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HONOURS PROJECT

TITLE: Recursively Constructing Tight and Economical Single-Change Covering Designs and Circular Single-Change Covering Designs (Some New (v,4) and (v,5) Designs)

AUTHOR: Amanda Chafee

SUPERVISOR: Brett Stevens

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Contents

1 Abstract

This paper examines how to recursively construct various types of designs. Specifically, we will examine single change covering designs (*sccd*) and circular single change covering designs (circular *sccd*). The blocks of two *sccd*s with a fixed block size can be joined with $\frac{v(v-k+1)}{k-1}$ new blocks to form a larger *sccd*. With this construction we discus the infinite families of *sccd*s with a fixed block size k. Similarly, the blocks of two *sccd*s with a fixed block size can be joined, deleting the first $k-1$ existing blocks and adding $\frac{v(v'-k+1)}{k-1}$ new blocks, to build a circular *sccd*. With this construction We discuss the infinite families of circular *sccd*s with a fixed block size k.

2 Introduction

Let $\lambda, v, k \in \mathbb{N}$ with $k < v$ and $X = \{1, 2, ..., v\}$. A k-**block**, B, is a k-subset of X [3]. A $(v, k, \lambda)-$**design**, $(X, \mathcal{B})$, is a set of k-blocks where each pair of elements occurs in exactly λ blocks, $B \in \mathcal{B}$.

Designs are great to use when constructing tests for the purpose of comparing results. Suppose there is an electronics company that has 7 variations of a component that it would like to test in some devices interacting over a network. Due to the cost of producing machines, they only have 3 devices available for initial testing. They would like to know how each component interacts with the others when operating at the same time on the network. The company could use the $(7, 3, 1)$-design shown in Table 1

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
Device 1:	1	1	1	2	2	3	3
Device 2:	2	4	6	4	5	4	5
Device 3:	3	5	7	6	7	7	6

Table 1: B=(7,3,1) [3]

Each device is represented by a row, each having 1 slot for the component (element) that is being tested. (Each pair of elements occurs in exactly one test.) We only need to run seven tests to get all the results we require to compare the components. Each column represents one test (block). Table 2 shows an example of what the design could look like if we increased the number of components (elements) being tested by 2 and need a $(9, 3, 1)$-design.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

Table 2: $(9, 3, 1)$-design, (X, B) [3]

As you can see adding 2 extra points to our design increased the number of blocks required by 5. If we add 4 more points we get a (13,3,1)-design, as seen below.

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	B21	B22	B23	B24	B25	B26
1	2	3	4	5	6	7	8	9	10	11	12	13	2	3	4	5	6	7	8	9	10	11	12	13	1
3	4	5	6	7	8	9	10	11	12	13	1	2	6	7	8	9	10	11	12	13	1	2	3	4	5
9	10	11	12	13	1	2	3	4	5	6	7	8	5	6	7	8	9	10	11	12	13	1	2	3	4

Table 3: (13,3,1)-design

Single-change covering designs are different from designs. A **single-change covering design** (*sccd*) is an ordered set of blocks $\mathcal{L} = (B_1, B_2, ..., B_b)$ of size k from

$X = \{1, 2, 3, ..., v\}$, $k < v$ [4]. Every consecutive block only differs by one element. Every pair of elements in X must occur on a block at least once.

An example of tsccd(7,3) can be found in Table 4. Each block is represented by a column, and for emphasis, we highlight each introduction with an asterisk, *. We can see that sccds have more blocks than $(v, k, 1)$-designs.

1*	1	1	6*	7*	7	4*	4	4	1*
2*	2	2	2	2	3*	3	3	7*	7
3*	4*	5*	5	5	5	5	6*	6	6

Table 4: Tight single change covering design (7,3)

tsccds can help us optimize situations where making changes between the k-blocks is expensive, but the test itself is cheap. They can help save us time and energy in electrical testing and statistical computing [1]. Recall the example from earlier: a company is testing 7 variations of a component on some device and they want to test every pair interacting. Suppose switching components is time-expensive, it takes 15 minutes, but the test itself is quick, lasting 1 minute. We can use a tsccd to optimize testing prototypes. In our design from Table 1, testing would cost $15*15 = 225$ minutes to switch components and 7 minutes to actually test. However, if we use the $tsccd(7, 3)$ in Table 2 the cost of testing is $15 * 12 = 180$ minutes to switch component and 10 minutes in running time. This totals to 190 minutes to test instead of 232 minutes. In this case a $tsccd(v, k)$ is better than the (v, k, λ)-design, but in another situation where testing is more expensive and switching components is relatively cheap, a (v, k, λ)-design may be better-suited to the task.

A circular single-change covering design is similar to a single-change covering design, with the additional property of a single change occurring between the first and last block of the design. For an example see Table 5.

6*	6	6	6	4*	4	2*	2
4	3*	3	3	3	5*	5	4*
2	2	5*$_\wedge$	1*	1	1$_\wedge$	1	1$_\wedge$

Table 5: Economic single change covering design (6,3) [5]

In this paper, we will be looking at tight and economical single change covering designs, and tight and economical circular single change covering designs. In particular, we prove they can be recursively constructed. In section 3 we discuss some background on design theory and go over some constructions for steiner triple systems. In Section 2 we formalize and prove a construction used by the authors of [2] for tight single change covering designs. We then generalize it to work with economical single change covering designs. We also go over some constructions for economic designs. In Section 3 we modify the method of section 2 to construct both tight and economical circular single change covering designs. The power of these methods is that only a small number of small sccds are sufficient to construct infinite families of *sccd*s. Although it was already known that circular $tsccd(v, 3)$ exist if $v \equiv 0, 1 \pmod 4$, our method gives a new proof. We use our construction to give the first infinite families of circular *tsccd*s with $k = 4, 5$. We finish by summarizing all known infinite families of v for a fixed k which are tight and economical *sccd*s and what the targets of our continuing research will be.

3 Background

We can determine some properties of designs.

Proposition 1. *[3] Let $(x, \mathcal{B})$ be a $(v, k, \lambda)-$design. Then,*

(1) every element occurs in exactly $r = \frac{\lambda(v-1)}{(k-1)}$ blocks.

(2) $\mathcal{B}$ has exactly $b = \frac{\lambda(v^2-v)}{(k^2-k)} = \frac{vr}{k}$ blocks.

Proof.

(1) Let $(X, \mathcal{B})$ be a $(v, k, \lambda) - design$ and suppose that $x \in X$. Let r_x be the number of blocks where $x \in B$. We define the set

$$I = \{(y, B) : y \in X, y \neq x, B \in \mathcal{B}, \{x, y\} \subseteq B\}.$$

We will show that $r = \frac{\lambda(v-1)}{(k-1)}$ by counting $|I|$ two distinct ways. First, there are $v - 1$ ways to choose a $y \in X$ such that $y \neq x$. For each y the pair $\{x, y\}$ will occur λ times. Therefore, $|I| = \lambda(v - 1)$.

We also know that there are r_x ways to choose a block B such that $x \in B$. Moreover, there are $(k - 1)$ ways to choose a $y \in B$, $x \neq y$ for each choice of B. So $|I| = r_x(k - 1)$ as well.

Combining these two equations we get that

$$r_x(k-1) = \lambda(v-1)$$
$$r_x = \frac{\lambda(v-1)}{k-1}$$

So r_x is independent of x, and $r = \frac{\lambda(v-1)}{k-1}$.

(2) To count the number of blocks, $b = |\mathcal{B}|$, we define the set

$$I = \{(x, B) : x \in X, x \in B, B \in \mathcal{B}\}$$

Again, we will count $|I|$ two different ways. First, there are v ways to choose $x \in X$, and for every x we choose this way, it can be found in r blocks. So $|I| = vr$. Similarly, there are b ways to choose a block $B \in \mathcal{B}$ and for each block choice of B there are k ways to choose x. So $|I| = bk$. Combining these two equations we get

$$\begin{aligned} bk &= vr \\ b &= \frac{vr}{k} \\ b &= \frac{v[\lambda(v-1)]}{k(k-1)} \\ b &= \frac{\lambda(v^2-v)}{k^2-k} \end{aligned}$$

□

From Proposition 1 we find necessary conditions for a $(v, k, \lambda)-$design to exist.

Corollary 2. *[3] If a $(v, k, \lambda)-$design exists, then $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$*

For example, a (8,3,1)-design cannot exist since $r = \frac{\lambda(v-1)}{(k-1)} = \frac{1*7}{2} = \frac{7}{2} \notin \mathbb{N}$. A (9,5,1)-design does not exist, even though $r = \frac{\lambda(v-1)}{(k-1)} = \frac{1*8}{4} = \frac{8}{4} \in \mathbb{N}$, because $b = \frac{\lambda(v^2-v)}{(k^2-k)} = \frac{1*(81-9)}{25-5} = \frac{72}{20} \notin \mathbb{N}$. When studying Design Theory, we frequently ask, "When are these necessary conditions sufficient for a design to exist?"

We will show that a $(v, 3, 1)$-designs exists wherever it is possible.

Theorem 3. *If there exists a $(v, 3, 1)$-design then a $(2v+1, 3, 1)$-design exists.*

Proof. Suppose $(v, 3, 1)$ is a design $(X, \mathcal{B})$, $|X| = v$, $\mathcal{B} = \{B_i : 1 \leq i \leq b\}$. Let $X' = X \times \{0,1\} \cup \{\infty\}$. Let $\mathcal{B}'_\infty = \{\{\infty, (x,0), (x,1)\} : x \in X\}$. $\forall B = (x,y,z) \in \mathcal{B}$, $\mathcal{B}'_B = \{\{(x,0),(y,0),(z,0)\}, \{(x,1),(y,1),(z,0)\}, \{(x,1),(y,0),(z,1)\}, \{(x,0),(y,1),(z,1)\}\}$. We will show that $(X', \mathcal{B}'_\infty \cup \mathcal{B}'_B)$ is a $(2v+1, 3, 1)$-design.

As there are v elements in X, we cover v blocks in $|\mathcal{B}'_\infty|$. For every block in the original $(v, 3, 1)$-design we will have 4 new blocks we will cover in $\mathcal{B}'_B$. So it follows that

$$\begin{aligned} b' &= v + 4b \\ b' &= v + \frac{4\lambda(v^2-v)}{k^2-k} = v + \frac{4(1)(v^2-v)}{9-3} \\ b' &= v + \frac{4(v^2-v)}{6} = \frac{6v + 4v^2 - 4v}{6} \\ b' &= \frac{4v^2+2v}{6} = \frac{(2v+1)^2 - (2v+1)}{6} \end{aligned}$$

Now we must show that every pair occurs in the new design.

$\mathcal{B}'_\infty$ contains the blocks where every element of X, now called $(x,0)$, sees its duplicate, $(x,1)$ and the new element ∞. So ∞ has seen every element in X' and the only pairs left to check are $\forall x, y \in X$, we must have (x, i) and (y, j) in a block for each $i, j \in \{0, 1\}$. In the original design $\{(x, y)\}$ appear on some block $B_i \in \mathcal{B}$, in the new design$\{(x, i), (y, j)\}$ appear in the blocks $\mathcal{B}'_B$. Since we have the exact number of required blocks, we must cover every pair exactly once.

□

Theorem 4. *If there exists a $(v, 3, 1)$-design then a $(2v + 7, 3, 1)$-design exists.*

The proof of this theorem is similar to that of the $(2v + 1, 3, 1)$-design existing, but more complicated and not the focus of this paper. For a more detailed proof please see [8].

Theorem 5. *[3] A $(v, 3, 1)$-design exists if and only if $v = 1, 3 \pmod 6$.*

Proof. We know that in a $(v, 3, 1)$-design there must be $b = \frac{(v^2-v)}{6}$ blocks. So $v \equiv 0, 1, 3, 4 \pmod 6$. We also know that $r = \frac{(v-1)}{2}$. So $r \equiv 1 \pmod 2$. Hence v must be equivalent to $1, 3 \pmod 6$.

We know the designs $(7, 3, 1)$, $(9, 3, 1)$, and $(13, 3, 1)$ exist. See the tables above for an example of each. Using our constructions of $(2v + 1, 3, 1)$ designs and $(2v + 7, 3, 1)$ designs and these three smaller ones we can build every design.

Using a $(7, 3, 1)$-design with the $(2v+1, 3, 1)$ construction we form a $(15, 3, 1)$-design. Using a $(9, 3, 1)$-design with the $(2v + 1, 3, 1)$ construction we form a $(19, 3, 1)$-design. Using a $(7, 3, 1)$-design with the $(2v + 7, 3, 1)$ construction we form a $(21, 3, 1)$-design. Using a $(9, 3, 1)$-design with the $(2v + 7, 3, 1)$ construction we form a $(25, 3, 1)$-design. Building designs in this order follows a pattern for all $x \geq 1$:

$$\begin{aligned}
2(6x + 1) + 1 &= 6(2x) + 3 \\
2(6x + 3) + 1 &= 6(2x + 1) + 1 \\
2(6x + 1) + 7 &= 6(2x + 1) + 3 \\
2(6x + 3) + 7 &= 6(2x + 2) + 1
\end{aligned}$$

For every $y \in \mathbb{N}$, $y \geq 15$ and $y \equiv 1, 3 \pmod 6$ we see that $y \in \{6(2x) + 3, 6(2x + 1) + 1, 6(2x + 1) + 3, 6(2x + 3) + 1\}$. Suppose this holds for $x = n$. When $x = n + 1$ we get:

$$\begin{aligned}
2(6(n + 1) + 1) + 1 &= 6(2(n + 1)) + 3 \\
2(6(n + 1) + 3) + 1 &= 6(2(n + 1) + 1) + 1 \\
2(6(n + 1) + 1) + 7 &= 6(2(n + 1) + 1) + 3 \\
2(6(n + 1) + 3) + 7 &= 6(2(n + 1) + 2) + 1
\end{aligned}$$

So the $(v, 3, 1)$-design can be constructed from a $(v', 3, 1)$-design, $v \geq v'$ for all $v \geq 15$, $v \equiv 1, 3 \pmod 6$.

