CARLETON UNIVERSITY

SCHOOL OF
MATHEMATICS AND STATISTICS

HONOURS PROJECT

TITLE: An Introduction to Inverse Semigroups and Their Connections To C*-Algebras

AUTHOR: Kyle Cheaney

SUPERVISOR: Dr. Charles Starling

DATE: August 24, 2020

AN INTRODUCTION TO INVERSE SEMIGROUPS AND THEIR CONNECTIONS TO C^*-ALGEBRAS

KYLE CHEANEY: 101012022

ABSTRACT. This paper sets out to provide the reader with a basic introduction to inverse semigroups, as well as a proof of the Wagner-Preston Representation Theorem. It then focuses on finite directed graphs, specifically the graph inverse semigroup and how properties of the semigroup correspond to properties of C^* algebras associated to such graphs.

1. INTRODUCTION

The study of inverse semigroups was founded in the 1950's by three mathematicians: Ehresmann, Preston, and Wagner, as a way to algebraically describe pseudogroups of transformations. The purpose of this paper is to introduce the topic of inverse semigroups and some of the basic properties and theorems. We use these basics to relate the inverse semigroup formed by a directed graph to the corresponding graph C^* algebra. The study of such C^* algebras is still an active topic of research in analysis, thus providing a connection between abstract algebraic objects and complex analysis.

In section 2, we outline some preliminary definitions and results which will be our basic tools for working with inverse semigroups. Section 3 provides a more detailed introduction to inverse semigroups and contains a proof of the Wagner-Preston representation theorem, a cornerstone in the study of inverse semigroups. In section 4, we present some detailed examples of inverse semigroups, including proving said sets are in fact inverse semigroups and discussing what their idempotents look like. We set out to relate familiar topics like sequences and square matricies to the topic of inverse semigroups. Section 5 is the core of the paper, where we introduce the graph inverse semigroup and investigate different properties the inverse semigroup has and can have if we impose conditions on the graph. These yield interesting results, summarized in section 6, where we connect the properties of the inverse semigroup to the corresponding graph C^* algebra.

2. PRELIMINARIES

We first want to recall what we mean by groups and semigroups.

Definition 2.1. Group[1, Pg 43]
Let G be a set together with a binary operation that assigns to each ordered pair (a, b), with $a, b \in G$, an element in G denoted ab. We say G is a *group* if the following are satisfied:

- The operation is associative
- There is an element $e \in G$ such that $ae = ea = a$
- For each element $a \in G$, there exists a unique $b \in G$ such that $ab = ba = e$

Definition 2.2. Semigroup
A *semigroup* is a set S with an associative binary operation.

One important thing to note about semigroups is that they need not contain an identity element. A semigroup that does contain an identity element is referred to as a *monoid*. Similarly, semigroups need not have a zero element. A zero element, or zero, is an element such that any element acting under the binary operation with the zero results in zero again. That is, for a in a semigroup, $a0 = 0a = 0$. Semigroups that contain a zero element are referred to as semigroups with zero.

Definition 2.3. Inverse Semigroup[3]
Let S be a semigroup, and let s be an element of S. We say that u in S is an *inverse* of s if $u = usu$ and $s = sus$. Then we say that S is a *regular semigroup* if, for every element s in S, there exists an inverse of s in S. Thus, we say S is an *inverse semigroup* if every element s in S has a unique inverse.

Definition 2.4. Partial Bijection[2, Pg 4]
Let X, Y be sets and define $f : dom(f) \subseteq X \to f(dom(f)) \subseteq Y$. Then we say f is a *partial function* from X to Y. We are particularly interested in partial functions that are bijective between their domains and images. We naturally refer to these functions as *partial bijections*.

Example 2.5. Symmetric Inverse Monoid
Let X be a set. Define the *symmetric inverse monoid*, $I(X)$ to be the set of all partial bijections from X to itself. Under function composition, where for f, $g \in I(X)$, we have

$$fg : g^{-1}(ran(g) \cap dom(f)) \to f(ran(g) \cap dom(f))$$

so that if the range of g and the domain of f are disjoint, then the composition fg is just the empty function which acts as the zero element of $I(X)$. With this operation, we claim $I(X)$ is a semigroup. Since the identity map is a partial bijection, we can say $I(X)$ is a monoid.
Let $\varphi \in I(X)$. Then φ is a partial bijection on X. This implies that on some subsets $D, R \subseteq X$, φ is a bijection from D to R. By definition of a bijection, there exists a map Γ such that $\Gamma\varphi = 1_D$.
Since identity maps are unique on the domains they are defined on, if there exists some other function $\psi \in I(X)$ such that $\psi\varphi = 1_D$, then it must be the case that $\psi = \Gamma$. So then $\varphi = \varphi\Gamma\varphi$.
Similarly, $\varphi\Gamma = 1_R$, and $\Gamma = \Gamma\varphi\Gamma$. Thus, the symmetric inverse monoid $I(X)$ is an inverse semigroup.

