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One-Relator Groups

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1 Introduction

As a finitely presented and a relatively simple case of the category of groups, one relator groups have been the subject of extensive research and discussion.

In regard to this research, three important algorithmic problems with respect to groups, posed by Max Dehn in 1912, are the most significant questions with regard to combinatorial group theory:

1. **Word Problem**: Is there an algorithm that, given a group G with a finite presentation $< X|R >$ and a non-trivial word $w \in (X^{\pm})^*$, can determine if $w = 1_G$?
   (a) **Generalized Word or Membership Problem**: Is there an algorithm that, given a finite presentation $< X|R >$ of a group G, a finite set $S = \{h_1, \ldots, h_m\}$, and a word $w \in (X^{\pm})^*$, can determine if $w \in H =< S >$?
2. **Conjugacy Problem**: Does there exist a finite algorithm such that for all words $x, y \in (X^{\pm})^*$, and a group G with finite presentation $< X|R >$, one can determine if x and y are conjugate in G?
   (a) **Conjugacy Problem with Constraint**: Let J be a subgroup of A. Does there exists a finite algorithm to determine, for any $g, h \in A$, if there exist $j \in J$ such that $j^{-1}gj = h$?
3. **Isomorphism Problem**: Is there an algorithm, given finite presentation of two groups G and H, that can determine whether G or H are isomorphic?

Note that all three problems are unsolvable for finitely presented groups in general. The focus of this text will be the discussion of the first two of Dehn's Problems with respect to one relator groups.

Definition 1.1. Let G be a group. We say that G is a **one-relator group** if $G =< X|r >$ for some generating set X and $r \in< X >$. We say that G is a **one-relator group with torsion** if $G =< X|r >$ with $r = u^m$ for some $u \in F(X), m \in \mathbb{Z}$.

First we deal with free groups, setting notation and context, then we move onto free products before discussing Higman-Neumann-Neumann extensions as a useful combinatorial tool for the discussion of one-relator groups. Consequentially the purpose of this expository paper will be to provide proofs for the following theorems.

Theorem 1. (Freiheitssatz) Let $G =< a_1, a_2, \ldots |r >$, where r is cyclically reduced, for a subset $L = \{a_{i_1}, a_{i_2}, \ldots\}$ that omits a generator from r, $< L >$ is a free subgroup of G.

Theorem 2. The word problem is solvable for groups defined by a one-relator presentation.

Theorem 3. The conjugacy problem is solvable with respect to one-relator groups with torsion.

The proofs of these theorems will be done by induction on the lengths of the relators. To do so, we will make use of free products and an amalgamated form of free products, Higman-Neumann-Neumann extensions or more commonly known as HNN extensions, to be able to reformulate one relator groups into groups that inductively satisfy our propositions. We will also make use of the malnormality of subgroups of one-relator groups with torsion to be able to seperate one-relator groups into a set of smaller subgroups that do not commute to prove the conjugacy problem with respect to one-relator groups with torsion.

We hope that the reader, bolstered with the knowledge of the theorems and enlightening examples, can come away with a solid understanding of the fundamental structure of one-relator groups.

1.1 Free Groups

Let X be a set, we call elements $x \in X$ **generators**. Let $X^{-1} = \{x^{-1} | x \in X\}$ be the set of inverses for X, and $X^{\pm 1} = X \cup X^{-1}$ be defined as the **alphabet** with its elements defined as **letters**.

For a set A and $n \in \mathbb{N}$, define A^n as the set of n-tuples whose elements are elements of A, where $A^1 = A$ and $A^0 = \{1_A\}$ (we write these n-tuples as $a_1...a_n, a_i \in A$). Let $w = w_1...w_n \in A^n$ and $s = s_1...s_k \in A^k$. The **concatenation** of w and s is the $n+k$-tuple $w_1...w_n s_1...s_k$, denoted as ws. With this operation, the closed set for this operation would include tuples of any finite length. Thus, let $A^* = \cup_{i \geq 0} A^i$ be the **Kleene star** of A and any element $w \in A^*$ be a **word** in A, where 1_A is the **empty word**. For a word w, a word a is a **subword** of w if there exists a **prefix word** b and a **suffix word** c such that $w = bac$. Note that the prefix or the suffix word may be the empty word.

We call a word $w \in (X^{\pm 1})^*$ **reduced** if w does not contain a subword aa^{-1} or $a^{-1}a$. Note that any word in $(X^{\pm 1})^*$ can be reduced by simply removing any instances of such subwords. Furthermore, no matter the order in which these subwords are removed, the full reduction will result in the same reduced word.

However, the operation of concatenation on the set of reduced words is not closed as the concatenated word may not be reduced. So, we redefine this operation $*$, where $a * b$ is the reduced word resulting from the concatenation of a and b (we will denote this as ab). By our construction, we have both a unique inverse for every element and an identity element. As well, given that disjoint elementary reductions are not affected by the order in which they are eliminated, and joint elementary reductions result in the same reduced word, concatenation is associative. Thus, the set of reduced words with concatenation is a group. We denote this group as $< X >$, also denoted as $F(X)$.

Definition 1.2. A group G is said to be a **free group** if there exists a (possibly infinite) set X such that $G = F(X)$ is equal to the set of reduced words in $(X^{\pm 1})^*$. We say that G is **freely generated** by X and that X is a **free basis** for G.

Note that, by our construction, we have shown that there is a finite algorithm that, given a word w in $(X^{\pm})^*$, determines if $w = 1_G$.

Theorem 1. The word problem is solvable with respect to any free group.

REMARK: More precisely, there is an algorithm that, for a given finite presentation $< X >$, and $w \in (X^{\pm})^*$ can determine whether $w = 1$ in $< X >$.

Example 1.1. If $|X| = 1$, $X = \{a\}$ and $< X >= \{\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots\}$ and thus $< a >$ is the infinite cyclic group, which implies $< a >\cong (\mathbb{Z}, +)$. For any word $w \in (X^{\pm})^*$, $w = 1$ in $< X >$ if and only if the sum of the exponents of a in w is 0.

Note that, in a free group, a free basis may not be unique.

Example 1.2. Let $H =< a, b >=< X >$, the free group on two letters. Let $G =< a, ab >=< Y >\leqslant H$. We wish to show $H \leqslant G$.

Define $\sigma : H \to G$ inductively on the length of a word $w \in H$ as $\sigma(a) = a$, $\sigma(b) = b$ (which is decomposed as $a^{-1}ab$ in G), and $\sigma(xw) = \sigma(x) * \sigma(w), x \in \{a, b\}$. As well, for $y, z \in H$, $\sigma(yz) = \sigma(y_1) \ldots \sigma(y_r)\sigma(z) = \sigma(y)\sigma(z)$.

Let $c = c_1...c_n, d = d_1...d_m \in H$ where $c_i \in X, d_i \in Y \forall i$. We wish to show that σ is one-to-one. Then $c = d$ if and only if $m = n$ and $\forall i, c_i = d_i$ if and only if $m = n$ and $\forall i, \sigma(c_i) = \sigma(d_i)$ if and only if $\sigma(c) = \sigma(d)$. So σ is both well-defined and one-to-one (and thus a one-to-one homomorphism), and thus $G \leqslant H$.

Definition 1.3. Given a free basis X of a free group G, we define the **length function** for a reduced word w, $|\cdot|_X : G \to \mathbb{N} \cup 0$ as $|w|_X = |x_1...x_n|_X = n, x_i \in X$, where $|1_G|_X = 0$.

Example 1.3. In this example, we show that the length of an element depends on the generating set set. By example 1.2, we know that the sets $X = \{a, b\}$ and $Y = \{a, ab\}$ generate the same group. Thus, for $w = b$, $|w|_X = |b|_X = 1$ and $|w|_Y = |b|_Y = |a^{-1}(ab)|_Y = 2$.

With this length function, we can now focus on the subgroups of free groups. Given some $G \leqslant< X >$, what can we say about G?

Let U be some finite subset of a free group $< X >$. We call U **Nielsen-reduced** (or **N-reduced**) if the following conditions hold $\forall u_1, u_2, u_3 \in U$

1. $u_1 \neq 1_{<X>}$
2. If $u_1u_2 \neq 1_{<X>}$ then $|u_1u_2|_X \geq |u_1|_X, |u_2|_X$
3. If $u_1u_2u_3 \neq 1_{<X>}$ then $|u_1u_2|_X \geq |u_1|_X - |u_2|_X + |u_3|_X$

We wish to define a N-reduced basis for G. Assume $X^{\pm}$ is well-ordered, and thus a well-ordering $u \ll g, u, g \in < X >$ (for example, for $G = < a, b >$, $b^{-1} \ll a^{-1} \ll a \ll b$ is one ordering). Extending this order lexicographically to $< X >$, we let 1 be the smallest word, and for $h = g$, $h \prec gx$ for all $x \in X$.

For non-trivial $w \in < X >$, let the left half of a word w, denoted $L(w)$, be the prefix of w of length $\lfloor \frac{|w|_X+1}{2} \rfloor$. Let $m(h) = \min\{L(h), L(h^{-1})\}$ and $M(h) = \max\{L(h), L(h^{-1})\}$ and define the well-ordering $\prec$ as follows: for $h, g \in < X >$, $h \prec g$ if and only if $m(h) \ll m(g)$ or $m(h) \equiv m(g)$ and $M(h) \ll M(g)$. This definition implies that, for two word w, h, if w is longer than h than $h \prec w$.

Example 1.4. In this example, we wish to calculate the order of two elements in a group. Let $G = < a, b >$ and let $b^{-1} \ll a^{-1} \ll a \ll b$. Define $w = aababaa, x = baaabab \in G$. Then $L(w) = aaba$ and $L(x) = baaa$. As well, $L(w^{-1}) = a^{-1}a^{-1}b^{-1}a^{-1}$ and $L(x^{-1}) = b^{-1}a^{-1}b^{-1}a^{-1}$. Thus, $m(w) = L(w^{-1})$ and $m(x) = L(x^{-1})$ where $m(x) \ll m(w)$ because x^{-1} is prefixed by b^{-1} and w^{-1} is prefixed by a^{-1}, where $b^{-1} \ll a^{-1}$ in our well-ordering. So, by our definition as $b^{-1} \ll a^{-1}$, $b^{-1}p \prec a^{-1}q$ for all $p, q \in G$. And so, $w \prec x$.

The proofs below, unless otherwise stated, are adapted from Lyndon, Schupp (1977).

Theorem 2. Let $G \leqslant < X >$. Then define the group $G_g = < \{h \in G | h \prec g\} >$ for each $g \in G$. Let A be the set of all elements not contained within their respective groups as defined above: $A = \{g \in G - \{1_G\} | g \notin G_g\}$. Then A is a Nielsen-reduced free basis for G, and by extension G is a free subgroup of $< X >$.

Proof. First, we will prove that G is generated by A. Given $A \subseteq G$, the smallest group generated by A, which we denote as $Gp(A)$, is a subgroup of G. Let $g \in G - Gp(A)$ be non-trivial, g minimal with respect to $\prec$ within $G - Gp(A)$. Thus, for $h \in G$ with $h \prec g$, $h \in Gp(A)$. As well, since $g \in G_g$, $g = h_1 \dots h_n \in G_g$ where $h_i \in < A > \forall i$ as $h_i \prec g$, which implies $g \in Gp(A)$, a contradiction. So, $G = Gp(A)$.

Now we'll show A is Nielsen-reduced. By definition, $1_G \notin A$. For (2), if, for some $x, y \in A, xy \prec x, y$, there are three cases.

CASE 1: $x \prec y$ and $xy \prec y$, $y \notin A$ as $y \in < x, xy >$, a contradiction. **CASE 2**: $y \prec x$ and $xy \prec y$, then $x \notin A$ as $x \in < y, xy >$, a contradiction. **CASE 3**: $x \equiv y$ and $xy \prec y$. Then if $x \prec y$, $y \in < x, xy >$ and $x \notin A$. If $y \prec x$, $x \in < y, xy >$ and $y \notin A$. If $y \equiv x$, then $y = ap$, $x = aq$, where $|p| = |q|$. Thus, if $|xy| < |x|, |y|$, then q must freely reduce a, and thus $q = a^{-1}$, a contradiction as then $x = 1_G$.

Finally, to show (3), let $x, y, z \in A$ where $x = ap^{-1}$, $y = pbq^{-1}$, and $z = qc$, where $|xy| \geq |x|$ and $|yz| \geq |z|$ by the above. Thus, $|xyz| = |abc| = |x| - |y| + |z| + |b| > |x| - |y| + |z|$, unless $b = 1_G$. If so, by the above inequalities, $|p| = |q| \leq \frac{|x|}{2}, \frac{|z|}{2}$, and $p \neq q$. If $p \prec q$, then $yz = pc \prec z = qc$ and if $q \prec p$, then $xy = aq^{-1} \prec x = ap^{-1}$, a contradiction unless $p \equiv q$, which would imply $p = q$, a contradiction.

Given that A is a Nielsen-reduced basis, for any $w = w_1 \dots w_n, n > 0, w_i \in A \forall i$, where $w_i w_{i+1} \neq 1 \forall i$, then for a generating set P of G, $|w|_P \geq t$, and thus every non-trivial word w of $(A^{\pm})^*$ is an element of G. Thus, $G =< A >$. □

Now we turn our attention to groups that are not free, and ask whether we can represent said groups using their respective generators. Let W be a set and $h : W \to G$ be a function. The **universal property of free groups** states that there exists a unique homomorphism $\bar{h} :< W >\to G$ extended from h. Let S be a generating set of G, and define the natural map $f : S \to G$. Extend natural map f to a group homomorphism $\sigma :< S >\to G$.

We note that σ is surjective as S is a generating set for G. Thus, by the First Isomorphism Theorem, $< S > /ker(\sigma) \cong G$. To better define $< S > /ker(\sigma)$, we let R be the generating set of the normal closure of $ker(G)$ in $< S >$, $< R >^{<S>}= ker(G)$. Thus, we use $< S|R >$ as notation to define the group with generating set S where $ker(G)$ is the normal closure of R in S. We call such $< S|R >$ a **presentation** of a group G.

