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1 Introduction

Topological group theory is the study of groups that have continuous group operations. Topological groups were first studied by Sophus Lie who began studying Lie groups, a particular class of topological groups, in 1873. In the year 1900, David Hilbert asked if the differentiability assumptions in the definition of a Lie group could be weakened to continuity. Specifically, Hilbert asked whether it is sufficient for a topological group to be a topological manifold for it to be a Lie group; this is Hilbert's fifth problem. In the first half of the 20^{th} century, the theory of topological groups was significantly developed, some major contributors being von Neumann, Pontryagin, and Weil. These developments would be important in fields like harmonic analysis, and the work of Gleason, Zippin, and Montgomery in 1952 would resolve Hilbert's fifth problem in the affirmative.

In this expository paper, we seek to give an account of the theory of topological groups. In the second section, we explore the basic theory of topological groups. Familiar examples are provided, and much of the basic theory of groups is generalized to the setting of topological groups, such as quotients, solvable and nilpotent groups, isomorphism theorems, and group actions. In the third section, we discuss the surprising separation and metrizability properties of topological groups. In particular, we show that in a topological group there is an abundance of non-trivial real-valued continuous functions, and that under very reasonable conditions a topological group has a left-invariant pseudo-metric. In the fourth section, we return to Lie groups; we briefly recount the basic theory of Lie groups, develop the tool of the matrix exponential, and explore the solution to Hilbert's fifth problem and its relation to the no small subgroups condition. In the fifth and final section we study in depth a translation invariant measure on locally compact a group, called the Haar measure. We give various examples of such measures as well as show that it exists on all locally compact groups and satisfies certain uniqueness properties.

2 Topological Groups

2.1 Examples and Basic Properties of Topological Groups

Definition 2.1.1. A topological group is a triple $(G, \tau, \cdot)$ where (G, τ) is a topological space and $(G, \cdot)$ is a group such that the multiplication map $(x, y) \mapsto xy$ and the inversion map $x \mapsto x^{-1}$ are both continuous.

Remark 2.1.2. The condition that the multiplication and inversion map both be continuous is equivalent to the requirement that the map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ be continuous.

We will frequently make use of nets, so we briefly mention how they function in topological groups and how we will write them. For the unfamiliar reader, nets are treated in detail in [11]. We will always omit the directed set for a net and instead write "(x_λ) is a net in G." If we need to refer to a subnet, then we shall use a subscript, so if (x_λ) is a net, then we will write a subnet as (x_{λ_γ}). We will use the notation $x_\lambda \to x$ to mean that x_λ converges to x, and the reader should recall that net convergence is in general not unique, it is only unique in a Hausdorff topological space. Continuity of the multiplication map translates to the following statement:

> If (x_α, y_α) is a net in $G \times G$ which converges to (x, y), then $x_\alpha y_\alpha \to xy$.

Continuity of the inversion maps translates to the following statement:

> If (x_λ) is a net in G which converges to x, then $x_\lambda^{-1} \to x^{-1}$.

A property that comes up frequently when discussing topological groups is local compactness. There are several distinct definitions for this property, all of which are equivalent in Hausdorff spaces, so for clarity we present ours here.

Definition 2.1.3. A topological space is called locally compact if every point has a neighbourhood base of compact sets. Similarly, a topological space is called locally connected if every point has a neighbourhood base of connected sets.

Before proceeding any further we will give some examples of topological groups.

Example 2.1.4. Let G be any group. When G is given the discrete topology, the topology where the collection of open sets is $\mathcal{P}(G)$ the power set of G, then G is a topological group. In a similar manner, the indiscrete topology on G, the topology where the collection of open sets is $\{\emptyset, G\}$, makes G into a topological group.

Example 2.1.5. The sets of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are all topological groups under addition when they are given their standard topologies. Similarly, if n is any positive integers, $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$ are all topological groups under addition when given their standard topologies.

Example 2.1.6. Let n be a positive integer, and let F denote either the real numbers or the complex numbers. Let $M_n(F)$ be the collection of all $n \times n$ matrices with entries in F which is made into a topological space by identifying the space with F^{n^2} via the map $(a_{ij})_{i,j=1}^n \mapsto (x_k)_{k=1}^{n^2}$ where $x_k = a_{ij}$ when $k = n(j-1)+i$. Then the linear group

$$\mathrm{GL}(n, F) = \{A \in M_n(F) : \det A \neq 0\}$$

is a topological group, under the matrix multiplication and the the topology it inherits as a subset of $M_n(F)$. That $\mathrm{GL}(n, F)$ is a group under matrix multiplication is standard in group theory, we only argue that the group operations are continuous. Let $A = (a_{ij})_{i,j=1}^n, B = (b_{ij})_{i,j=1}^n \in \mathrm{GL}(n, F)$. That the multiplication is continuous is apparent from the calculation of the entries of AB: we have that $AB = (\sum_{k=1}^n a_{ik}b_{kj})_{i,j=1}^n$, so the entries of AB are polynomials in the entries of A and B. Observe that the determinant is a continuous map from $M_n(F)$ to F as the determinant takes a matrix to a polynomial in its entries. From this it follows that the map given by $A \mapsto \frac{1}{\det A}$ is continuous on $\mathrm{GL}(n, F)$ and that the map $A \mapsto \mathrm{adj}(A)$, from A to the adjugate of A, is continuous on $M_n(F)$ (we are using here that the determinant map on $M_{n-1}(F)$ is continuous), from which we deduce that $A \mapsto \frac{\mathrm{adj}(A)}{\det A} = A^{-1}$ is continuous. The observation that the determinant is continuous shows that $\mathrm{GL}(n, F)$ is an open subset of $M_n(F)$, and so $\mathrm{GL}(n, F)$ is locally compact. A particular consequence of this result is that the set of non-zero real numbers and the set of non-zero complex numbers are topological groups under multiplication when given their standard topology.

Example 2.1.7. Let p be a prime number, and observe that for any non-zero rational number r, there is a unique integer m and rational number q with neither numerator nor denominator divisible by p such that $r = p^m q$; this is a consequence of the fundamental theorem of arithmetic. Hence we obtain a function called the p-adic absolute value: $|\cdot|_p : \mathbb{Q} \to [0, \infty)$ given by defining $|r|_p = p^{-m}$, when $r \neq 0$, and $|0|_p = 0$. Then the p-adic distance function $d_p : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)$, is given by $d_p(q, r) = |q - r|_p$ and is a metric on $\mathbb{Q}$. Then the completion of $(\mathbb{Q}, d_p)$ is called the set of p-adic numbers, it is denoted by $\mathbb{Q}_p$ and is a topological group where the addition is given by taking limits of rational numbers: if $x = \lim x_n$ and $y = \lim y_n$ then $x + y = \lim(x_n + y_n)$ (in fact the completion of any topological group under a metric is a topological group in this way). With some work it can be shown that the elements of $\mathbb{Q}_p$ are given by $\sum_{i=n}^{\infty} a_i p^i$, where $a_i \in \{0, 1, \ldots, p-1\}$ for every $i \geq n$.

We shall now elaborate on a standard piece of notation that we will use for the entirety of the paper. If G is a group and A and B are subsets of G, by the product of A and B we mean the set

$$AB = \{ab : a \in A,\ b \in B\}.$$

When $A = B$, we shall commonly use the notation A^2 to refer to AA. In greater generality, when $A_1, \ldots, A_n$ are subsets of G, we define $A_1 \cdots A_n$ by

$$A_1 \cdots A_n = \{a_1 \cdots a_n : a_j \in A_j,\ j = 1, \ldots, n\}.$$

Note that the product of sets is associative: if $A, B, C \subseteq G$, then $(AB)C = A(BC) = ABC$. If n is a positive integer, by A^n we mean $A \cdots A$, where there are n copies of A in the product. When $n = 0$, we define $A^n = \{e\}$, where e is the identity element of the group. When n and m are nonnegative integers $A^n A^m = A^{n+m}$. By A^{-1} we mean the set

$$A^{-1} = \{a^{-1} : a \in A\}.$$

When n is a negative integer we define $A^n = (A^{-1})^{-n}$, and again if n and m are nonpositive integers we have $A^n A^m = A^{n+m}$, however, we do not in general have $A^n A^m = A^{n+m}$ when n and m are arbitrary integers. When $a \in G$, we define $aB = \{a\}B$ and $Ba = B\{a\}$. The set aB is called the left translate of B by a, and the set Ba is the right translate of B by a. Observe that the product of two sets A and B can be written as the union of left or right translates:

$$AB = \bigcup_{a \in A} aB = \bigcup_{b \in B} Ab.$$

Furthermore, if $\mu : G \times G \to G$ is the multiplication map of the group G and $\iota : G \to G$ is the inversion map of the group G, then for any subsets A and B of G, we have $\mu(A \times B) = AB$ and $\iota(A) = A^{-1}$. With this notation explained we are able to more easily formulate some properties of topological groups.

Proposition 2.1.8. Let G be a group with a topology on it. Then G is a topological group if and only if the following two conditions are satisfied:

(i) For every neighbourhood U of xy, there is a neighbourhood V of x and a neighbourhood W of y such that $VW \subseteq U$.

(ii) For every neighbourhood U of x^{-1}, there is a neighbourhood of V of x such that $V^{-1} \subseteq U$.

Proof: It suffices to prove that (i) is equivalent to the multiplication map being continuous and that (ii) is equivalent to the inversion map being continuous. We shall prove that (i) is equivalent to the multiplication map being continuous; showing that (ii) is equivalent to the inversion map being continuous is equally simple to prove.

Suppose first that the multiplication map μ is continuous. Let U be a neighbourhood of xy. Since $\mu(x,y) = xy$ and μ is continuous at (x,y), there is a neighbourhood V of x and a neighbourhood W of y such that $VW = \mu(V \times W) \subseteq U$.

Now suppose that (i) holds. Let $(x,y) \in G \times G$, and let U be any neighbourhood of $\mu(x,y) = xy$. Then choose a neighbourhood V of x and a neighbourhood W of y such that $VW \subseteq U$. But $V \times W$ is a neighbourhood of (x,y) and $\mu(V \times W) = VW \subseteq U$. It follows that μ is continuous at (x,y). $\square$

The following proposition is mostly just a statement of the basic properties of the standard multiplication maps on topological groups.

Proposition 2.1.9. For a topological group, the inversion map is a homeomorphism, the maps of left and right multiplication by any element are homeomorphisms, the maps of conjugation by any element are homeomorphisms, and the multiplication map is open.

Proof: The inversion map is continuous and is its own inverse so is a homeomorphism. Fix $g \in G$. Then left multiplication by g is a continuous map, and its inverse, left multiplication by g^{-1}, is continuous so left multiplication by G is a homeomorphism. Similarly for multiplication on the right by g. Conjugation is simply the composition of left multiplication by g and right multiplication by g^{-1}, so is also a homeomorphism. When B is an open set and $x \in G$, then as left multiplication by x is a homeomorphism, xB is an open set, so if A is any subset of G, $AB = \bigcup_{a \in A} aB$ is open as well. That the multiplication map is open follows from the above fact that if A and B are open subsets of G, then $\mu(A \times B) = AB$ is open. $\square$

Part of what we have just shown is that when A or B is an open subset of a topological group, then their product AB is also open. There is a similar statement for closed sets.

Proposition 2.1.10. Let F be a closed subset of a topological group and C a compact set. Then both FC and CF are closed.

Proof: We show that FC is closed, the other proof is analogous. Let (x_λ) be a net in FC that converges to a point x. Since $x_\lambda \in FC$ for all λ, we find for each λ a $y_\lambda \in F$ and $z_\lambda \in C$ such that $x_\lambda = y_\lambda z_\lambda$. Since (z_λ) is a net in C and C is compact, there is a subnet (z_{λ_γ}) converging to $z \in C$. But then (y_{λ_γ}) is a net in F and $y_{\lambda_\gamma} = x_{\lambda_\gamma} z_{\lambda_\gamma}^{-1} \to xz^{-1}$ so $xz^{-1} \in F$ as F is closed. Hence, $x = (xz^{-1})z \in FC$. Since any net in FC which converges can only converge to points in FC, it follows that FC is closed. $\square$

One may wonder if the product of two closed sets is necessarily closed, but this is not necessarily the case as the next example shows.

Example 2.1.11. Let $F = \mathbb{Z}$ and $C = \sqrt{2}\mathbb{Z} = \{\sqrt{2}x : x \in \mathbb{Z}\}$, which are both closed in $\mathbb{R}$. Then $F + C = \{a + \sqrt{2}b : a, b \in \mathbb{Z}\}$ is not closed. This follows from the fact that $F + C$ is a dense subset of $\mathbb{R}$, yet can't be all of $\mathbb{R}$ as it is countable, so $F + C$ is not equal to its closure.

We now list some properties that will be useful in the next section.

Proposition 2.1.12. Let A and B be subsets of a topological group G and $x, y \in G$. Then we have the following properties:

(1) $\overline{A} \cdot \overline{B} \subseteq \overline{AB}$.

(2) $\overline{A}^{-1} = \overline{A^{-1}}$.

(3) $x\overline{A}y = \overline{xAy}$.

(4) If G is Hausdorff and $ab = ba$ for every $a \in A$ and $b \in B$, then $ab = ba$ for every $a \in \overline{A}$ and $b \in \overline{B}$.

Proof: Let μ denote the multiplication map. Since μ is continuous, $\overline{A} \cdot \overline{B} = \mu(\overline{A} \times \overline{B}) = \mu(\overline{A \times B}) \subseteq \overline{\mu(A \times B)} = \overline{AB}$. This proves (1). That (2) and (3) holds are immediate, since by Proposition 2.1.9, the inversion map and the map $a \mapsto xay$ are both homeomorphisms. To prove (4) we use nets. Let $a \in \overline{A}$ and $b \in \overline{B}$, and choose nets (a_λ) in A and (b_γ) in B so that $a_\lambda \to a$ and $b_\gamma \to b$. Then $a_\lambda b_\gamma \to ab$ and $b_\gamma a_\lambda \to ba$. But $a_\lambda b_\gamma = b_\gamma a_\lambda$ for every λ and every γ, so by uniqueness of convergence in a Hausdorff space, $ab = ba$. □

It is not difficult to find a counterexample to (4) when the assumption that G be Hausdorff is dropped. The one we provide is trivial.

Example 2.1.13. Let G be the linear group $\mathrm{GL}(2, \mathbb{R})$ and recall that G is not abelian. If we are to give G the indiscrete topology and let $A = B = \{I\}$, where I is the 2×2 identity matrix, then every element of A commutes with every element of B, but $\overline{A} = \overline{B} = G$, so we cannot have that every element of $\overline{A}$ commutes with every element of $\overline{B}$ since G is not abelian.

By Proposition 2.1.9, it is easy to see that many topological properties of topological groups that hold at one point, hold at every point. Often it is most simple to work around the identity element, and the next proposition justifies doing so in many scenarios.

Proposition 2.1.14. Let G be a topological group with identity element e and let $\mathscr{U}$ be a neighbourhood base of open sets at e. Then $\mathscr{U}_x^L = \{xU : U \in \mathscr{U}\}$ and $\mathscr{U}_x^R = \{Ux : U \in \mathscr{U}\}$ are open neighbourhood bases at x, for any $x \in G$.

Proof: We show that $\mathscr{U}_x^L$ is a neighbourhood base of open sets at x. By Proposition 2.1.9, left multiplication by x is a homeomorphism, and so $\mathscr{U}_x^L$ is a collection of open sets containing x. If W is a neighbourhood of x, then $x^{-1}W$ is a neighbourhood of e, so $U \subseteq x^{-1}W$ for some $U \in \mathscr{U}$. But then $xU \in \mathscr{U}_x^L$ and $xU \subseteq x(x^{-1}W) = W$. □

Corollary 2.1.15. A topological group is first-countable if and only if there is a countable neighbourhood base at the identity (or any point). A topological group is locally compact (respectively, locally connected) if and only if there is a neighbourhood base of compact (respectively, connected) sets at the identity (or any point).

The following proposition is immediate from the properties of continuous maps in topological spaces.

Proposition 2.1.16. Let A and B be subsets of a topological group. If both A and B are: compact, connected, or path connected, then so is AB.

Proof: Let μ be the multiplication map. If A and B are both compact, connected, or path connected then so is $A \times B$. The continuous image of a compact, connected, or path connected space is, respectively, compact, connected, or path connected. Since μ is a continuous map and $AB = \mu(A \times B)$, it follows that AB has the same property that both A and B do (of the ones listed here). □

The next proposition is very simple, but ends up having surprisingly deep applications in the theory of topological groups. It will play a key role later in showing that topological groups are completely regular, and that when a topological group is first-countable then its topology is generated by a left-invariant pseudo-metric. We first introduce a piece of terminology.

Definition 2.1.17. A subset A of a group is called symmetric if $A = A^{-1}$.

Remark 2.1.18. Whenever A is a subset of a group, both $A \cap A^{-1}$ and AA^{-1} are symmetric.

Proposition 2.1.19. Let U be a neighbourhood of the identity element e of a topological group. Then there is a symmetric neighbourhood of the identity V such that $V^2 \subseteq U$.

Proof: Since $e = e \cdot e$, and U is a neighbourhood of e, we can find neighbourhoods S and T of e such that $ST \subseteq U$. Now let $V = (S \cap T) \cap (S \cap T)^{-1}$. Note that $S \cap T$ is a neighbourhood of e since both S and T are, and that since $S \cap T$ is a neighbourhood of e, $(S \cap T)^{-1}$ is a neighbourhood of $e^{-1} = e$ and so V is a neighbourhood of e. But then $V^2 \subseteq ST \subseteq U$ which is what we wanted to prove. □

We will make an informal comment on how Proposition 2.1.19 is used that will perhaps illustrate why this simple fact is so useful. In the theory of metric spaces, it is often useful to start with the open ball around a point x of radius ε, $B_\varepsilon(x)$, and instead look at the open ball around x of radius $\frac{\varepsilon}{2}$, $B_{\frac{\varepsilon}{2}}(x)$. In this way, by looking at $\frac{\varepsilon}{2}$, we obtain that for any $y \in B_{\frac{\varepsilon}{2}}(x)$ that $x \in B_{\frac{\varepsilon}{2}}(y)$. Now, let x be a point in a topological group and let W be a neighbourhood of x. Then we may write W as xU for some neighbourhood U of the identity (take $U = x^{-1}W$). Using Proposition 2.1.19, we find a symmetric neighbourhood V of the identity with $V^2 \subseteq U$. But then for any $y \in xV$, we have $yV \subseteq (xV)V \subseteq xU = W$. What we see is choosing this subset V is like choosing $\frac{\varepsilon}{2}$.

An immediate application of Proposition 2.1.19 is the following.

Proposition 2.1.20. Let G be a topological group, U an open subset of G, and F a compact subset of G. If $F \subseteq U$, then there is a neighbourhood V of the identity such that $VFV \subseteq U$.

Proof: For every $x \in F$, choose an open set U_x with $x \in U_x$ and $U_x \subseteq U$. Since $x^{-1}U_x$ is a neighbourhood of the identity, using Proposition 2.1.19 we can find an open set V_x with $V_x^2 \subseteq x^{-1}U_x$. Note that the collection $\{xV_x\}_{x \in F}$ forms an open cover for F, so we obtain $x_1, \ldots, x_n \in F$ with $F \subseteq \bigcap_{k=1}^n x_k V_{x_k}$ by compactness. Now, let $V_1 = \bigcap_{k=1}^n V_{x_k}$, which is an open neighbourhood of e. Then suppose that $y \in FV_1$, where $y = fv$ with $f \in F$ and $v \in V_1$. Then $f \in x_k V_{x_k}$ for some k, so $y = fv \in x_k V_{x_k} V_{x_k} \subseteq x_k(x_k^{-1}U_{x_k}) = U_{x_k} \subseteq U$. Consequently, $FV_1 \subseteq U$. Next, we use Proposition 2.1.19 to find an open neighbourhood V_2 with $V_2^2 \subseteq V_1$. As FV_2 is an open set containing F, an analogous argument to the one above produces a neighbourhood V_3 of the identity with $V_3F \subseteq FV_2$. Then $V = V_2 \cap V_3$ is our desired set because $VFV \subseteq V_3FV_2 \subseteq FV_2^2 \subseteq FV_1 \subseteq U$. □

Yet another application of Proposition 2.1.19 yields a proposition which we will use later.

Proposition 2.1.21. [3, Theorem 4.9] Let G be a topological group, U a neighbourhood of the identity element, and F a compact subset of G. Then there is a neighbourhood V of the identity such that for every $x \in F$, $xVx^{-1} \subseteq U$.

Proof: First choose an open symmetric neighbourhood W of the identity such that $W^3 \subseteq U$, which can be done by a trivial modification of Proposition 2.1.19. Since $F \subseteq WF = \bigcup_{x \in F} Wx$, and F is compact, there are $x_1, \ldots, x_n \in F$ such that $F \subseteq \bigcup_{k=1}^n Wx_k$. Let $V = \bigcap_{k=1}^n x_k^{-1}Wx_k$. Let $x \in F$. If $y \in xVx^{-1}$, then $y = xvx^{-1}$ for some $v \in V$. Since $x \in F$, $x = wx_k$ for some $k = 1, \ldots, n$. But observe that $x_kVx_k^{-1} \subseteq W$, so that $x_kvx_k^{-1} = w' \in W$. Thus, $y = xvx^{-1} = wx_kvx_k^{-1}w^{-1} = ww'w^{-1} \in W^3 \subseteq U$. Hence, $xVx^{-1} \subseteq U$ for every $x \in F$. □

Corollary 2.1.22. When G is a compact group, and U is a neighbourhood of the identity of G, then there is a neighbourhood V of the identity element of G with $xVx^{-1} \subseteq U$ for every $x \in G$. As a consequence, in a compact group every neighbourhood of the identity contains a neighbourhood which is invariant under conjugation.