□

4 Single-Change Covering Designs

Single-change covering designs are different from designs. A **single-change covering design** (*sccd*) is an ordered set of blocks $\mathcal{L} = (B_1, B_2, ..., B_b)$ of size k from $X = \{1, 2, 3, ..., v\}$, $k < v$ [4]. In a sccd, each subsequent block differs from the previous block by exactly one element which is **introduced**, replacing an element from the previous block. Every pair of elements in X must occur on a block at least once. In the first block all k elements are introduced.

Let x, y be a pair of elements from the set X. We say that a block $\mathcal{B}$ of a design **covers** x, y if x and y are in the block and either x or y is introduced in this block. We say a *sccd* is **tight** if every pair is covered exactly once. We will use *tsccd*(v, k) to denote a tight single-change covering design over v elements.

An example of tsccd(7,3) can be found in Table 4. Each block is represented by a column, and for emphasis, we highlight each introduction with an asterisk, *. We can see that sccds have more blocks than $(v, k, 1)$-designs.

We can compute the exact number of blocks in a *tsccd*(v, k). There are $\binom{k}{2}$ pairs covered in the first block and in each subsequent block $k - 1$ new pairs are covered. Therefore $\binom{k}{2} + (b - 1)(k - 1) = \binom{v}{2}$ or equivalently $b = \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1$.

An economical single change covering design is similar to a tight *sccd*(v, k). A *sccd* is **economical** if $\binom{k}{2} + (b - 1)(k - 1) - \binom{v}{2} < k - 1$ or equivalently, $b = \left\lceil \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil$ [4]. We will use esccd(v, k) to denote an economical single-change covering design over v elements. The only difference between a tsccd(v, k) and an esccd(v, k) is that some pairs in an esccd may be covered more than once, but not many; in fact, no more than $k - 1$ pairs will be covered more than once. See Table 6 for an example of an economical single change covering design (14,4).

1*	1	1	1	1	1	1	1	1	1	1	1	13*	14*	4*	4	2*	8*	8	8	8	8	8	8	8	6*	6	6	6	6
2*	2	2	2	2	2	7*	8*	9*	9	9	9	9	9	9	9	9	9	9	13*	14*	10*	10	10	10	10	10	10	10	3*
13*	13	14*	3*	3	6*	6	6	6	10*	11*	12*	12	12	12	12	12	12	3*	3	3	3	11*	11	11	11	11	11	11	11
14*	4*	4∧	4	5*	5	5	5	5	5	5	5	5	5	∧5	7*	7	7	7	7	7	∧7	7	2*	4*	4	∧13*	14*	12*	12

Table 6: an esccd(14,4)

It is important to understand the distinctions between designs and single-change covering designs. Foremost, the number of times a pair of elements occur on a block in a design is always exactly λ. Whereas, in a *sccd* the number of times a pair of elements occur in a block is inconsequential, what matters is the number of times the pair is covered. Every pair will be covered once for a tight *sccd* and no more than $k-1$ pairs will be covered more than once in an economical *sccd*. Another distinction between designs and sccds is that, in designs, block order does not matter and there is no restriction on the number of elements in common between blocks. In a sccd, blocks are ordered and only one element changes between consecutive blocks.

Between two blocks B_i and B_{i+1} of a *sccd* we say that the set of elements that remain the same is the **unchanged subset**, $u_i = B_i \cap B_{i+1}$ [2]. An unchanged subset u_0 before the first block, B_1, or u_b after the last block B_b, can be any $k-1$ subset of B_1 or B_b respectively. For example, in Table 4, the unchanged subsets are u_1=$\{1,2\}$, u_2=$\{1,2\}$, u_3=$\{2,5\}$, u_4=$\{2,5\}$, u_5=$\{7,5\}$, u_6=$\{3,5\}$, u_7=$\{4,3\}$, u_8=$\{4,6\}$, u_9=$\{7,6\}$. Additionally u_0 could be $\{1,2\}$, $\{1,3\}$, or $\{2,3\}$, and u_{10} could be $\{1,7\}$, $\{6,7\}$, or $\{1,6\}$.

If the set X can be partitioned into $\frac{v}{(k-1)}$ unchanged subsets we say the sccd has an **expansion set**. If

$$X = \bigcup_{j=1}^{\frac{v}{k-1}} u_{i_j}$$

then we say that $\mathcal{E} = \{u_{i_1}, u_{i_2}, ..., u_{i_{\frac{v}{(k-1)}}}\}$ is an expansion set. If no $i_j = 0, b$ we say $\mathcal{E}$ is an **inner expansion set**. If $\mathcal{E}$ contains u_0, u_b or both, it is an **outer expansion set** and the u_0, u_b or both are called an **outer unchanged set**. Table 7 shows an example of a *sccd* with an outer expansion set. An example of an inner expansion set is given in Table 8. The expansion sets are indicated in each table with orange. Throughout the paper we will denote each expansion set location with a $\wedge$.

1*	1	1	1	1	1	1	1	1	4*	4	2*	8*	8	8	8	8	8	6*	6	6
2*	2	2	7*	8*	9*	9	9	9	9	9	9	9	9	10*	10	10	10	10	10	3*
3*	3	6*	6	6	6	10*	11*	12*	12	12	12	12	3*	3	11*	11	11	11	11	11
∧ 4*	5*	5	5	5	5	5	5	5	∧ 5	7*	7	7	7	∧ 7	7	2*	4*	4	∧ 12*	12

Table 7: tsccd(12,4) with outer expansion set [2]

In Table 7, the expansion set is $u_{i_1} = u_0 = \{1,2,4\}$, $u_{i_2} = u_9 = \{9,12,5\}$, $u_{i_3} = u_{14} = \{8,3,7\}$, and $u_{i_4} = u_{19} = \{6,10,11\}$.

1*	1	1	1	1	1	4*	4	4	4	4	4	3*	3	3	3	3	2*	2	5*	1*
2*	2	2	2	2	9*	9	9	10*	11*	11	12*	12	12	12	10*	10	10	10	10	10
3*	3	6*	7*	8*	8	8	8	8	8	7*	7	7	7	7	7	11*	11	11	11	11
4* ∧	5*	5	5	5	5	∧ 5	6*	6	6	6	∧ 6	6	8*	9*	9	9	9	12*	∧ 12	12

Table 8: *tsccd*$(12,4)$ with inner expansion set [2]

In Table 8, the expansion set is $u_{i_1} = u_1 = \{1,2,3\}$, $u_{i_2} = u_6 = \{9,8,5\}$, $u_{i_3} = u_{11} = \{4,7,6\}$, and $u_{i_4} = u_{19} = \{10,11,12\}$.

Expansion sets can be used to build new *sccd*s.

Proposition 6. *If a (tight, economical) sccd(v,k) with b blocks and an expansion set exists, then a (tight, economical) sccd$(v+1,k)$ with $\left(b+\frac{v}{k-1}\right)$ blocks exists.*

Proof.

Let $(X,\mathcal{L}=(B_1,...,B_b))$ be a *sccd*(v,k) with an expansion set $\mathcal{E}=\{u_{i_1},u_{i_2},...,u_{i_{\frac{v}{k-1}}}\}$. At each expansion location $u_i \in \mathcal{E}$ we will insert $B_i' = (B_i \cap B_{i+1}) \cup \{v+1\}$ between B_i and B_{i+1}. We will call this new $(v+1,k)$-design $(X',\mathcal{L}')$. We know that $|B_i| = k$, $|B_{i+1}| = k$ and that $|B_i \cap B_{i+1}| = k-1$ because $(X,\mathcal{L})$ is a (v,k) *sccd*. Because $B_i' = (B_i \cap B_{i+1}) \cup \{v+1\}$ we see that $|B_i'| = (k-1)+1 = k$. Since $\{v+1\} \notin B_i$ or B_{i+1}, and $B_i \cap B_{i+1} \subseteq B_i'$ we have $|B_i' \cap B_i| = |B_i' \cap B_{i+1}| = k-1$. Each B_i covers the exact same pairs it did in $(x,\mathcal{L})$. The only pairs covered in B_i' are $\{x, v+1\}$, for $x \in u_i$ and each is covered once because $\mathcal{E}=\{u_{i_1},u_{i_2},...,u_{i_{\frac{v}{k-1}}}\}$ is a partition of X. Therefore $(X',\mathcal{L}')$ is a *sccd*$(v+1,k)$. If $(X,\mathcal{L})$ was a *tight sccd*(v,k), then every pair from X is covered once in $(X,\mathcal{L})$ and covered once in $(X',\mathcal{L}')$. Thus, $(X',\mathcal{L}')$ will be a *tight sccd*$(v+1,k)$. If $(X,\mathcal{L})$ was an economic *sccd*(v,k), then $b = \left\lceil \frac{\binom{v}{2}-\binom{k}{2}}{k-1}+1 \right\rceil$ in $(X,\mathcal{L})$. Thus,

$$
\begin{aligned}
b' = b + \frac{v}{k-1} &\le \left\lceil \frac{\binom{v}{2}-\binom{k}{2}}{k-1} + 1 + \frac{v}{k-1} \right\rceil \\
&\le \left\lceil \frac{\frac{v(v-1)}{2}+\frac{v}{1}-\binom{k}{2}}{k-1} + 1 \right\rceil \le \left\lceil \frac{\frac{v(v-1)+2v}{2}-\binom{k}{2}}{k-1} + 1 \right\rceil \\
&\le \left\lceil \frac{\frac{v^2-v+2v}{2}-\binom{k}{2}}{k-1} + 1 \right\rceil \le \left\lceil \frac{\frac{v(v+1)}{2}-\binom{k}{2}}{k-1} + 1 \right\rceil \\
&\le \left\lceil \frac{\binom{v+1}{2}-\binom{k}{2}}{k-1} + 1 \right\rceil
\end{aligned}
$$

so $(X',\mathcal{L}')$ is economical.

□

For example, in Table 9 below, we show an *esccd*$(8,3)$, $(X,\mathcal{L})$, with an expansion set $\mathcal{E} = \{u_{i_1},u_{i_2},u_{i_3},u_{i_4}\} = \{u_2,u_7,u_{12},u_{14}\}$, taking $u_{i_4} = u_{14} = \{3,4\}$. We insert the blocks shown in blue to obtain the esccd shown in Table 10. A *sccd* generated this way cannot have an expansion set since element $v+1$ will never be in an unchanged subset by construction.

1*	1	1	1	1	1	3*	2*	4*	5*	5	5	5	5
2*	2	2	2	7*	7	7	7	7	7	6*	6	6	3*
3*	4*∧	5*	6*	6	8*	8 ∧	8	8	8	8	3*∧	4*	4 ∧

Table 9: *esccd*$(8,3)$ with repeated pair $(3,5)$

1*	1	1	1	1	1	1	3*	9*	2*	4*	5*	5	5	5	5	5	9*
2*	2	2	2	2	7*	7	7	7	7	7	7	6*	6	6	6	3*	3
3*	4*	9*	5*	6*	6	8*	8	8	8	8	8	8	3*	9*	4*	4	4

Table 10: *esccd*$(9,3)$

The authors of [2] implicitly use the following construction to show that $tsccd(v,4)$ exist, but never formally stated or proved it as a theorem.

Theorem 7. *[2] If there exists a $tsccd(v,k)$ with b blocks and a $tsccd(n(k-1),k)$ with b' blocks and an outer expansion set, then a $tsccd(v+(n-1)(k-1),k)$ with $b^* = b + b' + \frac{v(v-k+1)}{k-1}$ blocks exists. Furthermore, if the $tsccd(v,k)$ has an expansion set then the $tsccd(v+(n-1)(k-1),k)$ has an expansion set.*

Proof.

Suppose $(X,\mathcal{L})$ is a tsccd(v,k) and $(X',\mathcal{L}')$ a tsccd$(n(k-1),k)$ with an outer expansion set $\mathcal{E}' = \{u'_{i_j} : 1 \le j \le n\}$. If the only outer-set of $\mathcal{L}'$ is at $u'_{b'}$, we can simply reverse $(X,\mathcal{L}')$ so we can assume $\mathcal{L}'$ contains u'_0. To build $(X^*,\mathcal{L}^*)$, the $tsccd(v+(v-1)(k-1),k)$, we will append $(X,\mathcal{L})$ with the blocks of $(X',\mathcal{L}')$. In order to ensure we maintain the single change property we re-label the elements of $(X',\mathcal{L}')$ so that $B_b \cap B'_1 = X \cap X' = u'_0$. Therefore, $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ share $k-1$ elements. There are $v-(k-1) = v-k+1$ elements in $X \backslash X'$, we will refer to these as $\{e_1, e_2, ..., e_{v-k+1}\}$. There are $(n-1)(k-1)$ elements in $X' \backslash X$ and we will call them $\{h_1, h_2, ..., h_{(n-1)(k-1)}\}$.