2 Free Products and Extensions

In this chapter we build the foundation for the theory of one-relator groups. To be able to prove the solvability of the word and conjugacy problems with respect to one-relator groups, we need to consider free constructions of groups.

2.1 Free Products

Given presentations for two groups, $A =< X|R >$ and $B =< Y|S >$, where $X \cap Y = \emptyset$, we define the **free product** of A and B, denoted $A * B$, as the group with presentation $< X \cup Y|R \cup S >$.

Example 2.1. $< x_1 > * \dots * < x_n >=< \{x_1\} \cup \dots \cup \{x_n\} >=< x_1, \dots, x_n >$

Example 2.2. We note that the free product is not equivalent to the Cartesian product. For example, $\mathbb{Z}\times\mathbb{Z}$, commonly denoted as $\mathbb{Z}^2$, has a minimal generating set of two elements: $(0, 1)$ and $(1, 0)$ from the Fundamental Theorem of Fintely Generated Abelian Groups. However, we note that $(0, 1)+(1, 0)-(0, 1)-(1, 0) = (0, 0)$. Written in word notation, we have that $aba^{-1}b^{-1} = 1_{\mathbb{Z}^2}$. However, $aba^{-1}b^{-1}$ is freely irreducible, and thus $\mathbb{Z}^2$ is not free and is thus not isomorphic to $\mathbb{Z} * \mathbb{Z}$.

Given this notation, we want, for any groups A and B, to be able to represent elements in $A * B$ uniquely.

For $w \in A*B$, we let its **reduced sequence** or **normal form** be $g_1, \dots, g_n$, where $w = g_0 \dots g_n$ and $\forall i, g_i \neq 1, g_i \in A$ or $g_i \in B$, and

$$g_0 \in A \implies g_{2i} \in B, g_{2i+1} \in A \forall i$$

$$g_0 \in B \implies g_{2i} \in A, g_{2i+1} \in B \forall i$$

Thus, we write w as a sequences of elements alternating between A and B.

Theorem 3. Any element $w \in A * B$ or a word in $(X^{\pm} \cup Y^{\pm})^*$ can be written as a reduced sequence.

Proof. The reduced sequence for the identity element is the empty sequence. Write w, non-trivial, as a freely reduced sequence of letters in $X^{\pm} \cup Y^{\pm}$. If w starts with a letter from $X^{\pm}$, then find the largest prefix g_1 of w that is a word in A, and let $w = g_1 w_1$. Similarly, if w starts with a letter from $Y^{\pm}$, then find the largest prefix g_1 of w that is a word in B, and let $w = g_1 w_1$. If $w_1 = 1_{A*B}$, then we are done. If not, if w_1 starts with a letter from $X^{\pm}$, then find the largest prefix g_2 of w_1 that is a word in A, and let $w_1 = g_2 w_2$. Similarly, if w_1 starts with a letter from $Y^{\pm}$, then find the largest prefix g_2 of w that is a word in B, and let $w_1 = g_2 w_2$. Continue until for some n where $w_n = 1_{A*B}$. The reduced sequence is $g_1, \ldots, g_n$. □

Theorem 4. (Normal Form Theorem for Free Products) Let A and B be two groups. The following statements are equivalent and true:

1. A and B are embedded in $A * B$ by the natural map. If $w = g_1 \ldots g_n \in A * B$, $n > 0$, where $g_1, \ldots, g_n$ is a reduced sequence, then $w \neq 1$

2. Each element can be written uniquely in normal form.

Proof. We will show the statements are equivalent. If each element can be written uniquely in normal form, then for $w = g_1 \ldots g_n \in A * B$, $n > 0$, $g_1 \neq 1$ and reduction is not possible, so $w \neq 1$. So, (2) $\implies$ (1).

To show that (1) $\implies$ (2), let $w = g_1 \ldots g_n = h_1 \ldots h_m$, where $g_1, \ldots, g_n$ and $h_1, \ldots, h_m$ are reduced sequences. Then $g_1 \ldots g_n h_m^{-1} \ldots h_1^{-1} = 1$, and thus $g_1, \ldots, g_n, h_m^{-1}, \ldots, h_1^{-1}$ is not a reduced sequence. As $g_1, \ldots, g_n$ and $h_1, \ldots, h_m$ are reduced sequences, this means that g_n and h_m^{-1} are in the same factor. However, $g_1, \ldots, g_n h_m^{-1}, \ldots, h_1^{-1}$ represents ww^{-1}, and thus is still not a reduced sequence, implying $g_n h_m^{-1} = 1$, and that $g_1 \ldots g_{n-1} = h_1 \ldots h_{m-1}$. By induction, we can continue this process of elimination until we reach the reduced sequence 1_{A*B}. This implies that $n = m$ and $g_i = h_i\ \forall i$. Thus, we have shown the two statements are equivalent.

Now we prove (1). Let W be the set of all reduced sequences from $A * B$, and $S(W)$ be the set of permutations of W. For $a \in A$, define a permutation $\bar{a} \in S(W)$ as follows. If $a = 1$, $\bar{a}$ is the identity. If $a \neq 1$, then

$$\bar{a}(g_1, \ldots, g_n) = (a, g_1, \ldots, g_n), g_1 \in B$$

$$\bar{a}(g_1, \ldots, g_n) = (ag_1, \ldots, g_n), g_1 \in A, ag_1 \neq 1$$

$$\bar{a}(g_1, \ldots, g_n) = (g_2, \ldots, g_n), ag_1 = 1$$

Note that $\overline{a^{-1}} = \bar{a}^{-1}$ (since $a = (a^{-1})^{-1}$, we only need to prove $\bar{a}^{-1} \circ \overline{a^{-1}} = 1_{S(W)}$: for $g_1 \in B$,

$$\bar{a}(\overline{a^{-1}}(g_1 \ldots g_n)) = \bar{a}(a^{-1}, g_1 \ldots g_n) = (g_1 \ldots g_n)$$

If $g_1 \in A$ and , $a^{-1}g_1 \neq 1$, then

$$\bar{a}(\overline{a^{-1}}(g_1 \dots g_n)) = \bar{a}(a^{-1}g_1 \dots g_n) = (g_1 \dots g_n)$$

If $g_1 \in A$, and $g_1 = a$, then

$$\bar{a}(\overline{a^{-1}}(g_1 \dots g_n)) = \bar{a}(g_2 \dots g_n) = (g_1 \dots g_n)$$

We can similarly define a permutation $\bar{b} \in W$ for $b \in B$. So we can define a function $\Sigma : A * B \to S(W)$ as $\Sigma(1) = 1_{S(W)}$, $\Sigma(a) = \bar{a}, a \in A$, $\Sigma(b) = \bar{b}, b \in B$, and then $\Sigma(xy) = \Sigma(x) \circ \Sigma(y)$:

$$\bar{b}(\bar{a}(g_1 \dots g_n)) = \bar{b}(a, g_1 \dots g_n) = (b, a, g_1 \dots g_n) = b\bar{a}(g_1 \dots g_n), g_1 \in B$$

$$\bar{b}(\bar{a}(g_1 \dots g_n)) = \bar{b}(ag_1 \dots g_n) = (b, ag_1 \dots g_n) = b\bar{a}(g_1 \dots g_n), g_1 \in A, a^{-1}g_1 \neq 1$$

$$\bar{b}(\bar{a}(g_1 \dots g_n)) = \bar{b}(g_2 \dots g_n) = \overline{ba}(g_1 \dots g_n), g_1 \in A, g_1 = a$$

Similarly, one can show this result for $\overline{ab}$. Furthermore, any relations in $A*B$ are reduced to the identity map. Thus, Σ is a homomorphism. Writing $w = g_1 \dots g_n$ in normal form, $n \geq 1$, $\bar{w}$ sends the empty sequence to $g_1 \dots g_n$, so $\Sigma(w) \neq 1$, implying $w \neq 1$. □

Now that we know that reduced sequences are unique, for a free product, we can define the **length** of an element $w \in A * B$ being the length of its reduced sequence. If $w = g_1 \dots g_n$ where $g_1, \dots, g_n$ is a reduced sequence, then $l(w) = n$. Note that $l(1_{A*B}) = 0$.

Example 2.3. Let $w = a^5b^2 \in < a, b >$. Then $|w| = 5 + 2 = 7$. However, for $w \in < a > * < b >$, the reduced sequence for w is a^5, b^2, and thus $l(w) = 2$.

Corollary 4.1. If $A =< X|R >$ and $B =< Y|S >$ are finitely generated groups, and the word problem is solvable with respect to both A and B, then the word problem is solvable with respect to $A * B$.

Proof. Let $w \in (X^{\pm} \cup Y^{\pm})^*$ be a reduced word. If w is the empty word then we are done. If not, write w as a reduced sequence $g_1, \dots, g_n$. If $n > 1$, then $w \neq 1$. If $n = 1$, then $w \in A$ or $w \in B$, and thus by the assumption, we can calculate whether or not $w = 1_{A*B}$. □

Definition 2.1. Let $w = g_1 \dots g_n$ be a word in $(X^{\pm})^*$. A **cyclic permutation** of w is any word of the form $g_i \dots g_n g_1 \dots g_{i-1}, i \in \{1, \dots, n\}$.

We say that w is **cyclically reduced** if every cyclic permutation of w is freely reduced. For a normal form $g_1, ..., g_n$, it is **cyclically reduced** if g_n and g_1 are in different factors or if $n \leq 1$. We note that a cyclic permutation of a cyclically reduced normal form is a normal form as g_n and g_1 are in different factors.

If the word problem is solvable for free products, a naturally arising question is whether the conjugacy problem is solvable for free products.

Lemma 5. In a free group, two cyclically reduced words are conjugate if and only if they are cyclic permutations of each other.

Proof. Let $g = g_1 \dots g_n, x = x_1 \dots x_m, h = h_1 \dots h_t \in < X >$ be freely reduced. Suppose $x = hgh^{-1}$. Without loss of generality, assume x and g are cyclically reduced (if not, let $x = px'p^{-1}$, $g = qg'q^{-1}$, x', g' cyclically reduced, where $x' = (p^{-1}hq)g'(p^{-1}hq)^{-1}$). Thus, for hgh^{-1}, its cyclic reduction has either h or h^{-1} cancel out entirely. Say, without loss of generality, $h_j = g_j \forall j \in \{1, \dots, n\}$, then there cannot be any suffix reduction (else g is not cyclically reduced, a contradiction). Thus $g_i \dots g_n h_t^{-1} \dots h_{i+1}^{-1}$ is a cyclic permutation of g that is conjugate to x, and thus g is a cyclic permutation of x.

For cyclic permutations $g_1 \dots g_n$ and $g_i \dots g_n g_1 \dots g_{i-1}$,

$$g_i \dots g_n g_1 \dots g_{i-1} = (g_1 \dots g_{i-1})^{-1} g_1 \dots g_n (g_1 \dots g_{i-1}) = a g_1 \dots g_n a^{-1}$$

. □

Naturally, given two words in $A * B$, if they are conjugate, then their respective cyclically reduced sequences can be shifted accordingly to match each other.

Theorem 6. Let A and B be two groups with solvable conjugacy problems.

1. Let $u = g_1 \dots g_n$ and $v = h_1 \dots h_m$ be conjugate elements of $A * B$. If $g_1, \dots, g_n$ and $h_1, \dots, h_m$ are cyclically reduced normal forms, then $m = n$. Furthermore, if $n > 1$, then the normal forms are cyclic permutations of one another, and if $n = 1$, then $u, v \in A$ or $u, v \in B$.

2. The conjugacy problem is solvable with respect to $A * B$.

Proof. For the first proposition, let $u = cvc^{-1}$. The proof will be by induction on $l(c)$. If $l(c) = 0$, then $u = v$ and the Normal Form Theorem proves the result. So, let the claim be true for $l(c) < k$.

For $l(c) = 1$, we split into two cases.

Case 1: $n > 1$. Then without loss of generality, say $c \in A$ and $g_1 \in B$ (if $g_1 \in A$, then we consider u^{-1} and v^{-1}). Then $ch_1 \dots h_m c^{-1} = g_1 \dots g_n$, however $ch_1 \dots h_m c^{-1}$ is not freely reduced as it is not cyclically reduced but $g_1 \dots g_n$ is. Thus, given $h_1 \dots h_m$ is cyclically reduced and by the above assumption, $h_1 = c$ and $h_1 \dots h_m$ is a cyclic permutation of $g_1 \dots g_n$.

Case 2: $n = 1$. Again, assume $c \in A$. As $n = 1$, $u \in A$ or $u \in B$ (without loss of generality, let $u \in A$). Then for $u = cvc^{-1}$, if $v \in A$ we are done. If not, there must be some prefix and suffix $\bar{v}$ and $\bar{v}^{-1}$ of v that contains a subword in B that is freely reduced in cvc^{-1}, a contradiction as that would imply $m > 1$ and is not cyclically reduced (unless $u \in B$ and there is no reduction).

For $l(c) > 1$, let $c = c_1 ... c_k, k > 1$ be a reduced sequence. So

$$c_1 ... c_k h_1 \dots h_n c_k^{-1} ... c_1^{-1} = g_1 \dots g_m$$

There must be reduction in $c_k h_1$ or $h_n c_k^{-1}$, else g is not cyclically reduced, a contradiction. Without loss of generality, say $h_n c_k^{-1} = 1$. Thus, by induction,

$c_1...c_{k-1}(c_k h_1 \dots h_{n-1})c_{k-1}^{-1}...c_1^{-1} = g_1 \dots g_m$ implies that $c_k h_1...h_{n-1}$ and g are cyclic permutations. However we note that, by the above, $h_n = c_k$, and so $h_n h_1...h_{n-1}$ and g are cyclic permutations, and thus h and g are cyclic permutations (as $h_n h_1...h_{n-1}$ and h are cyclic permutations).