Proof: Using Proposition 2.1.21 with $F = G$, we obtain a neighbourhood V with $xVx^{-1} \subseteq U$ for all $x \in G$. The second statement follows by taking $W = \bigcup_{x \in G} xVx^{-1}$. □

As of yet, we have not produced any theorems which allow us to build topological groups from other topological groups. The next theorem is of this type.

Theorem 2.1.23. Let $\{G_\alpha\}_{\alpha\in A}$ be a family of topological groups. Then $\prod_{\alpha\in A} G_\alpha$ is a topological group when given pointwise multiplication and the product topology.

Proof: That $\prod_{\alpha\in A} G_\alpha$ is a group is a standard result from group theory, we shall only prove that the multiplication map and inversion map are continuous. For every $\alpha \in A$, let μ_α be the multiplication map in G_α, ι_α be the inversion map in G_α, and let π_α denote the projection map of $\prod_{\gamma\in A} G_\gamma$ onto G_α, i.e. if $f \in \prod_{\alpha\in A} G_\alpha$, then $\pi_\alpha(f) = f(\alpha)$. To show that μ, the multiplication map in $\prod_{\alpha\in A} G_\alpha$ is continuous, we show that if (f_λ, g_λ) is a net in $\prod_{\alpha\in A} G_\alpha \times \prod_{\alpha\in A} G_\alpha$ converging to (f, g), then $\mu(f_\lambda, g_\lambda) = f_\lambda g_\lambda \to fg$. It suffices to show that for every $\alpha \in A$, $f_\lambda(\alpha)g_\lambda(\alpha) \to f(\alpha)g(\alpha)$. But $f_\lambda \to f$ and $g_\lambda \to g$, so by continuity of π_α, $f_\lambda(\alpha) \to f(\alpha)$ and $g_\lambda(\alpha) \to g(\alpha)$. It follows by continuity of the multiplication map μ_α that $f_\lambda(\alpha)g_\lambda(\alpha) \to f(\alpha)g(\alpha)$. Hence, μ is continuous. We now show that the inversion map ι is continuous. Let (f_λ) be a net in $\prod_{\alpha\in A} G_\alpha$ which converges to f. As before it suffices to show that $f_\lambda(\alpha)^{-1} \to f(\alpha)^{-1}$ for every $\alpha \in A$ to show that ι is continuous. But $f_\lambda(\alpha) \to f(\alpha)$ and by continuity of ι_α it follows that $f_\lambda(\alpha)^{-1} \to f(\alpha)^{-1}$. So inversion is continuous as well. □

Example 2.1.24. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle in $\mathbb{C}$, which is a subgroup of the group of non-zero complex numbers under multiplication. As we shall see in the next section a subgroup of a topological group is also a topological group, so $\mathbb{T}$ is a topological group. When c is a cardinal, finite or infinite, $\mathbb{T}^c$ is a topological group by Theorem 2.1.23. Since $\mathbb{T}$ is compact Hasudorff and connected, it follows that $\mathbb{T}^c$ is also compact Hausdorff and connected. When c is countable, $\mathbb{T}$ being metrizable implies that $\mathbb{T}^c$ is also metrizable, when c is not countable, $\mathbb{T}^c$ is never metrizable. When $c = n$ is finite, then $\mathbb{T}^n$ is the n-dimensional torus, which is a compact Lie group. When c is infinite, then $\mathbb{T}^c$ is an infinite dimensional torus.

To finish this section, we give a theorem which characterizes an open neighbourhood base at the identity element of a topological group.

Theorem 2.1.25. Let G be a topological group and $\mathscr{U}$ an open neighbourhood base at the identity element of G. Then

1) for every $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ with $V^2 \subseteq U$;

2) for every $U \in \mathscr{U}$, there is a $V \in \mathscr{U}$ with $V^{-1} \subseteq U$;

3) for every $U \in \mathscr{U}$ and $x \in U$, there is a $V \in \mathscr{U}$ with $xV \subseteq U$;

4) for every $U \in \mathscr{U}$ and $x \in G$, there is a $V \in \mathscr{U}$ with $xVx^{-1} \subseteq U$;

5) for every $U, W \in \mathscr{U}$, there is a $V \in \mathscr{U}$ with $V \subseteq U \cap W$.

Conversely, if G is a group and $\mathscr{U}$ is a non-empty collection of sets containing the identity and satisfying 1)-5), then $(\{xU : U \in \mathscr{U}\})_{x\in G}$ is a system of open neighbourhood bases for a topology on G under which G is a topological group.

Proof: Assume first that G is a topological group. That 1) holds is due to the continuity of the multiplication map, that 2) holds is due to the continuity of the inversion map, that 3) holds is a consequence of Proposition 2.1.14, that 4) holds is easily seen from Proposition 2.1.21, and that 5) holds is due to $\mathscr{U}$ being a neighbourhood base.

Now assume that G is a group and $\mathscr{U}$ satisfies 1)-5). To show that $(\{xU : U \in \mathscr{U}\})_{x\in G}$ is a system of open neighbourhood bases for a topology on G it suffices to show that:

a) if $V \in \{xU : U \in \mathscr{U}\}$ then $x \in V$;

b) if $V_1, V_2 \in \{xU : U \in \mathscr{U}\}$ then there is a $V_3 \in \{xU : U \in \mathscr{U}\}$ with $V_3 \subseteq V_1 \cap V_2$;

c) if $V \in \{xU : U \in \mathscr{U}\}$ then there is a $V_0 \in \{xU : U \in \mathscr{U}\}$ such that if $y \in V_0$, then there is a $W \in \{yU : U \in \mathscr{U}\}$ with $W \subseteq V$.

A proof that this is sufficient can be found here: [11, Theorem 4.5]. Verifying a) is completely trivial, and b) follows from 5). We now verify c). Let $x \in G$ be arbitrary and choose any $V \in \mathscr{U}$. If $y \in xV$, then $x^{-1}y \in V$, so we can use 3) to find a $W \in \mathscr{U}$ with $x^{-1}yW \subseteq V$. But then $yW \subseteq xV$. Thus, $(\{xU : U \in \mathscr{U}\})_{x \in G}$ is a system of open neighbourhood bases for a topology on G.

It remains to show that with this topology G becomes a topological group. To show that the multiplication is continuous, it suffices to show that for all $x, y \in G$ and $U \in \mathscr{U}$ that there are $V, W \in \mathscr{U}$ with $xVyW \subseteq xyU$. So let $x, y \in G$ and $U \in \mathscr{U}$, and use 1) to find $W \in \mathscr{U}$ with $W^2 \subseteq U$. Now use 4) to find $V \in \mathscr{U}$ with $Vy \subseteq yW$ so that $xVyW \subseteq xyWW \subseteq xyU$. Since $\{xU : U \in \mathscr{U}\}$ is a neighbourhood base at each $x \in G$, to show that the inversion map is continuous it suffices to show that for each $x \in G$ and $U \in \mathscr{U}$ that there is a $V \in \mathscr{U}$ with $(xV)^{-1} \subseteq x^{-1}U$. To see this, use 2) to find a $W \in \mathscr{U}$ with $Wx^{-1} \subseteq x^{-1}U$ and then 4) to find a $V \in \mathscr{U}$ with $V^{-1} \subseteq W$. Then $(xV)^{-1} = V^{-1}x^{-1} \subseteq Wx^{-1} \subseteq x^{-1}U$. □

2.2 Subgroups in Topological Groups

Recall that a subgroup H of a group G is a subset of G which is a group under the same operation as G. Equivalently, a subset H of a group G is a subgroup if the following three conditions hold:

(a) $H \neq \emptyset$.

(b) For all $x, y \in H$, $xy \in H$.

(c) For every $x \in H$, $x^{-1} \in H$.

Using the notation we introduced in the last section, we can write this as:

(a') $H \neq \emptyset$.

(b') $H^2 \subseteq H$.

(c') $H^{-1} \subseteq H$.

As is standard, we can also replace conditions (b) and (c) with the condition that for every $x, y \in H$, $xy^{-1} \in H$. This amounts to replacing (b') and (c') with the condition that $HH^{-1} \subseteq H$.

When G is a topological group, any subgroup H becomes a topological group in the natural way, by giving H the subspace topology. The multiplication map and inversion map on H are the restrictions of the continuous multiplication and inversion maps on G, so are continuous. Hence H is indeed a topological group in this way.

Before exploring the theory of subgroups we present some familiar and important examples.

Example 2.2.1. Let n be a positive integer, F denote one of $\mathbb{R}$ or $\mathbb{C}$, and let $\mathrm{GL}(n, F)$ be as in Example 2.1.6. The topological group $\mathrm{GL}(n, F)$ has a variety of important subgroups, which are also topological groups by our discussion above. The special linear group is denoted by $\mathrm{SL}(n, F)$ and is given by

$$\mathrm{SL}(n, F) = \{A \in \mathrm{GL}(n, F) : \det A = 1\}.$$

Then $\mathrm{SL}(n, F)$ is a closed subgroup of $\mathrm{GL}(n, F)$ and is therefore locally compact. Let I_n be the $n \times n$ identity matrix. When $F = \mathbb{R}$, the orthogonal group is denoted by $\mathrm{O}(n)$ and is given by

$$O(n) = \{A \in GL(n, \mathbb{R}) : AA^T = I_n\}.$$

Then $O(n)$ is a compact subgroup of $GL(n, \mathbb{R})$. The special orthogonal group is denoted by $SO(n)$ and is the intersection of the orthogonal group and the special linear group, $SO(n) = O(n) \cap SL(n, \mathbb{R})$. Being the intersection of a compact set and a closed set in a Hausdorff space, $SO(n)$ is a compact subgroup of $GL(n, \mathbb{R})$. In the case where $F = \mathbb{C}$, the unitary group $U(n)$ is given by

$$U(n) = \{A \in GL(n, \mathbb{C}) : A\overline{A}^T = I_n\},$$

and is a compact subgroup of $GL(n, \mathbb{C})$. The special unitary group, $SU(n)$ is is the intersection of the unitary group and the special linear group, $SU(n) = U(n) \cap SL(n, \mathbb{C})$, and is a compact subgroup of $GL(n, \mathbb{C})$. All of these examples of subgroups are not just topological groups, they are examples of Lie groups and are also closed Lie subgroups of $GL(n, F)$.

Example 2.2.2. Recall from Example 2.1.7 that the p-adic numbers are the infinite series of the form $\sum_{i=n}^{\infty} a_i p^i$ where $a_i \in \{0, \ldots, p-1\}$. The set of p-adic integers, $\mathbb{Z}_p$, is the closure of the unit ball in $\mathbb{Q}_p$, $\mathbb{Z}_p = \overline{B_1(0)}$. The p-adic integers form a compact subgroup of the group of p-adic numbers.

Example 2.2.3. Let $\mathbb{T}^2$ be the 2-dimensional torus. Let $D = \{(e^{ix}, e^{i\sqrt{2}x}) : x \in \mathbb{R}\}$. This is a dense subgroup of the torus, but is not equal to $\mathbb{T}^2$. Hence, D is not a closed subgroup of $\mathbb{T}^2$.

Now that we have established some examples, we describe some of the general properties of subgroups. Once we have done this, we will be able to quickly introduce two subgroups which are useful in the study of topological groups.

Proposition 2.2.4. Let H be a subgroup of a topological group G. Then the closure of H in G, $\overline{H}$, is also a subgroup of G. Moreover, when H is a normal subgroup, so is $\overline{H}$, and when G is Hausdorff and H is abelian, $\overline{H}$ is also abelian.

Proof: The fact that $\overline{H}$ is a subgroup of G follows from (1) and (2) of Proposition 2.1.12, since we have $\overline{H} \cdot \overline{H} \subseteq \overline{HH} \subseteq \overline{H}$, and $\overline{H}^{-1} = \overline{H^{-1}} \subseteq \overline{H}$. That $\overline{H}$ is normal when H is follows from (3) of Proposition 2.1.12, since for every $g \in G$, we have $g\overline{H}g^{-1} = \overline{gHg^{-1}} = \overline{H}$. When G is Hausdorff and H is abelian, the fact that $\overline{H}$ is abelian follows from (4) of Proposition 2.1.12, every element of H commutes with every element of H so every element of $\overline{H}$ commutes with every elements of $\overline{H}$. □

Let G be a topological group and e the identity element of G. When G is not a T_1 topological space, then $\{e\}$ will not be closed and consequently, $\overline{\{e\}}$ will contain more than one element. If we have any neighbourhood U of e then Proposition 2.1.19 shows we can find a symmetric neighbourhood V of e with $V^2 \subseteq U$. Since V is a symmetric neighbourhood of e, it is easily seen that $\overline{V} \subseteq V^2$ so $\overline{\{e\}} \subseteq \overline{V} \subseteq V^2 \subseteq U$. From this it follows that $\overline{\{e\}}$ has the indiscrete topology, the topology where the only open sets are $\emptyset$ and $\overline{\{e\}}$. Since $\{e\}$ is always a normal subgroup, using Proposition 2.2.4 we have the following result.

Theorem 2.2.5. If G is a topological group, then $\overline{\{e\}}$ is a closed normal subgroup of G and its topology is the indiscrete topology.

Next, let G_e be the connected component of the identity in G. This turns out to also be a closed normal subgroup as the following theorem shows.

Theorem 2.2.6. If G is a topological group then G_e is a closed normal subgroup of G.

Proof: Of course, G_e is non-empty since it contains the identity element of the group. From Proposition 2.1.16, we have that G_eG_e is connected, and since it contains the identity element, $G_eG_e \subseteq G_e$. Since inversion is a homeomorhphism, G_e^{-1} is connected and contains the identity element of the group so $G_e^{-1} \subseteq$

G_e. So G_e is a subgroup of G. That G_e is closed is immediate from it being a connected component, all connected components in a topological space are closed (this follows from the fact that if a subset of a topological space is connected, then so is its closure). For any $g \in G$, gG_eg^{-1} is connected using the fact that conjugation is a homeomorphism (see Proposition 2.1.9), but we also have that the identity element of G is in gG_eg^{-1}, so $gG_eg^{-1} \subseteq G_e$. From this it follows that G_e is normal. □

In the next section we will examine the quotient groups $G/\overline{\{e\}}$ and G/G_e.

So far, all the subgroups we have discussed, either abstractly in our results, or the examples (besides Example 2.2.3) have been closed subgroups. Naturally, one is led to ask about open subgroups of a topological group as well. A somewhat surprising property of open subgroups is the content of the next proposition.

Proposition 2.2.7. An open subgroup H of a topological group G is closed.

Proof: Recall that the left cosets of a group partition the group, and that $xH = H$ only when $x \in H$. It follows that $G \setminus H = \bigcup_{x \in G \setminus H} xH$, so $G \setminus H$ is open and consequently H is closed. □

Since every open subgroup of a topological group is closed, a proper open subgroup can only exist when the group is not connected. Therefore we obtain the following corollary.

Corollary 2.2.8. If G is a connected topological group, then the only open subgroup of G is G itself.

We now talk about how we can derive topological properties of groups when they are generated by a sufficiently nice subset. Recall that the intersection of a family of subgroups is a group. From this, we make the following standard definition.

Definition 2.2.9. When A is a subset of a group, $\langle A \rangle$ denotes the intersection of all subgroups containing A. We call $\langle A \rangle$ the subgroup generated by A.

Remark 2.2.10. As is standard with such definitions, $\langle A \rangle$ is the smallest subgroup that contains A: if H is a subgroup which contains A, then $\langle A \rangle \subseteq H$.

It is not difficult to check that when A is subset of a topological group, then $\langle A \rangle = \{\prod_{i=1}^{n} x_i^{a_i} : n \in \mathbb{N},\ x_1, \ldots, x_n \in A,$ and $a_j = \pm 1$ for $j = 1, \ldots, n\}$ (since the empty product is the identity, this still works for the empty set). Using our notation introduced earlier, this translates to the following statement.

Remark 2.2.11. When A is a subset of a group, $\langle A \rangle = \bigcup_{n=0}^{\infty} (A \cup A^{-1})^n$.

The following proposition is almost immediate.

Proposition 2.2.12. Let G be a topological group and A a subset of G. If A is compact, then $\langle A \rangle$ is σ-compact. If A is connected and $e \in A$, then $\langle A \rangle$ is connected.

Proof: Suppose first that A is compact. By Remark 2.2.11, we just need to observe that when A is compact, then for $n = 0, 1, 2, \ldots$ that $(A \cup A^{-1})^n$ is also compact. When $n = 0$, then $(A \cup A^{-1})^n = \{e\}$ which is compact. When $n \geq 1$, both A and A^{-1} are compact, so their union is compact, and using Proposition 2.1.16 we conclude that $(A \cup A^{-1})^n$ is compact.

Now suppose that A is connected and $e \in A$. Using Proposition 2.1.16 as before, we see that for all $n = 0, 1, 2, \ldots$, $C_n = (A \cup A^{-1})^n$ is connected (note that $A \cup A^{-1}$ is connected since $e \in A \cap A^{-1}$). But $e \in \bigcap_{n=0}^{\infty} C_n$, so it follows that $\langle A \rangle = \bigcup_{n=0}^{\infty} C_n$ is connected as well. □

An easy application of Proposition 2.2.12 yields the following corollary.

Corollary 2.2.13. If A is a connected subset of a topological group G and $e \in A$, then $\langle A \rangle \subseteq G_e$.

We finish our discussion of subgroups by proving a simple result about the center of a group.

Proposition 2.2.14. The center $Z(G)$ of a Hausdorff topological group G is a closed normal subgroup.

Proof: That $Z(G)$ is a normal subgroup of G is standard, we only show that it is closed when G is Hausdorff. Let $x \in \overline{Z(G)}$. Then there is a net (x_λ) of elements of $Z(G)$ converging to x. If $g \in G$, then $gx_\lambda \to gx$ and $x_\lambda g \to xg$, but since (x_λ) is a net in $Z(G)$ we have $x_\lambda g = gx_\lambda$ for all λ. Therefore uniqueness of limits in a Hausdorff space implies that $xg = gx$. □

2.3 Quotients and Quotient Groups

Definition 2.3.1. Let G be a topological group, and H a subgroup of G. By G/H we mean the collection of left cosets $\{xH : x \in G\}$. We call G/H the quotient of G by H. Let $\pi : G \to G/H$ be the projection of G onto G/H, that is, the map that sends x to the left coset xH. We give G/H the quotient topology determined by π, that is, we declare a subset U of G/H to be open if and only if $\pi^{-1}(U)$ is an open subset of G.

Of course, we could also consider right cosets instead of left cosets to be the members of G/H. The theory is completely analogous.

Before considering any group theoretic properties of quotients, we first point out some topological properties.

Theorem 2.3.2. Let G be a topological group, H a subgroup of G, and let $\pi : G \to G/H$ be the projection. Then

1) The map $\pi : G \to G/H$ is open.

2) When H is compact, π is a closed map.

3) The quotient space G/H is Hausdorff if and only if H is closed.

Proof: We first prove 1). Let U be an open subset of G. Since $\pi^{-1}(\pi(U)) = UH$ is open, it follows that $\pi(U)$ is open. So π is an open map.

Assuming H is compact, we now show 2). Let F be a closed subset of G. Then $\pi^{-1}(\pi(F)) = HF$ is closed by Proposition 2.1.10, so $\pi(F)$ is closed. Consequently, π is a closed map.

We now prove 3). Suppose that G/H is Hausdorff. Then the singleton $\{H\}$ is a closed subset of G/H, so $H = \pi^{-1}(\{H\})$ is closed. Conversely, assume that H is closed. If $xH, yH \in G/H$ are distinct left cosets of H then $y^{-1}x \notin H$. As H is closed we can find an open set containing the identity, U, such that $y^{-1}xU \cap H = \emptyset$. Now, an easy application of continuity of the multiplication map shows that there is an open symmetric neighbourhood of the identity V with $Vy^{-1}xV \subseteq y^{-1}xU$. From 1) we know that $\pi(xV)$ and $\pi(yV)$ are neighbourhoods of xH and yH respectively, so it remains to show that they are disjoint. If $\pi(xV)$ and $\pi(yV)$ are not disjoint, then there are $v_1, v_2 \in V$ with $xv_1H = yv_2H$. But this would imply that $v_2^{-1}y^{-1}xv_1 \in H$ while we also have that $v_2^{-1}y^{-1}xv_1 \in Vy^{-1}xV \subseteq y^{-1}xU$ which contradicts disjointness of $y^{-1}xU$ and H. So $\pi(xV)$ and $\pi(yV)$ are disjoint. It follows that G/H is Hausdorff. □

Corollary 2.3.3. When H is a compact subgroup, the projection $\pi : G \to G/H$ is proper (i.e. the preimage of every compact subset of G/H is compact).

Proof: A closed map with compact fibers is proper, see page 119 of [5] for a proof. Using Theorem 2.3.2 we have that π is a closed map. The preimage of $xH \in G/H$ is compact since H is, so π is proper. □

Recall that when G is a group and H is a subgroup, the quotient G/H is a group with multiplication $(xH)(yH) = (xy)H$ precisely when H is a normal subgroup. The next theorem shows that when G is a topological group and H is a normal subgroup of G, the quotient group G/H is a topological group.

Theorem 2.3.4. Let G be a topological group and H a normal subgroup of G. Then G/H is a topological group, where G/H is given the quotient topology, and has multiplication $(xH)(yH) = (xy)H$.

Proof: It is standard in group theory that G/H with this multiplication is a group, we only show that the multiplication and inversion map are continuous. It is easier to show that inversion is continuous so we begin there.

Let $\iota : G/H \to G/H$ be the inversion map in G/H, that is, the map that sends xH to $x^{-1}H$. If $\widetilde{\iota}$ is the inversion map in the group G, then observe that $\iota \circ \pi = \pi \circ \widetilde{\iota}$. As both $\widetilde{\iota}$ and π are continuous, it follows that $\iota \circ \pi$ is continuous, and since π is the quotient map (which induces the quotient topology), it follows that ι is continuous.