Since $(X',\mathcal{L}')$ has an outer expansion set and the first expansion location of $(X',\mathcal{L}')$ is $u'_0 = B_b \cap B'_1$ we have that the remaining expansion locations partition $\{h_1, h_2, ..., h_{(n-1)(k-1)}\}$. For all $x \in \{e_1, ..., e_{v-k+1}\}$ and $2 \le j \le n$, we construct $B''_{i_j,x} = (B'_{i_j} \cap B'_{i_j+1}) \cup \{x\}$ and insert the $B''_{i_j,x}$ between B'_i and B'_{i+1} in any order.

The blocks of $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ cover the same pairs in $(X^*,\mathcal{L}^*)$ as they did in the original designs of $(X,\mathcal{L})$ and $(X',\mathcal{L}')$, with the exception of B'_1. In $(X',\mathcal{L}')$, B'_1 covered all of it's pairs. In $(X^*,\mathcal{L}^*)$, B'_1 only covers the pairs $\{x,y\}$ where $x = B'_1 \backslash B_b$, and $y \in u'_0 = B'_1 \cap B_b$; every other pair that occurs in B'_1 is covered once in $(X,\mathcal{L})$. The only pairs not covered in $(X,\mathcal{L})$ or $(X',\mathcal{L}')$ are $\{x,y\}$ where $x \in \{e_1, ..., e_{v-k+1}\}$ and $y \in \{h_1, h_2, ..., h_{(n-1)(k-1)}\} = \bigcup_{j=2}^{n} u'_{i_j}$. $B''_{i_j,x}$ covers $\{x,y\}$ for any $x \in \{e_1, ..., e_{v-k+1}\}$ and all $y \in u_{i_j}$ and no other $B''_{i_j,x}$ covers this pair. Therefore $(X^*,\mathcal{L}^*)$ is a tight *sccd* as every pair is covered exactly once.

Suppose now, that the tsccd(v,k), $(X,\mathcal{L})$, has an expansion set of $\mathcal{E} = \{u_{i_j} : 1 \leq j \leq \frac{v}{k-1}\}$ locations. Then tsccd$(v+(n-1)(k-1),k)$, $(X^*,\mathcal{L}^*)$, will have an expansion set

$$\mathcal{E}^* = \{\mathcal{E} \cup \mathcal{E}' \backslash u'_{i_0}\}.$$

□

For example, take $v = 6$, $k = 3$, and $n = 3$. Table 11 shows a tsccd$(6,3)$. $(X,\mathcal{L})$, and $(X',\mathcal{L}')$ have the same parameters and we will use a copy of $(X,\mathcal{L})$ with a different element set for $(X',\mathcal{L}')$. Shown in Table 12. In constructing the latter half of $(X^*,\mathcal{L}^*)$ based on $(X',\mathcal{L}')$, we note that: 1,2,3,4,5,6 become 7,2,5,8,9,10 respectively and $u'_0 = \{2,5\} = X \cap X'$. Following the proof of Theorem 7 we construct $(X^*,\mathcal{L}^*)$, a *tsccd*$(10,3)$, as seen in Table 13. The blocks of X are black, the blocks of X' are red, and the blocks of $B''_{i_j,x}$ are blue. Since $(X,\mathcal{L})$ has an outer expansion set, we can find an outer expansion set for $(X^*,\mathcal{L}^*)$.

1^*	1	1	1	3^*	3	2^*
2^*	2	5^*	6^*	6	6	6
$_\wedge 3^*$	$4^*_\wedge$	4	4	4	$5^*_\wedge$	5

Table 11: $D = tsccd(6,3)$ with outer expansion $u_{i_1} = \{2,3\}$, $u_{i_2} = \{1,4\}$, $u_{i_3} = \{5,6\}$

2^*	2	9^*	10^*	10	10	10
7^*	7	7	7	5^*	5	2^*
$_\wedge 5^*$	$8^*_\wedge$	8	8	8	$9^*_\wedge$	9

Table 12: $D' = tsccd(6,3)$ with outer expansion $u_{i_1} = \{2,5\}$, $u_{i_2} = \{7,8\}$, $u_{i_3} = \{9,10\}$

1^*	1	1	1	3^*	3	2^*	2	2	1^*	3^*	4^*	6^*	9^*	10^*	10	10	10	10	10	10	10
2^*	2	5^*	6^*	6	6	6	7^*	7	7	7	7	7	7	7	5^*	5	1^*	3^*	4^*	6^*	2^*
$_\wedge 3^*$	$4^*_\wedge$	4	4	4	$5^*_\wedge$	5	5	$8^*_\wedge$	8	8	8	8	8	8	8	$9^*_\wedge$	9	9	9	9	9

Table 13: $D^* = tsccd(10,3)$ with outer expansion $u_{i_1} = \{2,3\}$, $u_{i_2} = \{1,4\}$, $u_{i_3} = \{5,6\}$, $u_{i_4} = \{7,8\}$, $u_{i_5} = \{9,10\}$

Corollary 8. *If $v \geq 6$, a tsccd$(v,3)$ exists if and only if $v \equiv 2,3 \pmod 4$. [2]*

Proof. Suppose that $(X,\mathcal{L})$ is a *sccd*$(v,3)$. This has $b = \frac{\binom{v}{2}-\binom{k}{2}}{k-1} + 1 = \frac{(v-2)(v+1)}{4}$ blocks and $v = 2,3 \pmod 4$

We know that $(6,\mathcal{L}')$, a *tsccd*$(6,3)$ with an outer expansion set, exists (see Table 11). Applying Theorem 7 with a relabeled copy of $(6,\mathcal{L}')$, we can build a *tsccd*$(10,3)$ (see Table 13). Suppose that we have a *tsccd*$(v,3)$, $v \equiv 2 \pmod 4$, then we can build a

$tsccd(v+4,3)$ using $(6,\mathcal{L}')$ and Theorem 7. So we can construct a $v \equiv 2 \pmod 4$ $tsccd(v,3)$ for every v.

To construct the $tsccd(v',3)$, $v' \equiv 3 \pmod 4$, we can use Proposition 6 on each of the $tsccd(v,3)$ to get a $tsccd(v+1,3)$, $v+1 = 2+1 \equiv 3 \pmod 4$. We can use Proposition 6 because every $tsccd(v,3)$ we generate when $v \equiv 2 \pmod 4$ is built using designs with outer expansion sets, so the design built has an outer expansion set too. We have constructed a $tsccd(v,3)$ for every $v \equiv 2,3 \pmod 4$. □

Inspired by the recursive construction of tight *sccd*s, we considered whether we could do something similar for economic *sccd*s. We can. Before we get into the theorem and proof of this, we require a lemma. We remind the reader that a **single change covering design is tight** if it satisfies $b = \frac{\binom{v}{2}-\binom{k}{2}}{k-1}+1$ with equality. However, we say that a **block** B_i **in a** *sccd* **is tight** if the pairs it covers are not covered in any other block of the sccd.

Lemma 9. *Let $(X,\mathcal{L}=(B_1,...,B_b))$ be a $sccd(v,k)$ with $(k-1)b+\binom{k-1}{2}-\binom{v}{2}=d$. Suppose that $(X',\mathcal{L}'=(B_1,...,B_b,B'_{b+1},...,B'_{b'}))$ is a $sccd(v',k)$ and $X \subseteq X'$. If B'_i is tight $\forall\ i$, $b+1 \leq i \leq b'$, then $(X',\mathcal{L}')$ has $(k-1)b'+\binom{k-1}{2}-\binom{v'}{2}=d$.*

Proof.

Let $(X,\mathcal{L})$ be the esccd(v,k) and suppose that $(X,\mathcal{L})$ has been appended with $\mathcal{B}'=\{B'_{b+1},...,B'_{b'}\}$ tight blocks to form a $sccd(v',k)$, $(X',\mathcal{L}')$. There are $\binom{v'}{2}$ pairs we need to cover in $(X',\mathcal{L}')$. In the first block of $(X',\mathcal{L}')$ $\binom{k}{2}$ pairs are covered. The pairs from X are covered in $(B_1,...,B_b)$ possibly with some repetitions. Each block B'_i, $b+1 \leq i \leq b'$ covers exactly $k-1$ pairs not covered in any other block. If there are b blocks in the economical design $(X,\mathcal{L})$, and exactly b' blocks in the rest of $(X',\mathcal{L}')$ we find;

$$b = \left\lceil \frac{\binom{v}{2}-\binom{k}{2}}{k-1}+1 \right\rceil \tag{1}$$

$$b' = \frac{\binom{v'-v}{2}+(v'-v)v}{k-1} \tag{2}$$

and combining (1) and (2) we get

$$
\begin{aligned}
b'' &= b + b' \\
&= \left\lceil \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil + \frac{\binom{v'-v}{2} + (v'-v)v}{k-1} \\
&= \left\lceil \frac{\frac{v(v-1)}{2} - \binom{k}{2} + \frac{(v'-v)(v'-v-1)}{2} + \frac{2v(v'-v)}{2}}{k-1} \right\rceil + 1 \\
&= \left\lceil \frac{\frac{v^2-v+v'^2-v'v-v'-vv'+v^2+v+2vv'-2v^2}{2} - \binom{k}{2}}{k-1} \right\rceil + 1 \\
&= \left\lceil \frac{\frac{2v^2-2v^2+v-v+2vv'-2vv'-v'+v'^2}{2} - \binom{k}{2}}{k-1} \right\rceil + 1 \\
&= \left\lceil \frac{\frac{v'^2-v'}{2} - \binom{k}{2}}{k-1} \right\rceil + 1 \\
&= \left\lceil \frac{\frac{v'(1-v')}{2} - \binom{k}{2}}{k-1} \right\rceil + 1 \\
&= \left\lceil \frac{\binom{v'}{2} - \binom{k}{2}}{k-1} \right\rceil + 1.
\end{aligned}
$$

Therefore $(X', \mathcal{L}')$ is an economical *sccd*.

□

For example, consider the *economical sccd*$(4,3)$ found in Table 14. We can add the tight blocks from Table 15 to this *esccd*$(4,3)$ and form the *economical sccd*$(8,3)$ in Table 16. In Table 16 the blocks from the *esccd*$(4,3)$ are red and the tight single change blocks are black. Recall that Table 9 had another construction of an *esccd*$(8,3)$, so there is more than one way to build these designs.

1^*	4^*	4
2^*	2	1^*
$_\wedge 3^*$	3	$3_\wedge$

Table 14: esccd(4,3)

4^*	7^*	7	7	7	7	7	6^*	6	6	6
8^*	8	8	8	8	8	4^*	4	1^*	2^*	3^*
3^*	3	6^*	1^*	2^*	5^*	5	5	5	5	5

Table 15: Tight single change blocks

1^*	4^*	4	4	7^*	7	7	7	7	7	6^*	6	6	6
2^*	2	1^*	8^*	8	8	8	8	8	4^*	4	1^*	2^*	3^*
$_\wedge 3^*$	3	$3_\wedge$	3	3	$6^*_\wedge$	1^*	2^*	5^*	5	5	5	5	$5_\wedge$

Table 16: esccd(8,3)

Now that we have this lemma, we can use known *esccd* and *tsccd* (with expansion sets) to create bigger economic *sccd*s.

Theorem 10. *If there exists an economical sccd*(v,k) *with* b *blocks and a tight sccd*$(n(k-1),k)$ *with* b' *and an outer expansion set, then an economical sccd*$(v+(n-1)(k-1),k)$ *with* $b^* = b + b' + \frac{v(v-k+1)}{k-1}$ *blocks exists. Furthermore, if the economical sccd*(v,k) *has an expansion set then the economical sccd*$(v+(n-1)(k-1),k)$ *will have an expansion set.*

Proof.

Suppose $(X,\mathcal{L})$ is an economical *sccd*(v,k) and $(X',\mathcal{L}')$ a tight *sccd*$(n(k-1),k)$ with an outer expansion set $\mathcal{E}' = \{u'_{i_j} : 1 \le j \le n\}$. If the only outer-set of $\mathcal{L}'$ is at $u'_{b'}$, we can simply reverse $(X,\mathcal{L}')$; so we can assume $\mathcal{L}'$ contains u'_0. To build $(X^*,\mathcal{L}^*)$, the *tsccd*$(v+(v-1)(k-1),k)$, we will append $(X,\mathcal{L})$ with the blocks of $(X',\mathcal{L}')$. In order to ensure we maintain the single change property we re-label the elements of $(X',\mathcal{L}')$ so that $B_b \cap B'_1 = X \cap X' = u'_0$. Therefore, $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ share $k-1$ elements. Further, there are $v-(k-1) = v-k+1$ elements from X not in X', we will refer to these as $\{e_1, e_2, ..., e_{v-k+1}\}$. Also, there are $(n-1)(k-1)$ elements from X' not in X and we will call them $\{h_1, h_2, ..., h_{(n-1)(k-1)}\}$

Since $(X',\mathcal{L}')$ has an outer expansion set and the first expansion location of $(X',\mathcal{L}')$ is $u'_0 = B_b \cap B'_1$, we have that the remaining expansion locations partition $h_1, h_2, ..., h_{(n-1)(k-1)}$. Which means for every $x \in \{e_1, ..., e_{v-k+1}\}$ and $2 \le j \le n$, $B''_{i_j,x} = (B'_{i_j} \cap B'_{i_j+1}) \cup \{x\}$ we insert the $B''_{i_j,x}$ between B'_i and B'_{i+1} in any order to finish the construction.