The second proposition follow from the first. Recall that the word problem is solvable in A and in B because the conjugacy problem is solvable. □

2.2 Higman-Neumann-Neumann Extensions

We now turn our attention to the most fundamental construction for our work. Let G be a group, $A, B \subset G$ be subgroups, and let $\phi : A \to B$ be an isomorphism. The **HNN extension of G relative to A, B, and ϕ** is the group

$$G^* =< G, t | t^{-1}at = \phi(a), a \in A >$$

where $t \notin G$. G is called the **base** of G^*, t is called the **stable letter**, and A and B are the **associated subgroups**.

Extending from free products, we wish to be able to define elements of the group G^* uniquely. A sequence $g_0, t^{\epsilon_1}, g_1 \dots, g_{n-1}t^{\epsilon_n}$ where $g_n, g_i \in G$ and $\epsilon_i \in \{0, 1\}$ is called **reduced** if there does not exist a consecutive subsequence t, g_i, t^{-1} with $g_i \in B$ or t^{-1}, g_i, t with $g_i \in A$.

However, even with this reduction, two distinct reduced sequences may represent the same element in G^* (for $a \in A, b \in B$ where $t^{-1}at = b$, $at = tb$ and $t^{-1}a = bt^{-1}$; indeed, note that a, t and t, b are both reduced sequences). To ensure that each sequence represents a unique element in G^*, we choose right coset representatives of A and of B in G where 1_G is a representative for both subgroups.

Definition 2.2. A **normal form in** G^* is a reduced sequence $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n, n \geq 0$ where

1. $g_0 \in G$ is an arbitrary element;
2. If $\epsilon_i = -1$, g_i is a right coset representative for A;
3. If $\epsilon_i = 1$, g_i is a right coset representative for B.

REMARK: Unlike normal forms for free products, normal forms in HNN extension need not be from alternating coset representatives.

Lemma 7. Every element of G^* can be written in a normal form.

Proof. Consider the following operations, called **t-reductions**, for a word

$$w = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$$

that is freely reduced. We proceed as follows:

1. replace a subword tgt^{-1}, where $g \in A$, with $\phi(g) \in B$

2. replace a subword $t^{-1}gt$, where $g \in B$, with $\phi^{-1}(g) \in A$

If the obtained sequence starts with a t, we let $g_0 = 1 \in G$. We need to ensure conditions 2. and 3. of the normal form hold. If there is a subsequence t^{-1}, g where $g = wa, w \in A$, a is a right coset representative of A. We then replace $t^{-1}w$ with $\phi(w), t^{-1}, a$. Similarly, f there is a subsequence t, h, $h = wb, w \in B$, b a right coset representative of B, replace t, h with $\phi^{-1}(w), t, b$. Continue this process from right to left in the sequence until all such subsequences are replaced. □

Note that a sequence may be reduced in length through t-reductions but that the steps to ensure conditions 2. and 3. do not reduce the length of the sequence.

Example 2.4. Let $G =< a, b >$ be the free group and $G^* =< a, b, t|t^{-1}at = b^3 >$. Then for $H =< a >$, let the right coset representatives be any word in $b^{\pm}\{a^{\pm}, b^{\pm}\}^*$, and for $I =< b^3 >$, let the right coset representatives be any word that does not begin with a prefix of the form $b^{\pm 3}$. Let $w = t^{-1}a^3t^2abt^{-1}ab$. Going from right to left, $t^{-1}ab = b^3t^{-1}b$, and thus $w = t^{-1}a^3t^2ab^4t^{-1}b$. There is no reduction to be made for t^2ab^4, and $t^{-1}a^3t = b^9$ so $w = b^9tab^4t^{-1}b$. We see that b^9, t, ab^4, t^{-1}, b is a normal form for w.

Given this construction, we wish to prove that the normal forms for HNN extensions are unique for each word w. Similar for free products, we have such a property.

Theorem 8. (**The Normal Form Theorem for HNN extensions**) Let $G^* =< G, t|t^{-1}at = \chi(a), a \in A >$ be an HNN extension. The following are equivalent and true.

1. The group G is embedded in G^* by the natural map. The equality $g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n = 1, n \geq 1$ in G^* implies that the sequence $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ is not reduced.

2. **Britton's Lemma**: Every element $w \in G^*$ has a unique representation $w = g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n$, where $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ is a normal form.

Proof. We will first prove 1. and 2. are equivalent. Assume 2. It is clear G is embedded in G^* as the normal form sequence for $g \in G$ is g. Suppose $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ where $n > 1$ and $g_i \in G\forall$ is a reduced sequence. Then by the previous lemma, the normal form is $g_0', t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n'$ (if there is a reduction on the length of the sequence, then there exists some consecutive subsequence t, g_i, t^{-1} with $g_i \in B$ or t^{-1}, g_i, t with $g_i \in A$, a contradiction as we assumed the sequence was reduced). So, $g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n \neq 1$ because the normal form for the identity is the empty sequence.

Assume 1. holds and suppose

$$g_0t^{\epsilon_1} \dots t^{\epsilon_n}g_n = h_0t^{\delta_1} \dots t^{\delta_m}h_m$$

where both $g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$ and $h_0, t^{\delta_1}, \ldots, t^{\delta_m}, h_m$ are normal forms. Then $g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n h_m^{-1} t^{-\delta_m} \ldots t^{-\delta_0} h_0^{-1} = 1$. So the sequence

$$g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n h_m^{-1}, t^{-\delta_m}, \ldots, t^{-\delta_0}, h_0^{-1}$$

is not reduced. So, either $\epsilon_n = \delta_m = -1$ and $g_n h_m^{-1} \in A$ or $\epsilon_n = \delta_m = 1$ and $g_n h_m^{-1} \in B$. Without loss of generality, let $g_n h_m^{-1} \in B$, where we note that g_n and h_m are right coset representatives for B. Thus, $g_n = h_m$ and inductively, $m = n$, $g_i = h_i$, and $\epsilon_i = \delta_i$.

Having proved the equivalency of the two assertions, we only need to prove one of them. We wish to show that every normal form represents a nontrivial element of G^*. Let W be the set of all normal forms in G^* and $S(W)$ be the permutation group of W. We define the left action $\Psi : G^* \to S(W)$ by

$$\Psi(g)(g'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n) = gg'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n \forall g \in G$$

We wish to show that all defining relations in G^* will be mapped to the identity element in $S(W)$ and consequently that Ψ is well-defined.

For $\Psi(t)$, if $g'_0 \in B$ and $\epsilon_1 = -1$ then

$$\Psi(t)(g'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n) = \phi^{-1}(g'_0)g'_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g'_n$$

If $g'_0 \notin B$ or $\epsilon_1 = 1$

$$\Psi(t)(g'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n) = \phi^{-1}(b), t, h'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n$$

where $g'_0 = bh'_0$ where $b \in B$.

We also define $\Phi(t^{-1})$ as follows: for $g'_0 \in A$ and $\epsilon_1 = 1$,

$$\Psi(t^{-1})(g'_0, t^{\epsilon_1}, \ldots t^{\epsilon_n} g'_n) = \phi(g'_0)g'_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g'_n$$

If $g'_0 \notin A$ or $\epsilon_1 = -1$

$$\Psi(t^{-1})(g'_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g'_n) = \phi(a), t, h'_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g'_n$$

where $g'_0 = a * h'_0$ where $a \in A$.

We need to check that $\Psi(t^{-1})\Psi(t) = 1_{S(W)}$. There are 2 cases. Let $g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$ be a normal form.

CASE 1: $g_0 \in B$ and $\epsilon_1 = -1$. Then if $g_0 \neq 1$, $g_0 \notin A$ and $\epsilon_1 \neq 1$ as the sequence is in normal form. So

$$\Psi(t)(g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n) = \phi^{-1}(g_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n$$

Since g_1 is a coset representative and $\phi^{-1}(g_0) \in A$, the coset representative of $A\phi^{-1}(g_0)g_1$ is g_1. So

$$\Psi(t^{-1})(\phi^{-1}(g_0)g_1, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) = g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$$

. If $g_0 = 1$, then

$$\Psi(t)(g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n) = \phi^{-1}(1)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n = g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n$$

and

$$\Psi(t^{-1})(\phi^{-1}(1)g_1, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) = 1, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n = g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$$

CASE 2: If $g_0^{'} \notin B$ or $\epsilon_1 = 1$,

$$\Psi(t)(g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) = \phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$$

and thus as $\phi^{-1}(b) \in A$ and $\epsilon_1 = 1$,

$$\Psi(t^{-1})(\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n) = g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$$

In a similar way, one can show that $\Psi(t^{-1})\Psi(t) = 1_{S(W)}$. We now need to show that $\Psi(t^{-1}\phi^{-1}(b)t) = \Psi(b)$. Again, let $g_0', t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n'$ be a normal form.

CASE 1: If $g_0 \in B$ and $\epsilon_1 = -1$, then we note that g_1 is a coset representative for A, and thus unless $g_1 = 1$, $g_1 \notin A$. If $g_1 = 1$, then $\epsilon_2 = -1$ else the sequence is not a normal form and $\theta^{-1}(1_G) = 1_G$. So,

$$\Psi(t)(g_0', t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n') = \phi^{-1}(g_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n'$$
$$\Psi(\phi^{-1}(b))(\phi^{-1}(g_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n') = \phi^{-1}(bg_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n'$$
$$\Psi(t^{-1})(\phi^{-1}(bg_0)g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n') = bg_0, t^{-1}, g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n'$$

CASE 2: In the other case,

$$\Psi(t)(g_0', t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n') = \phi^{-1}(b'), t, \hat{g_0'}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n'$$
$$\Psi(\phi^{-1}(b))(\phi^{-1}(b'), t, \hat{g_0'}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n') = \phi^{-1}(bb'), t, \hat{g_0'}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n'$$
$$\Psi(t^{-1})(\phi^{-1}(bb'), t, \hat{g_0'}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n') = bb'\hat{g_0'}, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n'$$

where $g_0' = b'\hat{g_0'}$.

By the above, we have that $\Psi(gh)(m) = \Psi(\Psi(h))(\Psi(g, m))$. So Ψ is a left group action from G^* to $S(W)$. As well, note that for a normal form $g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$,

$$\Psi(g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n)(1) = g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$$

so the product of elements in distinct normal forms represent distinct elements in G^*. □

Corollary 8.1. Let $G^* =< G, t | t^{-1}at = \phi(a), a \in A >$ be an HNN extension. Suppose that G has a solvable word problem, the generalized word problems for A and B with respect to G are solvable, and there exists an algorithm such that $\phi(a)$ and $\phi^{-1}(b)$ can be calculated in finite time $\forall a \in A, b \in B$. Then G^* has a solvable word problem.

Proof. Let $w = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n \in G^*$. Again, consider t-reductions. If $n > 0$, by Britton's Lemma, the reduced $w' \neq 1$ implies that $w \neq 1$. And if $n = 0$ then $w = 1$ in G^* only if $w = 1$ in G. Thus, there is an algorithm to complete the t-reductions of a given sequence if there is an algorithm to tell whether an element in G is in A or B and if we can calculate ϕ and ϕ^{-1} effectively.

So, given a sequence $w \in G^*$, we need a solvable generalized word problem for G^* with respect to A and B to tell effectively whether each g_i is in A or B to be able to reduce the sequence further. □

Example 2.5. Let

$$G^* =< a, b, t | t^{-1} bab^{-1} t = aba^{-1} >$$

and let $w = b^2 ab^{-1} t^{-1} a^{b2} tab^3 a^{-1} t^{-1}$. Is $w = 1$?

$G =< a, b >$ is the free group, and thus has a solvable word problem, and with respect to $A =< bab^{-1} >$ and $B =< aba^{-1} >$, $A = \{(ba^{\pm n} b^{-1})^{\pm 1} | n \in \mathbb{Z}\}$ and $B = \{(ab^{\pm n} a^{-1})^{\pm 1} | n \in \mathbb{Z}\}$ and therefore for any element in G, we can determine whether or not the element is in A or B (and thus A and B have solvable generalized word problems relative to G). Both ϕ and ϕ^{-1} are easily calculable and thus, by the previous corollary, G^* has a solvable word problem.

Going from right to left, we see that $tab^3 a^{-1} t^{-1} \equiv ba^3 b^{-1}$, and thus we can write $w = b^2 ab^{-1} t^{-1} a^{b3} a^3 b^{-1}$. There are no more reductions possible and thus, as there cannot be further free reduction. Thus, w is not the identity in G^*.

2.2.1 Free Products with Amalgamation

Definition 2.3. We define the **free product with amalgamation** for groups $G =< X | S >, H =< Y, T > A \leq G, B \leq H$ where $\phi : A \to B$ is an isomorphism, as $G *_\phi H =< X \cup Y | S, T, a = \phi(a), a \in A >$. We define a **reduced sequence in** $G *_\phi H$ as a reduced sequence in $G * H$ where if $n > 1$, no c_i is in A or B.

And in a similar fashion, reduced sequences with respect to $G *_\phi H$ correspond to non-trivial words in $G *_\phi H$. To see this, let $(G * H)^*$ be the HNN extension of $G * H$ relative to G and H and define $\psi : G *_\phi H \to (G * H)^*$ as

$$\psi(g) = tgt^{-1}, g \in G$$

$$\psi(h) = h, h \in H$$

As ψ maps the defining relations of $G *_\phi H$ to 1, ψ is a homomorphism. For $a \in A - \{1\}$, $\psi(a) = b \neq 1$. For a sequence $c = c_1 \dots c_n$, $n > 1$, c is mapped to $\psi(c_1) \dots \psi(c_n)$ and the rest follows from the Normal Form Theorem for HNN extensions.

If the isomorphism ϕ is well understood, we identify the subgroup A of G with its image $\phi(A)$ in H and denoted the free product with amalgamation $\{G * H | A\}$.

3 One Relator Groups

3.1 Word Problem

Definition 3.1. Let G be a group. We say that G is a **one-relator group** if $G =< X|r >$ for some generating set X and $r \in F(X)$.

Before we can show the word problem is solvable for one relator groups, we first will apply HNN extensions to study subgroups of one relator groups.