Now let $\mu : G/H \times G/H \to G/H$ be the multiplication map. To show that μ is continuous we show that when U is open in G/H, that $\mu^{-1}(U)$ is open in $G/H \times G/H$. Let (xH, yH) be an element of $\mu^{-1}(U)$. Then $\pi^{-1}(U)$ is an open set containing xy, since $\pi(xy) = xyH = \mu(xH, yH) \in U$. Since $\widetilde{\mu}$, the multiplication map of G, is continuous, there are open sets V and W in G with $x \in V$ and $y \in W$ with $\widetilde{\mu}(V \times W) \subseteq \pi^{-1}(U)$. Then $\pi(V) \times \pi(W)$ is a neighbourhood of (xH, yH), and if $(aH, bH) \in \pi(V) \times \pi(W)$ with $a \in V$ and $b \in W$ is an arbitrary element of $\pi(V) \times \pi(W)$, then $\mu(aH, bH) = abH = \pi(ab) \in \pi(\widetilde{\mu}(V \times W)) \subseteq \pi(\pi^{-1}(U)) = U$, so $(aH, bH) \in \mu^{-1}(U)$, that is, $\pi(V) \times \pi(W) \subseteq \mu^{-1}(U)$. Hence, $\mu^{-1}(U)$ is open. So multiplication in G/H is continuous. $\square$

Now we prove what we mentioned about $\overline{\{e\}}$ and G_e in the last section.

Theorem 2.3.5. When G is a topological group, $G/\overline{\{e\}}$ is a Hausdorff topological group. Any continuous map $f : G \to X$ where X is a Hausdorff space factors through $G/\overline{\{e\}}$, that is $f = \widetilde{f} \circ \pi$ where $\widetilde{f} : G/\overline{\{e\}} \to X$ is a continuous map and $\pi : G \to G/\overline{\{e\}}$ is the quotient map.

Proof: That $G/\overline{\{e\}}$ is a Hausdorff topological group follows from Theorem 2.2.5, Theorem 2.3.2, and Theorem 2.3.4. To show that every continuous map $f : G \to X$ factors through $G/\overline{\{e\}}$, it suffices to show that f is constant on every left coset. Let $x\overline{\{e\}} \in G/\overline{\{e\}}$. As X is Hausdorff, $f^{-1}(\{f(x)\})$ is a closed set containing x so $x\overline{\{e\}} = \overline{\{x\}} \subseteq f^{-1}(\{f(x)\})$. So f is constant on $x\overline{\{e\}}$. $\square$

Theorem 2.3.6. When G is a topological group, G/G_e is a totally disconnected Hausdorff topological group.

Proof: Applying Theorem 2.2.6, Theorem 2.3.2, and Theorem 2.3.4 we immediately see that G/G_e is a Hausdorff topological group. We devote the remainder of the proof to showing that G/G_e is totally disconnected. Since G/G_e is a topological group, it is enough to show that $\{G_e\}$ is the connected component of G_e (if there was a connected set that was not a singleton, then as left multiplication is a homeomorphism we would obtain that G_e, the identity element of G/G_e, is contained in a connected set with more than one element).

Suppose that A is the connected component of G_e. Let $\pi : G \to G/G_e$ be the projection. Suppose that $aG_e \neq G_e$ is such that $aG_e \in A$ (meaning that A is not a singleton). Then $a \notin G_e$, so $\pi^{-1}(A)$ is not connected, since $G_e \subseteq \pi^{-1}(A)$ and $a \in \pi^{-1}(A)$. Therefore there are non-empty relatively open sets X and Y in $\pi^{-1}(A)$ with $X \cap Y = \emptyset$ and $X \cup Y = \pi^{-1}(A)$.

Note that if $x \in X$ and $y \in Y$, then $xG_e \subseteq X$ and $yG_e \subseteq Y$. Indeed, since $x \in X$, $\pi(x) \in A$, so $xG_e = \pi^{-1}(\{\pi(x)\}) \subseteq \pi^{-1}(A)$ we obtain that $xG_e \subseteq \pi^{-1}(A)$. If we do not have $xG_e \subseteq X$, then as $xG_e \subseteq \pi^{-1}(A) = X \cup Y$, we would then have $X \cap xG_e \neq \emptyset$ (as $x \in X \cap xG_e$) and $Y \cap xG_e \neq \emptyset$ so the subsets $X \cap xG_e$ and $Y \cap xG_e$ would form a disconnection for xG_e, which would be a contradiction. An analogous proof shows that the other claim is true.

Now, since X and Y are relatively open we can find non-empty open sets U and V with $X = U \cap \pi^{-1}(A)$ and $Y = V \cap \pi^{-1}(A)$. We now claim that $\pi(X) = \pi(U) \cap A$ and $\pi(Y) = \pi(V) \cap A$. It is clear that $\pi(X) \subseteq \pi(U)$ and $\pi(X) \subseteq \pi(\pi^{-1}(A)) = A$ so $\pi(X) \subseteq \pi(U) \cap A$. Now suppose that $z \in \pi(U) \cap A$. Since $z \in A$, $z = \pi(t)$ for some $t \in \pi^{-1}(A)$. As $\pi^{-1}(A) = X \cup Y$, either $t \in X$ or $t \in Y$. As $z \in \pi(U)$

we also have that $z = \pi(u)$ for some $u \in U$ and therefore $tG_e = \pi(t) = z = \pi(u) = uG_e$. If $t \in Y$, then $uG_e = tG_e \subseteq Y$ so U and Y are not disjoint. But this contradicts X and Y being disjoint since $\emptyset = X \cap Y = (U \cap \pi^{-1}(A)) \cap Y = U \cap (\pi^{-1}(A) \cap Y) = U \cap Y$, so we conclude that $t \in X$. So $z = \pi(t) \in \pi(X)$ and consequently $\pi(U) \cap A \subseteq \pi(X)$. Hence, $\pi(X) = \pi(U) \cap A$, and a similar proof shows that $\pi(Y) = \pi(V) \cap A$.

Note that $\pi(X) = \pi(U) \cap A$ and $\pi(Y) = \pi(V) \cap A$ are relatively open in A since π is an open map. We must also have $A = \pi(X) \cup \pi(Y)$ since $\pi(X) \cup \pi(Y) = \pi(X \cup Y) = \pi(\pi^{-1}(A)) = A$. Furthermore, $\pi(X)$ and $\pi(Y)$ are disjoint, if there was an element in their intersection, say zG_e, then zG_e is connected and $zG_e \subseteq \pi^{-1}(A) = X \cup Y$, yet $X \cap zG_e$ and $Y \cap zG_e$ are non-empty disjoint open subsets of zG_e whose union is zG_e, which contradicts zG_e being connected. Hence $\pi(X)$ and $\pi(Y)$ are disjoint, so they form a disconnection for A, which is a contradiction. So $A = \{G_e\}$, and therefore G/G_e is totally disconnected. □

Remark 2.3.7. The set G/G_e is precisely the collection of connected components of G. We could have instead proved the following more general statement: if X is a topological space and Y is its collection of connected components, then Y is totally disconnected, where Y is given the quotient topology induced by the projection of X onto the set of its connected components.

Definition 2.3.8. Let P be a property of topological groups (i.e. for a given topological group we can either say that it has the property P or it does not have the property P). We say that P is an extension property if for every topological group G and every normal subgroup H of G, if both H and G/H have the property P, then so does G.

Theorem 2.3.9. [8, Theorem 6.7] Let G be a topological group and H a subgroup of G.

(1) If G is connected then so is G/H. If both H and G/H are connected then G is connected.

(2) If G is compact then G/H is compact. If both H and G/H are compact then G is compact.

(3) If G is locally compact then G/H is locally compact. If both H and G/H are locally compact then G is compact.

Proof: Let $\pi : G \to G/H$ be the projection.

We first show (1) is true. If G is connected, then $\pi : G \to G/H$ is a continuous mapping of G onto G/H so G/H is connected. Now assume that both H and G/H are connected. As H is connected, it is contained in G_e. As H is contained in G_e, we have a continuous mapping φ of G/H onto G/G_e given by $\varphi(xH) = xG_e$, that it is continuous is simply because $\varphi \circ \pi$ is the projection of G onto G_e. As φ is a continuous mapping of G/H onto G/G_e, it follows that G/G_e is connected. But by Theorem 2.3.6, G/G_e is also totally disconnected, and hence must be a singleton, so $G = G_e$. We conclude that G is connected.

Now we prove (2). If G is compact, then G/H is the continuous image of G under π so is also compact. Now suppose that both H and G/H are compact. Since H is compact, π is a proper map by Corollary 2.3.3. Hence, $G = \pi^{-1}(G/H)$ is compact since G/H is.

The proof of (3) is significantly longer and is more technical than the others, so we will not present it here. It can be found on page 55 of [8]. □

The next theorem is a corollary of what we have just proved.

Theorem 2.3.10. The following properties are extension properties of topological groups: being connected, being compact, being locally compact.

2.4 Solvable and Nilpotent groups

In this section we will investigate the extent to which the classical theory of solvable and nilpotent groups changes for topological groups. In particular, we shall be concerned with how the theory generalizes for

Hausdorff topological groups. In the classical theory of solvable groups, we wish to approximate a group with a finite sequence of quotient groups, and if we should wish that our quotient groups are also Hausdorff, then by Theorem 2.3.2 we must impose the condition that we quotient by a closed subgroup. With this in mind, we make the following definition.

Definition 2.4.1. Let G be a topological group with identity element e. For $x, y \in G$, the commutator of x and y is $[x, y] = xyx^{-1}y^{-1}$. The derived group of G is $d(G) = \langle \{[x, y] : x, y \in G\} \rangle$, and the topological derived group is the closure of the derived group $D(G) = \overline{d(G)}$. We define, inductively $d^1(G) = d(G)$, $D^1(G) = D(G)$ and for $n > 1$, $d^n(G) = d(d^{n-1}(G))$ and $D^n(G) = D(D^{n-1}(G))$. We say that G is solvable if $d^n(G) = \{e\}$ for some positive integer n, and we say that G is topologically solvable if $D^n(G) = \{e\}$ for some positive integer n. When G is solvable, the least positive integer n for which $d^n(G) = \{e\}$ is called the derived length of G. When G is topologically solvable, the least positive integer n for which $D^n(G) = \{e\}$ is called the the topological derived length of G.

Remark 2.4.2. It is standard from group theory that $d(G)$ is a normal subgroup of G and that $G/d(G)$ is an abelian group. Moreover, H is a normal subgroup of G with G/H abelian if and only if H is a normal subgroup of G that contains $d(G)$. Using Proposition 2.2.4, we have that $D(G)$ is a closed normal subgroup of G such that $G/D(G)$ is a Hausdorff abelian topological group. Again, we see that H is a closed normal subgroup with G/H abelian if and only if H is a closed normal subgroup which contains $D(G)$.

Remark 2.4.3. [8, Remark 7.2] For every positive integer n, $D^n(G)$ is closed in G; this is not true by definition since the closure is taken within $D^{n-1}(G)$, but it is true since it follows that $D^{n-1}(G)$ is closed within G by induction. If G is topologically solvable, then $\{e\}$ is closed within G, this is equivalent to G being T_1, and as we shall see a T_1 topological group is Hausdorff. Hence, a topologically solvable topological group is necessarily a Hausdorff topological group.

Lemma 2.4.4. [8, Lemma 7.5] If H is a subgroup of a topological group G, then $d(\overline{H}) \subseteq \overline{d(H)}$.

Proof: Let $z \in d(\overline{H})$. Then $z = [x, y]$ for some $x, y \in \overline{H}$, so there are nets (x_α) and (y_β) in H with $x_\alpha \to x$ and $y_\beta \to y$. Then $([x_\alpha, y_\beta])$ is a net in $d(H)$ and $[x_\alpha, y_\beta] \to [x, y] = z$ so $z \in \overline{d(H)}$. □

Lemma 2.4.5. [8, Lemma 7.6] For every topological group G and every positive integer n, $d^n(G) \subseteq \overline{d^n(G)} = D^n(G)$.

Proof: The proof is by induction on n. When $n = 1$, the result is obvious from the definitions. Assume that $d^n(G) \subseteq \overline{d^n(G)} = D^n(G)$ holds for some positive integer n. Since $d^n(G) \subseteq D^n(G)$, it follows that $\overline{d^{n+1}(G)} \subseteq \overline{d(D^n(G))} = D^{n+1}(G)$. We also have, using Lemma 2.4.4, that $d(\overline{d^n(G)}) \subseteq \overline{d^{n+1}(G)}$, and hence, $D^{n+1}(G) = \overline{d}(\overline{d^n(G)}) \subseteq \overline{d^{n+1}(G)}$. Therefore, $d^{n+1}(G) \subseteq \overline{d^{n+1}(G)} = D^{n+1}(G)$. □

Theorem 2.4.6. [8, Theorem 7.7] Let G be a topological group. Then G is Hausdorff and solvable if and only if G is topologically solvable. Moreover, the derived length of G is the same as the topological derived length.

Proof: Simply apply Lemma 2.4.5. □

A standard result in group theory is the following: a group G is solvable if and only if we can find a sequence $(G_k)_{k=0}^n$ of subgroups of G such that $G_0 = G$, $G_n = \{e\}$, and G_{j+1} is a normal subgroup of G_j with G_j/G_{j+1} being an abelian group for $j = 0, \dots, n-1$. The corresponding result for topologically solvable groups is the following theorem.

Theorem 2.4.7. [8, Proposition 7.8] Let G be a topological group. Then G is topologically solvable if and only if we can find a sequence $(G_k)_{k=0}^n$ of closed subgroups of G such that $G_0 = G$, $G_n = \{e\}$, and G_{j+1} is a normal subgroup of G_j with G_j/G_{j+1} being an abelian Hausdorff topological group for $j = 0, \dots, n-1$.

Proof: First suppose that G is topologically solvable. Take $G_0 = G$, and $G_j = D^j(G)$ for $j = 1, \dots, n$. That $(G_k)_{k=0}^n$ is a sequence of subgroups of G such that $G_0 = G$, $G_n = \{e\}$, and G_{j+1} is a normal subgroup of G_j with G_j/G_{j+1} being an abelian Hausdorff topological group for $j = 0, \dots, n-1$ now follows from Remark 2.4.2.

Suppose that there is a sequence $(G_k)_{k=0}^n$ of subgroups of G such that $G_0 = G$, $G_n = \{e\}$, and G_{j+1} is a normal subgroup of G_j with G_j/G_{j+1} being an abelian Hausdorff topological group for $j = 0, \dots, n-1$. We will be done if we prove that $D^j(G) \subseteq G_j$ for $j = 1, \dots, n$, since we will have $D^n(G) = \{e\}$. When $j = 1$, this follows from Remark 2.4.2 since $G_0/G_1 = G/G_1$ is Hausdorff abelian so $D^1(G) = D(G) \subseteq G_1$. Now assume that $D^j(G) \subseteq G_j$ for some j. That G_j/G_{j+1} is Hausdorff abelian implies (using Remark 2.4.2) that $D(G_j) \subseteq G_{j+1}$. But $D(D^j(G)) \subseteq D(G_j)$ so $D^{j+1}(G) \subseteq G_{j+1}$ and our induction is complete. □

It is a well known fact that being a solvable group is an extension property. The exact same proof gives the following result.

Proposition 2.4.8. Being topologically solvable is an extension property.

Now we turn to the concept of a topologically nilpotent group.

Definition 2.4.9. Let G be a topological group with identity element e. The descending central series of G is the sequence $(\gamma_n(G))_{n=0}^\infty$ defined inductively by taking $\gamma_0(G) = G$ and for $n > 0$, $\gamma_n(G) = \langle\{[x, y] : x \in G,\ y \in \gamma_{n-1}(G)\}\rangle$. The topological descending central series of G is the sequence $(\Gamma_n(G))_{n=0}^\infty$ defined inductively by taking $\Gamma_0(G) = G$ and for $n > 0$, $\Gamma_n(G) = \overline{\langle\{[x, y] : x \in G,\ y \in \Gamma_{n-1}(G)\}\rangle}$. The ascending central series of G is the sequence $(Z_n(G))_{n=0}^\infty$ defined inductively by taking $Z_0(G) = \{e\}$ and for $n > 0$, $Z_n(G) = \pi_n^{-1}(Z(G/Z_{n-1}(G)))$ where $\pi_n : G \to G/Z_n(G)$ is the projection, and $Z(H)$ denotes the center of the group H. We say that G is nilpotent if $\gamma_n(G) = \{e\}$ for some nonnegative integer n and that G is topologically nilpotent if $\Gamma_n(G) = \{e\}$ for some nonnegative integer n. When G is nilpotent, the least integer n for which $\gamma_n(G) = \{e\}$ is called the nilpotency class of G. When G is topologically solvable, the least integer n for which $\Gamma_n(G) = \{e\}$ is called the topological nilpotency class of G.

Remark 2.4.10. [8, Remark 7.11] The ascending central consists only of normal subgroups of G and we always have that $\langle\{[x, y] : x \in G,\ y \in Z_n(G)\}\rangle \subseteq Z_{n-1}(G)$, these are standard results. When G is Hausdorff an application Proposition 2.2.14 shows that $Z_n(G)$ is closed for every nonnegative integer n. We also remark that as with topologically solvable groups, a topologically nilpotent group is necessarily Hausdorff.

Theorem 2.4.11. [8, Theorem 7.12] Let G be a topological group. Then we always have $\gamma_n(G) \subseteq \Gamma_n(G)$ for every nonnegative integer n. When G is Hausdorff and nilpotent with nilpotency class k, then $\Gamma_n(G) \subseteq Z_{k-n}(G)$ holds for $0 \le n \le k$. Thus, G is Hausdorff and nilpotent if and only if G is topologically nilpotent. Moreover, the nilpotency class of G is the same as the topological nilpotency class.

Proof: That $\gamma_n(G) \subseteq \Gamma_n(G)$ holds for all n is immediate from the definitions. Assume now that G is Hausdorff and nilpotent with nilpotency class k, we will prove that $\Gamma_n(G) \subseteq Z_{k-n}(G)$ holds by induction on n. That $Z_k(G) = G$ is well known, so $\Gamma_0(G) = G = Z_{k-0}(G)$. Now, assume that for some $n < k$, that $\Gamma_n(G) \subseteq Z_{k-n}(G)$. Using Remark 2.4.10 and $\Gamma_n(G) \subseteq Z_{k-n}(G)$ we have $\langle[x, y] : x \in G,\ y \in \Gamma_n(G)\rangle \subseteq \langle[x, y] : x \in G,\ y \in Z_{k-n}(G)\rangle \subseteq Z_{k-n-1}(G)$. Again referring to Remark 2.4.10, we have that $Z_{k-n-1}(G)$ is closed so $\Gamma_{n+1}(G) \subseteq Z_{k-n-1}(G)$ and the induction is complete. □

The main conclusion of this section is that when G is a Hausdorff topological group, the concepts of topological solvability and nilpotency are not essentially different from their classic algebraic counterparts. We end this section with an interesting proposition.

Proposition 2.4.12. Let G be a connected topological group. Then for every positive integer n, $d^n(G)$, $D^n(G)$, $\gamma_n(G)$, and $\Gamma_n(G)$ are all connected.

Proof: This is a simple application of the following: induction, the definitions of these groups, the continuous image of a connected set being connected, the product of connected sets is connected, the closure of a connected set is connected, and Proposition 2.2.12. □

2.5 Isomorphism Theorems and Open Mapping Theorems

In this section we prove topological versions of the classic isomorphism theorems from group theory.

Definition 2.5.1. Let G and H be topological groups. A topological isomorphism is a map $\varphi : G \to H$ that is both a homeomorphism and a group isomorphism. When such a map exists we say that G an H are topologically isomorphic.

As it turns out, when G and H are sufficiently regular topological groups, any continuous homomorphism of G onto H will be an open map. Before proving this open mapping theorem, we first prove a short lemma.

Lemma 2.5.2. A homomorphism of topological groups is an open map if and only if every neighbourhood of the identity is mapped onto a neighbourhood of the identity.

Proof: Let $\varphi : G \to H$ be an homomorphism of topological groups with e_G being the identity of G and e_H being the identity of H. If φ is open then it is obvious that every neighbourhood of e_G is mapped to a neighbourhood of e_H. Suppose φ maps neighbourhoods of e_G to neighbourhoods of e_H. Let U be a neighbourhood of $x \in G$. We need to show that $\varphi(U)$ is a neighbourhood of $\varphi(x)$. But $x^{-1}U$ is a neighbourhood of e_G, so $\varphi(x^{-1}U) = \varphi(x)^{-1}\varphi(U)$ is a neighbourhood of e_H. Hence $\varphi(U) = \varphi(x)(\varphi(x)^{-1}\varphi(U))$ is a neighbourhood of $\varphi(x)$. □

Theorem 2.5.3. [8, Theorem 6.19] Let G be a locally compact, σ-compact topological group and let H be a locally compact Hausdorff topological group. If $\varphi : G \to H$ is a continuous homomorphism of G onto H, then φ is an open map.

Proof: Write $G = \bigcup_{n=1}^{\infty} C_n$ where each C_n is a compact subset of G. To verify that φ is an open map it suffices to show that for every neighbourhood of e_G, the image under φ is a neighbourhood of e_H in H by Lemma 2.5.2. Let U be a neighbourhood e_G, and choose a symmetric compact neighbourhood V of e_G such that $V^2 \subseteq U$. As $C_n \subseteq C_nV = \bigcup_{x \in C_n} xV$, it follows that there is a finite set $F_n \subseteq C_n$ with $C_n \subseteq F_nV$. But then we have that $F = \bigcup_{n=1}^{\infty} F_n$ is a countable set so there is a sequence $(x_n)_{n=1}^{\infty}$ that lists the elements of F. But then we have $G = \bigcup_{n=1}^{\infty} x_nV$, and so $H = \bigcup_{n=1}^{\infty} \varphi(x_n)V$. Then $\varphi(V)$ is compact in H and hence closed since H is Hausdorff, so $\varphi(x_n)\varphi(V)$ is closed in H. From the Baire category theorem, we conclude that $\varphi(x_n)\varphi(V)$ has non-empty interior for some n, but this is a translate of $\varphi(V)$ so we conclude that $\varphi(V)$ has non-empty interior. Note if $y \in \operatorname{int}\varphi(V)$, then $y^{-1} \in (\operatorname{int}\varphi(V))^{-1} = \operatorname{int}\varphi(V^{-1}) = \operatorname{int}\varphi(V)$, so $(\operatorname{int}\varphi(V))^2$ is open and $e_H = yy^{-1} \in (\operatorname{int}\varphi(V))^2 \subseteq (\varphi(V))^2 = \varphi(V^2) \subseteq \varphi(U)$. So $\varphi(U)$ is a neighbourhood of e_H. □

Remark 2.5.4. A consequence of the proof shown above is that when G is a locally compact σ-compact topological group and U is any neighbourhood of the identity, we can find a symmetric compact neighbourhood V of the identity with $V^2 \subseteq U$ and a sequence $(x_n)_{n=1}^{\infty}$ in G with $G = \bigcup_{n=1}^{\infty} x_nV$.