The blocks of $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ cover the same pairs in $(X^*,\mathcal{L}^*)$ as they did in the original designs of $(X,\mathcal{L})$ and $(X',\mathcal{L}')$, with the exception of B'_1. In $(X',\mathcal{L}')$, B'_1 covered all of it's pairs. However in $(X^*,\mathcal{L}^*)$ B'_1 only covers the pairs x,y where $x = B'_1 \backslash B_b$, $y \in u'_0 = B'_1 \cap B_b$; every other pair that occurs in B'_1 are covered once in $(X,\mathcal{L})$. All the pairs not covered in $(X,\mathcal{L})$ or $(X',\mathcal{L}')$ are x,y where $x \in \{e_1, ..., e_{v-k+1}\}$ and $y \in \{h_1, h_2, ..., h_{(n-1)(k-1)}\} = \bigcup_{j=2}^{n} u_{i_j}$. $B''_{i_j,x}$ covers x,y for $x \in \{e_1, ..., e_{v-k+1}\}$ and $y \in u_{i_j}$ and no other $B''_{i_j,x}$ covers this pair. So by Lemma 9 we know that $(X^*,\mathcal{L}^*)$ is economical.

Suppose now, that the *esccd*(v,k), $(X,\mathcal{L})$, has an outer expansion set of $\mathcal{E} = \{u_{i_j} : 1 \le j \le \frac{v}{k-1}\}$ locations. Then *tsccd*$(v+(n-1)(k-1),k)$, $(X^*,\mathcal{L}^*)$, will have an expansion set

$$\mathcal{E}^* = \{\mathcal{E} \cup \mathcal{E}' \backslash u'_{i_0}\}.$$

□

It has previously been shown that $tsccd(v,3)$ exist. Our Theorem 7 reproved their existence.

Corollary 11. *If $v \geq 8$, an $esccd(v,3)$ exists if and only if $v \equiv 0,1 \pmod 4$.*

Proof. Suppose that $(X,\mathcal{L})$ is an $esccd(v,3)$. This has $b = \left\lceil \frac{\binom{v}{2}-\binom{k}{2}}{k-1} + 1 \right\rceil \geq \left\lceil \frac{(v-2)(v+1)}{4} \right\rceil$ blocks and $v = 0,1 \pmod 4$.

We know that $(8,\mathcal{L}')$, an $esccd(8,3)$ with an outer expansion set, exists (see Table 9). Applying Theorem 10 using $(6,\mathcal{L}')$ we build a $esccd(12,3)$. Suppose that we have an $esccd(v,3)$, $v \equiv 0 \pmod 4$, then we can build an $esccd(v+4,3)$, $v+4 \equiv 4 \pmod 4$, using a $(6,\mathcal{L}')$ and Theorem 10. This allows us to construct every $0 \equiv v \pmod 4$ $esccd(v,3)$.

To construct the $esccd(v,3)$, $v \equiv 1 \pmod 4$, we use Proposition 6 on each of the $esccd(v,3)$, $v \equiv 0 \pmod 4$, to get an $esccd(v+1,3)$, $v+1 \equiv 1 \pmod 4$. We can use Proposition 4 here because every $esccd(v,3)$ we generate when $v \equiv 0 \pmod 4$ is constructed using designs with outer expansion sets, so the design built has an outer expansion set too. So we have constructed every $esccd(v,3)$. □

Corollary 12. *Using $tsccd(12,4)$, $(15,4)$, $(18,4)$ with expansion sets and $esccd(14,4)$, $esccd(17,4)$, and $esccd(20,4)$; we can prove if $v \geq 12$ a $tsccd(v,4)$ exists if $v \equiv 0,1 \pmod 3$ and if $v \geq 14$ an $esccd(v,4)$ exists if $v \equiv 2 \pmod 3$. [2]*

We have not been able to find much on the creation of economical *sccd*s. So we thought it might be prudent to try and use some tight designs to create economical designs.

Theorem 13. *For $k \geq 4$ if a tight $sccd(v,k)$ with b blocks and an outer expansion set exists then a economical $sccd(v+2,k)$ with $\left(b + 2\frac{v}{k-1} + 1\right)$ blocks exists if $v+2$ is an admissible value for an economic design $(v+2,k)$.*

Proof. Let $(X,\mathcal{L} = (B_1,...,B_b))$ be a $tsccd(v,k)$ with an expansion set $\mathcal{E} = \{u_{i_1}, u_{i_2}, ..., u_{i_{\frac{v}{k-1}}}\}$. without loss of generality, assume the outer expansion is at u_0. At each expansion location $u_i \in \mathcal{E}$, with the exception of u_0, we will insert $B'_{j_1} = u_{i_i} \cup \{v+1\}$ and $B'_{j_2} = u_{i_i} \cup \{v+2\}$ between B_i and B_{i+1}. Let $x \in u_0$, before B_1 we will insert the 3 blocks $B'_1 = (u_0 \backslash x) \cup \{v+1, v+2\}$, $B'_2 = u_0 \cup \{v+1\}$, and $B'_3 = u_0 \cup \{v+2\}$. We will call this new $(v+2,k)$-design $(X',\mathcal{L}')$. As in the proofs of Theorem 7 and Theorem 10 $(X',\mathcal{L}')$ is a *sccd*.

Since $(X,\mathcal{L})$ was a tight $sccd(v,k)$, $b = \frac{\binom{v}{2}-\binom{k}{2}}{k-1} + 1$. Thus,

$$
\begin{aligned}
b' &= b + \frac{2v}{k-1} + 1 \\
&= \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1 + \frac{2v}{k-1} + 1 = \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + \frac{2v}{k-1} + 2 \\
&= \frac{\frac{v(v-1)}{2} + \frac{2v}{1} - \binom{k}{2}}{k-1} + 2 = \frac{\frac{v(v-1)+4v}{2} - \binom{k}{2}}{k-1} + 2 \\
&= \frac{\frac{v^2-v+4v}{2} - \binom{k}{2}}{k-1} + 2 = \frac{\frac{v^2-v+4v}{2} - \binom{k}{2} + (k-1)}{k-1} + 1
\end{aligned}
$$

Since $k \geq 2$, we have $k - 1 \geq 1$. Therefore,

$$
\begin{aligned}
b' &= \left\lceil \frac{\frac{v^2-v+4v}{2} - \binom{k}{2} + 1}{k-1} + 1 \right\rceil = \left\lceil \frac{\frac{v^2-v+4v+2}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil \\
&= \left\lceil \frac{\frac{(v+2)(v+1)}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil = \left\lceil \frac{\binom{v+2}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil
\end{aligned}
$$

□

To see an example of this see Table 6. The black blocks are from a tsccd(12,4). The blue blocks are the inserted blocks.

5 Single-Change Circular Covering Designs

A **single-change circular covering designs**, (*circular sccd*), is a cyclically ordered set of blocks $\mathcal{M} = (B_1, B_2, ..., B_b)$ of size k from $X = \{1, 2, 3, ..., v\}$, $k < v$ [5]. In a circular sccd, each subsequent block differs from the previous block by exactly one element which is **introduced**. This includes blocks B_b and B_1. So not only is $|B_i \cap B_{i+1}| = k - 1$, but $|B_b \cap B_1| = k - 1$ as well. Every pair of elements in X must be covered. Unlike a single change covering design, a single change circular covering design only covers $k - 1$ pairs in the first block. We will denote a *circular sccd*(v, k) by $(X, \mathcal{M})$.

Circular *sccd*s could be useful for running the same tests multiple times. Recall the company from earlier with only 3 machines to work with. To get more accurate results from testing, the company wants run these tests several times. In another scenario, this company may also have several versions of a second part it wishes to test all these components with. However, it does not care how these second parts preform compared with one another. So the company puts version a of this part in every machine and run the test for one complete cycle. Next they replace every a part with the next part b and run the second cycle. The company can do this for every version of this second party they have.

Lemma 14. *[5] For $v \geq 4$ and $k \geq 3$ the value of b, the minimum number of blocks in a circular $sccd(v,k)$ satisfies*

$$b \geq max\left\{v-1, \left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil\right\}$$

Proof. Exactly one element is introduced per block in a *circular sccd*. So if $b < v-1$, then at most $v-2$ distinct elements are introduced, and so at least 2 distinct elements are not introduced. Consequently this pair is never covered, a contradiction. So $b \geq v-1$

A $scccd(v,k)$ must cover every pair. There are $v(v-1)/2$ pairs of X and $k-1$ pairs covered per block. So $b \geq \frac{v(v-1)}{2(k-1)}$. □

If b equals $\left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil$ or $v-1$ we say that the *circular* $sccd(v,k)$ is **economical**. If $b = \frac{v(v-1)}{2(k-1)}$ then we say the *circular* $sccd(v,k)$ is **tight**.

If the set X can be partitioned into $\frac{v}{(k-1)}$ unchanged subsets we say the *scccd* has an **expansion set**. If

$$X = \bigcup_{j=1}^{\frac{v}{k-1}} u_{i_j}$$

then we say that $\mathcal{E} = \{u_{i_1}, u_{i_2}, ..., u_{i_{\frac{v}{(k-1)}}}\}$ is the expansion set. In the case of circular single change covering designs every design has both inner and outer expansion sets as you may rotate the first block until you get either. There is no choice regarding the unchanged subset at u_0 and u_b, as they are now equivalent and based on blocks B_1 and B_b. In Table 17 there is an example of a single change circular covering design with $v = 9$ and $k = 4$. The unchanged subsets are $u_0 = u_b = \{5,6,9\}, u_1 = \{1,5,6\}, u_2 = \{1,2,6\}, u_3 = \{1,2,7\}, u_4 = \{1,2,8\}, u_5 = \{1,2,3\}, u_6 = \{2,3,4\}, u_7 = \{3,4,9\}, u_8 = \{3,4,7\}, u_9 = \{3,4,5\}, u_{10} = \{4,5,6\}$, and $u_{11} = \{5,6,8\}$. The expansion set is $u_{i_1} = u_4$, $u_{i_2} = u_8$, and $u_{i_3} = u_{12}$.

1*	1	1	1	1	1	9*	9	5*	5	5	5
9	2*	2	2	2	2	2	7*	7	6*	6	6
5	5	7*	7	3*	3	3	3	3	3	8*	8
6	6	6	8*$_\wedge$	8	4*	4	4 $_\wedge$	4	4	4	9*$_\wedge$

Table 17: Tight single change circular covering design (9,4) with an expansion set [5]

We can see that circular *sccd*s have more blocks than *sccd*s.

In McSorley's paper discussing single-change circular covering designs [5], he states the following proposition without proof. The proof is similar to that of Proposition 6.

Proposition 15. *[5] If a (tight, economical) single change circular covering design-(v,k) with b blocks and an expansion set exists then a circular (tight, economical) sccd$(v+1,k)$ with $\lceil b + \frac{v}{k-1} \rceil$ blocks exists.*

Proof.

Let $(X, \mathcal{M})$ be a *circular sccd*(v,k) with an expansion set $\mathcal{E} = \{u_{i_1}, , u_{i_2}, ..., u_{i_{\frac{v}{k-1}}}\}$. At each expansion location $u_i \in \mathcal{E}$ we will insert $B_i' = (B_i \cap B_{i+1}) \cup \{v+1\}$ between B_i and B_{i+1}. If $B_i = B_b$ then $B_{i+1} = B_1$. We will call this new $(v+1,k)$-design $(X', \mathcal{M}')$. We know that $|B_i| = k$ and $|B_{i+1}| = k$ and that $|B_i \cap B_{i+1}| = k-1$ because $(X, \mathcal{M})$ is a (v,k) *circular sccd*. As $B_i' = (B_i \cap B_{i+1}) \cup \{v+1\}$ we get $|B_i'| = k$. Since $\{v+1\} \notin B_i$ or B_{i+1}, and $B_i \cap B_{i+1} \subseteq B_i'$ we have $|B_i' \cap B_i| = |B_i' \cap B_{i+1}| = k-1$. Each B_i covers the exact same pairs it did in $(X, \mathcal{M})$. The only new pairs covered in B_i' are $\{x, v+1\}$, for $x \in u_i$ and each is covered once because $\mathcal{E}$ is a partition of X. Therefore $(X', \mathcal{M}')$ is a *circular sccd*$(v+1,k)$

If $(X, \mathcal{M})$ was a tight single change circular covering design (v,k), then every pair from X is covered once in $(X, \mathcal{M})$ and therefore covered once in $(X', \mathcal{M}')$. Thus, $(X', \mathcal{M}')$ will be a tight circular *sccd*$(v+1,k)$. If $(X, \mathcal{M})$ was an economical circular *sccd*(v,k), then $b = \left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil$. Thus,

$$\begin{aligned} b' = b + \frac{v}{k+1} &\le \left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil + \frac{v}{k-1} \\ &\le \left\lceil \frac{v(v-1)}{2(k-1)} + \frac{v}{k-1} \right\rceil \le \left\lceil \frac{(v^2-v)(k-1) + 2v(k-1)}{2(k-1)^2} \right\rceil \\ &\le \left\lceil \frac{(v^2-v+2v)(k-1)}{2(k-1)^2} \right\rceil \le \left\lceil \frac{(v^2+v)(k-1)}{2(k-1)^2} \right\rceil \le \left\lceil \frac{v(v+1)}{2(k-1)} \right\rceil \end{aligned}$$

so $(X', \mathcal{M}')$ is economical. □

For example, consider the tight circular sccd(9,4) in Table 17. We can insert a block at each of the expansion locations with the new element 10. We do this in Table 18 to obtain a tight circular sccd(10,4). The inserted blocks are blue.