Definition 3.2. Let $w \in < X|R >$ be a reduced element, and let $t \in X$. The **exponent sum** of t in w, denoted as $\theta_t(w) = i_1 + \ldots + i_n$ where

$$w = w_1 t^{i_1} w_2 \ldots w_n t^{i_n} w_{n+1}$$

.

Lemma 9. Let $G =< X|r >, X = \{a_1, a_2, \ldots\}$ be a one-relator group where r is a cyclically reduced word in $(X^{\pm})^*$. Suppose that, for some $a \in X$, the exponent sum for a in r, $\theta_a(r) = 0$. Then

$$G \cong < a, B, D|K >,$$

where

$$B = \{b_m, ..., b_n\}$$

for some $m, n, b \in X, b_j = a^j b a^{-j}, j = m, \ldots, n$

$$D = \{c_i, \ldots | i \in \mathbb{Z}, c \in X - \{a, b\}\};$$

and

$$K = \{\hat{r}, ab_j a^{-1} b_{j+1} (j = m, \ldots, n-1), ac_j a^{-1} c_{j+1}, j \in \mathbb{Z}, c_0 = c, c \in X - \{a, b\}\}$$

where $|\hat{r}| < |r|$.

Proof. Without loss of generality, assume r starts with $b^{\pm 1}$ (if not, replace r with a cyclic permutation that starts with $b^{\pm 1}$), and define $b_i = a^i b a^{-i}$ and for any other generators c, define $c_i = a^i c a^{-i}$. In r, replace instances of b's and c's with $b_{\theta_a(r')}$ and $c_{\theta_a(r')}$, where r' is the prefix of r truncated at that specific instance of b and c, respectively. Furthermore, remove all instances of a from this word. Call this new word $\hat{r}$.

Example 3.1. For $r = bacb^2a^{-1} = bacbba^{-1}c^{-1}$, $\hat{r} = b_0 c_1 b_1^2 c_0^{-1}$

Given that a is a subword in r, by our assumption a^{-1} is also a subword where aa^{-1} and $a^{-1}a$ are not subwords in r. As well, a is not a subword in $\hat{r}$ where there are no additional letters added. So from the reduction, $|\hat{r}| < |r|$.

Let m and n denote the lowest and highest occurrences of b_i in $\hat{r}$, where $m \leq 0$ (by the assumption r starts with b and thus b_0 occurs in $\hat{r}$).

To prove the claim, denote $J =< a, B, D|K >$ and define maps $\phi : G \to J$ and $\mu : J \to G$ as $\phi(a) = a$, $\phi(b) = b_0$, $\phi(c) = c_0$, $\mu(a) = a$, $\mu(c_i) = a^i ca^{-i}$, and $\mu(b_i) = a^i ba^{-i}$. Note that $\phi(r) = \hat{r}$, $\mu(ac_j a^{-1}) = \mu(c_{j+1}) \forall c_j \in D$, and $\mu(\hat{r}) = r$, so both ϕ and μ are homomorphisms. As well, $\phi \circ \mu$ and $\mu \circ \phi$ are equal to the identity maps in J and G respectively, so ϕ is an isomorphism.

REMARK: We will be frequently using the constructions of G, H, ϕ and $\hat{r}$, where $\hat{r}$ may be denoted as r_0. Also note that we can write

$$G =< a, \hat{B}, D|ab_j a^{-1} = b_{j+1}, ac_j a^{-1} = c_{j+1} \dots (j \in \mathbb{Z}, c \in X - \{a, b\}) >$$

$$\hat{B} = \{b_i (i \in \mathbb{Z})\}$$

where $\widehat{a^i ra^{-i}}$ may be denoted as r_i. To see this, let $a^i Ga^{-i} = \{a^i ga^{-i} | g \in G\}$. $a^i Ga^{-i}$ is a group, and

$$a^i Ga^{-i} =< a^i aa^{-i}, a^i ba^{-i}, \dots |a^i ra^{-i} >=< a, a^i ba^{-i}, \dots |a^i ra^{-i} >$$

$$< a, a^i ba^{-i}, \dots |a^i ra^{-i} >=< a, b, \dots |r >= G$$

So, if we write $a^i Ga^{-i} =< a, b, \dots |a^i ra^{-i} >$, and write $G = \cup_{i \in \mathbb{Z}} a^i Ga^{-i}$, then using t-reductions we have the above presentation. □

Note that G is an HNN extension of a one-relator group $H =< B, D|s >$ where a is the stable letter and $\{b_{m+1}, ..., b_n, D\}$ and $\{b_m, ..., b_{n-1}, D\}$ are the associated subgroups.

Theorem 10. (Freiheitssatz) Let $G =< X|r >, X = \{a_1, a_2, \dots\}$ be a one-relator group where r is a cyclically reduced word written with the letters $\{x_1, \dots, x_n\} \subseteq X$. Let $L \subset X$ be such that $|L \cap \{x_1, \dots, x_n\}| < n$ so that at least one of the generators $x_1, \dots, x_n$ is missing in L. Then the subgroup generated by L is free.

Proof. By induction on the length of r. If $|r| = 0$ in $< X >$, then G is a free group and thus any subgroup of G is free by Theorem 2. If $|r| = 1$ or $r = a^k, a \in X$, then $G =< X - \{a\} > * < a|r >$ and $L \subseteq X - \{a\}$. As $< L >$ is a subgroup of a free group, $< L >$ is free. So, suppose that $|r| = p$ and the Freiheitssatz hold for all relators s with $|s| < p$. Suppose r contains $a, b \in \{x_1, \dots, x_n\}$ non-trivially.

CASE 1: The exponent sum for some generator in r, say $\theta_a(r)$, is 0.

By Lemma 9, we have that G is isomorphic to an HNN extension of the one-relator group $H =< B \cup D|s >$, where $|s| < |r|$. Thus, by the induction hypothesis, the Freiheissatz is true for any subset of $B \cup D$ that removes the at least one generator from s.

We will now prove the induction step. First suppose that $a \notin L$. Note that at least 1 generator in H with nonzero subscript occurs in $\hat{r}$, otherwise a could not occur in r given $\theta_a(r) = 0$. By the induction hypothesis, L, written as a subset of H, freely generates a subgroup of H and thus L freely generates a free subgroup of G (by the Normal Form Theorem for HNN extensions, H, and by extension $< L >$, are embedded in G by the natural map).

If $a \in L$, let $w \in < L >$ be freely reduced and non-trivial. In G, $\theta_a(p) = 0$ for all relators $p \in K$ and so $w = 1_G$ only if $\theta_a(w) = 0$. If so, rewrite w as a word on the set D. Thus, the freely reduced word w^* is nonzero as by the induction hypothesis $Gp(D)$ is a free group in H and as H is embedded in G, $< D >$ is a free subgroup of G.

CASE 2: Let the exponent sum for each generator in r be nonzero. In particular, let $\theta_a(r) = \alpha$ and $\theta_b(r) = \beta$. Define the group

$$C =< y, x, X - \{a, b\} | \Psi(r) >$$

and define a homomorphism $\Psi : G \to C$ where $\Psi(a) = yx^{-\beta}$, $\Psi(b) = x^{\alpha}$, and $\Psi(c) = c \ldots$. Let r_1 be the cyclic reduction of $\Psi(r)$. Then the exponent sum of x in r_1 is 0 and for s rewritten from r_1 in a similar manner to the above, the length of s is smaller than the length of r as all occurrences of x are removed, which only leave occurrences of y and $X - \{a, b\}$.

Thus, by the induction hypothesis, the subgroup of C, M generated by $\{x^{\alpha}, c, d, \ldots\}$ is freely generated, and as $\Psi : X - \{a\} \to M$ extends to a surjective homomorphism, $\{b, c, d, \ldots\}$ is free (for $\{a, c, d, \ldots\}$, we simply let $\Psi(a) = x^{\beta}$, $\Psi(b) = yx^{-\alpha}$). □

Example 3.2. Let $w = abab^2$. Then $\theta_a(w) = 2$ and $\theta_b(w) = 3$. So, $\Psi(w) = \Psi(abab) = yx^{-3}x^2yx^{-3}x^4 = yx^{-1}yx$. Thus $\theta_x(\Psi(w)) = 0$. $\Psi(w)$ is cyclically reduced so we let $r_1 = yx^{-1}yx$. Reducing as above yields $s = y_0y_{-1}$.

Let X be a countable set and $A \subseteq X$. A is a **recursive subset** of X if there exists an algorithm that takes an element $x \in X$ and, in finite time, can determine if $x \in A$.

Given that subgroups of one-relator groups that omit a single generator of the relator are free by the Theorem above, the word problem is solvable with respect to these subgroups by Theorem 1. Further extending this, we can show that the word problem is solvable for a group generated by any recursively generated subset of the generators of a one-relator group.

Recall the **generalized word problem**: Given a group G and a subgroup H generated by a recursive subset of G, does there exist an algorithm that can determine for a given word $w \in G$, if w is in H?

Theorem 11. (**Generalized Word Problem for One-Relator Groups**:) Let $G =< X|r >$ be a one-relator group where r is cyclically reduced and X is a countable set. Let $L \subseteq X$ be a recursive subset and M be the group generated by L. Then the generalized word problem with respect to M is solvable.

Proof. Proof by induction on the length of r. If $|r| = 1$, then G is a free product of a free group and a finite cyclic group. By the Normal Form Theorem for Free Products and Corollary 4.1, it is clear that the Theorem holds.

Suppose that the theorem holds for all one-relator groups such that the length of the relator is less than n. From the proof of Lemma 9 (see p. 15), we showed that G is isomorphic to the group $< H, t|s, tXt^{-1} = Y >$ where $H =< T|s >$ where $|s| < |r|$. We note by Corollary 8.1 and the induction hypothesis, G has a solvable word problem.

Corollary 11.1. The word problem is solvable with respect to one-relator groups.

We consider two cases: **CASE 1:** The exponent sum for some generator in r, say $\theta_a(r)$, is 0

Our inductive hypothesis implies that the generalized word problem is solvable for the free subgroups $X, Y \subseteq H$. If L contains all of the generators of r, then we consider G as the free product of the free group S, denoting all generators not in r, and the group generated by R, the set of generators occurring in r. For a word $w \in G$, since the word problem with respect to G is solvable, we write w in free product normal form, and using the Normal Form Theorem for Free Products, this implies that $w \in M$ if and only if $w_i \in < L \cap S >$ for a normal form component $w_i \in < S >$. So, given that $< L \cap S >$ is a free subgroup of $< S >$, one has to check if w_i is written as a word in $< L \cap S >$.

If L omits the stable letter, say t, we can simply check if $\theta_t(w) = 0$ by Britton's Lemma ($w \in M$ if and only if w omits the stable letter).

If L omits some other letter in r, say b, we let that letter have bounded subscript. Thus, by Britton's Lemma, if $\theta_t(w) = 0$ (if not, let $w_1 = wt^{-\theta_t(w)}$, where $w \in L$ if and only if $w_1 \in M$), then $w \in M$ iff, via t-reductions, the reduced word $w' \in < c_i, \dots, d_i, \dots (i \in \mathbb{Z}) >$.

CASE 2: Let the exponent sum for all generators in r be nonzero.

Recall the group C and homomorphism $\Psi : G \to C$ as from the proof of the Freiheissatz. By the induction hypothesis we have that the theorem holds for any subset L of the generating set of C. As the map is an embedding of G into C, a word w is in M if and only if the word $\Psi(w)$ is in $\Psi(M)$. Thus, for $\Psi(L) \subset C$, where C is defined by the generators of G, $\Psi(M)$ is solvable with respect to C and thus M is solvable with respect to G. □

Example 3.3. Let $G =< a, b|(ab)^2 >$. Then $\theta_a(r) = 2 = \theta_b(r)$. As well, let $w = ab^{-1}$. To show $\{w^n, n > 0\}$ is not the identity, we have to consider C and Ψ. $\Psi(a) = yx^{-2}$ and $\Psi(b) = x^2$. So $C =< y, x|y^2 >=< y|y^2 > * < x >$. $\Psi(w) = yx^{-4}$, and by the Normal Form Theorem for free products, the reduced sequence derived from w^n is not the identity.

Example 3.4. Let $G =< a, b|[a, b] >$ and $w = ab^{-1}$. We wish to show that $w \neq 1_G$.

As $\theta_a([a, b]) = 0$, a is the stable letter and we replace instances of b with $b_i's$, where it reduces to $r = b_1 b_0^{-1}$. Accordingly, $G =< a, b_0, b_1|r, ab_0a^{-1}b_1^{-1} >$, the HNN extension of the group $H =< b_0, b_1, a|r >$. As well, we rewrite w as $b_1^{-1}a$, which is irreducible in G. Thus, w is non-zero.

3.2 Newman's Spelling Theorem

Let G be a group. G is **torsion-free** if there does not exist non-trivial $w \in G, n \in \mathbb{Z} - \{0\}$ such that $w^n = 1_G$. The group G is a **group with torsion** otherwise.

We begin by focusing on one-relator group with torsion. In particular, we will show the following. If there exists some non-trivial cyclically reduced word in a one-relator group that has finite order, then that word to the power of its order must be a cyclic permutation to the relator. So, any words with finite order in a one-relator group must be roots of the cyclic permutations of the relator.

Theorem 12. A one-relator group $G =< X|r >$ is torsion-free if the relator r is not a power of another word. If r is a power, say $r = u^n$, where u itself is not a power of another word, then the only elements of finite order in G are all conjugates of powers of u.

Proof. Done by induction on the length of r. If $|r|_X = 0$, then the group is free and thus, for some N-reduced basis derived from X, for $w \in (X^{\pm})^*$, write $w^n = ax^na^{-1}$, where $|w^n| > |x^n|$ and thus proves $< X >$ is torsion free.

As well, if r involves only 1 letter or $|u| = 1$, say a, then G is the free product of a free group and the finite cyclic group $< a|a^n >$, and thus it follows from the Normal Form Theorem for Free Products that the only elements with torsion in G are conjugates of $a^k, 1 \leq k < n$.

