with the vertex.*

Proof. In relations in four or five classes, there are no points of weight six. We then obtain the equations $3Q_{iL} + D_{iL} = \frac{P}{i}$ (from the number of edges adjacent to that vertex) and $Q_{iL} + D_{iL} = 10$ (as the line in the net has ten points, the vertex in the graph is on ten cliques). Since $P + C = 10i\alpha$, where α is the number of classes in the relation, we get that $Q_{iL} + D_{iL} = \frac{P+C}{i\alpha}$. Then by Prop. 7.1, we can see that $(Q_i, D_i) = (iQ_{iL}, iD_{iL})$, and so $(Q_{iL}, D_{iL}) = (\frac{Q_i}{i}, \frac{D_i}{i})$. For a relation in six classes, consider a specific vertex in the graph, and let h be the number of K_6 incident with that vertex. We then get that $5h + 3Q_{iL} + D_{iL} = \frac{P}{i}$ and $h + Q_{iL} + D_{iL} = 10$. As these equations are linearly independent, there is a unique solution, and the result follows. □

Using this construction, we can find the values of Q_i, D_i, and (if relevant) Q_{iL}, and D_{iL} for each of the relations found in Section 5. For instance, the relation listed as (6) of type

$\{2, 2, 2, 2, 2, 6\}$ has $(Q_6, D_6) = (0, 60)$, $(Q_2, D_2) = (4, 16)$, and its complement analogue of type $\{2, 2, 2, 2, 4, 8\}$ has $(Q_8, D_8) = (8, 72)$, $(Q_4, D_4) = (12, 28)$, and $(Q_2, D_2) = (8, 12)$.

Before listing our results for this section, we first introduce an additional set of equations which can further determine the structure of the relations.

Given a relation over six classes, let $P_1, P_2, P_3, P_4, P_5, P_6$ be the six parts in the corresponding decomposition of a complete multipartite graph (note that the labels need not be in any particular order). Let a be the number of edges between two of P_1, P_2, P_3, and P_4, and let b be the number of edges between P_5 and one of P_1, P_2, P_3, and P_4. Now consider the configuration produced when P_6 is deleted. Let α and β be the number of K_4 and K_3, respectively, in the configuration and incident with P_5. From how we defined a and b, we get that

$$3\alpha + 6(Q - \alpha) + \beta + 3(T - \beta) + 6K \leq a$$

$$3\alpha + 2\beta + 4K \leq b$$

Together, these give that $6Q + 3T + 6K - a \leq 3\alpha + 2\beta \leq b - 4K$ for $\alpha \leq Q$ and $\beta \leq T$ (which must hold by how we defined α and β). Since the K_3 in the configuration must be K_4 in the original relation, we have that $\beta + K \leq |P_5| \times |P_6|$ (since the left hand side counts all cliques adjacent to both P_5 and P_6 in the original decomposition, and in a decomposition each edge can only belong to one clique). As the configuration has α K_4 incident with P_4, it must have $Q - \alpha$ K_4 incident with all of P_1, P_2, P_3 and P_4. Because any K_4 in the original relation adjacent to P_5 is either a K_4 or a K_3 adjacent to P_5 in the configuration, we have that $\alpha + \beta = Q_{|P_5|}$. Finally, since $Q - \alpha$ is the number of K_4 adjacent to neither P_5 nor P_6, we must have that $Q - \alpha$ is no greater than H in the relation formed by complementing P_5 and P_6.

With this additional structure in our relations, it now becomes possible to rule out one of the possible relations from Section 5. From that list, there must be five K_4 and eight K_3 in the corresponding decomposition of the complete multipartite graph $K_{2,2,2,2,4}$.

Proposition 7.3. *There is no packing of five copies of K_4 and eight copies of K_3 into the complete multipartite graph $K_{2,2,2,2,4}$ which corresponds to a configuration of type $\{2, 2, 2, 2, 4\}$.*

Proof. Label the parts of the graph P_1, P_2, P_3, P_4, and P_5, where P_5 is the part of cardinality four. From before, we have that $Q_4 = 12$ in the original relation, and so if such a packing exists, then $\alpha + \beta = 12$. Also, $a = (2)(2)\binom{4}{2} = 24$ and $b = 4(2 + 2 + 2 + 2) = 32$. This gives us that $6(5) + 3(8) + 6(0) - 24 \leq 3\alpha + 2\beta \leq 32 - 4(0)$, and so $30 \leq 3\alpha + 2\beta \leq 32$. Since

$\alpha \leq Q = 5$ and $\beta \leq T = 8$, for the lower bound to hold, we must have that $\alpha = 5$ and $\beta = 8$. But then $13 = 5 + 8 = \alpha + \beta = 12$, a contradiction. □