Remark 2.5.5. When G is a locally compact separable topological group, G is also σ-compact. This follows because if $(x_n)_{n=1}^{\infty}$ is a dense subset of G and V is a compact symmetric neighbourhood of the identity, then $G = \bigcup_{n=1}^{\infty} x_nV$. Hence Theorem 2.5.3 holds when G is locally compact and separable.

Before proving topological versions of the classical isomorphism theorems we briefly review the latter. If G and H are groups and $\varphi : G \to H$ an onto homomorphism of groups, then the first isomorphism theorem states that $G/\ker\varphi$ is isomorphic to H. If N is a normal subgroup of G and H is any subgroup of G, then the second isomorphism theorem states that HN/N is isomorphic to $H/(H \cap N)$. When K and N are normal subgroups of a group G, and $K \subseteq N$, then the third isomorphism theorem states that $(G/K)/(N/K)$ is isomorphic to G/N.

We first state and prove a version of the first isomorphism theorem for topological groups. This will be used to prove the remaining isomorphism theorems for topological groups.

Theorem 2.5.6. Let $\varphi : G \to H$ be a continuous open homomorphism of G onto H. Then $G/\ker\varphi$ is topologically isomorphic to H and the topological isomorphism is given by $\Phi : g\ker\varphi \mapsto \varphi(g)$.

Proof: That Φ is an isomorphism of groups is standard from group theory, we only show that this map is a homeomorphism. Let $\pi : G \to G/\ker\varphi$ be the quotient map. Since Φ is a mapping from $G/\ker\varphi$, it is continuous if and only if $\Phi \circ \pi$ is, but $\Phi \circ \pi = \varphi$ is continuous. Since π is an onto quotient map, it also follows from φ being open that Φ is open as well (since $\Phi(U) = \Phi(\pi(\pi^{-1}(U))) = \varphi(\pi^{-1}(U))$ is open when U is open). Hence, Φ is a homeomorphism. $\square$

Remark 2.5.7. It is not difficult to see that there is always a continuous mapping of $G/\ker\varphi$ onto H, we only needed that φ be open to prove that the map from $G/\ker\varphi$ to H is open. We also see without much effort that the assumption that φ be open is necessary, if G is a group, then the identity mapping from G with the discrete topology to G with any topology that is not discrete is never open.

Example 2.5.8. Let $\mathbb{T}$ be the unit circle, and let $f : \mathbb{R} \to \mathbb{T}$ be given by $f(x) = e^{2\pi i x}$. Then f is a continuous homomorphism of $\mathbb{R}$ onto $\mathbb{T}$ with $\ker f = \mathbb{Z}$, and by Theorem 2.5.3 and Theorem 2.5.6 we obtain that $\mathbb{R}/\mathbb{Z}$ is topologically isomorphic to $\mathbb{T}$. Similarly, for any integer $n \geq 1$, $\mathbb{R}^n/\mathbb{Z}^n$ is topologically isomorphic to $\mathbb{T}^n$.

Example 2.5.9. Let $n \geq 1$ be an integer. The map $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is a continuous onto homomorphism with $\ker\det = \mathrm{SL}(n, \mathbb{R})$. By Theorem 2.5.3 and Theorem 2.5.6 $\mathrm{GL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R})$ is topologically isomorphic to $\mathbb{R} \setminus \{0\}$.

We now turn to the second isomorphism theorem for topological groups. The assumptions that are needed for the second isomorphism theorem to hold in topological groups are quite a bit more involved than what is needed for the other isomorphism theorems to hold.

Theorem 2.5.10. [3, Theorem 5.33] Let H be a subgroup of a topological group G, and N a normal subgroup of G. Suppose that N is closed, H is locally compact and σ-compact, and HN is locally compact. Then HN/N is topologically isomorphic to $H/(H \cap N)$.

Proof: Let $\varphi : H \to HN/N$ be given by $\varphi(h) = hN$. It is obvious that φ is homomorphism, and it is continuous being the restriction of the quotient map $G \to G/N$ to H. Since $\ker\varphi = H \cap N$, by Theorem 2.5.6, we are done if we show that φ is open. But using Theorem, 2.5.3, since H is locally compact and σ-compact, it suffices to show that HN/N is locally compact and Hausdorff. But HN/N is Hausdorff because N being closed in G is also closed in HN. Since HN is locally compact, so is HN/N by Theorem 2.3.10. $\square$

Remark 2.5.11. There is always a continuous mapping of $H/(H \cap N)$ onto HN/N; it is just the map we obtain using the first isomorphism theorem on the map φ.

We finally prove the third isomorphism theorem. The third isomorphism theorem for topological groups requires no extra assumptions.

Theorem 2.5.12. Let G be a topological group with K a normal subgroup of G and N a normal subgroup of G contained in K. Then $(G/N)/(K/N)$ is topologically isomorphic to G/K.

Proof: The map $\varphi : gN \mapsto gK$ is an onto homomorphism with kernel K/N. Let $\pi_K : G \to G/K$ and $\pi_N : G \to G/N$ be the quotient maps. Then φ is continuous since $\varphi \circ \pi_N = \pi_K$. It is also an open map since π_k is and $\varphi \circ \pi_N = \pi_K$. Using Theorem 2.5.6 we are done. $\square$

2.6 Topological Transformation Groups and Group Actions

One of the central reasons for studying groups is the importance of group actions. The theory of group actions becomes even deeper when we also consider the topologies of the group and the space it is acting on.

Definition 2.6.1. Let G be a topological group with identity element e and X, a topological space such that $\mathrm{Homeo}(X)$, the group of all homeomorphisms of X, has been given a topology. A group action of G on X is a homomorphism $\sigma : G \to S_X$ where S_X is the set of all permutations of X. We say that $\delta : G \to \mathrm{Homeo}(X)$ is a topological group action of G on X when δ is a continuous group action. The triple

(G, X, ω) is called a topological transformation group when $\omega : G \times X \to X$ is a continuous map, with image written as $\omega(g, x) = g \cdot x$, which satisfies $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. Whenever (G, X, ω) is a topological transformation group, the notation $g \cdot x$ refers to $\omega(g, x)$.

Example 2.6.2. Let G be a topological group, and give $\mathrm{Homeo}(G)$ the topology that it inherits as a subset of G^G equipped with the product topology. The left regular representation of G is the map $\delta : G \to \mathrm{Homeo}(G)$ given by $\delta(g)(x) = gx$ for every $g, x \in G$, that is, $\delta(g)$ is the map of left multiplication by g. Then δ is continuous since if (g_λ) is a net converging to g and x is any fixed element of G, then $\delta(g_\lambda)(x) = g_\lambda x \to gx = \delta(g)(x)$, so $\delta(g_\lambda) \to \delta(g)$. It follows that δ is a topological group action.

Example 2.6.3. Let H be a subgroup of a topological group G. Then $(G, G/H, \omega)$ is a topological transformation group, where $\omega : G \times G/H \to G/H$, is given by $\omega(x, gH) = xgH$. Indeed, observe that $\gamma : G \times G \to G \times G/H$ given by $\gamma(x, y) = (x, yH)$ is a continuous open map and hence a quotient map. Therefore ω is continuous if and only if $\omega \circ \gamma$ is continuous. But, if $\pi : G \to G/H$ is the projection, and $\mu : G \times G \to G$ is the multiplication map of G, then $\omega \circ \gamma = \pi \circ \mu$ is continuous, so ω is continuous.

We shall introduce a topology on $\mathrm{Homeo}(X)$ which for any locally compact space X turns $\mathrm{Homeo}(X)$ into a topological group. We do this by modifying the compact-open topology which will also be introduced shortly. While the compact-open topology may seem an odd choice, it will at least have two obvious benefits which we state here. Firstly, when X is a metric space, the compact-open topology is the topology of uniform convergence on compact sets, a fact which we shall not prove here, but which is useful to know. Secondly we will see without much difficulty that with this topology the group multiplication in $\mathrm{Homeo}(X)$, function composition, is continuous.

Definition 2.6.4. [8, Definition 9.1] Let X and Y be topological spaces, and let $\mathscr{C}(X, Y)$ denote the set of all continuous functions from X to Y. Let $\lfloor U, V \rfloor = \{f \in \mathscr{C}(X, Y) : f(U) \subseteq V\}$. The compact-open topology on $\mathscr{C}(X, Y)$ is the topology where we take $S = \{\lfloor C, U \rfloor : C$ is a compact subset of X and U is an open subset of $Y\}$ to be a subbasis.

The next lemma serves the purpose of showing that the group of homeomorphisms of a locally compact topological space has a continuous multiplication when given the compact-open topology. There is no trouble in proving this in extra generality so we do so here.

Lemma 2.6.5. [8, Lemma 9.5] Let X, Y, and Z be topological spaces with Y locally compact. Then the map $T : \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$ given by $T(f, g) = f \circ g$ is continuous.

Proof: It suffices to show that for every compact subset C of X and open subset U of Z, there is an open subset V of Y and a compact subset D of Y with $T(\lfloor D, U \rfloor \times \lfloor C, V \rfloor) \subseteq \lfloor C, U \rfloor$ with $g \in \lfloor C, V \rfloor$ and $f \in \lfloor D, U \rfloor$. For every $y \in g(C)$, let V_y be a compact neighbourhood of y such that $V_y \subseteq f^{-1}(U)$. Then $\{V_y : y \in Y\}$ forms an open cover for G, which is compact, so there is a finite subset $\{y_1, \ldots, y_n\}$ of $g(C)$ such that $g(C) \subseteq \bigcup_{k=1}^n V_{y_k}$. Then $D = \bigcup_{k=1}^n V_{y_k}$ is compact and $V = \bigcup_{k=1}^n \mathrm{int}(V_{y_k})$ is open. Moreover, $g \in \lfloor C, V \rfloor$, $f \in \lfloor D, U \rfloor$, and for every $s \in \lfloor C, V \rfloor$ and $t \in \lfloor D, U \rfloor$, $s \circ t \in \lfloor C, U \rfloor$ since $(s \circ t)(C) = s(t(C)) \subseteq s(V) \subseteq s(W) \subseteq U$. Hence, $T(\lfloor D, U \rfloor \times \lfloor C, V \rfloor) \subseteq \lfloor C, U \rfloor$. □

With this topology, we are guaranteed that the group of homeomorphisms has a continuous multiplication map, however it is not guaranteed that the inversion map is also continuous. Therefore, we will modify the compact-open topology so that the inversion map is continuous. The next lemma shows how this can be done.

Lemma 2.6.6. [8, Lemma 9.13] Let G be a group and let τ be a topology on G with repect to which the multiplication is continuous. Then G with the topology $\widetilde{\tau}$ for which $\{S \cap T^{-1} : S, T \in \tau\}$ is a subbasis is a topological group. Moreover, if H is a topological group and $\varphi : H \to G$ is a continuous homomorphism with respect to τ, then it is also continuous with respect to $\widetilde{\tau}$.

Proof: That the inversion map is continuous is immediate from the fact that $(S \cap T^{-1})^{-1} = S^{-1} \cap T$ for any $S, T \subseteq G$. Suppose that $x, y \in G$, and that $S, T \in \tau$ with $xy \in S \cap T^{-1}$. Using the proof of Proposition 2.1.8, continuity of multiplication in τ allows us to find sets $U, V, W, X \in \tau$ with $UV \subseteq S$, $x \in U$, $y \in V$, and $WX \subseteq T$ with $x^{-1} \in W$ and $y^{-1} \in X$. Then $U \cap W^{-1}, V \cap X^{-1} \in \widetilde{\tau}$, $(U \cap W^{-1}) \times (V \cap X^{-1})$ is a neighbourhood of (x, y) in the product topology with respect to $\widetilde{\tau}$, and $(U \cap W^{-1})(V \cap X^{-1}) \subseteq S \cap T^{-1}$. So G with the topology $\widetilde{\tau}$ is a topological group.

Now suppose that $\varphi : H \to G$ is a continuous homomorphism with respect to the topology τ. Then for any $S, T \in \tau$, we have $\varphi^{-1}(S \cap T^{-1}) = \varphi^{-1}(S) \cap \varphi(T^{-1}) = \varphi(S) \cap \varphi(T)^{-1}$ is open in H (that $\varphi(T^{-1}) = \varphi(T)^{-1}$ is immediate from φ being a homomorphism). It follows that φ is continuous with respect to $\widetilde{\tau}$. □

Definition 2.6.7. [8, Definition 9.14] Let X be a locally compact space. The topology given to Homeo(X) by Lemma 2.6.6 applied to the compact-open topology is called the modified compact-open topology.

Corollary 2.6.8. [8, Corollary 9.15] When X is locally compact, Homeo(X) is a topological group when given the modified compact-open topology.

Proposition 2.6.9. [8, Lemma 9.16] Let X be locally compact and give Homeo(X) the modified compact-open topology. Then the map $\omega : \mathrm{Homeo}(X) \times X \to X$ given by $\omega(f, x) = f(x)$ is continuous, and so $(\mathrm{Homeo}(X), X, \omega)$ is a topological transformation group.

Proof: Let $f \in \mathrm{Homeo}(X)$, $x \in X$, and U be a neighbourhood of $f(x)$. Let C be a compact neighbourhood of x such that $f(C) \subseteq U$. Thus, $\lfloor C, U \rfloor \times C$ is a neighbourhood of (f, x), and $\omega(\lfloor C, U \rfloor \times C) \subseteq U$. It follows that ω is continuous. □

Example 2.6.10. Let G be any topological group. Then (G, G, μ) is a topological transformation group, where μ is the multiplication map. When G is locally compact, if we give Homeo(G) the modified compact-open topology, then the map $\delta : G \to \mathrm{Homeo}(G)$ given by $\delta(g)(x) = gx$ is a topological group action. Indeed, by Lemma 2.6.6, it suffices to check continuity with respect to the compact-open topology. So let C be a compact subset of G, U an open subset of G, and $\delta(g) \in \lfloor C, U \rfloor$. Since $gC = \delta(g)(C) \subseteq U$ and gC is compact, by Proposition 2.1.20, we can find a neighbourhood V of the identity such that $VgC \subseteq U$. But then Vg is a neighbourhood of g, and $\delta(Vg) \subseteq \lfloor C, U \rfloor$. Note that the topological transformation group (G, G, μ) is obtainable from δ, as $\mu(g, x) = \delta(g)(x)$, while δ is obtainable from (G, G, μ) since $\delta(g) = \mu(g, \cdot)$.

Example 2.6.11. Let X be a locally compact space. By Proposition 2.6.9, $(\mathrm{Homeo}(X), X, \omega)$ is a topological transformation group, where $\omega : \mathrm{Homeo}(X) \times X \to X$ is given by $f \cdot x = f(x)$. When $X = G$ is a locally compact group, we can also consider Aut(G), the subgroup of all topological automorphisms of G, that is the isomorphisms of G which are also homeomorphisms, which is a subgroup of Homeo(G). Then $(\mathrm{Aut}(G), G, \omega|_{\mathrm{Aut}(G) \times G})$ is a topological transformation group.

Recall the following facts from classical group theory. Let G be a group with identity element e. Suppose that $\omega : G \times X \to X$, with image written $\omega(g, x) = g \cdot x$, satisfies $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. Then $\delta_\omega : G \to S_X$ given by $\delta_\omega(g)(x) = \omega(g, x)$ is a group action. Conversely, if $\delta : G \to S_X$ is a group action, then $\omega_\delta : G \times X \to X$ given by $\omega_\delta(g, x) = \delta(g)(x)$, with image written $\omega_\delta(g, x) = g \cdot x$, satisfies $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for every $g, h \in G$ and $x \in X$. The next theorem asserts that the topological version of this is true when X is locally compact and Homeo(X) is given the modified compact-open topology. Indeed we show that when this is the case then every topological transformation group arises from a topological group action, and that every topological group action arises from a topological transformation group.

Theorem 2.6.12. [8, Lemma 10.4] Let X be a locally compact topological space, G a topological group, and give Homeo(X) the modified compact-open topology. If (G, X, ω) is a topological transformation group, then $\delta_\omega : G \to \mathrm{Homeo}(X)$ given by $\delta_\omega(g) = \omega_g$ where $\omega_g(x) = \omega(g, x)$, is a topological group action. If

$\delta : G \to \text{Homeo}(X)$ is a topological group action, then (G, X, ω_δ) is a topological transformation group where $\omega_\delta : G \times X \to X$ is given by $\omega_\delta(g, x) = \delta(g)(x)$.

Proof: The paragraph preceding the statement of the theorem means we need only show that δ_ω and ω_δ are continuous.

We first show that δ_ω is actually a map into Homeo(X). If $g \in G$, then $\delta_\omega(g) = \omega_g : X \to X$ is continuous, because ω is continuous and $\delta_\omega(g)(x) = \omega(g, x)$. The inverse is also continuous because $\delta_\omega(g)^{-1} = \delta_\omega(g^{-1})$, so $\delta_\omega(g) \in \text{Homeo}(X)$. We now show that δ_ω is continuous. Using Lemma 2.6.6, it is enough to show that δ_ω is continuous with respect to the compact-open topology. Let C be a compact subset of X, U be an open subset of X, and $\delta_\omega(g) \in \lfloor C, U \rfloor$. As $\delta_g \in \lfloor C, U \rfloor$ is equivalent to $\omega(\{g\} \times C) \subseteq U$. Continuity of ω, allows us to find, for each $c \in C$, a neighbourhood V_c of g and a neighbourhood W_c of c such that $\omega(V_c \times W_c) \subseteq U$. Now use compactness to choose $c_1, \ldots, c_n \in C$ such that $\bigcup_{k=1}^n W_{c_k} \supseteq C$. Take $V = \bigcap_{k=1}^n V_{c_k}$, which is a neighbourhood of g, and then note that $\omega(V \times C) \subseteq U$, so $\delta_\omega(V) \subseteq \lfloor C, U \rfloor$. So δ_ω is continuous.

We now show that ω_δ is continuous. Since δ is continuous, so is $\omega_1 : G \times X \to \text{Homeo}(X) \times X$ given by $\omega_1(g, x) = (\delta(g), x)$. But Proposition 2.6.11 shows that the map $\omega_2 : \text{Homeo}(X) \times X \to X$ given by $\omega_2(f, x) = f(x)$ is continuous. Then observe that $\omega = \omega_2 \circ \omega_1$. □

Remark 2.6.13. If we give Homeo(X) the compact-open topology, then the proof above shows that every topological transformation group induces a topological group action of G on X.

Recall the following concepts, some of which should be familiar from classical group theory. Let (G, X, ω) be a topological transformation group. The orbit of an element $x \in X$ is the set $G \cdot x = \{g \cdot x : g \in G\}$. If $G \cdot x = X$ then we say that ω is transitive. The stabilizer of an $x \in X$ is the subgroup G_x of elements of G which fix x: $G_x = \{g \in G : g \cdot x = x\}$. Then, for any fixed $x \in X$, there is a natural bijection $\varphi : G/G_x \to G \cdot x$ given by $\varphi(yG_x) = y \cdot g$. That this map is well defined and one-to-one is immediate from the definition of G_x. If (G, Y, ω') is another topological transformation group, then we say that a map $F : X \to Y$ is G-equivariant if $F(\omega(g, x)) = \omega'(g, F(x))$ for every $x \in X$ and $g \in G$. If we write both the image of ω and ω' using the dot notation from before, then the condition for F to be G-equivariant is that $F(g \cdot x) = g \cdot F(x)$ holds for every $x \in X$ and $g \in G$. Finally, we say that $F : X \to Y$ is a G-equivariant isomorphism between topological transformation groups (G, X, ω) and (G, Y, ω') if F is a G-equivariant homeomorphism.

Theorem 2.6.14. Let (G, X, ω) be a topological transformation group where G is locally compact and σ-compact, X is a Hausdorff space, and let $x \in X$. Let $\varphi : G/G_x \to G \cdot x$ be the natural bijection. Then φ is continuous and if $G \cdot x$ is locally compact, then φ is a homeomorphism.