Let r involve at least 2 letters and $|r| = n$. If there exists a letter in the word r with exponent sum zero, then define H and $\hat{r}$ from p. 15. r is a power iff $\hat{r}$ is a power (from the isomorphisms ϕ and μ as in the Lemma 9, again see p. 15). So, by the induction hypothesis any element of H, and thus of G by the Normal Theorem for HNN extensions is torsion free unless $\hat{r}$ is a power, which if it is, then any word of finite length in H, and thus G (the normal form for any word in G implies the power cannot be equal to 1_G unless it is an element in G itself), is a cyclically conjugate of the relator.

If there does not exist a letter in r that has exponent sum zero, we consider the group C as in the Freiheitssatz. r_1 is a power iff r is: $\Phi(r)$ is a power if r is, and for $r = st^ns^{-1}$ where t^n is cyclically reduced, we have that $r_1 = t^n(yx^{-\beta}, x^{\alpha}, c, \ldots)$, as the transformation is simply replacing letters. Since Φ is an embedding, as before G is torsion-free unless r is a proper power. If it is, then for $r = u^n$, $\Phi(u)$ has finite order and thus u has finite order.

We then need to show that all elements which are of finite order in $\Phi(G)$ are conjugate in G if they are conjugate in C, thus showing that, for any element of finite order in $\Phi(G)$, it is conjugate to u in G.

We can write $C = \{G* < x > |b = x^{\alpha}\}$, and suppose $cgc^{-1} = g'$ where $g, g' \in G$. Let $c = c_1 \ldots c_k$, where $c_1 \ldots c_k$ is in normal form. Without loss of generality, we assume $c_k = x^j$. By the Normal Form Theorem for Free Products, cgc^{-1} can be reduced to g' only if g has a prefix or suffix containing b. Say, without loss of generality, it's the prefix. Then $c_k = b_1$ and $c_1 \ldots c_{k-1}(g_1gg_1^{-1})c_{k-1}^{-1} \ldots c_1^{-1} = g'$. The result follows inductively on the length of c. ☐

Now that we have specified the groups in question, we now turn and focus on a theorem concerning the length of words in a one-relator group with torsion. If we have two freely reduced words that are equal where one omits a letter contained in the other, then clearly there must be some further reduction in order to reduce the former word. However, in the case without torsion, this "reduction" may in fact only increase the length of the word.

Example 3.5. Let $G =< a, b|ab^{-2} >$, and $g = a^5b$. We can also write g equivalently as $g' = b^{11}$. So g' omits a from g, but g has a shorter length than g'.

This is not the case with torsion. In fact, Newman bounds the reduction from above in the following Theorem.

Theorem 13. (**Newman's Spelling Theorem**) Let

$$G =< a_1, a_2, \ldots |r^n >=< S|r^n >$$

where r is cyclically reduced, not a proper power, and $n > 1$. Suppose $w = v$ in G where w is written with the generators $\{a_{i_1}, \ldots, a_{i_m}\}$, denoted as $w = w(a_{i_1}, \ldots, a_{i_m})$ and one of the generators, a_{i_j}, which we denote by t, is omitted in v. Then w contains a subword z such that z is also a subword of $r^{\pm n}$ and $|z| > (n-1)/n * |r^{\pm n}|$.

Proof. We assume that v is a freely reduced word. Suppose the generator t does not occur in r. Let K be the subgroup of G generated by all the generators except t. Then $G = K* < t >$ and $v \in K$. Writing $w = w_1 * \ldots * w_n$ in normal form with respect to the free product $K* < t >$, some w_j must be a non-trivial power of t, a contradiction to the Normal Form Theorem for Free Products. So, we only consider equations where the omitted generator occurs in r^n. This will be done by induction on the length of r.

If r involves only one generator then necessarily $r = t$. Then $G = K* < t|t^n >$ with $v \in K, w \notin K$, again in contradiction to the Normal Form Theorem for Free Products. So we assume r involves at least 2 generators.

CASE 1: The exponent sum for some generator in r, say $\sigma_a(r)$, is 0. Let b be the other generator that occurs in r. We consider G as an HNN extension with base $H =< R|s >$ and $X = \{b_m, \ldots, b_{n-1}, D\}$ and $Y = \{b_{m+1}, \ldots, b_n, D\}$ the associated subgroups (see p. 15). As well, the sole relator s of H is a power and $|s| < |r|$. Our goal is to represent an equivalent equality in H, thus being able to use the induction hypothesis on the length of r.

Suppose a occurs in w but not v, so that $a = t$. Then $w(a, b_0, \ldots) = v(b_0, \ldots)$ (replacing b with b_0) implies w can be reduced to an a-free word w^* by a series of a-reductions and replacing a subword $a^\epsilon u(b_i, c_i)a^{-\epsilon}, \epsilon = \pm 1$ where a does not occur in u with $u(b_{i+\epsilon}, c_{i+\epsilon}) = u'$, where all of the generators of u' occur among the generator X $(\epsilon = 1)$ or Y $(\epsilon = -1)$.

Suppose we do such reductions and we reach a word w' which we cannot reduce further but is not a-reduced. Then w' contains a word $a^\epsilon ua^{-\epsilon}$ where u is a-free and not a word on the generators of X or Y but u is equal to a word z on the given generators. Thus z omits a generator of H and $u = z$ in H, and so by the induction hypothesis, u contains a subword q of $s^{\pm n}$ of the required length. We can recover w from w' by replacing b_i's with a^iba^{-i} and freely reducing. Given that we have only shifted subscripts to obtain s from r, and that $|q| > (n-1/n) * |s^n|$, the part s in u recovered from q will contain a subword of r^n of the desired length if we ignore the $a^{\pm 1}$ occurring at the ends.

If w can be reduced to a a-free word w^* by shifting subscripts then w^* must contain a generator with non-zero subscripts and $w^* = v$ in H. By the induction hypothesis, w^* contains a subword q of $s^{\pm n}$ of the required length and implies w will contain a subword of r^n of the desired length, where we note the subword may not end with $a^{\pm 1}$.

Next let $w(a, b_0, c_0, \ldots) = v(a, c_0, \ldots)$ where b occurs in w but not in v. By Britton's Lemma, $\sigma_a(w) = \sigma_a(v) = \alpha$ as both words have the same normal form with respect to a-reductions; this implies $w * a^{-\alpha} = v * a^{-\alpha}$. Since $v * a^{-\alpha}$ has exponent sum 0 for a and does not contain b_0, we can reduce v to a a-free word v^* where $w * a^{-\alpha} = v^*$ holds. By the previous argument, if $wa^{-\alpha}$ cannot be reduced to a t-free word then there exists a subword s which does not end with a $a^{\pm 1}$ to ensure s is a subword of w. If $wa^{-\alpha}$ can be reduced to a w^*, then some b_i occurs in w and $w^* = v^*$ in H, which by the induction hypthesis says w^* contains a subword of r^n of the desired length, where we note the subword may not end with $a^{\pm 1}$.

CASE 2: The exponent sum of any generator in r is nonzero. We define the group C and map Ψ as per the Freiheissatz. Let w be the freely reduced word obtained from $\Psi(w)$ and v the freely reduced word obtained from $\Psi(v)$. So w contains y and v does not. As in Case 1, w contains a subword q of $\Psi(r^n)$ of desired length that does not end in $x^{\pm 1}$. So w contains a suitable subword of $r^{\pm n}$. □

Note that the Spelling Theorem is an alternative proof for the word problem with respect to one-relator groups with torsion: for a freely reduced word $w \in (X^{\pm})^*$, if $w = 1$ then by the Spelling Theorem there must exist a subword s of $r^{\pm n}$ where $|s| \geq (n-1)/n * |r^{\pm n}|$. If such a subword exists, replace the subword s with the subword u where $su \equiv 1_G$, such that the length of w is reduced. Continue to reduce until w is of minimal length. If such subword does not exist and w is freely reduced and non-trivial, we can say that $w \neq 1_G$.

3.3 Conjugacy Problem for One-Relator Groups with Torsion

Now we begin the proof of the conjugacy problem. Like the other proofs, we need to split our problem into cases. Given a one-relator group with torsion, we need to show that there are certain conditions necessary for two elements in a one-relator group to belong to one and the same conjugacy class.

A subgroup $H < G$ is **malnormal** in G if $g \in G - H$ implies $gHg^{-1} \cap H = \{1_G\}$.

Example 3.6. Clearly the identity and the entire group are malnormal subgroups of a group. Furthermore, malnormality is transitive: a malnormal subgroup K of a malnormal subgroup H of a group G is malnormal in G. To see this, let $g \notin K$. If $g \notin H$ then $gKg^{-1} \cap H = \{1_G\}$ as $K \subseteq H$ and H is malnormal in G. If $g \in H$ then $gKg^{-1} \cap K = \{1_G\}$ as K is malnormal in H. Specifically, if a subgroup J of a group A is malnormal in A, J is malnormal in $A * B$.

This can help us show that a subgroup generated by a subset of a free group's generating set is malnormal: given some Nielsen-reduced set $X = \{a_1, a_2, \dots, a_n\}$, we need only consider, without loss of generality, $Y = \{a_2, \dots, a_n\}$ since by transitivity, the subgroup of $< X >$ generated by Y is malnormal in X. So, $a_1^n < Y > a_1^{-n} \cap < Y >= \{1_X\}$ as any word $w \in < Y >$ does not contain any letter a_1 and so there cannot be any further free reduction that completely eliminates a_1.

Note that some subgroups of free groups are not malnormal. For instance, let $G =< a, b >$ and $H =< a^2 >$. Then $aa^2a^{-1} = a^2 \in H$, while $a \notin H$.

The proofs below, unless otherwise stated, are adapted from Newman (1968).

Lemma 14. Let $C = \{A * B|J\}$ be the free product of groups A and B with the amalgamated subgroup J, malnormal in both A and B. Then A and B are malnormal subgroups of C.

Proof. From symmetry we will prove A is malnormal in C. Say there exists in $g \in C, a_1, a_2 \in A$ such that $g^{-1}a_1g = a_2$. Let left coset representatives be chosen for J in A and B and write g, a_1, a_2 as elements of these left cosets: $g = s_1...s_nj$, $a_1 = t_1j_1$, $a_2 = t_2j_2$, $j, j_1, j_2 \in J$, $t_1, t_2 \in A$ and $s_1, \dots, s_n$ be coset representatives in A or B.

CLAIM: $g \in A$. Done by induction on $|g|$. If $|g| = 0$ then $g \in J$ and so $g \in A$. Say the claim is true for all $|g| < n$, and let $|g| = n$. Then we have that

$$t_1j_1s_1...s_nj = a_1g = ga_2 = s_1...s_njt_2j_2$$

If $t_1 = 1$ then $(s_1^{-1}j_1s_1)s_2...s_nj = s_2...s_njt_2j_2$. Note that, by the assumption of J being malnormal in A, $(s_1^{-1}j_1s_1) = w \in (A \cup \{1\}) - J$. And so, this means that for $g' = s_2...s_nj$, $g' = g't_2j_2$ in C which implies $a_1, a_2 \in J$, which is a contradiction as J is malnormal in $A * B$. This implies $s_1 \in J$ which is a contradiction as s_1 is a non-trivial coset representative of J in A or B.

Then we let $t_1 \neq 1$ and $(s_1^{-1}t_1j_1s_1)s_2...s_nj = s_2...s_njt_2j_2$ which implies s_1 and t_1 are in the same factor (else, for $s_1 \in A, t_1 \in B$ without loss of generality, $t_1j_1 \in B$ and so by induction, $s_1^{-1}t_1j_1s_1 \notin B$ and thus $s_1^{-1}t_1j_1s_1 \notin J$, so $s_1^{-1}t_1j_1s_1 \neq 1_C$, and so for $g' = s_2 \dots s_nj$, then $g' = bg't_1$ where b, t_1 are non-trivial in C, a contradiction) and $s_1^{-1}t_1j_1s_1 = j_3 \in J$ and $j_3s_2...s_nj = s_2...s_njt_2j_2$. Hence $s_{1n}j \in A$. □

Now that we have this powerful lemma, we turn our attention back to one-relator groups. By Lemma 14, for a free product with amalgamation $\{A * B|J\}$ if we let J be the group generated by a single relator, this would imply that, so long as we prove that the relator subgroup is malnormal with respect to the factors, that two words from two different factors cannot commute. Thus, we come to the following lemma

Lemma 15. Let $G =< a, b, \dots, c, t | r^n >$ and $n > 1$ with r cyclically reduced.

1. Any subset of the generators of G generates a malnormal subgroup of G.
2. Suppose $w_1(a, b, \dots, c)$ is a word that contains a non-trivial subword of the form $a^{\pm n}$ and $w_2(b, \dots, c, t)$ is a word that contains a non-trivial subword of the form $t^{\pm m}$ where both words are cyclically reduced. If r has non-trivial subwords that contain a and t, then w_1, w_2 are not conjugate.

Proof. Proof by induction on the length of r^n. If $|r^n| < 4$ or $r^n = wx^n w^{-1}, x \in \{a, b, \dots, c, t\}$ then the lemma is trivially true as r can be written as the product of one generator, and G is thus a free product between a free group and a finite cyclic group. To simplify the proof, we assume G involves at most the generators a, b, c, and t.

We'll consider the first proposition. It suffices to prove this for $< b, c, t >$ by Example 3.6.

CASE 1: Suppose G has two generators a, b and $\sigma_a(r) = 0$, and $g^{-1} b^m g = b^t, g \in G, m, t \in \mathbb{Z}, t \neq 0$. Without loss of generality, assume r is cyclically reduced. We will show g is a power of b. Let N be the normal closure of $< b >$, written as $N =< b >^G$. Using a-reductions to rewrite G as $G =< a, b_i, \dots (i \in \mathbb{Z}) | r_0, a b_i a^{-1} = b_{i+1} (i \in \mathbb{Z}) >$ defined in Lemma 9, where r_0 is r written using a-reductions. Let μ be the highest subscript of b_i in r_0. Define

$$r_i = a^i r a^{-i}$$

$$N_0 =< b_0, \dots, b_\mu | r_0^n >$$

$$N_i =< b_i, \dots, b_{\mu+i} | r_i^n >$$

where, without loss of generality, all b_i appear in r_0 (if not for some j, let $\hat{r}_0 = b^j r_0 b^{-j}$ be the sole relator). Note that, for any missing b_j, then for a subword $b_k^{\pm n}$ of r_0 where $|k - j|$ is minimum in $\{0, \dots, \mu\}$, there exists a corresponding subword $a^i b^{\pm n}$ or $b^{\pm n} a^i$ of r where $i > |k - j|$ by our assumption. As $k \neq j$, $|k - j| \geq 2$, and as r is cyclically reduced, the corresponding subwords occur at least two times. So, $|r_0| \leq |r| - 4$ which implies $|\hat{r}_0| = |r_0| + 2 \leq |r| - 2 < |r|$.