We can construct an inverse semigroup with zero by simply adjoining a zero to an inverse semigroup. That is, if S is an inverse semigroup, we can define $S \cup \{0\}$ to be an inverse semigroup with zero.

Definition 2.6. Idempotent
An element $e \in S$ is said to be an *idempotent* if $e = e^2$. The set of all idempotents of a semigroup S is denoted $E(S)$.

Corollary 2.7. *If e is an idempotent in an inverse semigroup S, then $e = e^{-1}$*

Proof. Let e be an idempotent. Then $e = e^2$. So:

$$\begin{aligned} e &= ee^{-1}e \\ &= (ee)e^{-1}(ee) \\ &= e(ee^{-1}e)e && \text{By associativity} \\ &= eee \end{aligned}$$

By the uniqueness of inverses, $e = e^{-1}$. □

Definition 2.8. Semigroup Homomorphism
Let S, T be semigroups. Then $\theta : S \to T$ is a homomorphism if θ preserves the semigroup operation. That is, if $s, t \in S$, then $\theta(st) = \theta(s)\theta(t)$.
If both S and T are monoids, then we insist θ maps the identity of S to the identity of T. Similarly, if S and T are semigroups with zero, then we insist θ maps the zero of S to the zero of T.

3. Inverse Semigroups

Since the identity need not be in an inverse semigroup, it is natural to wonder what it means to multiply an element by its inverse.

Lemma 3.1. *Let S be an inverse semigroup. Then for all $s \in S$, ss^{-1} and $s^{-1}s$ are idempotents*

Proof. Let S be a semigroup and let $s \in S$. Consider $ss^{-1} \in S$. Then:

$$\begin{aligned}(ss^{-1})^2 &= (ss^{-1})(ss^{-1}) \\ &= s(s^{-1}ss^{-1}) \\ &= ss^{-1}\end{aligned}$$

□

We note that in fact, every idempotent is of the form ss^{-1}, since if e is an idempotent, then $e = ee^{-1}$. Now that we have an idea of how we can make idempotents, we wish to explore the importance of idempotents in a semigroup.

Theorem 3.2. [3, Proposition 2.2] *A regular semigroup is inverse if and only if its idempotents commute.*

Proof. Let S be a regular semigroup. Suppose its idempotents commute. Let $s \in S$ and assume u and v are inverses of s. That is, $s = sus$, $u = usu$ and $s = svs$, $v = vsv$. Then,

$$\begin{aligned}u &= u(svs)u \\ &= (us)(vs)u && \text{By associativity} \\ &= (vs)(us)u && \text{By Lemma 3.1} \\ &= vs(usu) \\ &= (vsv)su \\ &= v(sv)(su) \\ &= v(su)(sv) && \text{By Lemma 3.1} \\ &= v(sus)v \\ &= vsv \\ &= v\end{aligned}$$

Conversely, suppose S is a regular inverse semigroup. Let $e, f \in E(S)$. We want to show that $ef \in E(S)$.

Since S is regular, $\exists s \in S$ such that $ef = efsef$
We note that:

$$\begin{aligned} ef(fse)ef &= e(ff)s(ee)f && \text{By associativity} \\ &= efsef && \text{Since } e, f \in E(S) \end{aligned}$$

Since S is inverse, we have $s = fse$
By definition of s, we have:

$$\begin{aligned} s &= sefs \\ &= (fse)ef(fse) \\ &= fs(ee)(ff)se && \text{By associativity} \\ &= (fse)(fse) = (fse)^2 \end{aligned}$$

Thus, $s = fse \in E(S)$
And since $s \in E(S)$, $\implies s = s^{-1} = ef$
$\implies ef \in E(S)$
This implies that $E(S)$ is closed under multiplication.

$$\implies fe \in E(S)$$

Now consider:

$$\begin{aligned} (ef)(fe)(ef) &= e(ff)(ee)f && \text{By associativity} \\ &= (ef)(ef) \\ &= (ef)^2 \\ &= ef && \text{Since } ef \in E(S) \\ &\implies ef \text{ is an inverse of } ef. \end{aligned}$$

Since S in an inverse semigroup, inverses are unique.
$\implies fe = ef$
$\implies$ idempotents commute. □

We now have all the tools we need to prove the Wagner-Preston representation theorem, which provides a framework for working with the algebraic inverse semigroup as a set of functions.