Proof: Let $\pi : G \to G/G_x$ be the natural map. The map $\widetilde{\varphi} : G \to G \cdot x$ defined by $\widetilde{\varphi} = \varphi \circ \pi$ is continuous since $\widetilde{\varphi} = \omega(\cdot, x)$, so φ is also continuous. Now assume that $G \cdot x$ is locally compact. To show that φ is a homeomorphism, it suffices to prove that $\widetilde{\varphi}$ is open, since this will show that φ is open. We show that for every $y \in G$, any neighbourhood of y is mapped onto a neighbourhood of $\widetilde{\varphi}(y)$. Let U be any neighbourhood of y, and use Remark 2.5.4 to find a compact symmetric neighbourhood of the identity V such that $yV^2 \subseteq U$ and a sequence $(x_n)_{n=1}^\infty$ with $G = \bigcup_{n=1}^\infty x_n V$. Since $\widetilde{\varphi}$ maps onto $G \cdot x$, $G \cdot x = \bigcup_{n=1}^\infty \widetilde{\varphi}(x_n V)$. But $G \cdot x$ is a Hausdorff space, and each $\widetilde{\varphi}(x_n V)$ is compact as $\widetilde{\varphi}$ is continuous, so each $\widetilde{\varphi}(x_n V)$ is closed. Since $G \cdot x$ is a locally compact Hausdorff space, the Baire category theorem implies that for some n that $\widetilde{\varphi}(x_n V)$ has non-empty interior. But $\widetilde{\varphi}(x_n V) = x_n \cdot \widetilde{\varphi}(V)$ is homeomorphic to $\widetilde{\varphi}(V)$ by the map $G \cdot x \to G \cdot x$ defined by $z \mapsto x_n^{-1} \cdot z$, so it follows that $\widetilde{\varphi}(V)$ has non-empty interior. But then $\widetilde{\varphi}(V)$ is a neighbourhood of $\widetilde{\varphi}(v)$ for some $v \in V$. Consequently, $yv^{-1} \cdot \widetilde{\varphi}(V) = \widetilde{\varphi}(yv^{-1}V) \subseteq \widetilde{\varphi}(yV^{-1}V) = \widetilde{\varphi}(yV^2) \subseteq \widetilde{\varphi}(U)$ is a neighbourhood of $(yv^{-1}) \cdot \widetilde{\varphi}(v) = \widetilde{\varphi}(y)$ which finishes the proof. □

Corollary 2.6.15. Let (G, X, ω) be a topological transformation group where G is locally compact and σ-compact, X is locally compact and Hausdorff, and ω is transitive. Then for every $x \in X$, the map $g \mapsto \omega(g, x)$ is open. Moreover, if $(G, G/G_x, \omega')$ is the topological transformation group given by $\omega(g, yG_x) = gyG_x$, then the natural bijection $\varphi : G/G_x \to G \cdot x = X$ is a G-equivariant isomorphism between (G, X, ω) and $(G, G/G_x, \omega')$.

Proof: Let $\pi : G \to G/G_x$ be the projection, $\varphi : G/G_x \to G \cdot x = X$ the natural bijection, and let $\widetilde{\varphi} = \varphi \circ \pi$. By Theorem 2.6.14, φ is a homeomorphism and is thus open, and since π is open, $\widetilde{\varphi}$ is also open. That φ is a G-equivariant isomorphism is an immediate consequence of Theorem 2.6.14. $\square$

When X is a topological space, we say that X is homogeneous space of G if there is a topological transformation group (G, X, ω) where ω is transitive. If G is locally compact and σ-compact and X is locally compact Hausdorff, then the above shows that there is a G-equivariant isomorphism between X and a coset space G/H where H is the stabilizer of any $x \in G$. It follows that a large class of homogeneous spaces can be identified with coset spaces.

3 Separation Axioms and Metrizability in Topological Groups

3.1 Separation Axioms

The separation properties of topological groups are quite strong. We will show that every topological group is completely regular and that a T_0 topological group is Tychonoff. Some of the terms for separation axioms are not completely standard, so we clarify them here.

Definition 3.1.1. Let X be a topological space. We say that X is regular if for every closed set A and $a \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $a \in V$. We say that X is completely regular if for every non-empty closed set A and every $a \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{1\}$ and $f(a) = 0$. We say that X is normal if for every pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. We say that X

1. is T_0 if for all distinct $x, y \in X$, there is an open set U such that $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$;
2. is T_1 if for all distinct $x, y \in X$, there is an open set U such that $x \in U$ and $y \notin U$;
3. is T_2 (Hausdorff) if for all distinct $x, y \in X$, there are disjoint open sets U and V with $x \in U$ and $y \in V$;
4. is T_3 if it is T_1 and regular;
5. is Tychonoff if it is T_1 and completely regular;
6. is T_4 if it is T_1 and normal.

Remark 3.1.2. It is standard that the class of T_i spaces is contained in the class of T_{i-1} spaces ($i = 1, 2, 3, 4$). It is also not difficult to see that a Tychonoff space is a T_3 space. That a T_4 space is a Tychonoff space is a consequence of Urysohn's lemma.

With these definitions in place, we can now begin our examination of the separation axioms in topological groups.

Theorem 3.1.3. A T_0 topological group is T_1.

Proof: Suppose that x and y are distinct points in a topological group. Then there is an open set U which either contains x but not y, or contains y but not x. If U contains x but not y then the open set $yU^{-1}x$ contains y but not x. Similarly if U contained y but not x we can obtain an open set containing x but not y. It follows that any T_0 topological group is T_1. □

Without much more difficulty we can show that a topological group that is T_0 is actually T_3 and hence also Hausdorff. As it turns out, every topological group is completely regular, and hence by Theorem 3.1.3, every T_0 topological group is Tychonoff. We will instead prove that every topological group is completely regular, and deduce that every T_0 group is Hausdorff. Therefore, in the preceding section any result that required a topological group to be Hausdorff can be replaced with the weaker assumption that it is a T_0 space. To show that topological groups are completely regular, we will construct a particular countable family of open sets containing the identity that will be used to create the desired continuous function. Although it is not necessary for proving that topological groups are completely regular, we shall do the construction with a sufficient amount of generality, so that we may use it later to show how metrics can be defined on certain topological groups. Our construction closely resembles and is based on the construction given on pages 28 and 29 of [6].

Let $(S_n)_{n=0}^{\infty}$ be a sequence of neighbourhoods of the identity. Then choose the sequence $(U_n)_{n=0}^{\infty}$ of open neighbourhoods of the identity by induction: choose U_0 to be an open symmetric set containing the identity and contained in S_0, and for $n > 0$ choose U_n to be an open symmetric set containing the identity so that $U_n^2 \subseteq U_{n-1} \cap S_{n-1}$ (using Proposition 2.1.19).

Now, let K be the set of dyadic rationals in $(0, 1]$, i.e. K is the set of numbers of the form $r = \frac{k}{2^n}$ where $n \geq 0$ is an integer and k is an integer with $1 \leq k \leq 2^n$. We construct a family of open sets $(V_r)_{r \in K}$ inductively:

For $N = 0$, we set $V_1 = U_0$. For $N \geq 1$, if the V_r $(r = \frac{k}{2^n})$ has been constructed for all $n < N$ we consider three cases for $1 \leq k \leq N$: if $k = 1$ we set $V_{\frac{1}{2^N}} = U_N$, if $k = 2m$ for an integer m we set $V_{\frac{k}{2^N}} = V_{\frac{m}{2^{N-1}}}$, and if $k = 2m + 1$ then set $V_{\frac{k}{2^N}} = V_{\frac{1}{2^N}} V_{\frac{m}{2^{N-1}}}$ (so for a fixed n we do induction on k). It should be noted that the definition of V_r does not depend on the choice of dyadic rational r.

Hence, we obtain a countable family of open neighbourhoods of the identity $(V_r)_{r \in K}$. We claim that if $r < s$ are dyadic rationals then we also have $V_r \subseteq V_s$. To do this we first show that for a fixed N, we have

$$V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} \subseteq V_{\frac{k+1}{2^N}} \text{ whenever } 1 \leq k < k + 1 \leq 2^N.$$

This will be done by induction on N.

When $N = 0$, there is nothing to show. So assume that $N > 0$ and that for all $n < N$ we have $V_{\frac{1}{2^n}} V_{\frac{k}{2^n}} \subseteq V_{\frac{k+1}{2^n}}$ whenever $1 \leq k < k + 1 \leq 2^n$. Suppose that $1 \leq k < k + 1 \leq 2^N$. If $k = 1$, then

$$V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} = V_{\frac{1}{2^N}} V_{\frac{1}{2^N}} = U_N^2 \subseteq U_{N-1} = V_{\frac{1}{2^{N-1}}} = V_{\frac{2}{2^N}} = V_{\frac{k+1}{2^N}}.$$

If k is even, then $k = 2m$ for some integer m and by definition we have

$$V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} = V_{\frac{1}{2^N}} V_{\frac{m}{2^{N-1}}} = V_{\frac{k+1}{2^N}}.$$

If k is odd, then $k = 2k + 1$ for some integer m and using our induction assumption we have

$$V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} = V_{\frac{1}{2^N}} (V_{\frac{1}{2^N}} V_{\frac{m}{2^{N-1}}}) \subseteq V_{\frac{1}{2^{N-1}}} V_{\frac{m}{2^{N-1}}} \subseteq V_{\frac{m+1}{2^{N-1}}} = V_{\frac{2(m+1)}{2^N}} = V_{\frac{k+1}{2^N}}.$$

So the result holds for any N. But then we have for any $N \geq 0$ and any $1 \leq k < k+1 \leq 2^N$ that $V_{\frac{k}{2^N}} \subseteq V_{\frac{k+1}{2^N}}$. Since V_r and V_s do not depend on the choice of representation as dyadic rationals, we may assume that their denominators are the same from which it follows that if $r < s$, then $V_r \subseteq V_s$. We summarize this with the following proposition.

Proposition 3.1.4. Let G be a topological group and let $(S_n)_{n=0}^{\infty}$ be a sequence of neighbourhoods of the identity. Then there is a family of open neighbourhoods of the identity $(V_r)_{r \in K}$ with the following properties:

1. We have $V_{\frac{1}{2^N}}^2 \subseteq S_N$ for every integer $N \geq 0$, and $V_{\frac{1}{2^N}}$ is symmetric.
2. If N is a non negative integer and $1 \leq k < k + 1 \leq 2^N$ then $V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} \subseteq V_{\frac{k+1}{2^N}}$. We also have that if r and s are dyadic rationals with $r < s$ then $V_r \subseteq V_s$.

Now we prove that every topological group is completely regular.

Theorem 3.1.5. [6, Section 1.18] Every topological group is completely regular.

Proof: Let F be a closed subset of a topological group G, and $x \notin F$. We need to find a continuous function $f : G \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ when $y \in F$. Since left multiplication is a homeomorphism of G, we can assume that x is the identity element, $x = e$.

Consider the open neighbourhood of the identity $S = G \setminus F$. Then Proposition 3.1.4 asserts that there is a family of open neighbourhoods of the identity indexed by the set of dyadic rationals in $(0,1]$, $(V_r)_{r\in K}$, such that $V_1 \subseteq S$ and $V_r \subseteq V_s$ whenever $r < s$. In particular the entire family is contained in S. For every $g \in G$ let $Q_g = \{0\} \cup \{r \in (0,1] : g \notin V_r\}$ Define the function $f : G \to [0,1]$ by

$$f(g) = \begin{cases} 0 & \text{when } g \in \bigcap_{r\in K} V_r \\ 1 & \text{when } g \notin V_1 \\ \sup Q_g & \text{otherwise} \end{cases}.$$

Clearly $f(e) = 0$ and $f(y) = 1$ when $y \in F$ since $F \subseteq G \setminus V_1$. So it remains to show that f is continuous. We will show continuity pointwise. We remark that if $z \in V_s$ and $z \notin V_r$ then $r \le f(z) \le s$.

Suppose $g \in G$ is such that $f(g) = 1$ and let $\varepsilon > 0$. Choose an integer $N \ge 2$ so that $\frac{1}{2^{N-1}} < \varepsilon$. We claim that if $h \in V_{\frac{1}{2^N}} g$ then $|f(g) - f(h)| < \varepsilon$. We first show that $h \notin V_{\frac{2^n-2}{2^N}}$. If $h \in V_{\frac{2^N-2}{2^N}}$ then $g \in V_{\frac{1}{2^N}} h \subseteq V_{\frac{1}{2^N}} V_{\frac{2^N-2}{2^N}} \subseteq V_{\frac{2^N-1}{2^N}}$ so that $f(g) \le \frac{2^N-1}{2^N} < 1$ a contradiction. So $h \notin V_{\frac{2^N-2}{2^N}}$ and consequently $1 - \frac{1}{2^{N-1}} \le f(h) \le 1$ so $|f(g) - f(h)| \le \frac{1}{2^{N-1}} < \varepsilon$. So f is continuous at every g with $f(g) = 1$.

Now suppose $g \in G$ is such that $0 < f(g) < 1$ and let $\varepsilon > 0$. Choose integers N and $3 \le k \le 2^N$ so that $\frac{k-1}{2^N} \le f(g) \le \frac{k}{2^N}$ and $\frac{1}{2^N} < \varepsilon$. Then $g \in V_{\frac{k}{2^N}}$ but $g \notin V_{\frac{k-1}{2^N}}$. We claim that if $h \in V_{\frac{1}{2^N}} g$ then $|f(g) - f(h)| < \varepsilon$. First note that $h \in V_{\frac{k+1}{2^N}}$ since $V_{\frac{1}{2^N}} g \subseteq V_{\frac{1}{2^N}} V_{\frac{k}{2^N}} \subseteq V_{\frac{k+1}{2^N}}$. Hence, $f(h) \le \frac{k+1}{2^N}$. We cannot have $h \in V_{\frac{k-2}{2^N}}$, otherwise we would have $g \in V_{\frac{1}{2^N}} h \subseteq V_{\frac{1}{2^N}} V_{\frac{k-2}{2^N}} \subseteq V_{\frac{k-1}{2^N}}$. Hence, $f(h) \ge \frac{k-2}{2^N}$. Since $\frac{k-1}{2^N} \le f(g) \le \frac{k}{2^N}$ and $\frac{k-2}{2^N} \le f(h) \le \frac{k+1}{2^N}$ we have $|f(g) - f(h)| \le \frac{1}{2^{N-1}} < \varepsilon$. So f is continuous at every g with $0 < f(g) < 1$.

Now suppose that $g \in G$ is such that $f(g) = 0$, that is, $g \in V_r$ for all r. Let $\varepsilon > 0$, and choose N so that $\frac{1}{2^N} < \varepsilon$. If $h \in V_{\frac{1}{2^N}}$ (which is a neighbourhood of g) then $0 \le f(h) \le \frac{1}{2^N}$, so $|f(g) - f(h)| < \frac{1}{2^N} < \varepsilon$. So it follows that f is continuous at every g with $f(g) = 0$.

Since f is continuous it follows that G is completely regular. □

Corollary 3.1.6. Every T_0 topological group is Tychonoff.

Remark 3.1.7. It is not true that every topological group is normal. For example, $\mathbb{Z}^{\mathbb{R}}$ is a topological group but is not normal, this takes some work to show.

If we assume a little more, that our topological group is locally compact, then we can deduce that our group is in fact normal. Before we do that, we first prove a lemma about general topological spaces.

Lemma 3.1.8. Let X be a topological space and suppose that π is a partition of X into normal subspaces that are open in X. Then X is normal.

Proof: Suppose that A and B are closed disjoint subsets of X. Then for every $P \in \pi$, $P \cap A$ and $P \cap B$ are closed disjoint subsets of P. Since P is normal, there are disjoint open sets in P, U_P and V_P containing $A \cap P$ and $B \cap P$ respectively. But U_P and V_P are also open in X since P is open in X, so $U = \bigcup_{P\in\pi} U_P$ and $V = \bigcup_{P\in\pi} V_P$ are open disjoint sets containing A and B respectively. □

From Lemma 3.1.8 we immediately deduce:

Corollary 3.1.9. If a topological group has an open subgroup that is topologically normal, then the topological group is normal.

Proof: The cosets of the subgroup form a partition of G. □

Theorem 3.1.10. A locally compact topological group is normal.

Proof: We show that G has an open normal subgroup from which the result follows by Corollary 3.1.9. Let U be a compact neighbourhood of the identity element of G. Then $H = \bigcup_{n=1}^{\infty}(U \cup U^{-1})^n$. Then H is open since for all $x \in H$, $xU \subseteq H$ is a neighbourhood of x in H. Moreover H is σ-compact and hence Lindelöf. Since H is a topological group itself, it is completely regular and hence regular. But since H is a regular Lindelöf space, H is normal (for a proof of this fact see [11, Theorem 16.8]). So G is normal as well. □

Corollary 3.1.11. Every T_0 locally compact topological group is T_4.

3.2 Metrizability

In this section we show that a first-countable Hausdorff topological group is metrizable. It is a more surprising fact that we can actually assume that such a metric is invariant under left (or right) translations by group elements. To make this notion precise, we introduce the following definition.

Definition 3.2.1. Let G be a group, and ρ a pseudo-metric or metric on G. We say that ρ is left-invariant if for every $x, y, g \in G$, $\rho(gx, gy) = \rho(x, y)$.

We first construct a left-invariant pseudo-metric on a first-countable topological group that generates the original topology and then we will deduce that this is a metric when the group is Hausdorff. Note that since group multiplication is a homeomorphism, being first-countable is equivalent to having a countable neighbourhood base at the identity.

Theorem 3.2.2. [6, Section 1.22] Every topological group that has a countable neighbourhood base at the identity has its topology generated by a left-invariant pseudo-metric.

Proof: Let $(S_n)_{n=0}^{\infty}$ be a countable neighbourhood base at the identity. Let $(V_n)_{r \in K}$ be as in Proposition 3.1.4 with respect to the neighbourhood base $(\bigcap_{j=0}^{n} S_j)_{n=0}^{\infty}$. Let $f : G \times G \to [0, 1]$ be given by

$$f(x,y) = \begin{cases} 0 & \text{if } e \in \bigcap_{r \in K} xV_rV_r^{-1}y^{-1} \\ \sup\{r : e \notin xV_rV_r^{-1}y^{-1}\} & \text{otherwise} \end{cases}.$$

Note that for every $g \in G$, $e \in xV_rV_r^{-1}y^{-1}$ if and only if $e \in (gx)V_rV_r^{-1}(gy)^{-1}$ from which it follows that $f(gx, gy) = f(x, y)$. Also, $V_rV_r^{-1}$ is symmetric, so $e \in xV_rV_r^{-1}y^{-1}$ if and only if $e \in yV_rV_r^{-1}x^{-1}$. Hence, $f(x, y) = f(y, x)$. Now we define our pseudo-metric $\rho : G \times G \to [0, \infty)$ by

$$\rho(x, y) = \sup_{a \in G} |f(x, a) - f(y, a)|.$$

Note that the supremum exists since f is bounded above, so ρ is well defined. It is immediately obvious from the definition of ρ that $\rho(x, x) = 0$ and $\rho(x, y) = \rho(y, x)$ for all $x, y \in G$. The triangle inequality is satisfied because if $x, y, z \in G$, then $\rho(x, y) = \sup_{a \in G} |f(x, a) - f(z, a)| \leq \sup_{a \in G}(|f(x, a) - f(y, a)| + |f(y, a) - f(z, a)|) = \sup_{a \in G} |f(x, a) - f(y, a)| + \sup_{a \in G} |f(y, a) - f(z, a)| = \rho(x, y) + \rho(y, z)$. If $g, x, y \in G$

then we have

$$\begin{aligned}
\rho(gx, gy) &= \sup_{a\in G} |f(gx, a) - f(gy, a)| \\
&= \sup_{ga\in G} |f(gx, ga) - f(gy, ga)| \text{ since left multiplication is a bijection} \\
&= \sup_{ag\in G} |f(x, a) - f(y, a)| \\
&= \sup_{a\in G} |f(x, a) - f(y, a)| \\
&= \rho(x, y).
\end{aligned}$$

So ρ is left invariant. We now show that ρ generates the topology on G. To show that the topologies are the same, we will show that for every open set in G that contains the identity, U, there is a $\varepsilon > 0$ such that $B_\varepsilon(e) \subseteq U$ and we will show for every $\varepsilon > 0$, there is an open set U in G such that $e \in U \subseteq B_\varepsilon(e)$ (where $B_\varepsilon(e)$ is the open ball of radius ε about e). It suffices to do this at the identity because of the translation invariance of the pseudo-metric and the properties that G has as a topological group.

Let U be an open set in G containing the identity. Since $(S_n)_{n=0}^\infty$ form a neighbourhood base at e, there is an N with $S_N \subseteq U$. We claim that $B_{\frac{1}{2^{N+1}}}(e) \subseteq U$. Indeed, if $x \in B_{\frac{1}{2^{N+1}}}(e)$, then $\frac{1}{2^{N+1}} > \rho(x, e) = \sup_{a\in G} |f(x, a) - f(e, a)| \geq |f(x, x) - f(e, x)| = f(e, x)$ so $e \in V_{\frac{1}{2^{N+1}}} V_{\frac{1}{2^{N+1}}}^{-1} x^{-1}$ so $x \in V_{\frac{1}{2^{N+1}}} V_{\frac{1}{2^{N+1}}}^{-1} = V_{\frac{1}{2^{N+1}}}^2 \subseteq V_{\frac{1}{2^N}} \subseteq S_N \subseteq U$.

Now suppose $\varepsilon > 0$ is given. Choose $N > 0$ so that $\frac{1}{2^N} < \varepsilon$. We show that when $x \in V_{\frac{1}{2^{N+1}}}$, then $x \in B_{\frac{1}{2^N}}(e)$ which will complete the proof. Let $x \in V_{\frac{1}{2^{N+1}}}$ and pick an arbitrary $a \in G$. We are going to estimate $|f(ax, e) - f(a, e)|$. We will show that $f(ax, e) - f(a, e) \leq \frac{1}{2^{N+1}}$. Suppose that $a \in V_r V_r^{-1}$ for some r. Then $ax \in V_r V_r^{-1} x \subseteq V_{\frac{1}{2^{N+1}}} V_r V_r^{-1} V_{\frac{1}{2^{N+1}}}^{-1} = (V_{\frac{1}{2^{N+1}}} V_r)(V_{\frac{1}{2^{N+1}}} V_r)^{-1} \subseteq V_{\min\{\frac{1}{2^{N+1}}+r,1\}} V_{\min\{\frac{1}{2^{N+1}}+r,1\}}^{-1}$. It follows that $ax \notin V_{\min\{\frac{1}{2^{N+1}}+r,1\}} V_{\min\{\frac{1}{2^{N+1}}+r,1\}}^{-1}$ implies $a \notin V_r V_r^{-1}$. Taking the supremum over r, we obtain $f(ax, e) - f(a, e) \leq \frac{1}{2^{N+1}}$. Since $x^{-1} \in V_{\frac{1}{2^{N+1}}}$, using the fact that f is left-invariant, a similar computation shows that $f(a, e) - f(ax, e) \leq \frac{1}{2^{N+1}}$. Hence, $|f(x, a^{-1}) - f(e, a^{-1})| = |f(ax, e) - f(a, e)| \leq \frac{1}{2^{N+1}}$. Taking the supremum over all $a \in G$ we obtain $\rho(x, e) \leq \frac{1}{2^{N+1}} < \frac{1}{2^N} < \varepsilon$. □

Corollary 3.2.3. If a topological group has a countable neighbourhood base at the identity and is T_0, then it has its topology generated by a left-invariant metric.