Then we have that

$$N = \cup_{k=0}^{\infty} K_k$$

$$K_0 = N_0$$

$$K_{2k} = \{K_{2k-1} * N_{-k} | J_{-k}\}$$

$$K_{2k+1} = \{K_{2k} * N_{k+1} | J_{k+1}\}$$

where $n > 0$ and

$$J_i =< b_i, \dots, b_{\mu+i-1} >, i > 0$$

$$J_i =< b_{i+1}, \dots, b_{\mu+i} >, i < 0$$

where in G, we can write $N =< \dots, b_i, \dots | \dots, r_i, \dots (i \in \mathbb{Z}) >= \cup_{k=0}^{\infty} K_k$. Furthermore, $|r_i| < |r| \forall i \in \mathbb{Z}$.

Example 3.7. $K_1 = \{N_0 * N_1 | J_1\}$ where

$$N_0 =< b_0, \ldots, b_\mu | r_0^n >$$

$$N_1 =< b_1, \ldots, b_{\mu+1} | r_1^n >$$

$$J_i =< b_1, \ldots, b_\mu >$$

. Rewrittten, $K_1 =< b_0, \ldots, b_\mu, b_{\mu+1} | r_0^n, r_1^n >$.

$K_2 = \{K_1 * N_{-1} | J_{-1}\}$ where

$$N_{-1} =< b_{-1}, \ldots, b_{\mu-1} | r_{-1}^n >$$

$$J_{-1} =< b_0, \ldots, b_{\mu-1} >$$

. Rewritten, $K_2 =< b_{-1}, b_0, \ldots, b_\mu, b_{\mu+1} | r_{-1}^n, r_0^n, r_1^n >$

$J_1 =< b_1, \ldots, b_\mu >$ is a malnormal subgroup of N_0 and N_1 by the induction hypothesis, and so by Lemma 14, J_1 is a malnormal subgroup of K_1 and thus N_0 and N_1 are malnormal subgroups of K_1. Inductively, for $K_{2i+1} = \{K_2 i * N_{i+1} | J_{i+1}\}$, J_{i+1} is malnormal in N_i and N_{i+1} and as N_i is malnormal in K_i by the induction hypothesis, by example 3.6 J_{i+1} is malnormal in K_i. So, by Lemma 14, K_{2i} and N_{i+1} are malnormal subgroups of K_1. Furthermore, by example 3.6, $K_0, \ldots, K_{2i-1}$ are malnormal in K_{2i+1}. Similarly, $K_0, \ldots, K_{2i+1}, N_{-i-1}$ is malnormal in K_{2i+2}.

As well, one can show that K_i is a malnormal subgroup of N for all integers i: for K_{i+1}, we have that K_i and N_i are malnormal subgroups of K_k for all $k > i$. Say some words $w \in N - K_{i+1}, g, h \in K_{i+1}$ exist where $wgw^{-1} = h$. Then as $N = \cup_{k=0}^{\infty} K_k$, then $w \in K_j$ for some j. So, as K_{i+1} is malnormal in K_j by the previous paragraph, $wgw^{-1} \in K_j - K_{i+1}$, a contradiction.

For $g^{-1} b^m g = b^t$, let $\sigma_a(g) = s$ and define $\bar{g} = a^{-s} g$ and rewrite the equation above as $\bar{g}^{-1} b_s^m \bar{g} = b_0^t$. If $s = 0$, then $\bar{g} \in N_0$ as N_0 is a malnormal subgroup of N. As $< b_0 >$ is malnormal in N_0 by the induction hypothesis ($|r_0| < |r|$), $g = b_0^p$ for some integer p.

If $0 < s \leq \mu$ then again $\bar{g} \in N_0$ but by the induction hypothesis, $\bar{g}^{-1} b_s^m \bar{g} = b_0^t$ is impossible as b_s and b_0 occur non-trivially in b_s^m and b_0^t, respectively and both occur in r_0 non-trivially.

If $s = \mu + k$ then $g \in K_k$ which is a malnormal subgroup of N. We can write K_k as $\{N_0 * < N_l, \ldots N_k > | J_1\}$ where $b_0 \in N_0$ and $b_s \in < N_l, \ldots N_k >$. This implies $\bar{g}^{-1} b_s^m \bar{g} = b_0^t$ being possible only if b_0^t being conjugate to an element in J_1 in N_0, impossible by the induction hypothesis.

CASE 2: Suppose G has two generators a, b and $\sigma_a(r) \neq 0$ and without loss of generality, $\sigma_b(r) = 0$ (if not, embed into a larger subgroup C with this property as in the Freiheitssatz). For $g^{-1} b^m g = b^t$ as above, where we assume $\sigma_b(g) = 0$ (if not, let $\bar{g} = b^{-s} g$, where $g^{-1} b^m g = \bar{g}^{-1} b^{m-s+s} \bar{g} = \bar{g}^{-1} b^m \bar{g} = b^t$), and that m is non-zero, re-write the equation as $b^m g = g b^t$ and multiply from the right b^{-m} on both sides: $b^m g b^{-m} = g b^{t-m}$. Thus, $\sigma_b(b^m g b^{-m}) = 0$ and so, given the only possible reductions are free reductions and replacing subwords of the relator with its equivalent in N, both of which do not affect the exponent sum of b in g (as $\sigma_b(r) = 0$ implies for $r = uv$, $\sigma_b(u) = -\sigma_b(v) = \sigma_b(v^{-1})$), the only way this equality is possible is if $t = m$. So, $b^m g b^{-m} = g$.

Let $N =< a >$ and construct it from copies of $N_i =< a_i, \ldots, a_{\mu+i} | r_i >$ similar to the above case. Denote g_i as g written in N, where one may assume that out of any of the possible ways to write g in N, g_i is the shortest. If g_i contains some a_i we assume g_i is a word in $\{a_0, \ldots, a_\lambda, \lambda > 0\}$ which implies that, for the above equation re-written in H, $g_i(a_m, \ldots, a_{\lambda+m}) = g_i(a_0, \ldots, a_\lambda)$. If $\lambda + m < \mu$, then $g_0 \in N_0 = K_0$ this implies that we can reduce g_i by the Spelling Theorem, a contradiction. Inductively, say this property holds for K_j. Then for $K_{j+1} = \{N_0 * Gp\{N_1, \ldots, N_{j+1}\} | J_1\}$, $g_i(a_m, \ldots, a_{\lambda+m}) \in Gp\{N_1, \ldots, N_{j+1}\}$ and thus $g_i(a_0, \ldots, a_\lambda) \in Gp\{N_1, \ldots, N_{j+1}\}$. Write $g_i(a_0, \ldots, a_\lambda)$ as a reduced sequence on $\{N_0 * Gp\{N_1, \ldots, N_{j+1}\} | J_1\}$. For some element in the sequence $g_{i_k} \in Gp\{N_1, \ldots, N_{j+1}\} - J_1$, by the present induction hypothesis, can be reduced as a conjugate $g_{i_k}(a_0, \ldots, a_j)$, a contradiction. So, say that some subword g_i' is in J_1, then by the present induction hypothesis, g_i' can be reduced as $g_i'(a_0, \ldots, a_\mu)$, a contradiction.

CASE 3: Suppose G has more than 2 generators and without loss of generality $\sigma_b(r) = 0$ (again, embedding into a larger subgroup C with this property as in the Freiheitssatz if this is not the case). Without loss of generality, assume that, for every generator there exists a non-trivial subword s of r that contains said generator (if, for instance, t does not appears in r, then replace r with $q = crc^{-1}$ as the sole relator of the group).

Construct $N =< a, c, t >$ from $N_i =< a_i, \ldots, a_{\mu+i}, c_j, t_j (j \in \mathbb{Z}) >$ where $|r_i| < |r| \forall i \in \mathbb{Z}$. Inductively we know that $< t_j, c_j (j \in \mathbb{Z}) >$ is a malnormal subgroup of N_i for all i by the theorem's induction hypothesis, and similar to Case 1, $J_i =< a_i, \ldots, a_{\mu+i}, c_j, t_j (j \in \mathbb{Z}) >$ being a malnormal subgroup of K_{i-1} and N_i implies that N_i (and specifically the subgroup $< t_j, c_j (j \in \mathbb{Z}) >$), is malnormal in K_i by Theorem 14 (and thus $< t_j, c_j (j \in \mathbb{Z}) >$ is malnormal in N).

Let $g^{-1} w_1(b, c, t) g = w_2(b, c, t)$ where, for $\sigma_b(g) = s$, $\sigma_b(w_1) = t$, and $\sigma_b(w_2) = u$, we let $\bar{g} = b^{-s} g$, $\bar{w_1} = b^{-t} w_1$, $\bar{w_2} = b^{-u} w_2$, we have that $\bar{g}, \bar{w_1}, \bar{w_2} \in N$. Thus, we substitute and re-write the first equation as $b^{-u} \bar{g}^{-1} b^t b^{-s} \bar{w_1} b^s \bar{g} = \bar{w_2}$. We re-arrange the formula to get $b^{-u} \bar{g}^{-1} b^t = \bar{w_2} (b^{-s} \bar{w_1} b^s \bar{g})^{-1}$. Note that $\bar{w_2} (b^{-s} \bar{w_1} b^s \bar{g})^{-1} \in N$. Consequently, $b^{-u} \bar{g}^{-1} b^t \in N$, and thus the conjugation holds only if $t = s$ as in Case 2. Applying b-reductions and re-writing in N we have

$$\bar{g_0}^{-1}(a_{i+t}, c_{j+t}, t_{k+t}) = \bar{w_2}(c_i, t_i) \bar{g_0}(a_i, c_j, t_k) \bar{w_1}^{-1}(c_i, t_i)$$

If $t = 0$, then

$$\bar{g_0}^{-1}(a_i, c_j, t_k) \bar{w_1}(c_i, t_i) \bar{g_0}^{-1}(a_i, c_j, t_k) = \bar{w_2}(c_i, t_i)$$

and $g_0 \in < t_i, c_i (i \in \mathbb{Z}) > \subset < b, c, t >$. For $t \neq 0$ and assuming $\bar{g_0}$ has the minimal length of g as written and reduced in N, the equation above shows some a_i can be removed from g_0 by the Spelling Theorem, contradicting the choice of g_0.

Now we'll prove the second proposition:

CASE 1: Suppose G involves 2 generators a and b and $g^{-1}a^m g = b^s$ where $\sigma_a(r) \neq 0$ and $\sigma_b(r) \neq 0$. Without loss of generality, assume s is large, and let $\sigma_a(r) = -\alpha$ and $\sigma_b(r) = \beta$. Embed G in $C =< x, y|r(x^\beta, x^\alpha y) >$ where $a \to x^\beta$ and $b \to x^\alpha y$. Thus in C we have $g_1^{-1}x^{m\beta}g_1 = (x^\alpha y)^s$ where g_1 is g in C. Again, without loss of generality assume $\alpha s > 0$ and $\sigma_x(g_1) = 0$ (if not, let $\bar{g_1} = x^{-t}g_1$, where $g_1^{-1}x^{m\beta}g_1 = \bar{g_1}^{-1}b^{m\beta-t+t}\bar{g_1} = \bar{g_1}^{-1}x^{m\beta}\bar{g_1} = (x^\alpha y)^s$), and that m is large (where we multiply m and t by some constant i if necessary). Let $N =< y >$, constructed from $N_i =< y_i, \ldots, y_{\mu+i}|r_i^n >$. Since $m\beta = s\alpha$ we can write the above equation as $g_0^{-1}(y_{i+r\alpha})g_0(y_i) = y_{r\alpha-r}y_{r\alpha-2r}\ldots y_0$, where $g_0(y_i)$ is the smallest word in N representing g_1. Either the minimal or maximal y_j in $g_0^{-1}y_{i+r\alpha}g_0(y_i)$ can be removed, hence the word has some xx^{-1} or more than half of $r_i^{\pm n}$ for some i. By choosing a sufficiently large s we assume the subword occurs entirely within either $g_0^{-1}y_{i+r\alpha}$ or $g_0(y_i)$, which contradicts the minimality of $g_0(y_i)$. So g_1 is a power of x, which means g is a power of a, which means $g^{-1}a^m g = b^r$ is impossible by the Spelling Theorem.

CASE 2: Suppose G involve more than 2 generators and assume without loss of generality that $\sigma_b(r) = 0$ (again, embedding into a larger subgroup C with this property as in the Freiheitssatz if this is not the case).

Let $N =< a, c, t >$, constructing N from copies of $N_i =< a_i, \ldots, a_{\mu+i}, c_i, t_i (i \in \mathbb{Z}) >$. Let $g^{-1}w_1 g = w_2$, $g, w_1, w_2 \in G$ where, cyclically reduced, w_1 contains some non-trivial subword of a_i, c_i but has no non-trivial subword of t_i and, when cyclically reduced, w_2 contains some non-trivial subword of t_i, c_i but has no non-trivial subword of a_i. Given the only possible reductions are free reductions and replacing subwords of the relator with its equivalent in N, both of which do not affect the exponent sum of b in g (as $\sigma_b(r) = 0$ implies for $r = uv$, $\sigma_b(u) = -\sigma_b(v) = \sigma_b(v^{-1})$), let $w_1 = b^s w_1'$ $w_2 = b^s w_2'$, $w_1', w_2' \in N$.