1*	1	1	1	1	1	1	9*	9	10*	5*	5	5	5	5
9	2*	2	2	2	2	2	2	7*	7	7	6*	6	6	6
5	5	7*	7	10*	3*	3	3	3	3	3	3	8*	8	10*
6	6	6	$8^*_\wedge$	8	8	4*	4	$4_\wedge$	4	4	4	4	$9^*_\wedge$	9

Table 18: Tight single change circular covering design (10,4)

Another examples of this is in Table 19. Using an economic single change circular covering design (6,3) we create the economic circular *sccd*$(7,3)$ in Table 20 using the expansion set u_3, u_6, and u_8. Again the new blocks are emphasized in blue.

6*	6	6	6	4*	4	2*	2
4	3*	3	3	3	5*	5	4*
2	2	5* ∧	1*	1	1 ∧	1	1 ∧

Table 19: Economic single change covering design (6,3) [5]

6*	6	6	6	6	4*	4	7*	2*	2	2
4	3*	3	3	3	3	5*	5	5	4*	4
2	2	5* ∧	7*	1*	1	1 ∧	1	1	1 ∧	7*

Table 20: Economic single change covering design (7,3) [5]

It would be nice if we could recursively join tight circular designs the way we did with tight single change. However, it is precisely because of their circular nature that we cannot. Our idea would be to use a *tsccd* $(X, \mathcal{L})$ with an outer expansion set at both u_0 and u_b as well as a second *tsccd* $(X', \mathcal{L}')$ that share the $2(k-1)$ elements at u_0 and u_b to create a circular *tsccd*. If we join any tsccd $(X, \mathcal{L})$ with another, $(X', \mathcal{L}')$, to make the design $(X^*, \mathcal{L}^*)$ such that $X^* = X \cup X'$, $\mathcal{L}^* = \{B_1, B_2, ..., B_b, B'_1, B'_2, ...B'_{b'}\}$, $X \cap X' = \{u_0 \cup u_b\}$ we do get a circular design. However, the blocks where $x \in u_b$ are reintroduced in $(X', \mathcal{L}')$ will re-cover the pairs $\{x, y\}$ such that $x, y \in u_0, x \neq y$. This means we have a design that while every pair is covered it is too big.

For example, consider the circular tight *sccd*$(9,4)$ in Table 17 and a copy of it where $1,2,7,8,3,4$ become $10,11,12,13,14,15$ respectively, see Table 21. Applying Theorem 7 we get the design in Table 22. The pairs $\{5,6\}$, $\{5,9\}$, and $\{6,9\}$ are covered twice, highlighted in green in the example. $\{5,6\}$ is in blocks 10 and 34 while pairs $\{5,9\}$ and $\{6,9\}$ are in blocks 12 and 36. We have a circular sccd with 36 blocks but a tight circular sccd would only have 35 blocks. Every time we do this we will end up with some extra blocks. To avoid this duplication we need to use this idea of joining two tsccd with some extra properties that allow us to easily remove any duplicated blocks.

5	5	12*	12	14*	14	14	14	14	14	13*	13
6	6	6	13*	13	15*	15	15	15	15	15	9*
10*	10	10	10	10	10	9*	9	5*	5	5	5
9	11*	11	11 ∧	11	11	11	12* ∧	12	6*	6	6 ∧

Table 21: Modified copy of a tight single change circular covering design (9,4) with an expansion set

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	
1*	1	1	1	1	1	9*	9	5*	5	5	5	5	5	12*	12	1*	2*	3*	4*	
2	2	2	2	2	7*	7	6*	6	6	6	6	6	13*	13	13	13	13	13	13	
5	5	7*	7	3*	3	3	3	3	3	8*	8	10*	10	10	10	10	10	10	10	
6	6	6	8* ∧	8	4*	4	4 ∧	4	4	4	9* ∧	9	11*	11	11 ∧	11	11	11	11	con't below
B21	**B22**	**B23**	**B24**	**B25**	**B26**	**B27**	**B28**	**B29**	**B30**	**B31**	**B32**	**B33**	**B34**	**B35**	**B36**					
7*	8*	14*	14	14	14	14	14	14	14	14	14	14	14	13*	13					
9	2*	13	15*	15	15	15	15	15	15	15	15	15	15	15	9*					
10	10	10	10	9*	9	1*	2*	3*	4*	7*	8*	5*	5	5	5					
11	11	11	11	11	12* ∧	12	12	12	12	12	12	12	6*	6	6 ∧					

Table 22: Almost tight single change circular covering design (15,4) with an expansion set. Pairs $\{5,6\}$, $\{5,9\}$, and $\{6,9\}$ are covered twice.

This means that we need to be smart about what designs we use to build larger circular ones. If we use two tight single change designs where both have expansion sets and one has a few extra properties, we can combine them to create a circular design.

If a $sccd(v,k)$ has an outer expansion set using both u_0 and u_b, the first $k-1$ unchanged sub sets equal u_0, and every point in u_b is introduced in the first $k-1$ blocks, we say that the $sccd(v,k)$ has the **disjoint-capable outer set** property. For example consider the tight $sccd(10,3)$ in Table 23. $u_0 = \{a,b\}$, and $u_b = \{c,d\}$. The first two blocks of this $tsccd(10,3)$ introduce c and d respectively.

The elements a and b only see c and d in the first two blocks; so only these two blocks cover the points $\{ac, ad, bc, bd\}$. By removing the first $k-1$ blocks from the design, we eliminate all the repeated pairs without eliminating any pairs that would need to be covered elsewhere. If we eliminated blocks in a design that didn't satisfy this property, we would also eliminate pairs that needed to be covered there. Consider block 34 in Table 22, if we eliminated it we would need to cover the pairs $\{6,14\}$ and $\{6,15\}$ else where.

a*	a	a	a	a	4*	5*	6*	6	6	6	6	6	6	6	4*	4	4	4	3*	3	3
b*	b	b	2*	3*	3	3	3	c*	d*	d	d	d	b*	a*	a	c*	2*	2	2	2	d*
∧ c*	d*	1*	1	1	1	1	1 ∧	1	1	2*	4*	5*	5	5	5 ∧	5	5	b*	b ∧	c*	c ∧

Table 23: Tight sccd(10,3) [2] with disjoint-capable outer set property

Lemma 16. *Let $(X, \mathcal{L} = (B_1, ..., B_b))$ be an economical $sccd(v,k)$. Suppose that $(X', \mathcal{M}' = (B_1, ..., B_b, B'_{b+1}, ..., B'_{b'}))$ is a circular $sccd(v',k)$ and $X \subseteq X'$. If B'_i is tight $\forall i$, $1 \le i \le b'$, and a tsccd(v,k) exists in $B'_{b+1}, ..., B'_{b'}$ then $(X', \mathcal{M}')$ is economical.*

Proof. Let $(X, \mathcal{L})$ be the $esccd(v,k)$. Next we appended $(X, \mathcal{L})$ with $\mathcal{B}' = \{B'_{b+1}, ..., B'_{b'}\}$ tight blocks to form a circular $sccd(v',k)$, $(X', \mathcal{M}')$.

There are $\binom{v'}{2}$ pairs we need to cover in $(X', \mathcal{M}')$. The pairs from X are covered in $(B_1, ..., B_b)$, possibly with some repetitions, with the exception of B_1. B_1 now only covers $\{x, y\}$ such that $x \in B_1 \cap B_2$, $y \in B_1 - B_2$. So B_1 covers $k-1$ pairs instead of $\binom{k}{2}$.

Each block B'_i, for $b+1 \leq i \leq b'$ covers exactly $k-1$ pairs not covered in any other block. There are $b = \left\lceil \frac{\binom{v}{2}-\binom{k}{2}}{k-1} \right\rceil + 1$ blocks in the economical design $(X, \mathcal{L})$. Every pair of elements $\{x, y\}$, such that $x, y \in (X' \backslash X \cup u_0 \cup u_b), x \neq y$, occur in $b'' = \frac{\binom{v'-v+2(k-1)}{2}-\binom{k}{2}}{k-1} + 1$ blocks, as only one element is ever introduced per block. Every other pair of elements $\{x, y\}$ such that $y \in X, x \in X' \backslash X$ will occur at the expansion set locations in $(X', \mathcal{M}') \backslash (X, \mathcal{M})$ which will add $b''' = \frac{(v'-v)(v-2(k-1))}{(k-1)}$ blocks. We are also going to be losing $k-1$ blocks to remove the duplicated pairs from u_0 and u_b in each design. Adding these blocks together we find;

$$
\begin{aligned}
b' &= b + b'' + b''' \\
&= \left\lceil \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1 \right\rceil + \left(\frac{\binom{v'-v+2(k-1)}{2} - \binom{k}{2}}{k-1} + 1 \right) - (k-1) + \left(\frac{(v'-v)(v-2(k-1))}{k-1} \right) \\
&= \left\lceil \frac{\frac{v(v-1)-k(k-1)}{2} + \frac{(v'-v+2k-2)(v'-v+2k-3)-k(k-1)}{2}}{k-1} + \frac{(v'-v)(v-2(k-1))}{k-1} + 3 - k \right\rceil \\
&= \left\lceil \frac{\frac{v'^2-v'}{2} + k^2 - 4k + 3}{k-1} + 3 - k \right\rceil \\
&= \left\lceil \frac{\frac{v'(v'-1)}{2} + k^2 - k^2 - 4k + 3k + k + 3 - 3}{k-1} \right\rceil \\
&= \left\lceil \frac{\frac{v'(v'-1)}{2}}{k-1} \right\rceil \\
&= \left\lceil \frac{v'(v'-1)}{2(k-1)} \right\rceil
\end{aligned}
$$

Therefore there are $b' = b''' + b'' + b$ blocks in $(X', \mathcal{M}')$ so the economical bound for the number of blocks in $(X', \mathcal{M}')$ holds. □

Theorem 17. *If a tight sccd(v, k) with b blocks satisfies the disjoint-capable outer set property exists and a second tight (economical) sccd(v', k) with b' blocks exists, then a tight (economical) circular sccd$(v + v' - 2(k-1), k)$ with $b + b' - (k-1) + (\frac{v}{k-1})(v' - 2(k-1))$ blocks exists. Furthermore, if the sccd(v', k) has an outer expansion set using both u'_0 and $u'_{b'}$ then the the circular sccd$(v + v' - 2(k-1), k)$ has an expansion set.*

Proof. Suppose that $(X, \mathcal{L})$ is a *tsccd*(v, k) with the outer expansion set $\mathcal{E}' = \{u_{i_j}$:

$1 \leq j \leq \frac{v'}{k-1}\}$ that satisfies the disjoint-capable outer set property and $(X', \mathcal{L}')$ is a *tsccd*(v', k). To build $(X^*, \mathcal{M}^*)$, the tight circular *sccd*$(v+v'-2(k-1), k)$, we will delete the first $k-1$ blocks from $(X, \mathcal{L})$. We then append $(X, \mathcal{L})$ with the blocks of $(X', \mathcal{L}')$. To keep the single change property of $(X^*, \mathcal{M}^*)$ we re-label the elements of $(X', \mathcal{L}')$ so that $B_b \cap B'_1 = u'_0$, $B'_{b'} \cap B_k = u_0$ and $u'_0 = (X \cap X') \backslash u_0$, $u'_{b'} = (X \cap X') \backslash u_b$. We can always do this because $u_0 \cap u'_0 = u_b \cap u'_b = \{\}$. So no two elements have the same name, and a permutation does not change how pairs are covered. Therefore, $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ share $2(k-1)$ elements. Further there are $v - 2(k-1)$ points from X not in X'. We will refer to these as $\{e_1, e_2, ..., e_{v-2(k-1)}\}$. There are also $v' - 2(k-1)$ elements from X' not in X and we will refer to these as $\{h_1, h_2, ..., h_{v'-2(k-1)}\}$.

Since $(X, \mathcal{L})$ has an outer expansion set at both the beginning, $u_0 = u_1 = ... = u_{k-1} = B'_b \cap B_k$, and end, $u_b = B'_0 \cap B_b$, of the set of expansion locations we get that the rest of the expansion locations partition $\{e_1, e_2, ..., e_{v-2(k-1)}\}$. For all $x \in \{h_1, h_2, ..., h_{v'-2(k-1)}.\}$ and $2 \leq j \leq \frac{v'}{k-1} - 2$, we construct $B''_{i_j,x} = (B_{i_j} \cap B_{i_j+1}) \cup \{x\}$ and insert $B''_{i_j,x}$ between B_i and B_{i+1} in any order.