Example 3.2.4. For any positive integer n, Corollary 3.2.3 shows that we can find a left-invariant metric on $\mathrm{GL}(n, \mathbb{R})$. Note that the metric given by identifying $\mathrm{GL}(n, \mathbb{R})$ as an open subset of $\mathbb{R}^{n^2}$ in Example 2.1.6 is not left-invariant.

Let G be a group with topology generated by the left-invariant metric d. When H is a closed subgroup of G, the coset space G/H is metrizable, and has a metric which is very easy to describe in terms of d. If we define, for non-empty subsets A and B of G, $d(A, B) = \inf\{d(a, b) : a \in A, \text{ and } b \in B\}$, then defining $d(aH, bH)$ to be the distance between aH and bH defines a metric on G/H. Seeing that it is a metric is not difficult, and it is not difficult to show that this metric generates the topology on G/H. This metric is also invariant under the action by left multiplication, that is, for any $g \in G$ and $aH, bH \in G/H$ we have $d(aH, bH) = d(gaH, gbH)$.

We close this section with a proposition on metrizable locally compact topological groups.

Proposition 3.2.5. Let G be a locally compact topological group and d a left-invariant metric which generates the topology of G. Then (G, d) is a complete metric space.

Proof: Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in G with respect to d and let e denote the identity element of G. Since G is locally compact, there is a $\delta > 0$ such that the closed ball $B = \{x \in G : d(x, e) \leq \delta\}$ is compact. Choose N so that $d(x_N, x_n) < \delta$ when when $n \geq N$. Observe that $x_N B$ is compact, and that $x_N B = \{x_N x : d(x, e) \leq \delta\} = \{x_N x : d(x_N x, x_N) \leq \delta\} = \{y \in G : d(y, x_N) \leq \delta\}$. But $(x_{N+n})_{n=1}^{\infty}$ is then a sequence contained in $x_N B$ so by compactness has a convergent subsequence. But this will also be a subsequence of $(x_n)_{n=1}^{\infty}$ so $(x_n)_{n=1}^{\infty}$ has a convergent subsequence. Any Cauchy sequence with a convergent subsequence converges so we are done. $\square$

4 Lie Groups

4.1 Lie Group Definitions

Any study of topological groups would be incomplete without a discussion of one of the most important classes of such groups, Lie groups. Lie groups arise naturally in several areas like analysis, geometry, and physics, but we shall not be exploring those topics here. Our main concern will be with exploring the solution to Hilbert's fifth problem and some related concepts. We will not explore deeply the theory of Lie groups through the view of differential geometry as we have not developed sufficiently the relevant theory, but will state some interesting results which only require a very basic knowledge of differential geometry to understand.

As stated in the introduction, Hilbert's fifth problem asks whether or not topological groups which are topological manifolds are Lie groups. At the time that this was stated, the expected answer was that Hilbert's fifth problem would be resolved in the negative. Yet 33 years later, in 1933, von Neumann proved that all compact topological groups which are topological manifolds are Lie groups; a year later, in 1934, Pontryagin proved this for locally compact abelian groups, and finally, 18 years later, in 1952, Zippin, Montgomery, and Gleason proved the case for an arbitrary topological group that is a topological manifold.

We spend the rest of this section defining the concepts seen above. The goal of this section is not to develop the theory or provide motivation for the definitions, but rather to clarify our definitions that will be used in the sequel.

Definition 4.1.1. We say that a topological space X is locally Euclidean of dimension m, if for every $x \in X$, there is an open set U containing x and a homeomorphism $\varphi : U \to V$ onto an open subset V of $\mathbb{R}^m$. When X is a Hausdorff space which is locally Euclidean of dimension m, then we say that that X is an m-dimensional topological manifold. A topological space which is an m-dimensional topological manifold for some m is called a topological manifold.

Remark 4.1.2. Some authors require that topological manifolds be second countable, but no such assumption will be made here. Doing so would cause a needless loss of generality in our results to come.

Given an m -dimensional topological manifold X, we obtain coordinate charts on our manifold; pairs (U, φ) where U is an open subset of the manifold and φ is a homeomorphic mapping of U onto an open subset of $\mathbb{R}^m$. In this case, we call U the domain of the coordinate chart. Note that any topological manifold is covered by the union of the domains of all the coordinate charts.

Given two coordinate charts, (U, φ) and (V, ψ), if $U \cap V \neq \emptyset$, the transition function from φ to ψ is the function $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. Recall that a function $F : A \to \mathbb{R}^m$, where A is an open subset of $\mathbb{R}^n$, is said to be smooth if the partial derivatives of all orders of F exist. When F is invertible and the inverse function is also smooth we say that F is a diffeomorphism. We say that (U, φ) and (V, ψ) are smoothly compatible if either $U \cap V = \emptyset$, or the transition function from φ to ψ is a diffeomorphism. A smooth atlas $\mathscr{A}$ on X is a collection of coordinate charts that are smoothly compatible, and whose domains cover X.

Definition 4.1.3. A pair $(X, \mathscr{A})$ is called an m-dimensional smooth manifold if X is a topological manifold of dimension m and $\mathscr{A}$ is a maximal smooth atlas on X (so if $\mathscr{B}$ is another smooth atlas on X and $\mathscr{B} \supseteq \mathscr{A}$ then $\mathscr{B} = \mathscr{A}$). We will generally omit the maximal smooth atlas and simply say X is a smooth manifold.

If we are given a function $f : X \to Y$ between smooth manifolds X and Y, then we say that f is smooth if for every pair of coordinate charts (U, φ) on X and (V, ψ) on Y with $f(U) \subseteq V$, the function $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth (as a map between Euclidean spaces). A bijective smooth function between smooth manifolds with smooth inverse is called a diffeomorphism.

When X is a smooth manifold of dimension m and Y is a smooth manifold of dimension n, $X \times Y$ becomes an $(m+n)$-dimensional smooth manifold in a natural way, essentially by taking the product of the atlases and extending it to a maximal smooth atlas. Whenever we mention the product of two smooth manifolds, we will always assume that $X \times Y$ has this smooth manifold structure. We have now discussed enough to define a Lie group.

Definition 4.1.4. Let G be a group such that G is a smooth manifold. If the maps $(x, y) \mapsto xy$ from $G \times G \to G$ and $x \mapsto x^{-1}$ from $G \to G$ are smooth then we say that G is a Lie group.

Remark 4.1.5. Any smooth map is automatically continuous, so a Lie group is a topological group. Hence all of our theory developed in the previous sections applies to Lie groups.

We spend the remainder of this section stating some interesting properties of Lie groups, some of which we have not developed the necessary theory to prove. The first observation is that part of our definition of a Lie group is redundant.

Proposition 4.1.6. Let G be a group that is also a smooth manifold. If the multiplication map is smooth, then G is a Lie group, that is the inversion map is also smooth.

The proof of Proposition 4.1.6 is a consequence of applying the inverse function theorem to the smooth map $G \times G \to G \times G$ given by $(g, h) \mapsto (g, gh)$ (this map is a smooth bijection, so proving that the map has constant full rank allows us to conclude it is a diffeomorphism; the inverse function is then smooth, from which it easy to see that the inversion map is also smooth).

That a Lie group is locally Euclidean has a fair amount of consequences. Some of these, as well as ones that can be deduced from the preceding sections, are summarized within the next proposition.

Proposition 4.1.7. Every Lie group is a locally compact, locally connected, T_4 space.

In general, for a topological group G, the identity component G_e may not be open, $G = \mathbb{Q}$ is an easy example. However, this is always the case when G is a Lie group.

Proposition 4.1.8. If G is a Lie group, then the identity component G_e is open.

Proof: This is immediate since G is locally connected by Proposition 4.1.7 and the components of a locally connected space are open. □

Corollary 4.1.9. A totally disconnected Lie group is discrete, and hence 0-dimensional.

When discussing topological isomorphisms between topological groups it is not enough to require that the map be a continuous isomorphism. However, a routine calculation shows that smooth group homomorphisms between Lie groups have constant rank. An application of this fact along with the inverse function theorem yields the following.

Theorem 4.1.10. A smooth isomorphism between Lie groups has a smooth inverse, that is, every smooth group isomorphism is a diffeomorphism.

4.2 Hilbert's Fifth Problem

In this section we discuss Hilbert's fifth problem, and in particular the property of having "no small subgroups" which is integral to its solution. We also talk about the exponential function for matrices and how this relates to having "no small subgroups".

Definition 4.2.1. A topological group is said to have no small subgroups if there is a neighbourhood of the identity which contains no subgroup other than the trivial subgroup which consists of only the identity element.

Example 4.2.2. Let n be a positive integer. Then $\mathbb{R}^n$ has no small subgroups. Indeed, if H is a subgroup of $\mathbb{R}^n$, then for every positive integer k and $x \in H$, $kx \in H$. In particular, we see that if $x \neq 0$ for some $x \in H$, then H is unbounded. Thus, the only subgroup of a bounded open set containing 0, such as the open unit ball, is the trivial group $\{0\}$.

For the remainder of this section, let n be a positive integer and $F = \mathbb{R}$ or $F = \mathbb{C}$. We let $M_n(F)$ denote the set of all $n \times n$ matrices with entries in F. We now give $M_n(F)$ the operator norm, that is, given $A \in M_n(F)$ we define $||A|| = \inf\{C \in \mathbb{R} : ||Ax||_2 \leq C||x||_2 \text{ for every } x \in F^n\}$ where $|| \cdot ||_2$ is the Euclidean norm on F^n. Since any two norms on a finite dimensional vector space are equivalent, this is the same topology as the one on $M_n(F)$ defined in Example 2.1.6.

Remark 4.2.3. The operator norm satisfies the following inequality for every $A, B \in M_n(F)$: $||AB|| \leq ||A||||B||$. In particular, for any nonnegative integer k, $||A^k|| \leq ||A||^k$, where $A^0 = I$ is the $n \times n$ identity matrix and by convention we define $0^0 = 1$.

Lemma 4.2.4. For every $A \in M_n(F)$, the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges absolutely. Moreover, the function $f : M_n(F) \to M_n(F)$, given by $f(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$, converges uniformly on compact sets, so f is continuous.

Proof: That the series converges absolutely is an application of Remark 4.2.3: we have $\sum_{k=0}^{\infty} ||A^k|| \leq \sum_{k=0}^{\infty} \frac{1}{k!} ||A||^k = e^{||A||}$. Since $(M_n(F), || \cdot ||)$ is a Banach space it follows that $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges for every $A \in M_n(F)$. Hence, f is defined. On compact sets K, we can find an $M > 0$ with $||A|| \leq M$ for every $A \in K$ since K is bounded. Then $\sum_{k=1}^{\infty} \frac{1}{k!} M^k = e^M < \infty$, so by the Weierstrauss M-test we conclude that $f(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges uniformly on K. □

Definition 4.2.5. The function $\exp : M_n(F) \to M_n(F)$ given by $\exp(A) = \sum_{k=1}^{\infty} \frac{1}{k!} A^k$, is called the matrix exponential. We will also use the notation $e^A = \exp(A)$.

In many ways the matrix exponential behaves like the regular exponential, in fact, when we take $n = 1$ we recover the exponential function on $\mathbb{R}$ and $\mathbb{C}$. Here are some of the familiar properties of the matrix exponential.

Theorem 4.2.6. Let $A, B \in M_n(F)$. Then

(i) if A and B commute, we have $e^A e^B = e^{A+B}$;

(ii) we have $\det e^A = e^{\operatorname{tr} A}$ where $\operatorname{tr} A$ denotes the trace of A.

Proof: To prove (i) suppose that A and B commute. Then for every nonnegative integer k the binomial theorem holds: $(A + B)^k = \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} A^m B^{k-m}$. Absolute convergence implies that the following manipulations are justified:

$$
\begin{aligned}
e^{A+B} &= \sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{1}{k!} \frac{k!}{m!(k-m)!} A^m B^{k-m} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{1}{m!(k-m)!} A^m B^{k-m} \\
&= \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{1}{m!(k-m)!} A^m B^{k-m} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!k!} A^m B^{k} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} A^m \sum_{k=0}^{\infty} \frac{1}{k!} B^k \\
&= e^A e^B.
\end{aligned}
$$

To prove (ii) we first show that the result holds for upper triangular matrices. Let $B = (b_{ij})_{i,j=1}^n \in M_n(F)$ be upper triangular, and note that for each nonnegative integer k, B^k is upper triangular and the entries on the diagonal of B^k are $b_{11}^k, \ldots, b_{nn}^k$, so it follows that e^B is upper triangular and has entries on the diagonal $e^{b_{11}}, \ldots, e^{b_{nn}}$. Consequently $\det e^B = e^{b_{11}+\cdots+b_{nn}} = e^{\operatorname{tr} B}$. Now, if $A \in M_n(F)$, then A is similar to an upper triangular matrix in $M_n(\mathbb{C})$, say $A = CBC^{-1}$ where B is upper triangular. But it is easily seen that $e^A = Ce^BC^{-1}$ as $A^k = CB^kC^{-1}$ for every nonnegative integer k, so $\det e^A = \det e^B = e^{\operatorname{tr} B} = e^{\operatorname{tr}(C^{-1}AC)} = e^{\operatorname{tr}((AC)C^{-1})} = e^{\operatorname{tr} A}$. □

Corollary 4.2.7. The matrix exponential is a map of $M_n(F)$ into $\mathrm{GL}(n, F)$. Moreover, for any $A \in M_n(F)$, the inverse of e^A is e^{-A}.

Proof: If $A \in M_n(F)$, then using Theorem 4.2.6 we see that $e^Ae^{-A} = e^{A+(-A)} = e^0 = I$. Consequently $e^A \in \mathrm{GL}(n, F)$ and the inverse of e^A is e^{-A}. □

We are almost ready to discuss the role that the no small subgroups property plays for Lie groups, but before we do so we need an extra property of the matrix exponential. From now on we restrict our attention to $F = \mathbb{R}$.

Theorem 4.2.8. [6, Section 2.1.5] The matrix exponential $\exp : M_n(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is a local homeomorphism at 0, that is there is a neighbourhood U of 0 such that $\exp|_U$ is a homeomorphism onto an open subset of $\mathrm{GL}(n, \mathbb{R})$.

Proof: We identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, and are now in a position to calculate the derivative of exp at 0. Let I denote the $n \times n$ identity matrix. We claim that the derivative at 0 is the identity linear transformation $\lambda : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $\lambda(A) = A$. This computation is straightforward, since

$$
\frac{||e^H - e^0 - \lambda(H)||}{||H||} = \frac{||e^H - I - H||}{||H||} = \frac{||\sum_{k=2}^{\infty} \frac{1}{k!} H^k||}{||H||} \leq \sum_{k=2}^{\infty} \frac{1}{k!} \frac{||H||^k}{||H||} = \sum_{k=2}^{\infty} \frac{1}{k!} ||H||^{k-1} \to 0,
$$

as $H \to 0$. Since λ is invertible, using the inverse function theorem we obtain that exp is a local homeomorphism at 0 (and in fact is a local diffeomorphism). □

We are ready to give a (very) partial proof of the main result in the case that our Lie group is $\mathrm{GL}(n, \mathbb{R})$. Afterwards we will briefly discuss how this proof is essentially the same as the proof for a general Lie group.

Theorem 4.2.9. [6, Section 3 and Section 4] A locally compact Hausdorff topological group is a Lie group if and only if it has no small subgroups.

Proof: We will show that $\mathrm{GL}(n, \mathbb{R})$ has no small subgroups. Using Theorem 4.2.8 we can find a neighbourhood U of 0 in $M_n(\mathbb{R})$ such that $\exp|_U$ is a homeomorphism onto an open subset of $\mathrm{GL}(n, \mathbb{R})$. Let $\varepsilon > 0$ be such that $V = \{A \in M_n(\mathbb{R}) : ||A|| < \varepsilon\} \subseteq U$. Now, let $W = \{A \in M_n(\mathbb{R}) : ||A|| < \frac{\varepsilon}{2}\}$ and let I denote the $n \times n$ identity matrix. We claim that there is no nontrivial subgroup of $\exp(W)$, (which is a neighbourhood of I since W is an open subset of U and exp is a homeomorphism of U onto an open subset of $\mathrm{GL}(n, \mathbb{R})$). It suffices to show that if $A \in \exp(W)$ and $A \neq I$, then $A^k \notin \exp(W)$ for some integer k, so that no subset of $\exp(W)$ which contains an element other than I can be closed under multiplication. Assume towards contradiction that there is an $A \neq I$ for which $A^k \in \exp(W)$ for all positive integers k. Write $A = e^w$ for some $w \in W$, and note that $w \neq 0$ since $\exp(0) = I$. Since $w \neq 0$, there is a least positive integer n with $nw \notin W$, as $||kw|| \to \infty$ when $k \to \infty$. Note that $n \geq 2$ since $w \in W$ and therefore $2(n-1) \geq n$, so $||nw|| \leq 2||(n-1)w|| < \varepsilon$. Consequently $nw \in U \setminus W$. But $A^n \in \exp(W)$ so $e^{nw} = A^n = e^{w'}$ for some $w' \in W$. Thus, $nw, w' \in U$, $nw \neq w'$, and $\exp(nw) = \exp(w')$, which is a contradiction since $\exp|_U$ is injective. □

Remark 4.2.10. When we say that a locally compact Hausdorff topological group with no small subgroups is a Lie group, what we mean is that there is a Lie group structure compatible with the topology of the topological group.

As for proving that any Lie group has no small subgroups, there is very little to change in the above proof. To every Lie group G, there is a Lie algebra $\mathfrak{g}$ associated to G, and an exponential function $\exp : \mathfrak{g} \to G$. This exponential function is a local homeomorphism, maps the 0 of $\mathfrak{g}$ to the identity element of G, and satisfies $\exp(A + B) = \exp(A)\exp(B)$ when A and B commute, and in particular we have $\exp(A)^k = \exp(kA)$ for every integer k. Therefore, the only part of the proof that must be adjusted is replacing $M_n(\mathbb{R})$ with $\mathfrak{g}$ and $\mathrm{GL}(n, \mathbb{R})$, with G.

Theorem 4.2.11. [6, Section 4.10] A topological group which is a topological manifold is a Lie group.

To prove the "if" part of Theorem 4.2.9 requires far more preparation and effort than we can undertake here. However, we can discuss without difficulty the main approximation theorem which is used to prove it.

Definition 4.2.12. Let G be a topological group. Then G is said to be almost connected if G/G_e is compact.

Theorem 4.2.13. [6, Section 4.6] Let G be a locally compact Hausdorff topological group which is almost connected and let U be any neighbourhood of the identity element of G. Then U contains a compact normal subgroup H of G, such that G/H has no small subgroups and is a Lie group.

In particular we see that when a connected locally compact Hausdorff group has no small subgroups, then we can choose a neighbourhood U of the identity which forces $H = \{e\}$, from which we conclude that G is a Lie group.

The consequences of Theorem 4.2.13 for the theory of locally compact groups turns out to be far more important than the solution of Hilbert's fifth problem. With the aid of this theorem we are able to approximate a large class of locally compact Hausdorff topological groups with Lie groups, and thereby use techniques from Lie group theory in the study of locally compact groups. Ending our discussion of Lie groups, the next example shows how Theorem 4.2.13 works, and illustrates that groups that have too many dimensions are not Lie groups.

Example 4.2.14. Let c be an infinite cardinal. We claim that the infinite torus $\mathbb{T}^c$ is not a Lie group. Indeed, if U is any neighbourhood of the identity, then it contains a set of the form $\prod_{\alpha \in c} U_\alpha$, where each U_α

is open and all but finitely many are equal to $\mathbb{T}$. In particular, choosing a γ for which $U_\gamma = \mathbb{T}$, which exists because c is infinite, $X = \prod_{\alpha \in c} X_\alpha$ where $X_\gamma = \mathbb{T}$ and all the other factors are equal to $\{1\}$, is a subgroup which is contained in U. We can also take $H = \prod_{\alpha \in c} H_\alpha$ where $H_\alpha = \mathbb{T}$ when $U_\alpha = \mathbb{T}$ and $H_\alpha = \{1\}$ when $U_\alpha \neq \mathbb{T}$. Then $H \subseteq U$, and H is a compact normal subgroup with T^c/H being topologically isomorphic to T^n where n is the finite number of α for which $H_\alpha = \{1\}$. A similar reasoning shows that when G is any compact Lie group with more than one element, then G^c has small subgroups and is not a Lie group. If G is connected and compact then we can also find a normal subgroup H of G^c so that G^c/H is topologically isomorphic to G^n for some integer n, and is therefore a Lie group.

5 Integration on Topological Groups

5.1 Haar Measure

In this section we discuss integration on topological groups, specifically on locally compact Hausdorff topological groups. For this reason, we will now refer to a locally compact Hausdorff topological group as a locally compact group. In a similar way, a compact group is a compact Hausdorff topological group.

Before we make precise the measures we will discuss, we give a quick review of the terminology and notation we will use.

Definition 5.1.1. Let X be a topological space. We will always let $\mathscr{B}(X)$ denote the σ-algebra of Borel sets. A Radon measure on X is a Borel measure $\mu : \mathscr{B}(X) \to [0, \infty]$ that is finite on compact sets, inner regular on open sets, and outer regular on Borel sets, that is,

1. when K is compact, $\mu(K) < \infty$;
2. when U is open, $\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ is compact}\}$;
3. when B is a Borel set, $\mu(B) = \inf\{\mu(U) : U \supseteq B \text{ and } U \text{ is open}\}$.