If $s = 0$, then

$$g^{-1}w_1(a_{i_1}, \ldots, a_{i_j})g = w_2(c_i, t_i)$$

where, cyclically reduced, they contain some a_i, t_i terms non-trivially. If $i_1, \ldots, i_j$ is in $\{0, \ldots, \mu\}$, then $w_1.w_2, g \in N_0$, impossible by the induction hypothesis. Assuming $g^{-1}w_1(a_{i_1}, \ldots, a_{i_j})g = w_2(c_i, t_i)$ takes place in $K_k =< N_0, \ldots, N_k >$, without loss of generality, assume that all of the words representating conjugates of the word w_1, $w_1(a_0, \ldots, a_{\mu+k}, c_i)$ has minimal length, and also $a_0, a_{\mu+k}$ occur non-trivially in w_1' (if not, for a_0, this implies $w_1 = b^i w_1' b^j$ where w_1 has a prefix and suffix of a or c; we let $\hat{w_1} = b^j w_1 b^{-j}$ and $\hat{g} = b^{-j}g$). Inductively assume $g^{-1}w_1(a_{i_1}, \ldots, a_{i_j})g = w_2(c_i, t_i)$ is impossible in K_{k-1} (as it is impossible in $K_0 = N_0$ by the above statement). Let $K_k = \{N_0 * Gp(N_1, \ldots, N_k)|J_1\}$. Now $w_2' \in J_1$. If w_1 is cyclically reduced then it must be in one of the factors. This however implies $w_1(a_0, \ldots, a_{\mu+k}, c_i) = W(a_0, \ldots, a_\mu, c_i, t_i)$ or $w_1(a_0, \ldots, a_{\mu+k}, c_i) = W(a_1, \ldots, a_{\mu+k}, c_i, t_i)$ for some word W. In either case, w_1 is not reduced, a contradiction by our assumption.

So let $s \neq 0$. So $g^{-1}b^s w_1' g = b^s w_2$. For $g^{-1}(b^s w_1')^m g = (b^s w_2)^m$, both sides still satisfy the hypothesis. Thus we assume s is large. Let

$$w_1 = b^r w_1'(a_{i_1}, \ldots, a_{i_l}, c_i)$$

$$w_2 = b^r w_2'(t_i, c_i)$$

$$g = g(a_{j_1}, \dots, a_{j_n}, c_i, t_{k_1}, \dots, t_{k_p})$$

, where every cyclic reduction of a $b^r w_1'$ term contains some non-trivial subword of a_i. Assume $i_1 + r > i_l + \mu$. The equation now is

$$w_1' g(a_{j_1}, \dots, a_{j_n}, c_i, t_{k_1}, \dots, t_{k_p}) = g(a_{j_1+s}, \dots, a_{j_n+s}, c_{i+s}, t_{k_1+s}, \dots, t_{k_p+s}) w_2'$$

where $w_1' = w_1'(a_{i_1}, \dots, a_{i_l}, c_i)$ and $w_2' = w_2'(t_i, c_i)$. Now the a_i with $i < j_1 + r$ appearing on the left side are removable. Without loss of generality, assume the left hand side is freely reduced, hence $w_1' g$ contains a subword of r_i^n where the length of this subword is greater than $|r_i^n|/2$. One may assume w_1' and g are words written as minimal length. Given the use of t_j in r_i, we are restricted in our choice of i to such r_i that only contain $t_{k_1}, \dots, t_{k_p}$. Note that no new letters are introduced, and the only a_i removed are those in the subword of r_i. If there is some a_i outside of this range with $i < j_1 + r$, we are done. If not, the only removable a_i are at the end of the word, so let $w_1' = w_1 x, g = x^{-1} h$ where X involves only a and c, and h only involves c and t. This implies $h^{-1}(x b^s w_1) h = b_s w_2'$. Then a is removable from the left hand side, and thus from $x b^r w_1$, and since no t occurs in the subword, it is only removable by free reduction. But this implies a is removable from a cyclic permutation of $b^s w_1'$, a contradiction by our earlier assumption. □

Given this base, we will need stronger conditions to be able to ensure we can define the syllables of the words and find the conjugate letters, if they exist.

Definition 3.3. Let J be a subgroup of A. J is **strongly malnormal** in A if

- J is malnormal in A.
- Given the presentation of J, the word problem is solvable for J.
- There exists a finite algorithm such that, given any words $g, h \in A$, one can determine if there exists $j_1, j_2 \in J$ such that $g j_1 = j_2 h$.

Example 3.8. A free factor is malnormal in the free product by assumption, however if the word problem is not solvable in the free product or the third condition is not satisfied, then the free factor is not strongly malnormal.

Example 3.9. Let G be a group and H a strongly malnormal subgroup of G. Then, $\forall g \in G - H$, gHg^{-1} is strongly malnormal.

Let $k \in G - (H \cup \{g\})$. Then if $k^{-1} g h_1 g^{-1} k = g h_2 g^{-1}$, $h_1, h_2 \in H$, then $(g^{-1} k g) h_1 (g k^{-1} g^{-1})^{-1} = h_2$, which implies that $g^{-1} k g \in H$ as H is malnormal, and thus $k \in gHg^{-1}$. So gHg^{-1} is malnormal with respect to G.

Given the word problem is solvable with respect to H, for $H = < X | R >$ and $w \in (X^{\pm})^*$ freely reduced, $w \in gHg^{-1} \iff g^{-1} w g \in H$, for which there exists a finite algorithm. So the word problem is solvable with respect to gHg^{-1}.

For $w, k \in G$, if we can find some word $g h_1 g^{-1}, g h_2 g^{-1}, h_1, h_2 \in H$ such that $k(g h_1 g^{-1}) = (g h_2 g^{-1}) w$, then we can rearrange the equation as $g^{-1} k g h_1 =$

$h_2g^{-1}wg$. By the assumption of H being a strongly malnormal subgroup, there exists an algorithm to prove the existence and find such h_1 and h_2. Furthermore, this is true for $(g^jkg^i)h_1 = h_2(g^jwg^i), i, j \in \{1, -1\}$, which shows that there exists a finite algorithm to show whether or not there exists $t, s \in gHg^{-1}$ such that $kt = sw$.

Note that if j_1, j_2 exist, they are unique so long as $g \notin J$. If not, let $k_1, k_2 \in J$ such that $k_2^{-1}gk_1 = h = j_2^{-1}gj_1$. Then we have that $g^{-1}(j_2k_2^{-1})g = (k_1j_1^{-1}) = 1$ as J is malnormal in A.

Furthermore, we can determine if there exists $j_1, j_2 \in J$ such that $gj_1 = j_2 1_A$ and thus if $g = j_2j_1^{-1} \in J$ which means the generalized word problem for J with respect to G is solvable.

Lemma 16. A strongly malnormal subgroup K of a strongly malnormal subgroup H of G is a strongly malnormal subgroup of G.

Proof. Firstly K is a malnormal subgroup of G by Example 3.6 and the word problem is solvable with respect to K by our assumption that K is strongly malnormal in H. Let $g_1, g_2 \in G$. Then there exists an algorithm that can determine if there exist $h_1, h_2 \in H$ such that $g_1h_1 = h_2g_2$ and so long as $g_1 \notin H$ then h_1, h_2 are unique. We only need to see whether or not $h_1, h_2 \in K$ which is possible given that the generalized word problem for K in H is solvable. If $g_1 \in H$, which we can determine algorithmically, then $h_2^{-1}g_1h_1 = g_2 \in H$ and thus by our hypothesis, there is an algorithm to show if there exists $k_1, k_2 \in K$ such that $g_1k_1 = k_2g_2$. □

For the rest of the paper, when writing the free product with amalgamation $C = \{A * B|J\}$, we will assume that there exists a finite algorithm to write an element $j \in J$ as a word in A or B. This allows us to write an element $g \in C$ in reduced form where for $g = g_1 \ldots g_n$, $g_i \notin J \forall i$.

Lemma 17. Let $C = \{A * B|J\}$ where J is a strongly malnormal subgroup of A and B. Then A and B are strongly malnormal in G.

Proof. Without loss of generality we will only prove this for A. Firstly A is malnormal in C by Lemma 13. As J is strongly malnormal in A and by our assumption, the word problem is solvable in A (for a freely reduced word $g = g_1 \ldots g_n \in (A^{\pm} \cup B^{\pm})^*$, determine if $g_i \in J \forall i$ and, if so, concatenate such g_i with g_{i-1}, writing g_i on the generators of A or B until one has a word $g' = g'_1 \ldots g'_t$. The rest follows from Corollary 4.1).

Let $g, h \in C$ where we assume $l(g) \leq l(h)$. Let $l(g) \geq 2$. Without loss of generality, let $g = g_Ag_B \ldots$ and $h = h_Ah_B \ldots$ where a subscript of A means the subword is in A and a subscript of B means the subword is in B, where g_A or h_A may be 1_C. If there exist $a_1, a_2 \in A$ such that $a_1g = ha_2$, then there exist $j_1, j_2 \in J$ such that $a_1g_A = h_Aj_1$ (as $(h_A^{-1}a_1g_A)g_B \ldots = h_B \ldots$ implies $h_A^{-1}a_1g_A \in J$) and $j_1^{-1}h_B = g_Bj_2$ and so on, which we can find algorithmically (we only need calculate j_1, j_2 from the second equation to determine a_1 by the strong malnormality of J). If a_1 exists, it is unique and $a_2 = h^{-1}a_1g$. The remaining cases below are similar.

If $|g| < 2$ and $g \in A$, then $h \in A$ by $a_1ga_2^{-1} = h$. If $g \in B - J$, then one of the following holds:

- $h = h_B$. Then $a_1g = h_Ba_2$ implies $a_1, a_2 \in J$ (as above where $h_A = g_A = 1$, which implies $a_1 = j_1$ and $j_1^{-1}h_B = g_Bj_2 \implies j_2 = g_B^{-1}a_1^{-1}h_B = a_2^{-1}$).

- $h = h_Bh_A$. Then, similar to the first case, $a_1g = h_Bh_Aa_2$ implies $a_1 \in J$ (similar to the case where $h = h_B$) and $a_1g = h_Bj_1, j_1 \in J$ (as $h_B^{-1}a_1g = h_Aa_2 \in A$. So one can determine a_1 by the malnormality of J and $a_2 = h^{-1}a_1g$.

- $a_1g = h_Ah_B \dots a_2$. Then, similar to the case where $|g| \geq 2$, this implies $a_1 = h_Aj_1$ and $j_1^{-1}h_B = gj_2$ for $j_1, j_2 \in J$. Hence one can determine a_1 by the malnormality of J and $a_2 = h^{-1}a_1g$.

□

With Lemma 17, the natural question arises as to whether a group $\{A*B|J\}$ has a solvable conjugacy problem. Recall that the conjugacy problem asks if there exist a finite algorithm such that for all words $x, y \in (X^{\pm})^*$, and a group G with finite presentation $< X|R >$, that one can determine if x and y are conjugate in G? Recall that, for J a subgroup of A, J has a solvable extended conjugacy problem with respect to A if there exists a finite algorithm to determine if there exists $j \in J$ for $g, h \in A$ such that $j^{-1}gj = h$.

Lemma 18. Let $C = \{A*B|J\}$ where J is a strongly malnormal subgroup of A and B. If the conjugacy problem of A and B and extended conjugacy problem of J relative to A and B areis solvable, then the conjugacy problem for C and the extended conjugacy problem of A and B with respective of C is solvable.

Proof. Note that as J is strongly malnormal, the extended word problem is solvable for A and B with respect to C. Let $g, h \in C$ and write in reduced form $g = g_1...g_n$, $h = h_1...h_m$, and assume g and h are cyclically reduced (if not, say $g = tg't^{-1}, h = a^{-1}h'a$, then if $h' = acgc^{-1}a^{-1} = actg'(act)^{-1}$, so we can consider h' and g' instead).

CASE 1: Suppose $|g| > 1$. If g and h are conjugates, then h must have the same length as g (similar proof as Theorem 6) and a cyclic permutation of one must be conjugate to another by an element of J (similar to Lemma 6, in reducing cgc^{-1}, we get some prefix or suffix word that is reducible in C: WLOG we assume reduction is for the suffix and thus have that $1_C = g_i \dots g_nc_t^{-1} \dots c_1^{-1}$ where we note $n \leq t$ else the reduced form is not cyclically reduced-a contradiction-and thus $h = j^{-1}g_i \dots g_ng_1 \dots g_{i-1}j^{-1}$). So we check if there exists j_1, j such that $j^{-1}g_1 = h_1j_1$ as in Lemma 17 and if such j exist (which is unique), if $j^{-1}gj = h$.

CASE 2: Suppose $|g| = 1$ and g is not conjugate to an element of J in A or B. Without loss of generality, let $g \in A$. If g and h are conjugate, then $h \in A$

(if not, then there must be some prefix and suffix of h that contains subwords in B reduced by c, a contradiction as we let h be cyclically reduced) and by the hypothesis we can determine if g and h are conjugate in A.

CASE 3: Suppose $|g| \leq 1$ and is conjugate to an element $j_1 \in J$ in either A or B. If g and h are conjugate, then h must lie within A or B (if $h \notin A$, then then there must be some prefix and suffix of h that contains subwords in B that are removed by c, a contradiction as h is cyclically reduced unless we let $h \in B$) and be conjugate to an element in J in one of the factors (without loss of generality, let $g \in A$. If $h \in A$, then h is conjugate to j_1 by a word $a \in A$. If $h \in B$, then $h = cgc^{-1}$ implies, as in Case 1, that $l(c) \leq 1 \implies c \in A$ or $c \in B$. If $c \in A$ then we are done. If $c \in B$ then $c^{-1}hc \in A \cap B = J$ by the definition of the amalgamated subgroup), and since J is strongly malnormal, we can find such a j_1, j_2. Thus j_1, j_2 must be conjugate in C and since J is malnormal in C, then j_1, j_2 must be conjugate in J, where the conjugacy problem is solvable. □

Lemma 19. Any subgroup of a one-relator group with torsion is strongly malnormal.