The blocks of $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ cover the same pairs in $(X^*, \mathcal{M}^*)$ as they did in the original designs of $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ with the exception of $B_1, ..., B_{k-1}$ and B'_0. In $(X^*, \mathcal{M}^*)$ we no longer cover the pairs in blocks $B_1, ..., B_{k-1}$ of $(X, \mathcal{L})$ as we omitted these blocks. These omitted pairs are $\{x, y\}$ where $x \in u_0$, $y \in u_b$ and the pairs $\{x, x'\}$ where $x, x' \in u_0$ such that $x \neq x'$. However, this is exactly what we need as they are covered in the blocks of $(X', \mathcal{L}')$. These are the only possible repeated pairs when we combine these designs, so every one of these pairs is covered exactly once. We also only cover the pairs $\{x, y\}$ where $x \in B'_1 \backslash B_b$ and $y \in u'_0 = B'_1 \cap B_b$ in B'_0; all other pairs that occur in B'_1 are covered once in $(X, \mathcal{L})$ when they are introduced. The only pairs not covered in $(X, \mathcal{L})$ or $(X', \mathcal{L}')$ are $\{x, y\}$ where $x \in \{e_1, e_2, ..., e_{v-2(k-1)}\} = \bigcup_{j=2}^{\frac{v'}{k-1}-2} u_{i_j}$ and $y \in \{h_1, h_2, ..., h_{v'-2(k-1)}\}$. $B''_{i_j,x}$ covers $\{x, y\}$ for all $x \in \{h_1, h_2, ..., h_{v'-2(k-1)}\}$ and $y \in u_{i_j}$ and no other $B''_{i_j,x}$ covers this pair. Moreover, since $u_k = u'_{b'}$ there is a single change between the last block of the design and the first block. Therefore $(X^*, \mathcal{M}^*)$ is a circular tight *sccd* since every pair is covered exactly once and the unchanged subset from the start of the design and end are equivalent.

Suppose further that the *tsccd*(v', k), $(X', \mathcal{L}')$ has an expansion set of $\mathcal{E}' = \{u'_{i_j} : 1 \leq j \leq \frac{v'}{k-1}\}$ locations. Then the tight circular *sccd*$(v+v'-2(k-1), k)$, $(X^*, \mathcal{M}^*)$, will have an expansion1 set

$$\mathcal{E}^* = \{(\mathcal{E} \cup \mathcal{E}') \backslash \{u'_{i_0}, u'_{i_{b'}}\}\}.$$

For the economical version of the proof, we use the same construction and apply Lemma 16. □

For a tight example, consider the tight $sccd(10,3)$ in Table 23 and the tight $sccd(7,3)$ in Table 24. Applying Theorem 17 we construct the tight circular $sccd(13,3)$ using these two designs. The black blocks are from the tight $sccd(10,3)$ in Table 23, the red blocks are from the tight $sccd(7,3)$ in Table 24, and the blue blocks are the inserted $B''_{i_j,x}$ blocks. We inserted the block number above each block to aid in readability.

12*	13*	13	13	13	13	11*	11	b*	b
d*	d	11*	11	12*	12	12	d*	d	c*
c*	c	c	b*	b	a*	a	a	a	a

Table 24: A tight sccd(7,3) that has been re-labeled from [2]

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	
a	a	a	4*	5*	6*	6	6	6	6	6	6	6	6	6	6	4*	4	4	4	
b	2*	3*	3	3	3	11*	12*	13*	c*	d*	d	d	d	b*	a*	a	11*	12*	13*	
1*	1	1	1	1	1	1	1	1	1	1	2*	4*	5*	5	5	5	5	5	5	con't be-low

B21	B22	B23	B24	B25	B26	B27	B28	B29	B30	B31	B32	B33	B34	B35	B36	B37	B38	B39
4	4	4	3*	3	3	3	3	3	12*	13*	13	13	13	13	11*	11	b*	b
c*	2*	2	2	2	2	2	2	*d**	d	d	11*	11	12*	12	12	d*	d	c*
5	5	b*	b	11*	12*	13*	c*	c	c	c	c	b*	b	a*	a	a	a	a

Table 25: Tight circular sccd(13,3) constructed from the tsccd(10,3) in Table 23 and the tsccd(7,3) in Table 24. Since the tsccd(7,3) did not have an expansion set, our tight circular sccd(13,3) does not either.

For an economic example, consider the $tsccd(10,4)$ in Table 23 and the from Table 9 relabeled as shown in Table 26. Applying the economic version of Theorem 17 we construct the economic circular $sccd(v,k)$ in Table 27. The black blocks are from the $tsccd(10,3)$ in Table 23, the red blocks are from the $escd(8,3)$ in Table 26, and the blue blocks are the inserted $B''_{i_j,x}$ blocks. We inserted the block number above each block to aid in readability.

11*	11	11	11	11	11	c*	d*	b*	a*	a	a	a	a
d*	d	d	d	13*	13	13	13	13	13	12*	12	12	c*
$_\wedge$c*	b*	a*	12*	$_\wedge$12	14*	14	$_\wedge$14	14	14	14	c*	b*	b $_\wedge$

Table 26: Economic $sccd(8,3)$

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	
a	a	a	4*	5*	6*	6	6	6	6	6	6	6	6	6	6	6	4*	4	4	
b	2*	3*	3	3	3	11*	12*	13*	14*	c*	d*	d	d	d	b*	a*	a	11*	12*	
1*	1	1	1	1	1 ∧	1	1	1	1	1	1	2*	4*	5*	5	5	5 ∧	5	5	con't below

B21	B22	B23	B24	B25	B26	B27	B28	B29	B30	B31	B32	B33	B34	B35	B36	B37	B38	B39	B40	
4	4	4	4	4	3*	3	3	3	3	3	3	11*	11	11	11	11	11	c*	d*	
13*	14*	c*	2*	2	2	2	2	2	2	2	d*	d	d	d	d	13*	13	13	13	
5	5	5	5	b*	b ∧	11*	12*	13*	14*	c*	c ∧	c	b*	a*	12* ∧	12	14* ∧	14	14	con't below

B41	B42	B43	B44	B45	B46
b*	a*	a	a	a	a
13	13	12*	12	12	c*
14	14	14	c*	b*	b ∧

Table 27: Economic circular sccd(14,3)

McSorley constructs three infinite families of circular single-change covering designs in his paper [5]. Theorem 17 gives an alternate proof of two of these.

Theorem 18. *We know that there exists a* $tsccd(4t-2,3)$, $t \geq 2$, *with an outer expansion set,* $(X', \mathcal{L}')$. *Using a* $tsccd(6,3)$ *with an outer expansion set* $(X', \mathcal{L}')$ *and Theorem 7 we can build a* $tsccd(4t+2,3)$ *with an outer expansion set.*

Proof. If $t = 2$, then the $tsccd(6,3)$, $(X, \mathcal{L})$ is the first design we consider. We do know that this has an outer expansion set; see Table 11. So using $(X, \mathcal{L})$ and a $tsccd(6,3)$, $(X', \mathcal{L}')$, with an outer expansion set, with Theorem 7 we build a $tsccd(10,3)$, $(X', \mathcal{L}'')$. Recall, that as $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ have outer expansion sets, $(X'', \mathcal{L}'')$ does as well. Suppose this holds for $t = n$ and we build a $tsccd(4n+2,3)$. If $t = n+1$ then we build a $tsccd(4(n+1)-2,3)$, we see that $4(n+1)-2 = 4n+4-2 = 4n+2$. So we may build all $tsccd(4t+2,3)$ so that they have an outer expansion set.

□

Corollary 19. *A circular* $tsccd(v,3)$ *exists if and only if* $v \equiv 0, 1 \pmod 4$.

Proof. Suppose that $(X, \mathcal{M})$ is a circular $tsccd(v,3)$. This has $b = \frac{v(v-1)}{2(k-1)} = \frac{v(v-1)}{4}$ blocks and $v = 0, 1 \pmod 4$.

Using a $tsccd(10,3)$, $(X, \mathcal{L})$, with the disjoint-capable outer expansion set property and the $tsccd(4t-2,3)$ used in Theorem 18, with Theorem 17 we can build a circular tight single-change covering design where $v \equiv 0 \pmod 4, v \geq 12$ for every admissible v.

When $t = 2$, using $(X, \mathcal{L})$ and $tsccd(4t-2,3)$ with Theorem 17 we have the circular $tsccd(12,3)$. Suppose this is true for $t = n$. Using $(X, \mathcal{L})$ with $tsccd(4n-2,3)$ and

Theorem 17 we get a circular $tsccd(10 + 4n - 6, 3) = tsccd(4(n + 1), 3)$. If $t = n + 1$ then we find that $v = 10 + 4(n + 1) - 6 = 10 + 4n + 4 - 6 = 8 + 4n = 4(n + 2)$ and we can build a circular $tsccd(v, 3)$, $(X^*, \mathcal{M}^*)$, for all $v \equiv 0 \pmod 4$. Since both $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ have an outer expansion set so does $(X^*, \mathcal{M}^*)$. So we use Proposition 15 to build all the designs where $v \equiv 1 \pmod 4$. Further, we do know the tight circular single change covering designs $(4, 3)$, $(5, 3)$, $(8, 3)$, and $(9, 3)$ [5].

□

Theorem 20. *We know that there exists an* $esccd(4t, 3)$*,* $t \geq 1$*, with an outer expansion set,* $(X', \mathcal{L}')$*. Using a* $tsccd(6, 3)$ *with an outer expansion set* $(X', \mathcal{L}')$ *and Theorem 10 we can build a* $tsccd(4t + 4, 3)$ *with an outer expansion set.*

Proof. If $t = 1$, then the $esccd(4, 3)$, $(X, \mathcal{L})$ is the first design we consider. We do know that this has an outer expansion set; see Table 14. So using $(X, \mathcal{L})$ and an $tsccd(6, 3)$, $(X', \mathcal{L}')$, with an outer expansion set, with Theorem 10 we build a $esccd(8, 3)$, $(X', \mathcal{L}'')$. Recall, that as $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ have outer expansion sets, $(X'', \mathcal{L}'')$ does as well. Suppose this holds for $t = n$, so we get a $esccd(4n + 4, 3)$. If $t = n + 1$ for the $esccd(4(n + 1) + 4, 3)$, we see that $4(n + 1) + 4 = 4n + 4 + 4 = 4(n + 2)$. So we may build all $esccd(4t, 3)$ so that they have an outer expansion set.

□

Corollary 21. *A circular* $esccd(v, 3)$ *exists if and only if* $v \equiv 2, 3 \pmod 4$.

Proof. Suppose that $(X, \mathcal{M})$ is a circular $tsccd(v, 3)$. This has $b = \left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil = \left\lceil \frac{v(v-1)}{4} \right\rceil$ blocks and $v = 2, 3 \pmod 4$.

Using a $tsccd(10, 3)$, $(X, \mathcal{L})$, with the disjoint-capable outer expansion set property and the $tsccd(4t, 3)$ used in Theorem 20, with Theorem 17 we can build a circular tight single-change covering design where $v \equiv 2 \pmod 4$, $v \geq 14$ for every admissible v.

When $t = 2$, using $(X, \mathcal{L})$ and $tsccd(4t, 3)$ with Theorem 17 we have the circular $tsccd(14, 3)$. Suppose this is true for $t = n$. Using $(X, \mathcal{L})$ with $tsccd(4n, 3)$ and Theorem 17 we get a circular $tsccd(6 + 4n), 3) = tsccd(4(n + 1) + 2, 3)$. If $t = n + 1$ then we find that $v = 6 + 4(n + 1) = 4(n + 2) + 2$ and we can build a circular $tsccd(v, 3)$, $(X^*, \mathcal{M}^*)$, for all $v \equiv 2 \pmod 4$. Since both $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ have an outer expansion set so does $(X^*, \mathcal{M}^*)$. So we use Proposition 15 to build all the designs where $v \equiv 3 \pmod 4$. Further, we do know the economic circular single change covering designs $(6, 3)$, $(7, 3)$, $(10, 3)$, and $(11, 3)$ [5]. □

McSorley was able to build all circular $tsccd(9, 4)$ and circular $tsccd(10, 4)$. However, he could not construct any infinite families of this size. With Theorem 17 we can now produce the first infinite families for block size 4. The $tsccd(21, 4)$ shown in Table 28 has the disjoint-capable outer set property. Using this we can now construct circular $tsccd(v, 4)$ for $v \geq 27$. With McSorely's circular $tsccd(9, 4)$ and circular $tsccd(10, 4)$, if we can find a circular $tsccd(v, 4)$ for $v = 12, 13, 15, 16, 18, 19, 21, 22, 24, 25$ then we will have a circular $tsccd(v, 4)$ for every admissible v.

Theorem 22. *If we have a $tsccd(v, k)$ with an outer expansion set that uses both u_0 and u_b, we can build a $tsccd(2v - k + 1, k)$ with the disjoint-capable outer expansion set property.*

Proof. Suppose that $(X, \mathcal{B})$ is a $tsccd(v, k)$ that has an outer expansion set using both u_0 and u_b. Applying Theorem 7 to $(X, \mathcal{B})$ and a re-labeled copy of $(X, \mathcal{B})$ we get a $tsccd(v + v - k + 1, k)$, $(X', \mathcal{B}')$, with an outer expansion set using u_0 and u'_b. Since, we can insert the blocks at the outer expansion set in any order we like, we choose to insert the last $k - 1$ blocks in such a way that the last $k - 1$ elements introduced are $x \in u_0$. From paper [4] we know that the reverse of a tsccd is a tsccd. So we reverse $(X', \mathcal{B}')$ and get a $tsccd(v', k)$ with the disjoint capable outer expansion set. $\square$

We build the following design based on the last $tsccd(12, 4)$ constructed in [2] using Theorem 22. It has that disjoint-capable outer set property.