We now state without proof a result of great importance. It will be important for constructing measures as well as verifying certain uniqueness conditions. When X is a topological space, the support of a function $f : X \to \mathbb{R}$ is the closure of the set $\{x \in X : f(x) \neq 0\}$, and we denote it by $\text{supp}(f)$. Let $C_c(X) = \{f : X \to \mathbb{R} : f \text{ is continuous and } \text{supp}(f) \text{ is compact}\}$ and let $C_c^+(X) = \{f \in C_c(X) : f(x) \geq 0 \text{ for all } x \in X, \text{and } f \text{ is non-zero}\}$. Observe that $C_c(X)$ is a real vector space and that when X is a locally compact Hausdorff space, Urysohn's lemma for locally compact Hasdorff spaces shows that $C_c^+(X)$ is non-empty. We call a linear functional $\psi : C_c(X) \to \mathbb{R}$ positive if $f \in C_c^+(X) \cup \{0\}$ implies $\psi(f) \geq 0$. The following theorem is called the Riesz representation theorem (it is one of many Riesz representation theorems).

Theorem 5.1.2. [1, Theorem 7.2] Let X be a locally compact Hausdorff space and ψ a positive linear functional on $C_c(X)$. Then there is a unique Radon measure μ on X such that $\psi(f) = \int_X f d\mu$ for every $f \in C_c(X)$.

The following proposition states that every Borel function $f : G \to [-\infty, \infty]$, where G is a topological group, factors through $G/\overline{\{e\}}$.

Proposition 5.1.3. Let G be a topological group, and let e be the identity element of G. Then any Borel function $f : G \to [-\infty, \infty]$ is constant on the cosets $x\overline{\{e\}}$. Thus, if $\pi : G \to G/\overline{\{e\}}$ is the projection, there is a Borel function $\psi : G/\overline{\{e\}} \to [\infty, \infty]$ with $f = \psi \circ \pi$.

Proof: We verify this for characteristic functions, the general result will follow since simple functions will be constant on the cosets, and then so will Borel functions since they are the pointwise limits of simple functions. Let $\mathscr{B} = \{B \subseteq G : \text{for every } x \in G,\ B \supseteq x\overline{\{e\}} \text{ or } B \cap x\overline{\{e\}} = \emptyset\}$. This is clearly a σ-algebra of subsets of G, consequently, showing that every closed set is a member of $\mathscr{B}$ will show that χ_B, the characteristic function of B, is constant on the cosets $x\overline{\{e\}}$ whenever B is a Borel set. Let $F \subseteq G$ be closed. If $x \in F$, then $x\overline{\{e\}} = \overline{\{x\}} \subseteq F$ because F is closed. If $x \in G \setminus F$, then Theorem 2.2.5 asserts that $\overline{\{e\}}$ has the indiscrete topology. Consequently $x\overline{\{e\}}$ also has the indiscrete topology, so $x\overline{\{e\}} \subseteq G \setminus F$ as $G \setminus F$ is open. Thus, $F \in \mathscr{B}$. □

We are now prepared to give the definition of the measures we will be looking at.

Definition 5.1.4. Let G be a topological group. We say that a measure μ on a σ-algebra $\mathscr{A} \supseteq \mathscr{B}(G)$ of subsets of G is left-invariant if for every $A \in \mathscr{A}$ and $x \in G$, $xA \in \mathscr{A}$ and $\mu(xA) = \mu(A)$, and right-invariant if for every $A \in \mathscr{A}$ and $x \in G$, $Ax \in \mathscr{A}$ and $\mu(Ax) = \mu(A)$. Now suppose that G is a locally compact group. A left Haar measure on G is a non-zero Radon measure that is left-invariant. A right Haar measure on G is a non-zero Radon measure that is right-invariant. If we call a measure a Haar measure we mean that it is either a left Haar measure or a right Haar measure.

Example 5.1.5. The counting measure is a left and right Haar measure on any topological group that has the discrete topology.

In fact, Example 5.1.5 has a partial converse.

Proposition 5.1.6. Let G be a locally compact group with left Haar measure μ, such that $\mu(\{x\}) > 0$ for some $x \in G$. Then G has the discrete topology.

Proof: It is immediate that $\mu(\{x\}) = \mu(\{g\}) > 0$ for all $g \in G$ since μ is left-invariant. Consequently for every Borel set A, $\mu(A)$ is finite if and only if A is finite. In particular, compact sets are finite, and since there is a compact neighbourhood of the identity e, there is a finite neighbourhood which contains e in its interior. Since G is Hausdorff it follows that $\{e\}$ is open and from this it follows that G is discrete. □

Before we give more examples of Haar measures, we will develop a small amount of theory. We introduce the following piece of notation: if $g \in G$ and $f : G \to [-\infty, \infty]$, the left translate of f through g is the function $L_g f$ defined by $L_g f(x) = f(g^{-1}x)$; the right translate of f through g is the function $R_g f$ defined by $R_g f(x) = f(xg)$ (both L_g and R_g are chosen this way so that $L_h \circ L_g = L_{hg}$ and $R_h \circ R_g = R_{hg}$).

Definition 5.1.7. Let G be a topological group, and for a real-valued function $\psi : G \to \mathbb{R}$, we let $||\psi||_\infty = \sup_{x \in G} |\psi(x)|$. We say that a function $f : G \to \mathbb{R}$ is left uniformly continuous if for every $\varepsilon > 0$, there is a neighbourhood U of the identity such that $||L_g f - f||_\infty < \varepsilon$ for every $g \in U$. Similarly, f is right uniformly continuous if for every $\varepsilon > 0$ we can find a neighbourhood U of the identity such that $||R_g f - f||_\infty < \varepsilon$ whenever $g \in U$.

Proposition 5.1.8. [2, Proposition 2.6] Let G be a locally compact group. Then every $f \in C_c(G)$ is both left and right uniformly continuous.

Proof: We show that f is left uniformly continuous, showing that it is also right uniformly continuous is completely analogous. Let $\varepsilon > 0$, and let $K = \text{supp}(f)$. Since f is continuous, for every $x \in K$, there is an open neighbourhood of the identity, U_x, such that $|f(ux) - f(x)| < \frac{\varepsilon}{2}$ when $u \in U_x$. Let V_x be a symmetric neighbourhood of the identity such that $V_x^2 \subseteq U_x$ which can be found using Proposition 2.1.19. Since K is compact, we can choose $x_1, \ldots, x_n \in K$ with $K \subseteq \bigcup_{j=1}^n V_{x_j} x_j$. Let $V = \bigcap_{j=1}^n V_{x_j}$, then we claim that when $y \in V$, that $||L_y f - f||_\infty \leq \varepsilon$. When $x \in K$, we have $x \in V_{x_j} x_j$ for some j, so $xx_j^{-1} \in V_x$ and therefore $y^{-1}xx_j^{-1} \in V_{x_j} V_{x_j} \subseteq U_{x_j}$ so that $y^{-1}xx_j^{-1} \in U_{x_j}$, and we consequently have that $|f(y^{-1}x) - f(x)| \leq |f(y^{-1}x) - f(x_j)| + |f(x_j) - f(x)| = |f((y^{-1}xx_j^{-1})x_j) - f(x_j)| + |f((xx_j^{-1})x_j) - f(x_j)| < \varepsilon$. If $y^{-1}x \in K$, then we may do the same argument with $y^{-1} \in V$ to show that $|f(y^{-1}x) - f(x)| = |f(y(y^{-1}x)) - f(y^{-1}x)| < \varepsilon$. Otherwise x and $y^{-1}x$ lie outside of K so $|f(y^{-1}x) - f(x)| = 0$. Thus, $||L_y f - f||_\infty \leq \varepsilon$. □

Proposition 5.1.9. [2, Proposition 2.9] Let G be a locally compact group, μ a Radon measure on G, and $\widetilde{\mu} : \mathscr{B}(G) \to [0, \infty]$ be given by $\widetilde{\mu}(A) = \mu(A^{-1})$.

(i) If μ is a left Haar measure then $\widetilde{\mu}$ is a right Haar measure, and if $\widetilde{\mu}$ is a right Haar measure then μ is a left Haar measure.

(ii) The measure μ is a left Haar measure if and only if $\int_G L_g f d\mu = \int_G f d\mu$ for every $g \in G$ and $f \in C_c^+(G)$, and μ is a right Haar measure if and only if $\int_G R_g f d\mu = \int_G f d\mu$ for every $g \in G$ and $f \in C_c^+(G)$.

Proof: The proof of (i) is simple. First note that A^{-1} is a Borel set when A is. Then if $x \in G$, and μ is a left Haar measure, $\widetilde{\mu}(Ax) = \mu((Ax)^{-1}) = \mu(x^{-1}A^{-1}) = \mu(A^{-1}) = \widetilde{\mu}(A)$. It follows that $\widetilde{\mu}$ is right-invariant, and the rest of the properties are straightforward to verify (for example, $\widetilde{\mu}$ is non-zero since $\widetilde{\mu}(G) = \mu(G) > 0$). So $\widetilde{\mu}$ is a right Haar measure. Similarly when $\widetilde{\mu}$ is a right Haar measure one deduces that μ must be a left Haar measure.

We now show (ii). Note that when μ is a Radon measure, for any $g \in G$, μ_g is also a Radon measure, where $\mu_g(A) = \mu(gA)$ for every $A \in \mathscr{B}(G)$. By the change of variable formula we have $\int_G f d\mu_g = \int_{gG} L_g f d\mu = \int_G L_g f d\mu$ for every $g \in G$ and $f \in C_c^+(G)$. If G is a left Haar measure then $\mu_g = \mu$ for all $g \in G$ so $\int_G f d\mu = \int_G f d\mu_g = \int_G L_g f d\mu$ for all $g \in G$ and $f \in C_c^+(G)$. If $\int_G f d\mu_g = \int_G L_g f d\mu = \int_G f d\mu$ for every $f \in C_c^+(G)$, then this equality also holds for every $f \in C_c(G)$, so by the uniqueness in Theorem 5.1.2 we have $\mu = \mu_g$ and so μ is a left Haar measure. The other proof is the same. □

Remark 5.1.10. From (ii) in Proposition 5.1.9, we see that μ is a left Haar measure if and only if $\int_G f(gx)d\mu(x) = \int_G f(x)d\mu(x)$ holds for every $g \in G$ and $f \in C_c^+(G)$.

Note that (i) of Proposition 5.1.9 shows that we can restrict our attention to left Haar measures when proving any theoretical results about Haar measures, and we shall do so in the sequel.

Here are two examples of Haar measure on familiar groups which we have quoted from [2].

Example 5.1.11. Let m_n denote the Lebesgue measure on the Borel sets of $\mathbb{R}^n$. Then this is both a left and right Haar measure. Note that if γ is the counting measure on $\mathbb{R}^n$, then this is not a Haar measure even though it is left-invariant and right-invariant. Indeed, the compact set $[0,1]$ has $\gamma([0,1]) = \infty$.

Example 5.1.12. Let $\mathrm{GL}(n,\mathbb{R})$ be the general linear group, and identify $\mathrm{GL}(n,\mathbb{R})$ with open subset of $\mathbb{R}^{n^2}$ by identifying $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Then λ given by $d\lambda(X) = |\det X|^{-n} dm_{n^2}(X)$ is a left and right Haar measure on $\mathrm{GL}(n,\mathbb{R})$. If A is a Borel subset of $\mathrm{GL}(n,\mathbb{R})$, and g is any element of $\mathrm{GL}(n,\mathbb{R})$, then

$$\begin{aligned}
\lambda(gA) &= \int_{gA} |\det X|^{-n} dm_{n^2}(X) \\
&= \int_A |\det(gX)|^{-n} |\det \operatorname{diag}(g,\ldots,g)| dm_{n^2}(X) \\
&= \int_A |\det g|^{-n} |\det X|^{-n} |\det g|^n dm_{n^2}(X) \\
&= \int_A |\det X|^{-n} dm_{n^2}(X) \\
&= \lambda(A).
\end{aligned}$$

The computation above uses the fact that the Jacobian matrix of the map $X \mapsto gX$ is the diagonal block $n^2 \times n^2$ matrix $\operatorname{diag}(g,\ldots,g)$. This shows that λ is left-invariant, showing that λ is right-invariant is essentially the same computation.

The next result is something that we shall use later.

Proposition 5.1.13. [2, Proposition 2.19] Let G be a locally compact group and μ a Haar measure on G. Then $\mu(U) > 0$ for every non-empty open set U and $\int_G f d\mu > 0$ for every $f \in C_c^+(G)$.

Proof: Suppose that $\mu(U) = 0$ for some non-empty open set U. Every compact set K is covered by finitely many translates of U by compactness, and thus also has $\mu(K) = 0$. But then $\mu(G) = \sup\{\mu(K) : K \text{ is compact}\} = 0$ which would imply that μ is the zero measure, but this is not the case since μ is a Haar measure. So $\mu(U) > 0$ for every non-empty open set U. Now let $f \in C_c^+(G)$, and let $W = \{x \in G : f(x) > \frac{1}{2}||f||_\infty\}$. Then W is open and $\int_G f d\mu \geq \frac{1}{2}||f||_\infty \mu(W) > 0$. □

Another result along the lines of Proposition 5.1.6 is the following.

Proposition 5.1.14. Let G be a locally compact group and μ a left Haar measure on G. Then $\mu(G) < \infty$ if and only if G is compact.

Proof: If G is compact, then $\mu(G) < \infty$ since μ is a Radon measure. Now suppose that $\mu(G) < \infty$ and suppose towards contradiction that G is not compact. Let U be a non-empty compact neighbourhood of the identity. Then G cannot be covered by finitely many left translates of U, otherwise there is a finite sequence $(x_n)_{n=1}^N$ such that $G = \bigcup_{n=1}^N x_n U$ and G would be compact. Hence, we may inductively choose a sequence $(x_n)_{n=1}^\infty$ in G with $x_n \notin \bigcup_{k=1}^{n-1} x_k U$. Using Proposition 2.1.19 we may choose an open symmetric neighbourhood V of the identity with $V^2 \subseteq U$. We claim that $x_n V \cap x_k V = \emptyset$ when $n \neq k$. Indeed, assume that $n > k$ and $x_n V \cap x_k V \neq \emptyset$, then there are $v_1, v_2 \in V$ with $x_n v_1 = x_k v_2$, so $x_n = x_k v_2 v_1^{-1} \subseteq x_k V V^{-1} = x_k V^2 \subseteq x_k U$ which contradicts the way the sequence was chosen. But now we have that $\mu(G) \geq \mu(\bigcup_{n=1}^\infty x_n V) = \sum_{n=1}^\infty \mu(x_n V) = \sum_{n=1}^\infty \mu(V) = \infty$ as $\mu(V) > 0$ by Proposition 5.1.13, a contradiction. So G is compact. □

If we make some adjustments to the argument above, we obtain the following.

Proposition 5.1.15. Let G be a locally compact group and μ a left Haar measure on G. Then μ is σ-finite if and only if G is σ-compact.

Proof: If G is σ-compact, then μ being a Radon measure implies every compact set has finite-μ measure, so μ is σ-finite. Now suppose that μ is σ-finite and G is not σ-compact. Let U be a compact neighbourhood of the identity, and using Proposition 2.1.19, choose an open symmetric neighbourhood V of the identity with $V^2 \subseteq U$. Then there is no sequence $(x_n)_{n=1}^\infty$ with $G = \bigcup_{n=1}^\infty x_n U$, otherwise G would be σ-compact. But using Zorn's lemma, one easily obtains that there is a maximal set $S \subseteq G$ with $sV \cap tV = \emptyset$ whenever $s \neq t$. It is easily seen that $G = \bigcup_{s \in S} sU$: if there were $x \notin \bigcup_{s \in S} sU$ then by maximality, $xV \cap sV \neq \emptyset$ for some $s \in S$, so $x \in sVV^{-1} \subseteq sU$, a contradiction. It follows that S is uncountable. Now, since μ is σ-finite and $\{sV : s \in S\}$ is a family of pairwise disjoint sets, we must have that the set $\{s \in S : \mu(sV) > \frac{1}{n}\}$ is countable for each positive integer n. But $\mu(sV) = \mu(V) > 0$ for every $s \in G$, so $S = \bigcup_{n=1}^\infty \{s \in S : \mu(sV) > \frac{1}{n}\}$ is countable, a contradiction. So G is σ-compact. □

5.2 The Modular Function

In the next section we will prove that every locally compact group G has a left Haar measure μ, and that this measure is unique in the sense that if λ is another left Haar measure on G, then there is a $c > 0$ with $c\mu = \lambda$. Let us assume these results for now, and see how we can use them to measure the extent to which a left Haar measure fails (or succeeds) to be a right Haar measure.

Let G be a locally compact group and fix a left Haar measure μ on G. If $x \in G$, then define μ_x by setting $\mu_x(A) = \mu(Ax)$. Then μ_x is also a left Haar measure on G, so by uniqueness, there is a $\Delta(x) > 0$ with $\Delta(x)\mu = \mu_x$. That is, for any Borel set A, we have the equality $\mu(Ax) = \Delta(x)\mu(A)$. Thus we obtain a function $\Delta : G \to (0, \infty)$ which satisfies $\mu(Ax) = \Delta(x)\mu(A)$ for every $x \in G$ and every Borel set A. Note that by uniqueness of Haar measure, Δ will not depend on the left Haar measure μ and that μ is a right Haar measure if and only if $\Delta(x) = 1$ for every $x \in G$.

Definition 5.2.1. Let G be a locally compact group. Then the function $\Delta : G \to (0, \infty)$ which satisfies $\Delta(x)\mu(A) = \mu(Ax)$ for every Borel set A, $x \in G$, and left Haar measure μ, is called the modular function of G. When $\Delta(x) = 1$ for every $x \in G$ we say that G is unimodular.

A unimodular group is just a locally compact group for which every left Haar measure is also a right Haar measure. These are the only types of groups which we have discussed thus far, and naturally one is curious about examples of groups which are not unimodular. Before we present an example of such a group, we first prove a proposition which states some of the most basic properties of the modular function and will make it easier to calculate Δ.

Theorem 5.2.2. [2, Proposition 2.24] Let G be a locally compact group with left Haar measure μ and let Δ be the modular function of G. Then Δ is a continuous homomorphism of G into group $(0, \infty)$ of positive real numbers, and for any $f \in C_c(G)$,

$$\int_G R_g f d\mu = \Delta(g^{-1}) \int_G f d\mu.$$

Proof: Let U be a compact neighbourhood of e. Then $0 < \mu(U) < \infty$ by Proposition 5.1.13 and the fact that U is compact. If $x, y \in G$, then $\Delta(xy)\mu(U) = \mu(Uxy) = \Delta(y)\mu(Ux) = \Delta(y)\Delta(x)\mu(U)$ so $\Delta(xy) = \Delta(x)\Delta(y)$. Thus Δ is a homomorphism. Let $\mu_{g^{-1}} = \Delta(g^{-1})\mu$. Then by the change of variables formula, for any $f \in C_c(G)$,

$$\Delta(g^{-1}) \int_G f d\mu = \int_G f d\mu_{g^{-1}} = \int_{Gg} R_g f d\mu = \int_G R_g f d\mu.$$

Now, choosing an $f \in C_c^+(G) \subseteq C_c(G)$ and using the fact that Δ is a homomorphism we have $\Delta(g) = \frac{\int_G f d\mu}{\int_G R_g f d\mu}$, so to prove that Δ is continuous, it suffices to show that the map $g \mapsto \int_G R_g f d\mu$ is continuous (it is never zero by Proposition 5.1.13). Let $K = \operatorname{supp} f$ and U be a compact symmetric neighbourhood of the identity, and note that $\mu(KxU) < \infty$ for any $x \in G$ since KxU is compact by Proposition 2.1.16. For $g \in G$ and $h \in Ug$ we have $h^{-1} \in g^{-1}U$ and so,

$$\begin{aligned}
\left| \int_G R_g f d\mu - \int_G R_h f d\mu \right| &= \left| \int_{Kg^{-1} \cup Kh^{-1}} R_g f - R_h f d\mu \right| \\
&\leq ||R_g f - R_h f||_\infty \mu(Kg^{-1} \cup Kh^{-1}) \\
&\leq (\mu(Kg^{-1}) + \mu(Kh^{-1}))||R_{h^{-1}g} f - f||_\infty \\
&\leq 2\mu(Kg^{-1}U)||R_{h^{-1}g} f - f||_\infty.
\end{aligned}$$

Since $f \in C_c^+(G)$, by Proposition 5.1.8, f is right uniformly continuous so $||R_{h^{-1}g} f - f||_\infty$ can be made arbitrarily small by choosing an appropriate neighbourhood of g, and if we also make this neighbourhood a subset of Ug then $|\int_G R_g f d\mu - \int_G R_h f d\mu|$ can be made arbitrarily small. □

Here is an example of a group that is not unimodular.

Example 5.2.3. Let $\{(a, b) \in \mathbb{R}^2 : a > 0\}$ be the open right half plane. This is a topological group when given the multiplication $(a, b)(c, d) = (ac, ad + b)$ and is called the $ax + b$ group. Then $\frac{1}{a^2} dadb$ is a left Haar measure, and $\frac{1}{a} dadb$ is a right Haar measure. We only verify that $\frac{1}{a^2} dadb$ is left-invariant. Note that left multiplication by a fixed $g = (x, y)$ is given by $(a, b) \mapsto (xa, xb + y) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$, which has Jacobian matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ which in turn has determinant x^2. Hence, if A is any Borel set, we have $\int_{gA} \frac{1}{a^2} dadb = \int_A \frac{1}{(xa)^2} x^2 dadb = \int_A \frac{1}{a^2} dadb$. It is easy to show that the modular function is given by $\Delta(a, b) = \frac{1}{a}$, just compute the integral over a sufficiently nice region, like a square in the open right half plane, and then do the same for the image after right multiplication by (a, b). The $ax + b$ group is solvable since the normal subgroup $\{(1, b) : b \in \mathbb{R}\}$ has quotient isomorphic to the group of positive real numbers which is abelian. Consequently, not every solvable group is unimodular.