Proof. Let $H =< a, b, c, \dots, t|r^n >$ where we assume that there exists subwords of r^n that contain $a, b, c, \dots, t$ non-trivially (if not for a generator s, let $r' = srs^{-1}$ be the relator). By Lemma 16, we thus only need to prove that $Y = \{b, c, \dots, t\}$ is strongly malnormal. Y is malnormal by Lemma 15, is solvable with respect to the word problem by Corollary 11.1. So let $g_1, g_2 \in G$ be of minimal length. For some $h_1, h_2 \in H$, $g_1 h_1 g_2^{-1} = h_2$ only if $|h_1|, |h_2| \leq 2|r^n|(|g_1| + |g_2|)$ where h_1 and h_2 minimal. If not, say without loss of generality, $|h_1| > 2|r^n|(|g_1| + |g_2|)$. Then note, for $g_1 h_1$, that there are at most $2|g_1|$ reductions possible with each reduction taking at most $|r^n|$ letters by the Spelling Theorem. So for h_1, there is a subword of h_1 not eliminated in $g_1 h_1 g_2^{-1}$ and thus if some a is eliminated in $g_1 h_1 g_2^{-1}$ it is eliminated in either $g_1 h_1$ or $h_1 g_2^{-1}$ which implies that $g_1 \in H$ or $g_2 \in H$. So one simply checks all words h_1, h_2 where $|h_1|, |h_2| \leq 2|r^n|(|g_1| + |g_2|)$. □

Theorem 20. The conjugacy problem is solvable in one-relator groups with torsion.

Proof. The proof will be done by induction on the length of the generator.

Let $G =< a, b, c \dots |r^n >=< X|r^n >, n > 1$. If r is a power of some single letter $u \in X$, then G is the free product of a free group and a finite cyclic group, and so the conjugation problem is solvable by Theorem 5. So we assume r contains at least 2 generators, a and b, and that the theorem is proved for all relators t where $|t| < |r|$.

Let $g, h \in G$ be cyclically reduced. To simplify notation, we shall assume that $G =< a, b, c|r^n >$.

CASE 1: Suppose $\sigma_a(r) = \sigma_a(g) = 0$. Let $N =< b, c >$ and construct N from copies of $N_i =< b_i, \dots, b_{\mu+i}, c_i(i \in \mathbb{Z}|r_i^n >$ as in Lemma 15, where not all b_i or c_i may occur in r_0. Note that since $a \notin N_i \forall i$, the relations $ab_i a^{-1} b_{i+1}^{-1}$

with respect to the supergroup $H =< R|s >$ from Lemma 9 cannot be satisfied in $N_i \forall i$.

If g and h are conjugate in G, say $x^{-1}gx = h$, then $x = a^s x_0$, $x_0 \in N$, without loss of generality we assume $s \geq 0$. Then $x_0^{-1}a^{-s}ga^s x_0 = h$, and reducing we have $x_0^{-1}g_s x_0 = h$, where g_s is the word $a^{-s}ga^s$ written as a word in N. Thus, for g and h to be conjugate, $\sigma_a(h) = 0$, so $g, h \in N$. Note that s could be arbitarily large. However, we shall show that s is bounded.

Lemma 21. Let N be as above and let $j, l \in N$,

$$g(b_0, \ldots, b_s, c_0, \ldots, c_r)$$

$$h(b_m, \ldots, b_{m+\mu}, c_0, \ldots, c_{m+\mu})$$

where g is non-trivial and both words are of minimal length. If $s + \mu < m$ then g and h are not conjugate in N.

Proof. Assume $s + \mu < m$. By Lemma 15, construct $N =< b, c >$ from $N_i =< b_i, \ldots, b_{\mu+i}, c_j (j \in F)|r_i^n >$. In particular we can write $K_k = \{K_{m-1} * Gp(N_m, ..., N_k)|J_m\}$ where $J_i = \{b_i, \ldots, b_{\mu+i-1}\}$ or $J_i = \{b_{i+1}, \ldots, b_\mu\}$. As $s < m$, g is in the first factor K_{m-1} and h is in the second. By our construction, g is conjugate to h only if g is conjugate to an element in J_m, which implies g is conjugate to an element of N_{m-1}. Similarly, we write $K_{m-1} =< K_{m-2} * N_{m-1}|J_{m-1} >$ where g is in the first factor (as $s < m - \mu$). So g is conjugate to an element in N_{m-1} only if g is conjugate to an element of J_{m-1}. Inductively, we conclude g is conjugate to an element of $J_{m-\mu}, \ldots, J_m$ Hence there is an element $j \in J_{m-\mu}$ conjugate to $J_{m-\mu}, \ldots, J_m$

Claim: If $j_0 \in J_0$ is cyclically reduced and is conjugate to an element in J_1 then $j_0 \in J_0 \cap J_1$

Let $j_0(b_0, \ldots, b_\mu, c_0, \ldots)$ and $j_1(b_1, \ldots, b_{\mu+1}, c_0, \ldots)$ be conjugate where b_0 and c_0 are not trivial in j_0. Clearly the elements are conjugate in N_0, so if all of the generators in r_0 don't occur in j_0 and j_1 then they lie in a free subgroup of N_0 and b_0 and c_0 can be cyclically removed from j_0, a contradiction. If all of the generators in r_0 do occur in j_0 and j_1, by Lemma 14, the elements are not conjugate. So, to remove b_0, c_0, $j \in J_0 \cap J_1$.

Inductively, we can say $j \in J_{m-\mu} \cap \ldots \cap J_m$ and so $j = 1$. Hence, g is trivial, a contradiction. □

Example 3.10. Let $G =< a, b|ab^3a^{-1} >$ and $g = b^2$, $h = ab^{-1}a^2b^2a^{-3}b$. So $N_0 =< b_0, b_1|b_1b_1b_1 >$, and $g_0 = b_0^2$ and $h_0 = b_1b_3^2b_0$. So $s + \mu = 1 + 1 = 2 < m = 3$. So g and h are not conjugate.

Since s is bounded, it suffices to show the conjugacy problem is solvable in N for all possible g_s. Say $g_s, h \in K_l$ for some large enough l. Since K_l is malnormal in N, g_s must be conjugate to h in K_l and so it suffices to show the conjugacy problem is solvable for all K_k, done by induction on k. Since $|r_0| < |r|$ (and by extension $|r_i| < |r| \forall i$), $K_0 = N_0$ has a solvable conjugacy problem by the initial induction hypothesis and so does $N_i \forall i$. So $K_1 = \{N_0 * N_1; J_1\}$ where J_1 is strongly malnormal in N_0 and N_1 by Lemma 19, N_0 and N_1 have solvable conjugacy problems and the extended conjugacy problem with respect to J_1 in both N_0 and N_1 is solvable. So by Lemma 18, the conjugacy problem is solvable in K_1 and we can inductively show this for K_l.

CASE 2: Suppose $\sigma_a(r) = 0, \sigma_a(g) = s \neq 0$, where we assume without loss of generality that $s > 0$. Then if h is conjugate to g, $\sigma_a(h) = s$ (as, for $\sigma_a(x) = t$ and $x^{-1}gx = h$, $x'^{-1}a^{-t}a^s g' a^t x' = h$). Let $g = a^s g_1(b_0, \dots b_l, c_0, \dots, c_l)$ and $h = a^s h_1(b_0, \dots, b_l, c_0, \dots, c_l)$ where not all b_i, c_i will occur in g and h and suppose $x^{-1}gx = h$ where all words are of minimal length. Then $x_s^{-1} g_1 x_0 = h_1$ where x_s is $a^{-s}xa^s$ written as a word in N. For

$$h_1(b_0, \dots, b_l, c_0, \dots, c_l) g_1^{-1}(b_0, \dots b_l, c_0, \dots, c_l) x_s(b_{i+r}, \dots, b_{j+r}) = x_0(b_i, \dots, b_j)$$

we have that $i < 0$ or $j + s > l$, else we have that some $b_m, m < 0$ or $m > l - s$ can be eliminated from the right hand side, a contradiction of the minimality of x_0. Thus $x_0(b_0, \dots, b_{l-s})$. Choose an integer m such that $(m-1)s > l$. Then let $g^m = a^{ms} g_2(b_0, \dots, b_{l+(m-1)s}, c_0, \dots, c_{l+(m-1)s})$ and $h^m = a^{ms} h_2(b_0, \dots, b_{l+(m-1)s}, c_0, \dots, c_{l+(m-1)s})$ where the bottom l and top l generators cannot be removed from g_2 or h_2 else it would be possible to do the same with g_1 and h_1, contradicting the minimality of both.

Consider the equation

$$x^{-1}(b_{ms}, \dots, b_{l+(m-1)s}, c_{ms}, \dots, c_{l+(m-1)s}) g_2 x(b_0, \dots, b_{l-s}, c_0, \dots, c_{l-s}) = h_2$$

and suppose it takes place in $K_k = \{A * B | J\}$ where

$$A =< b_0, \dots, b_{\mu+l}, c_0, \dots, c_{\mu+l} >$$

$$B =< b_{l+1}, \dots, b_{l+(m-1)r}, c_{l+1}, \dots, c_{b_{l+(m-1)r}} >$$

$$J =< b_{l+1}, \dots, b_{\mu+l}, c_{l+1}, \dots, c_{\mu+l} >$$

Let $g_2 = s_1 ... s_u$ and $h_2 = t_1 ... t_v$, $u, v > 0$. Then

$$x^{-1}(b_{mr}, \dots, b_{l+(m-1)r}, c_{mr}, \dots, c_{l+(m-1)r}) \in B$$

$$x(b_0, \dots, b_{1-r}, c_0, \dots, c_{1-r}) \in A$$

In most cases, by Lemma 16, one can determine if an x exists. The case of g_2, h_2 being in different factors is eliminated since g_2, h_2, if not trivial, involve letters in $(A \cup B) - J$

The exceptional case is when $g_2 = s_B s_A$ and $h_2 = t_B t_A$, $t_A, s_A \in A$, $s_B, t_B \in B$, where both s_A, t_A non-trivial and s_B, t_B non-trivial. Then $x_m r^{-1} s_B s_A = t_B t_A x_0^{-1}$ which implies

$$x_{mr}^{-1} s_B = t_B j(b_{l+1}, \dots, b_{l+\mu}, c_{l+1}, \dots, c_{l+\mu}) \ (1)$$

and

$$j^{-1}(b_{l+1}, \dots, b_{l+\mu}, c_{l+1}, \dots, c_{l+\mu}) t_A x_0^{-1} = s_A \ (2)$$

. A translation of (1) by $-mr$ yields

$$x_0^{-1}s_B(-mr) = t_B(-mr)j(b_{l+1-mr},\dots,b_{l+\mu-mr},c_{l+1-mr},\dots,c_{l+\mu-mr})\ (3)$$

. Combining (2) and (3) one gets $t_A^{-1}js_A = t_B(-mr)j(-mr)s_B(-mr)^{-1}$. By examining the generators on both sides of the equation, both sides can be reduced to a word in $\{b_0,\dots,b_{l-r},c_0,\dots,c_{l-r}\}$, so all the letters of j on the left hand side can be removed. So j is not long, and can only involve letters in $R_0,\dots,R_l,t_A,s_A$. So, there are only finitely many possibilities for j.

Example 3.11. Let $G =< a,b|(aba^{-1}b^{-1})^2 >$ and $g = ab, h = b^{-1}a^{-1}baba$. Note that $\sigma_a(g) = 1 = \sigma_a(h)$. Thus we let $g_1 = a^{-1}ab = b$ and thus $g = b_0$ and $h_1 = a^{-1}b^{-1}a^{-1}baba$ and thus $h_1 = b_{-1}^{-1}b_{-2}b_{-1}$. To avoid $l < 0$ we conjugate h by a^2 to get $h_1' = b_1^{-1}b_0b_1$ (where if g is conjugate to h, it is conjugate to h'). Note that we can see that g_1 and h_1' are conjugate with $x = b_1$.

Thus $l = 1, s = 1$ implies that x is some power of b_0 and we choose $m = 3$ so that $2s = 2 > 1 = l$. So

$$g^3 = ababab$$

$$h'^3 = a^2(b^{-1}a^{-1}babab^{-1}a^{-1}babab^{-1}a^{-1}baba)a^{-2}$$

and $g_2' = b_1b_2b_3$ and $h_2' = b_1^{-1}b_0b_1b_2^{-1}b_1b_2b_3^{-1}b_2b_3$, where for $x^{-1}g_2x = h_2$, $x^{-1} \in A =< b_0,b_1,b_2 >$, $x \in B =< b_2,b_3 >$ and $J =< b_2 >$. Thus, through trial and error one can find that $x = ab^{-1}a^{-1}ba$.

CASE 3: Suppose there does not exist any letter that has exponent sum zero in r. We can embed G in a supergroup

$$C = \{G* < \hat{b} > |\hat{b}^s = b\}$$

as in Lemma 9 for some s. If g_1,g_2 are elements of G and are conjugate in H, they must be conjugate in G: if not, then g_1,g_2 are conjugates of elements in $< \hat{b} >$, say j_1,j_2, which are conjugate in H. However, since j_1,j_2 are invariant with respect to conjugation for any element in $< \hat{b} >$, then it is conjugated by an element in G and thus g_1,g_2 are conjugates in G. □

4 Conclusion

We have seen that one-relator groups can be seen as an HNN extension, and that certain subgroups of one-relator groups are free. We have also proved that the generalized word problem is solvable with respect to one relator groups. In particular this shows that the word problem is solvable with respect to one relator groups. As well, using the properties of malnormality we were able to prove that any one-relator group with torsion are solvable with respect to the conjugacy problem.

We note that Newman also proves the extended conjugacy problem for one-relator groups with torsion. As well, Pietrowski proved the isomorphism problem for one-relator groups with non-trivial center, and Dahmani and Guiradel have proved the isomorphism problem for all hyperbolic groups and thus for one-relator groups with torsion. Finally, in general, it is very difficult to determine the solvability of the conjugacy and isomorphism problems for one-relator groups with no torsion, and the problem has not been solved. We hope that the reader will continue to explore the topic further given the references below.

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