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20	
1*	1	1	1	1	1	1	1	1	1	1	1	1	1	1	13*	14*	15*	16*	17*	
2*	2	2	2	2	2	2	2	2	2	2	2	2	8*	8	8	8	8	8	8	
3*	3	3	3	3	3	3	3	3	3	3	3	7*	7	7	7	7	7	7	7	
∧13*	14*	15*	16*	17*	18*	19*	20*	21*	4*	5*	6*	6	6	9* ∧	9	9	9	9	9	con't below

B21	B22	B23	B24	B25	B26	B27	B28	B29	B30	B31	B32	B33	B34	B35	B36	B37	B38	B39	B40	
18*	19*	20*	21*	3*	10*	10	11*	11	11	11	11	11	11	11	11	11	11	11	11	
8	8	8	8	8	8	8	8	8	8	8	9*	10*	10	10	10	10	10	10	10	
7	7	7	7	7	7	7	7	7	7	2*	2	2	13*	14*	15*	16*	17*	18*	19*	
9	9	9	9	9	9	5*	5	4*	12*	12	12	12 ∧	12	12	12	12	12	12	12	con't below

B41	B42	B43	B44	B45	B46	B47	B48	B49	B50	B51	B52	B53	B54	B55	B56	B57	B58	B59	B60	
11	11	11	11	11	4*	4	4	4	4	4	4	4	4	6*	19*	19	20*	20	20	
10	10	10	10	10	10	5*	5	5	5	5	5	17*	17	17	17	17	17	17	17	
20*	21*	1*	3*	6*	6	6	6	6	6	6	16*	16	16	16	16	16	16	16	16	
12	12	12	12	12	12	12	9*∧	13*	14*	15*	15	15	18*	18 ∧	18	14*	14	13*	21*	con't below

B61	B62	B63	B64	B65	B66	B67	B68	B69
20	20	20	20	20	20	13*	13	13
17	18*	19*	19	19	19	19	14*	14
5*	5	5	4*	6*	15*	15	15	15
21	21	21 ∧	21	21	21	21	21	18*∧

Table 28: A $tsccd(21,4)$ with disjoint-capable outer set property. The expansion set is $\{u_{i_1} = \{1,2,3\}, u_{i_2} = \{7,8,9\}, u_{i_3} = \{10,11,12\}, u_{i_4} = \{4,5,6\}, u_{i_5} = \{16,17,18\}, u_{i_6} = \{19,20,21\}, u_{i_7} = \{13,14,15\}\}$.

The following Table is the first circular $tsccd(27,4)$ built; constructed using $tsccd(21,4)$ found in Table 28 and a $tsccd(12,4)$. The black columns are the blocks of Table 28. The red columns are the $tsccd(12,4)$ and the blue columns are the inserted $B''_{i_j,x}$ blocks.

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15
1	1	1	1	1	1	1	1	1	1	1	1	22*	23*	24*
2	2	2	2	2	2	2	2	2	2	8*	8	8	8	8
3	3	3	3	3	3	3	3	3	7*	7	7	7	7	7
∧16*	17*	18*	19*	20*	21*	4*	5*	6*	6	6	9*∧	9	9	9

con't below

B16	B17	B18	B19	B20	B21	B22	B23	B24	B25	B26	B27	B28	B29	B30
25*	26*	27*	13*	14*	15*	16*	17*	18*	19*	20*	21*	3*	10*	10
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
9	9	9	9	9	9	9	9	9	9	9	9	9	9	5*

con't below

B31	B32	B33	B34	B35	B36	B37	B38	B39	B40	B41	B42	B43	B44	B45
11*	11	11	11	11	11	11	11	11	11	11	11	11	11	11
8	8	8	8	9*	10*	10	10	10	10	10	10	10	10	10
7	7	7	2*	2	2	22*	23*	24*	25*	26*	27*	13*	14*	15*
5	4*	12*	12	12	12 ∧	12	12	12	12	12	12	12	12	12

con't below

B46	B47	B48	B49	B50	B51	B52	B53	B54	B55	B56	B57	B58	B59	B60
11	11	11	11	11	11	11	11	11	4*	4	4	4	4	4
10	10	10	10	10	10	10	10	10	10	5*	5	5	5	5
16*	17*	18*	19*	20*	21*	1*	3*	6*	6	6	6	6	6	6
12	12	12	12	12	12	12	12	12	12	12	9*∧	22*	23*	24*

con't below

B61	B62	B63	B64	B65	B66	B67	B68	B69	B70	B71	B72	B73	B74	B75
4	4	4	4	4	4	4	4	4	6*	22*	23*	24*	25*	26*
5	5	5	5	5	5	5	17*	17	17	17	17	17	17	17
6	6	6	6	6	6	16*	16	16	16	16	16	16	16	16
25*	26*	27*	13*	14*	15*	15	15	18*	18 ∧	18	18	18	18	18

con't below

B76	B77	B78	B79	B80	B81	B82	B83	B84	B85	B86	B87	B88	B89	B90
27*	19*	19	20*	20	20	20	20	20	20	20	20	20	20	20
17	17	17	17	17	17	17	18*	19*	19	19	19	19	19	19
16	16	16	16	16	16	5*	5	5	22*	23*	24*	25*	26*	27*
18	18	14*	14	13*	21*	21	21	21 ∧	21	21	21	21	21	21

con't below

B91	B92	B93	B94	B95	B96	B97	B98	B99	B100	B101	B102	B103	B104	B105
20	20	20	13*	13	13	13	13	13	13	13	13	13	13	13
19	19	19	19	14*	14	14	14	14	23*	24*	25*	25	25	25
4*	6*	15*	15	15	15	15	15	1*	1	1	1	26*	3*	27*
21	21	21	21	21	18*∧	2*	22*	22	22	22	22	22	22	22 ∧

con't below

B106	B107	B108	B109	B110	B111	B112	B113	B114	B115	B116	B117
2*	2	14*	24*	24	24	24	24	24	1*	1	1
25	25	25	25	25	26*	26	26	26	26	26	15*
27	27	27	27	15*	15	3*	3	3	3	3	3
22	23*	23	23	23	23 ∧	23	14*	2*	2	27*	27

Table 29: A circular *tsccd*(27, 4). The expansion set is $\{u_{i_1} = \{1,2,3\}, u_{i_2} = \{7,8,9\}, u_{i_3} = \{10,11,12\}, u_{i_4} = \{4,5,6\}, u_{i_5} = \{16,17,18\}, u_{i_6} = \{19,20,21\}, u_{i_7} = \{13,14,15\}, u_{i_8} = \{22,25,27\}, u_{i_9} = \{23,24,26\}\}$.

Corollary 23. *For $v \geq 27$ a circular tsccd$(v, 4)$ exists if and only if $v \equiv 0, 1 \pmod 3$,*

Proof. If a circular *tsccd*$(v, 4)$ exists then $b = \frac{v(v-1)}{6} \in \mathbb{N}$, so $v \equiv 0, 1, 3, 4 \pmod 6$. From Corollary 12 for all $v \geq 12, v \equiv 0, 1 \pmod 3$ there exits a *tsccd*$(v, 4)$. Using the *tsccd*$(21, 4)$ with the disjoint-capable outer set property from Table 28 and Theorem 17 we can construct a circular *tsccd*$(v+15, 4)$ therefore we can construct a circular *tsccd*$(v, 4)$ for every $v \geq 27, v \equiv 0, 1 \pmod 3$.

□

Corollary 24. *For $v \geq 29$ a circular esccd$(v, 4)$ exists if and only if $v \equiv 2 \pmod 3$,*

Proof. If a circular *esccd*$(v, 4)$ exists then $b = \left\lceil \frac{v(v-1)}{6} \right\rceil \in \mathbb{N}$, so $v \equiv 2, 5 \pmod 6$. From Corollary 12 for all $v \geq 14, v \equiv 2 \pmod 3$ there exits an *esccd*$(v, 4)$. Using the *tsccd*$(21, 4)$ with the disjoint-capable outer set property from Table 28 and Theorem 17 we can construct a circular *esccd*$(v+15, 4)$ therefore we can construct a circular *tsccd*$(v, 4)$ for every $v \geq 29, v \equiv 2 \pmod 3$. □

There are only two tight *sccd*$(v, 5)$ known, both with $v = 20$. Phillips, [7], found them by using a computer in a staged search. Phillip's second example, shown in Table 30 has an outer expansion set using both u_0 and u_b. Using this we may apply Theorem 7 and Proposition 6 to find *tsccd*$(v, 5)$ for every $v \equiv 4, 5 \pmod{16}$. These two infinite families are enough to find half of the admissible designs for v. To find the rest of the admissible designs we would need a *tsccd*$(28, 5)$.

Using Theorem 22 once we find that we can build a *tsccd*$(36, 5)$ with 156 blocks that has the disjoint-capable outer set property, as seen in Table 33. With it, we may now use Theorem 17 to find circular *tsccd*$(v, 5)$, for all $v \geq 48$, $v \equiv 0 \pmod{16}$. See Table 34 for an example of a circular *tsccd*$(48, 5)$ which has 282 blocks. We may also apply Proposition 15 to find a circular *tsccd*$(v, 5)$ for all $v \equiv 1 \pmod{16}$. These two infinite families cover half the admissible tight circular *sccd*$(v, 5)$ for $v \geq 48$. We require a *tsccd*$(28, 5)$ to know a construction for all admissible designs $v \geq 48$. To find a circular *tsccd*$(v, 5)$ for all admissible v we would need to construct a design for everything below 48. These possible $(v, 5)$s have $v = 16, 17, 24, 25, 32, 33, 40, 41$.

Corollary 25. *If $v \equiv 4, 5 \pmod{16}$ then a tsccd$(v, 5)$ exists, $v \geq 20$.*

Corollary 26. *If $v \equiv 6 \pmod{16}$ then a tsccd$(v, 5)$ exists, $v \geq 22$.*

Corollary 27. *If $v \equiv 0, 1 \pmod{16}$ then a circular tsccd$(v, 5)$ exists, $v \geq 48$.*

Corollary 28. *If $v \equiv 2 \pmod{16}$ then a circular tsccd$(v, 5)$ exists, $v \geq 50$.*

1*	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5*	5	5	2*
2*	2	2	2	9*	9	11*	12*	12	12	12	12	12	12	12	12	12	12	12	12
3*	3	3	8*	8	10*	10	10	13*	14*	14	14	14	18*	19*	19	19	19	19	19
4*	4	7*	7	7	7	7	7	7	7	15*	16*	17*	17	17	20*	20	20	20	20
∧5*	6*	6	6	6	6∧	6	6	6	6	6	6	6	6	6	6	6	8*	9*	9

con't below

11*	11	11	11	11	11	7*	15*	15	15	15	15	15	15	15	15	15	15	15	10*
12	12	12	13*	14*	18*	18	18	10*	16*	16	16	16	16	16	16	16	16	16	16
19	19	19	19	19	19	19	19	19	19	17*	17	17	17	17	17	17	17	17	17
20	20	20	20	20	20	20	20	20	20	20	9*	8*	8	8	2*	5*	5	5	5
9∧	3*	4*	4	4	4	4	4	4	4	4	4	4∧	3*	11*	11	11	7*	13*	13

con't below

10	10	10	10	10	9*
16	14*	14	14	14	14
18*	18	18	18	18	18
5	5	2*	8*	3*	3
13	13	13	13	13	13∧

Table 30: tsccd(20,5) with outer expansion set $\{u_{i_1} = \{2,3,4,5\}, u_{i_2} = \{1,10,7,6\}, u_{i_3} = \{11,12,19,20\}, u_{i_4} = \{15,16,17,8\}, u_{i_5} = \{9,14,18,13\}\}$

9*	9	23*	23	23	23	23	23	23	23	31*	32*	33*	33	33	36*	36	36	36	36
14*	14	14	14	25*	25	27*	28*	28	28	28	28	28	28	28	28	28	28	28	28
18*	18	18	24*	24	26*	26	26	29*	30*	30	30	30	34*	35*	35	35	35	35	35
21*	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	13*	13	13	14*
∧13*	22*	22	22	22	22∧	22	22	22	22	22	22	22	22	22	22	22	24*	25*	25

36	36	36	36	36	36	36	36	36	36	36	25*	24*	24	24	14*	13*	13	13	13
28	28	28	29*	30*	34*	34	34	26*	32*	32	32	32	32	32	32	32	32	32	32
35	35	35	35	35	35	35	35	35	35	33*	33	33	33	33	33	33	33	33	33
27*	27	27	27	27	27	23*	31*	31	31	31	31	31	31	31	31	31	31	31	26*
25∧	18*	9*	9	9	9	9	9	9	9	9	9	9∧	18*	27*	27	27	23*	29*	29

13	13	14*	24*	18*	18
32	30*	30	30	30	30
34*	34	34	34	34	34
26	26	26	26	26	25*
29	29	29	29	29	29∧

Table 31: relabeled tsccd(20,5) with outer expansion set $\{u_{i_1} = \{9,14,18,13\}, u_{i_2} = \{23,26,21,22\}, u_{i_3} = \{36,28,35,27\}, u_{i_4} = \{24,32,33,31\}, u_{i_5} = \{30,34,25,29\}\}$. Row 1 and 4 have switched places.