It is obvious that any abelian locally compact group is unimodular. It is less obvious that a compact group is unimodular, but the modular function makes this very easy to prove.

Proposition 5.2.4. [2, Corollary 2.28] A compact group is unimodular.

Proof: Let G be a compact group with modular function Δ. By Theorem 5.2.2, Δ is a continuous homomorphism and therefore $\Delta(G)$ is a compact subgroup of the positive real numbers. The only such subgroup is $\{1\}$ so we deduce that G is unimodular. ☐

We have already seen that not every solvable group is unimodular, so naturally we ask whether or not every nilpotent group is unimodular. To this question the answer is yes. It follows from this proposition, the proof of which can be found on page 92 of [7].

Proposition 5.2.5. [7, Chapter 3 Proposition 25] Let G be a locally compact group and $Z(G)$ its center. Then G is unimodular if and only if $G/Z(G)$ is unimodular.

Corollary 5.2.6. A locally compact nilpotent group is unimodular.

5.3 Existence and Uniqueness

In this final section, we show that every locally compact group has a Haar measure, and, moreover, that it is essentially unique. What is to come is far more technical than anything thus far, and is based on the presentation given in [2]. We begin by introducing an important technical tool for the construction of the measure.

Definition 5.3.1. Let G be a locally compact group, and $f, g \in C_c^+(G)$. Then we define

$$(f:g) = \inf\{\sum_{j=1}^{n} c_j : f \leq \sum_{j=1}^{n} c_j L_{y_j} g,\ c_j \geq 0\}.$$

Proposition 5.3.2. We always have that $(f:g)$ is finite (i.e., the set above is not empty).

Proof: Let $f, g \in C_c^+(G)$. Let $U = \{x \in G : g(x) > \frac{1}{2}||g||_\infty\}$, then U is open and non-empty. The support of f is compact, so can be covered by finitely many of the xU where $x \in G$, say $\text{supp}(f) \subseteq \bigcup_{j=1}^n x_j U$. We claim that $f \leq \sum_{j=1}^n \frac{2||f||_\infty}{||g||_\infty} L_{x_j} g$. If $x \in \text{supp}(f)$, then $x \in x_j U$ for some j and hence $x_j^{-1}x \in U$ so that $\frac{2||f||_\infty}{||g||_\infty} L_{x_j} g(x) = \frac{2||f||_\infty}{||g||_\infty} g(x_j^{-1}x) \geq ||f||_\infty \geq f(x)$. Thus, $(f:g) \leq 2n\frac{||f||_\infty}{||g||_\infty}$. ☐

Next we establish some properties which will be useful to us later in the proof of the existence of a Haar measure.

Lemma 5.3.3. Let G be a locally compact group, $f_1, f_2, f, g, h \in C_c^+(G)$, and $c > 0$. Then we have the following:

(i) $(f:g) = (L_y f : g)$ for any $y \in G$;

(ii) $(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g)$;

(iii) $(cf : g) = c(f : g)$;

(iv) $(f_1 : g) \leq (f_2 : g)$ when $f_1 \leq f_2$;

(v) $(f : g) \geq \frac{||f||_\infty}{||g||_\infty}$;

(vi) $(f : g) \leq (f : h)(h : g)$.

Proof: We begin by proving (i). Suppose that $y \in G$. If $f \leq \sum_{j=1}^n c_j L_{y_j} g$, then $L_y f \leq \sum_{j=1}^n c_j L_{yy_j} g$ so it follows that $(L_y f : g) \leq (f : g)$. Applying this result to the function $L_y f$ with $y^{-1} \in G$ yields $(f : g) \leq (L_y f : g)$.

Suppose that $f_1 \leq \sum_{j=1}^n a_j L_{x_j} g$ and $f_2 \leq \sum_{k=1}^m b_j L_{y_j} g$. Then $f_1 + f_2 \leq \sum_{j=1}^n a_j L_{x_j} g + \sum_{k=1}^m b_j L_{y_j} g$, so $(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g)$ and so (ii) holds.

Suppose that $f \leq \sum_{j=1}^n c_j L_{y_j} g$, then $cf \leq \sum_{j=1}^n cc_j L_{y_j} g$ so $(cf : g) \leq c(f : g)$, and doing the same with $\frac{1}{c}$ to cf we obtain $c(f : g) \leq (cf : g)$ so (iii) is true.

Suppose that $f_2 \leq \sum_{j=1}^n c_j L_{y_j} g$, then $f_1 \leq \sum_{j=1}^n c_j L_{y_j} g$, so $(f_1 : g) \leq (f_2 : g)$ which yields (iv).

We now prove (v). Let $f \leq \sum_{j=1}^n c_j L_{y_j} g$. Then $||f||_\infty \leq \sum_{j=1}^n c_j L_{y_j} g \leq ||g||_\infty \sum_{j=1}^n c_j$, so $\sum_{j=1}^n c_j \geq \frac{||f||_\infty}{||g||_\infty}$ and consequently $(f : g) \geq \frac{||f||_\infty}{||g||_\infty}$ by taking the infimum.

We finally prove (vi). Suppose that $f \leq \sum_{j=1}^n c_j L_{x_j} h$ and $h \leq \sum_{k=1}^m d_k L_{y_k} g$, then $f \leq \sum_{j=1}^n \sum_{k=1}^m c_j d_k L_{x_j y_k} g$, so $(f : g) \leq \sum_{j=1}^n \sum_{k=1}^m c_j d_k = (\sum_{j=1}^n c_j)(\sum_{k=1}^m d_k)$. Taking infimums we obtain $(f : g) \leq (f : h)(h : g)$. $\square$

Definition 5.3.4. Let G be a locally compact group and fix $a \in C_c^+(G)$. For any $g \in C_c^+(G)$, we obtain a function $I_g : C_c^+(G) \to \mathbb{R}$ by defining

$$I_g(f) = \frac{(f : a)}{(a : g)}.$$

This is defined since Lemma 5.3.3 (v) shows that the denominator is non-zero. From Lemma 5.3.3 we may also deduce the content of the following lemma.

Proposition 5.3.5. For any $y \in G$, and $f, h, g \in C_c^+(G)$ and $c > 0$

1. $I_g(L_y f) = I_g(f)$ (I_g is left-invariant);
2. $I_g(f + h) \leq I_g(f) + I_g(h)$ (I_g is subadditive);
3. $I_g(cf) = cI_g(f)$ (I_g is homogeneous of degree 1);
4. $I_g(f) \leq I_g(h)$ when $f \leq h$ (I_g is monotone);
5. $(a : f)^{-1} \leq I_g(f) \leq (f : a)$.

Lemma 5.3.6. [2, Lemma 2.18] If $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$, there is a neighbourhood V of the identity such that $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \varepsilon$ whenever $\text{supp}(g) \subseteq V$.

Proof: Let $h \in C_c^+(G)$ be such that $h(x) = 1$ when $x \in \text{supp}(f_1 + f_2)$ (since G is a locally compact Hausdorff space, such an h is guaranteed to exist by Urysohn's lemma for locally compact Hausdorff spaces). Choose $\delta > 0$ so that $2\delta(f_1 + f_2 : a) + \delta(1 + 2\delta)(h : a) < \varepsilon$. Now let $k = f_1 + f_2 + \delta h$ and for $j = 1, 2$ let $k_j = \frac{f_j}{k}$ where $k_j = 0$ when $f_j = 0$ (so that there are no issues with the denominator being 0). Since $k_j \in C_c^+(G)$, there is an open neighbourhood V of the identity such that $y^{-1}x \in V$ implies $|k_j(x) - k_j(y)| < \delta$ for both $j = 1$ and $j = 2$. Now suppose that $g \in C_c^+(G)$ and that $\text{supp}(g) \subseteq V$. If we have that $k \leq \sum_{i=1}^n c_i L_{x_i} g$, then $f_j(x) = k(x)k_j(x) \leq \sum_{i=1}^n c_i g(x_i^{-1}x) k_j(x) \leq \sum_{i=1}^n c_i g(x_i^{-1}x)[k_j(x_j) + \delta]$, since $|k_j(x) - k_j(x_i)| < \delta$ whenever $x_i^{-1}x \in \text{supp}(g)$ and when $x_i^{-1}x \notin \text{supp}(g)$, $g(x_i^{-1}x) = 0$. Since we also have $k_1 + k_2 \leq 1$ (it is obvious from the way we defined them), we then have that $(f_1 : g) + (f_2 : g) \leq \sum_{j=1}^n c_j(k_1(x_j) + \delta) + \sum_{j=1}^n c_j(k_2(x_j) + \delta) \leq (1 + 2\delta)\sum_{j=1}^n c_j$. Taking the infimum over all such $\sum_{j=1}^n c_j$, we obtain $(f_1 : g) + (f_2 : g) \leq (1 + 2\delta)(k : g) \leq (1 + 2\delta)(f_1 + f_2 : g) + \delta(1 + 2\delta)(h : g)$, so dividing by $(a : g)$ yields $I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + 2\delta I_g(f_1 + f_2) + \delta(1 + 2\delta)I_g(h) \leq I_g(f_1 + f_2) + 2\delta(f_1 + f_2 : a) + \delta(1 + 2\delta)(h : a) \leq I_g(f_1 + f_2) + \varepsilon$. $\square$

We are now in a position to prove the existence of the Haar measure. We first prepare our notation. For each $f \in C_c^+(G)$, let X_f be the interval $X_f = [(a : f)^{-1}, (f : a)]$, which is non-empty by Proposition 5.3.5 and compact. Then $X = \prod_{f \in C_c^+(G)} X_f$ is a compact Hausdorff space by Tychonoff's theorem and for every $g \in C_c^+(G)$, $I_g \in X$ since $(a : f)^{-1} \leq I_g(f) \leq (f : a)$ for all $f \in C_c^+(G)$, by Proposition 5.3.5.

Theorem 5.3.7. [2, Theorem 2.10] Every locally compact group has a left Haar measure.

Proof: For every neighbourhood V of the identity in G, let $K(V)$ be the closure of $\{I_g \in X : g \in C_c^+(G)$, and $\text{supp}(g) \subseteq V\}$. The collection K of all the sets of the form $K(V)$ has the finite intersection property since $\bigcap_{j=1}^n K(V_j) \supseteq K(\bigcap_{j=1}^n V_j) \neq \emptyset$ holds because if g has $\text{supp}(g) \subseteq \bigcap_{j=1}^n V_j$ then $\text{supp}(g) \subseteq V_j$ for $j = 1, \ldots, n$, and taking closures yields the result (using the fact that $\overline{A} \cap \overline{B} \supseteq \overline{A \cap B}$, non-emptiness is a consequence of Urysohn's lemma for locally compact Hausdorff spaces). Since X is compact, there is an $I \in X$ that is an element of each $K(V)$. Using Lemma 5.3.6, for any neighbourhood V of the identity in G, $\varepsilon > 0$, and $f_1, \ldots, f_n \in C_c^+(G)$, there exists $g \in C_c^+(G)$ with $\text{supp}(g) \subseteq V$ and $|I(f_j) - I_g(f_j)| < \varepsilon$ (as this is the form of a basic neighbourhood in the product topology and I is in the intersection of the $K(V)$). This will prove that I satisfies $I(cf) = cI(f)$ for $c > 0$, $f \in C_c^+(G)$, $I(f_1 + f_2) = I(f_1 + f_2)$ for $f_1, f_2 \in C_c^+(G)$, and that $I(L_y f) = I(f)$ for every $y \in G$ and $f \in C_c^+(G)$. Indeed, for any $c > 0$ and $f \in C_c^+(G)$, for a suitable choice of g we have that $|I(cf) - I_g(cf)|$ and $c|I(f) - I_g(f)|$ can be made less than ε so $|I(cf) - cI(f)| < 2\varepsilon$ for any $\varepsilon > 0$, so $I(cf) = cI(f)$ (note that we have used 3. of Proposition 5.3.5). The same can be done to show $I(L_y f) = I(f)$ (using 1. of Proposition 5.3.5). If $f_1, f_2 \in C_c^+(G)$, then we can find a $g \in C_c^+(G)$ with $|I(f_1+f_2) - I_g(f_1+f_2)| < \varepsilon$, $|I(f_1) - I_g(f_1)| < \varepsilon$, $|I(f_2) - I_g(f_2)| < \varepsilon$, and $|I_g(f_1) + I_g(f_2) - I_g(f_1+f_2)| < \varepsilon$ so that $|I(f_1 + f_2) - I(f_1) - I(f_2)| \leq |I(f_1 + f_2) - I_g(f_1 + f_2)| + |I_g(f_1 + f_2) - I_g(f_1) - I_g(f_2)| + |I_g(f_1) - I(f_1)| + |I_g(f_2) - I(f_2)| < 4\varepsilon$, so $I(f_1 + f_2) = I(f_2) + I(f_2)$. Therefore we may extend I to a function $\widetilde{I}$ on $C_c(G)$ by writing any $f \in C_c(G)$ as $f = f_1 - f_2$ where $f_1, f_2 \in C_c^+(G)$ and defining $\widehat{I}(f) = I(f_1) - I(f_2)$. Note that this function is well defined, if $f_1 - f_2 = f_3 - f_4$ for $f_1, f_2, f_3, f_4 \in C_c^+(G)$, then $f_1 + f_4 = f_2 + f_3$ so $I(f_1) + I(f_4) = I(f_2) + I(f_3)$ from which it follows that $I(f_1) - I(f_2) = I(f_3) - I(f_4)$. But then $\widehat{I}$ is a positive linear functional on $C_c(G)$ where G is a locally compact Hausdorff space so the Riesz representation theorem gives that there is a Radon measure μ on G such that $I(f) = \int_G f d\mu$. But μ is non zero since $I(f) \neq 0$ for any $f \in C_c^+(G)$ since $I \in X$. Lastly, μ is in fact a Haar measure by Proposition 5.1.9, as $\int_G L_y f d\mu = I(L_y f) = I(f) = \int_G f d\mu$ for every $y \in G$. □

We have now determined that every locally compact group has a Haar measure, so a natural question is how many distinct Haar measures can a locally compact group have? Without much trouble one realizes that for every positive constant c and every Haar measure μ, $c\mu$ is also a Haar measure. It is a pleasant fact that these are the only exceptions when it comes to the uniqueness of the Haar measure (in fact this was already stated in the last section).

Theorem 5.3.8. [2, Theorem 2.20] Let G be a locally compact group and μ and λ left Haar measures on G. Then there is a $c > 0$ such that $c\mu = \lambda$.

Proof: We first show that the expression $\dfrac{\int_G f d\lambda}{\int_G f d\mu}$ does not depend on the choice of $f \in C_c^+(G)$, and note that the denominator here is non-zero by Proposition 5.1.13. Let $f, g \in C_c^+(G)$, choose a symmetric compact neighbourhood V_0 of the identity, and let $A = (\text{supp}\, f)V_0 \cup V_0(\text{supp}\, f)$, $B = (\text{supp}\, g)V_0 \cup V_0(\text{supp}\, g)$, and note that A and B are compact and thus have finite measure (with respect to either μ or λ). Let $\varepsilon > 0$, and find a symmetric neighbourhood V of the identity contained in V_0 such that $|f(xy) - f(yx)| < \varepsilon$ and $|g(xy) - g(yx)| < \varepsilon$ whenever $x \in G$ and $y \in V$, which can be done by Proposition 5.1.8. Now, choose an $h \in C_c^+(G)$ with $h(x) = h(x^{-1})$ and $\text{supp}\, h \subseteq V$ (this can be done, choose an $s \in C_c^+(G)$ with $\text{supp}\, s \subseteq V$ using Urysohn's lemma for locally compact Hausdorff spaces, and let $h(x) = s(x) + s(x^{-1})$). We have that

$$\int_G h d\mu \int_G f d\lambda = \int_G \int_G h(y) f(x) f d\lambda(x) d\mu(y) = \int_G \int_G h(y) f(yx) d\lambda(x) d\mu(y).$$

We also have

$$
\begin{aligned}
\int_G hd\lambda \int_G fd\mu &= \int_G \int_G h(x)f(y)d\lambda(x)d\mu(y) \\
&= \int_G \int_G h(y^{-1}x)f(y)d\lambda(x)d\mu(y) \\
&= \int_G \int_G h(x^{-1}y)f(y)d\lambda(x)d\mu(y) && \text{since } h(y^{-1}x) = h((y^{-1}x)^{-1}) = h(x^{-1}y) \\
&= \int_G \int_G h(x^{-1}y)f(y)d\mu(y)d\lambda(x) && \text{using Fubini's theorem} \\
&= \int_G \int_G h(y)f(xy)d\mu(y)d\lambda(x) && \text{using the fact that } \mu \text{ is a Haar measure} \\
&= \int_G \int_G h(y)f(xy)d\lambda(x)d\mu(y) && \text{using Fubini's theorem.}
\end{aligned}
$$

Note that Fubini's theorem applies since the functions have compact support and hence we are effectively integrating over a set with finite measure with respect to μ or λ. Hence we have

$$
\begin{aligned}
\left| \int_G hd\mu \int_G fd\lambda - \int_G hd\lambda \int_G fd\mu \right| &= \left| \int_G \int_G h(y)[f(yx) - f(xy)]d\lambda(x)d\mu(y) \right| \\
&\le \int_G h(y) \int_G |f(yx) - f(xy)|d\lambda(x)d\mu(y) \\
&\le \varepsilon\lambda(A) \int_G h(y)d\mu.
\end{aligned}
$$

Note that we had $\int_G |f(yx) - f(xy)|d\lambda(x) \le \varepsilon\lambda(A)$ for $y \in \operatorname{supp} h$ since $\operatorname{supp} h \subseteq V$, so $|f(yx) - f(xy)| < \varepsilon$ for $x \in G$ and $y \in \operatorname{supp} h$. If we divide this inequality by $\int_G hd\mu \int_G fd\mu$, we obtain

$$
\left| \frac{\int_G hd\lambda}{\int_G hd\mu} - \frac{\int_G fd\lambda}{\int_G fd\mu} \right| \le \varepsilon \frac{\lambda(A)}{\int_G fd\mu}.
$$

Similarly we also obtain

$$
\left| \frac{\int_G hd\lambda}{\int_G hd\mu} - \frac{\int_G gd\lambda}{\int_G gd\mu} \right| \le \varepsilon \frac{\lambda(B)}{\int_G gd\mu}.
$$

Now we obtain

$$
\left| \frac{\int_G fd\lambda}{\int_G fd\mu} - \frac{\int_G gd\lambda}{\int_G gd\mu} \right| \le \left| \frac{\int_G hd\lambda}{\int_G hd\mu} - \frac{\int_G fd\lambda}{\int_G fd\mu} \right| + \left| \frac{\int_G hd\lambda}{\int_G hd\mu} - \frac{\int_G gd\lambda}{\int_G gd\mu} \right| \le \varepsilon \left(\frac{\lambda(A)}{\int_G fd\mu} + \frac{\lambda(B)}{\int_G gd\mu} \right).
$$

As $\varepsilon > 0$ was arbitrary, and A and B do not depend on ε, we conclude that $\dfrac{\int_G fd\lambda}{\int_G fd\mu} = \dfrac{\int_G gd\lambda}{\int_G gd\mu}$ and thus $\dfrac{\int_G fd\lambda}{\int_G fd\mu} = c$ for some $c \in \mathbb{R}$. That $c > 0$ follows from Proposition 5.1.13 which implies that the quantity $\dfrac{\int_G fd\lambda}{\int_G fd\mu}$ is positive for any $f \in C_c^+(G)$. Consequently we have $c\int_G fd\mu = \int_G fd\lambda$ for every $f \in C_c(G)$, and since both sides of the equations are positive linear functionals, by the uniqueness statement in Theorem 5.1.2 we conclude that $c\mu = \lambda$. □

Our discussion of Haar measure only scratches the surface of the theory of measure and integration on groups. For locally compact groups, there is also a theory of amenable groups, groups which possess a left-invariant finitely additive Borel probability measure. When the group is compact, we can choose a left Haar measure (which is also a right Haar measure by Proposition 5.2.4) to be a probability measure and in this way we obtain a left-invariant finitely additive Borel measure. As a consequence, compact groups are amenable. But not all locally compact groups are amenable, and for those that are, it is not possible (with the exception of compact groups) to obtain an explicit construction of a left-invariant finitely additive Borel probability measure. There is also a theory of measurable groups, groups with a σ-algebra of subsets, with respect to which the group operations are measurable. Every second-countable topological group is a measurable group with respect to the σ-algebra of Borel sets. This is because $\mathscr{B}(G \times G)$ is equal to the product σ-algebra of $\mathscr{B}(G)$ with itself when G is second-countable.

In general, it is not easy to determine which topological groups have a left Haar measure (extending the definition to groups which are not locally compact). It is shown by Weil in [10] that a non-zero left-invariant measure on a measurable group G which satisfies certain extra conditions determines a topology on G, which turns G into a locally compact topological group. He does this by taking subsets A with finite positive measure and letting $\mathscr{U}$ denote the collection of sets of the form AA^{-1}. He shows that with the conditions he has put in place, $\mathscr{U}$ satisfies the conditions of Theorem 2.1.25, from which we obtain a topological group structure on G. Moreover, Weil shows that for topological groups G which satisfy these extra conditions, the process above actually generates the original topology on G, and so G must be locally compact.

Weil's work was subsequently refined by Mackey. We conclude by stating the following more concrete version of Weil's result, combining Mackey's theorem [9, Theorem 5.4.1] and a theorem from descriptive set theory [4, Theorem 9.10]. Recall that a Polish space is a separable completely metrizable topological space, and that a Polish group is topological group which is also a Polish space.

Theorem 5.3.9. Let G be a Polish group. If there is a non-zero σ-finite left-invariant measure on the Borel sets of G, then G is locally compact.

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