Corollary 7.4. *There can be no relation of type* $\{2, 2, 2, 2, 4, 8\}$ *in a net* N_6 *of order 10.*

Proof. If such a relation exists, then by removing the parallel class of weight eight, we have that a configuration of type $\{2, 2, 2, 2, 4\}$ exists in the subnet N_5. Then by combinatorial equivalence, we must have the corresponding decomposition of the complete multipartite graph $K_{2,2,2,2,4}$ which includes five copies of K_4 and eight copies of K_3, contradicting Prop. 7.3. □

With these sets of equations, we can now extend the list from Section 5 by elaborating on the structures of the relations. Note that we can also determine what a line in a parallel class not in the relation can look like, as each such line must intersect all lines in the relation at one point, and there are 10 points on the line. This gives that $Q + D + Z = 10$ and $4Q + 2D = l$, where l is the number of lines in the relation.

We begin with the relations in four or five classes, as in these $k = 0$, which simplifies the systems of equations. In the following lists, when describing the various Q, a number indicates that the point is incident with that parallel class, while a $*$ indicates that the point is not incident with that parallel class. (This list is again a summary of the results from various non-trivial analysis and computations, which are omitted for the sake of brevity)[1]

1. Relation of type $\{4, 4, 4, 4\}$. Four Q of type 4444, the remaining pairs are covered by D. Each line in the fifth or sixth class meets one of: $(4Q, 6Z), (3Q, 2D, 5Z), (2Q, 4D, 4Z), (1Q, 6D, 3Z), (8D, 2Z)$.

2. Relation of type $\{2, 2, 2, 4, 6\}$. Four Q of type 2224*, the remaining pairs are covered by D. Each line in the sixth class meets one of: $(1Q, 6D, 3Z), (8D, 2Z)$. Complement analogue of type $\{2, 2, 4, 4, 8\}$. Twelve Q, four each of types 2244*, *2448, 2*448. Each line in the sixth class meets one of: $(2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)$.

3. Relation of type $\{2, 2, 4, 4, 4\}$. Five Q of types 2244*, 224*4, 22*44, 2*444, *2444, the remaining pairs are covered by D. Each line in the sixth class meets one of: $(3Q, 2D, 5Z), (2Q, 4D, 4Z), (1Q, 6D, 3Z), (8Q, 2Z)$.

4. Relation of type $\{2, 4, 4, 4, 6\}$. Fourteen Q, two each of types 2444*, 24*46, 244*6, and six of type *4446, the remaining pairs are covered by D. Each line in the sixth class

meets one of: $(4Q, 2D, 4Z), (3Q, 4D, 3Z), (2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)$. Complement analogue of type $\{4, 4, 4, 4, 8\}$. 26 Q with two of type 4444* and six each of types *4448, 4*448, 44*48, 444*8, the remaining pairs are covered by D. Each line in the sixth class meets one of: $(4Q, 4D, 2Z), (3Q, 6D, Z), (2Q, 8D)$.

5. Relation of type $\{4, 4, 4, 4, 4\}$. Fifteen Q, three each of types *4444, 4*444, 44*44, 444*4, 4444*, the remaining pairs are covered by D. Each line in the sixth class meets one of: $(5Q, 5Z), (4Q, 2D, 4Z), (3Q, 4D, 3Z), (2Q, 6D, 2Z), (1Q, 8D, 1Z), (10D)$.

6. Relation ruled out above

7. Relation of type $\{2, 2, 2, 2, 2, 2\}$. All possible pairs covered by D.

For relations in six classes, the existence of points of weight six means that we now have a range of possibilities for structures. It is also far more difficult to list the arrangement of the Q. Let $a, b, \ldots, o$ represent the number of Q of the following forms:

$$a:\ 1111**\ \ b:\ 111*1*\ \ c:\ 111**1\ \ d:\ 11*11*\ \ e:\ 11*1*1$$

$$f:\ 11**11\ \ g:\ 1*111*\ \ h:\ 1*11*1\ \ i:\ 1*1*11\ \ j:\ 1**111$$

$$k:\ *1111*\ \ l:\ *111*1\ \ m:\ *11*11\ \ n:\ *1*111\ \ o:\ **1111$$

Then $a + b + \ldots + o = Q$. Also, each type of Q has an upper bound by complementing the two nonadjacent classes. For example, a point of weight four of type a becomes a point of weight six in the relation formed by complementing the last two classes, and so $a \leq H$ where H is taken from the new relation. Finally, we will require that $j \geq n \geq o$ and $b \geq c$ (where possible), in order to try to minimize relations which are simply permutations of the parallel classes of other relations.