25*	25	39*	39	39	39	39	39	39	39	44*	45*	46*	46	46	48*	48	48	48	48
30*	30	30	30	4*	4	42*	43*	43	43	43	43	43	43	43	43	43	43	43	43
34*	34	34	40*	40	41*	41	41	5*	2*	2	2	2	3*	47*	47	47	47	47	47
37*	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	29*	29	29	30*
∧29*	38*	38	38	38	38∧	38	38	38	38	38	38	38	38	38	38	38	40*	4*	4

48	48	48	48	48	48	48	48	48	48	48	4*	40*	40	40	30*	29*	29	29	29
43	43	43	5*	2*	3*	3	3	41*	45*	45	45	45	45	45	45	45	45	45	45
47	47	47	47	47	47	47	47	47	47	46*	46	46	46	46	46	46	46	46	46
42*	42	42	42	42	42	39*	44*	44	44	44	44	44	44	44	44	44	44	44	41*
4∧	34*	25*	25	25	25	25	25	25	25	25	25	25∧	34*	42*	42	42	39*	5*	5

29	29	30*	40*	34*	34
45	2*	2	2	2	2
3*	3	3	3	3	3
41	41	41	41	41	4*
5	5	5	5	5	5∧

Table 32: Second relabeled tsccd(20,5) with outer expansion set $\{u_{i_1} = \{25, 30, 34, 29\}, u_{i_2} = \{39, 41, 37, 38\}, u_{i_3} = \{48, 43, 47, 42\}, u_{i_4} = \{40, 45, 46, 44\}, u_{i_5} = \{2, 3, 4, 5\}\}$

25*	29*	30*	34*	21*	22*	23*	24*	26*	27*	28*	31*	32*	33*	35*	36*	1*	1	1	1	
2*	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	
3*	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	8*	
4*	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	7*	7	
∧5*	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	6*	6	6	con't below

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
9*	9	21*	22*	23*	24*	25*	26*	27*	28*	29*	30*	31*	32*	33*	34*	35*	36*	11*	12*	
8	10*	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	
6	6∧	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	con't below

1	1	1	1	1	1	1	1	5*	5	5	2*	11*	11	11	11	11	11	11	11	
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	
13*	14*	14	14	14	18*	19*	19	19	19	19	19	19	19	19	19	19	19	19	19	
7	7	15*	16*	17*	17	17	20*	20	20	20	20	20	20	20	20	20	20	20	20	
6	6	6	6	6	6	6	6	6	8*	9*	9	9∧	21*	22*	23*	24*	25*	26*	27*	con't below

11	11	11	11	11	11	11	11	11	11	11	11	11	11	7*	15*	15	15	15	15	
12	12	12	12	12	12	12	12	12	12	12	13*	14*	18*	18	18	10*	16*	16	16	
19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	17*	17	
20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	9*	
28*	29*	30*	31*	32*	33*	34*	35*	36*	3*	4*	4	4	4	4	4	4	4	4	4	con't below

15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	
8*	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	2*	
4∧	21*	22*	23*	24*	25*	26*	27*	28*	29*	30*	31*	32*	33*	34*	35*	36*	3*	11*	11	con't below

15	15	15	10*	10	10	10	10	10	9*	9	9	23*	23	23	23	23	23	23	23	
16	16	16	16	16	14*	14	14	14	14	14	14	14	14	25*	25	27*	28*	28	28	
17	17	17	17	18*	18	18	18	18	18	18	18	18	24*	24	26*	26	26	29*	30*	
5*	5	5	5	5	5	2*	8*	3*	3	21*	21	21	21	21	21	21	21	21	21	
11	7*	13*	13	13	13	13	13	13	13∧	13	22*	22	22	22	22∧	22	22	22	22	con't below

31*	32*	33*	33	33	36*	36	36	36	36	36	36	36	36	36	36	36	36	36	36	
28	28	28	28	28	28	28	28	28	28	28	28	28	29*	30*	34*	34	34	26*	32*	
30	30	30	34*	35*	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	
21	21	21	21	21	21	13*	13	13	14*	27*	27	27	27	27	27	23*	31*	31	31	
22	22	22	22	22	22	22	24*	25*	25	25∧	18*	9*	9	9	9	9	9	9	9	con't below

36	25*	24*	24	24	14*	13*	13	13	13	13	13	14*	24*	18*	18
32	32	32	32	32	32	32	32	32	32	32	30*	30	30	30	30
33*	33	33	33	33	33	33	33	33	33	34*	34	34	34	34	34
31	31	31	31	31	31	31	31	31	26*	26	26	26	26	26	25*
9	9	9∧	18*	27*	27	27	23*	29*	29	29	29	29	29	29	29∧

Table 33: tsccd(36,5) with disjoint-capable outer expansion set $\{u_{i_1} = \{2,3,4,5\}, u_{i_2} = \{1,10,7,6\}, u_{i_3} = \{11,12,19,20\}, u_{i_4} = \{15,16,17,8\}, u_{i_5} = \{9,14,18,13\}, u_{i_6} = \{21,26,23,22\}, u_{i_7} = \{27,28,35,36\}, u_{i_8} = \{31,32,33,24\}, u_{i_9} = \{25,30,34,29\}\}$

21*	22*	23*	24*	26*	27*	28*	31*	32*	33*	35*	36*	1*	1	1	1	1	1	1	1	1	1	1	1	1	
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	9*	9	37*	38*	39*	40*	41*	42*	43*	
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	8*	8	10*	10	10	10	10	10	10	10	
4	4	4	4	4	4	4	4	4	4	4	4	4	4	7*	7	7	7	7	7	7	7	7	7	7	
∧5	5	5	5	5	5	5	5	5	5	5	5	5	6*	6	6	6	6∧	6	6	6	6	6	6	6	con't below

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
44*	45*	46*	47*	48*	21*	22*	23*	24*	25*	26*	27*	28*	29*	30*	31*	32*	33*	34*	35*	36*	11*	12*	12	12	
10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	13*	14*	
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	con't below

1	1	1	1	1	1	5*	5	5	2*	11*	11	11	11	11	11	11	11	11	11	11	11	11	11	11	
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	
14	14	14	18*	19*	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	
15*	16*	17*	17	17	20*	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	
6	6	6	6	6	6	6	8*	9*	9	9∧	37*	38*	39*	40*	41*	42*	43*	44*	45*	46*	47*	48*	21*	22*	con't below

11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	7*	15*	15	15	15	15	
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	13*	14*	18*	18	18	10*	16*	16	16	
19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	17*	17	
20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	9*	
23*	24*	25*	26*	27*	28*	29*	30*	31*	32*	33*	34*	35*	36*	3*	4*	4	4	4	4	4	4	4	4	4	con't below

15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	
16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	
8*	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	
4∧	37*	38*	39*	40*	41*	42*	43*	44*	45*	46*	47*	48*	21*	22*	23*	24*	25*	26*	27*	28*	29*	30*	31*	32*	con't below

15	15	15	15	15	15	15	15	15	15	10*	10	10	10	10	10	9*	9	9	9	9	9	9	9	9	
16	16	16	16	16	16	16	16	16	16	16	16	14*	14	14	14	14	14	14	14	14	14	14	14	14	
17	17	17	17	17	17	17	17	17	17	17	18*	18	18	18	18	18	18	18	18	18	18	18	18	18	
8	8	8	8	8	8	2*	5*	5	5	5	5	5	2*	8*	3*	3	37*	38*	39*	40*	41*	42*	43*	44*	
33*	34*	35*	36*	3*	11*	11	11	7*	13*	13	13	13	13	13	13	13∧	13	13	13	13	13	13	13	13	con't below

9	9	9	9	9	9	23*	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	
14	14	14	14	14	14	14	14	25*	25	37*	38*	39*	40*	41*	42*	43*	44*	45*	46*	47*	48*	27*	28*	28	
18	18	18	18	18	18	18	24*	24	26*	26	26	26	26	26	26	26	26	26	26	26	26	26	26	29*	
45*	46*	47*	48*	21*	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	
13	13	13	13	13	22*	22	22	22	22∧	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	con't below

23	31*	32*	33*	33	33	36*	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	
28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	
30*	30	30	30	34*	35*	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	
21	21	21	21	21	21	21	13*	13	13	14*	27*	27	27	27	27	27	27	27	27	27	27	27	27	27	
22	22	22	22	22	22	22	22	24*	25*	25	25∧	37*	38*	39*	40*	41*	42*	43*	44*	45*	46*	47*	48*	18*	con't below

36	36	36	36	36	36	36	36	36	25*	24*	24	24	24	24	24	24	24	24	24	24	24	24	24	24	
28	29*	30*	34*	34	34	26*	32*	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	
35	35	35	35	35	35	35	35	33*	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	
27	27	27	27	23*	31*	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	
9*	9	9	9	9	9	9	9	9	9	9∧	37*	38*	39*	40*	41*	42*	43*	44*	45*	46*	47*	48*	18*	27*	con't below

14*	13*	13	13	13	13	13	14*	24*	18*	18	37*	37	37	37	37	37	37	37	37	37	37	37	37	37	
32	32	32	32	32	32	30*	30	30	30	30	30*	30	30	30	4*	4	42*	43*	43	43	43	43	43	43	
33	33	33	33	33	34*	34	34	34	34	34	34*	34	34	40*	40	41*	41	41	5*	2*	2	2	2	3*	
31	31	31	31	26*	26	26	26	26	26	25*	25*	25	39*	39	39	39	39	39	39	39	44*	45*	46*	46	
27	27	23*	29*	29	29	29	29	29	29	29∧	29*	38*	38	38	38	38∧	38	38	38	38	38	38	38	38	con't below

37	37	29*	29	29	30*	42*	42	42	42	42	42	39*	44*	44	44	44	44	44	44	44	44	44	44	44	
43	43	43	43	43	43	43	43	43	5*	2*	3*	3	3	41*	45*	45	45	45	45	45	45	45	45	45	
47*	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	46*	46	46	46	46	46	46	46	46	
46	48*	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	4*	40*	40	40	30*	29*	29	29	
38	38	38	40*	4*	4	4∧	34*	25*	25	25	25	25	25	25	25	25	25	25∧	34*	42*	42	42	39*	5*	con't below

41*	41	41	41	41	41	4*
45	45	2*	2	2	2	2
46	3*	3	3	3	3	3
29	29	29	30*	40*	34*	34
5	5	5	5	5	5	5∧

Table 34: circular tsccd(48,5)

Table 34: circular tsccd(48,5) with outer expansion set $\{u_{i_1} = \{2,3,4,5\}, u_{i_2} = \{1,10,7,6\}, u_{i_3} = \{11,12,19,20\}, u_{i_4} = \{15,16,17,8\}, u_{i_5} = \{9,14,18,13\}, u_{i_6} = \{21,26,23,22\}, u_{i_7} = \{27,28,35,36\}, u_{i_8} = \{31,32,33,24\}, u_{i_9} = \{25,30,34,29\}, u_{i_{10}} = \{39,41,37,38\}, u_{i_{11}} = \{48,43,47,42\}, u_{i_{12}} = \{40,45,46,44\}$.

6 Conclusion

In conclusion the following infinite families of single-change covering designs exist:

(i) There exists a tight *sccd*$(v,3)$ for all $v \equiv 2,3 \pmod 4$, $v \geq 6$ [2, 5]

(ii) There exists an economic *sccd*$(v,3)$ for all $v \equiv 0,1 \pmod 4$, $v \geq 4$

(iii) There exists a tight *sccd*$(v,4)$ for all $v \equiv 0,1 \pmod 3$, $v \geq 12$ [2]

(iv) There exists an *esccd*$(v,4)$ for all $v \equiv 2 \pmod 3$, $v \geq 14$

(v) There exists a tight *sccd*$(v,5)$ for all $v \equiv 4,5 \pmod{16}$, $v \geq 20$

(vi) There exists an economic *sccd*$(v,5)$ for all $v \equiv 6 \pmod{16}$, $v \geq 20$

(vii) There exists a tight circular *sccd*$(v,3)$ for all $v \equiv 0,1 \pmod 4$, $v \geq 4$ [5].

(viii) There exists an economic circular *sccd*$(v,3)$ for all $v \equiv 2,3 \pmod 4$, $v \geq 6$

(ix) There exists a tight circular *sccd*$(v,4)$ for all $v \equiv 0,1 \pmod 3$, $v \geq 27$

(x) There exists an economic circular *sccd*$(v,4)$ for all $v \equiv 2 \pmod 3$, $v \geq 29$

(xi) There exists a tight circular *sccd*$(v,5)$ for all $v \equiv 0,1 \pmod{16}$, $v \geq 48$

(xii) There exists a economic circular *sccd*$(v,5)$ for all $v \equiv 2 \pmod{16}$, $v \geq 50$

With McSorely's circular *tsccd*$(9,4)$ and circular *tsccd*$(10,4)$, if we can find a circular *tsccd*$(v,4)$ for $v = 12,13,15,16,18,19,21,22,24,25$ then we will have a circular *tsccd*$(v,4)$ for every admissible v.

If we can find a *tsccd*$(28,5)$ and circular *tsccd*$(v,5)$ for $v = 16,17,24,25,32,33,40,41$ then we will have a circular *tsccd*$(v,5)$ for every admissible v.

If we can find an *esccd*$(v,5)$ for $v = 23,24,25,26,27,30,31,32,33,34,35$ then we can find an *esccd*$(v,5)$ for every admissible v.

If we can find all the *esccd*$(v,5)$ and circular *tsccd*$(v,5)$ for $v = 16,17,24,25,32,33,40,41$ with the disjoint-capable outer set property, then we will have a circular *esccd*$(v,5)$ for every admissible v.

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