Since the number of possibilities is too great to list outright, we will simply provide the number of possibilities for each relation. (This list is again a summary of the results of various non-trivial analysis and computations, which are omitted for the sake of brevity)[1]

1. Relation of type $\{2, 2, 2, 2, 4, 4\}$ has analogues of types $\{2, 2, 2, 2, 6, 6\}$, $\{2, 2, 2, 4, 6, 8\}$, and $2, 2, 4, 4, 8, 8$, each having $0 \leq H \leq 2$. Solving for $a, b, \ldots, o$ gives 46 solutions.

2. Relations of type $\{2, 2, 2, 4, 4, 6\}$ have analogues of types $\{2, 4, 4, 6, 8, 8\}$, $\{2, 2, 4, 6, 6, 8\}$, $\{2, 2, 4, 4, 4, 8\}$, $2, 2, 2, 6, 6, 6$, and $\{2, 2, 2, 4, 4, 6\}$, each having $0 \leq H \leq 3$. Solving for $a, b, \ldots, o$ gives 266 solutions.

3. Relations of type $\{2, 2, 4, 4, 4, 4\}$ have analogues of types $\{4, 4, 4, 4, 8, 8\}$, $\{2, 4, 4, 4, 6, 8\}$, and $\{2, 2, 4, 4, 6, 6\}$, each having $0 \leq H \leq 4$. Solving for $a, b, \ldots, o$ gives 1546 solutions.

4. Relations of type $\{2, 4, 4, 4, 4, 6\}$ have analogues of types $\{4, 4, 4, 6, 6, 8\}$, $\{4, 4, 4, 4, 4, 8\}$, $\{2, 4, 4, 6, 6, 6\}$, and $\{2, 4, 4, 4, 4, 6\}$, each having $0 \leq H \leq 5$. Solving for $a, b, \ldots, o$ gives 5658 solutions.

5. Relations of type $\{4, 4, 4, 4, 4, 4\}$ have analogues of type $\{4, 4, 4, 4, 6, 6\}$ with $0 \leq H \leq 6$. Solving for $a, b, \ldots, o$ gives 470821 solutions.

This gives us a total of $6 + 46 + 266 + 1546 + 5658 + 470821 = 478337$ different possible structures for a relation in a net N_6 of order 10, of which only 7522 are not of type $\{4, 4, 4, 4, 4, 4\}$.

8 Conclusion

In this paper, we have been considering the possible existence of a net N_6 of order 10, which would imply the existence of four $MOLS(10)$. We have determined that should such a net exist, we would have that $35 \leq \dim C_2(N_6) \leq 53$, and that there would be at least two relations in the net, which have 478337 possible structures.

It is clear that these restrictions alone are insufficient to determine whether or not a net N_6 of order 10 exists. They do, however, suggest the possibility of using a computer search to answer this question conclusively.

Such a computer search would involve taking each potential structure for a relation R in a net N_6, determining the corresponding matrix $C_2(R)$, and then seeing if $C_2(R)$ could be extended to $C_2(N_6)$ for a net N_6. Such a search would resemble the one used by Lam, Thiel and Swiercz to show that there is no projective plane of order 10.

Is such a search currently feasible? The search by Lam, Thiel and Swiercz for weight 19 codewords involved taking 45 initial cases of a 6×19 matrix, expanding it to a 111×19 matrix, and then attempting to expand the number of columns [7]. These 45 initial cases resulted in 639624 possible 111×19 matrices, which is similar in scope to the number of possible structures of relations we obtained in the last section. The subsequent expansion of the matrix is also similar in scope. As such, if it can be shown that there are few possibilities for $C_2(R)$ for a given structure of relation R (up to permutation of row and column vectors), then such a search is entirely possible (especially considering the increase in computing power since 1989, when Lam, Thiel and Swiercz conducted their search). For example, there is only one possible $C_2(R)$ (up to permutation of row and column vectors) for a relation R of type $\{2, 2, 2, 2, 2, 2\}$.

If there are too many possibilities for $C_2(R)$ for a given structure of relation R, it may still be useful to conduct a search for all structures of relation except those of type $\{4, 4, 4, 4, 4, 4\}$. This limits us to 7522 possible structures, which should make for a feasible search unless the number of possible matrices is enormous for each structure. Assuming no net N_6 was found, this would then allow us to use the results of Section 6 to drastically limit the dimension of a potential net N_6 of order 10.

In either case, a computer search which would, at worst, drastically limit the scope of search for a net N_6 of order 10 does seem feasible, and worthy of consideration.

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