Theorem 3.3. *Wagner-Preston Representation Theorem*
Every inverse semigroup can be embedded in a symmetric inverse monoid.

Proof. Let S be an inverse semigroup. We want to show that there exists an injective homomorphism between S and $I(S)$. We first note that $\forall a \in S$,

$$aS = aa^{-1}aS \subseteq aa^{-1}S \subseteq aS$$

$$(\star) \qquad \implies aS = aa^{-1}S$$

By a similar argument, $\implies a^{-1}aS = a^{-1}S$.
Define $\Theta_a : a^{-1}aS \to aa^{-1}S$ by $\Theta_a(x) = ax$
and $\Theta_{a^{-1}} : aa^{-1}S \to a^{-1}aS$ by $\Theta_{a^{-1}}(x) = a^{-1}x$

Consider $\Theta_{a^{-1}}\Theta_a : a^{-1}aS \to a^{-1}aS$. Then,

$$\begin{aligned}\Theta_{a^{-1}}\Theta_a(x) &= \Theta_{a^{-1}}(ax)\\ &= a^{-1}ax\\ &= a^{-1}a(a^{-1}as) \text{ for some } s \in S\\ &= a^{-1}as\\ &= x\end{aligned}$$

Thus, $\Theta_{a^{-1}}\Theta_a$ is the identity map on $a^{-1}aS$. By a symmetric argument, $\Theta_a\Theta_{a^{-1}}$ is the identity map on $aa^{-1}S$. Thus, Θ_a is bijective with $\Theta_a^{-1} = \Theta_{a^{-1}}$
So, we have $\Theta : S \ni a \mapsto \Theta_a \in I(S)$.
Now consider $e, f \in E(S)$. By Theorem 3.2, $efS = feS$. Let $x \in efS$. Then we also have $x \in feS$

$$\begin{aligned}&\implies x \in eS \text{ and } x \in fS\\ &\implies x \in eS \cap fS\\ &\implies efS \subseteq eS \cap fS\end{aligned}$$

Now consider $x \in eS \cap fS$. Then $x = es$ and $x = ft$ for some $s, t \in S$. Then,

$$\begin{aligned}x &= xx^{-1}x\\ &= (es)(es)^{-1}(ft)\\ &= ess^{-1}eft\\ &= eefss^{-1}t && \text{by Theorem 3.2}\\ &= ef(ss^{-1}t)\end{aligned}$$

Thus, $x \in efS$.
$\implies eS \cap fS \subseteq efS$, and therefore $efS = eS \cap fS$.
So, letting $e = a^{-1}a$ and $f = bb^{-1}$, then

$$\begin{aligned}dom(\Theta_a) \cap im(\Theta_b) &= a^{-1}aS \cap bb^{-1}S\\ &= a^{-1}abb^{-1}S\\ \implies dom(\Theta_a\Theta_b) &= b^{-1}a^{-1}aS\\ &= b^{-1}a^{-1}S && \text{By } (\star)\\ &= (ab)^{-1}S\\ &= (ab)^{-1}(ab)S && \text{By } (\star)\\ &= dom(\Theta_{ab})\end{aligned}$$

It is clear that $\Theta_a\Theta_b(x) = \Theta_{ab}(x)$. Therefore Θ is a homomorphism. Now we want to show that Θ is injective.

Suppose $\Theta_a = \Theta_b$. Then

$$\begin{aligned} &\Theta_a(b^{-1}) = \Theta_b(b^{-1}) \\ \implies &ab^{-1} = bb^{-1} \\ \implies &ab^{-1} \in E(S) \\ \implies &ab^{-1} = (ab^{-1})^{-1} = ba^{-1} \end{aligned}$$

So,

$$\begin{aligned} &\Theta_a(a^{-1}a) = \Theta_b(a^{-1}a) \\ \implies &aa^{-1}a = ba^{-1}a \\ \implies &a = ab^{-1}a \end{aligned}$$

By a symmetric argument, we also have $b = ba^{-1}b$, and so $b^{-1} = b^{-1}ab^{-1}$. Thus,

$$\begin{aligned} \implies &a^{-1} = b^{-1} && \text{By uniqueness of inverses} \\ \implies &a = b \end{aligned}$$

Therefore, Θ is injective. □

Now that we are familiar with the elements in an inverse semigroup, there is a natural way to compare the elements that is implied by the Wagner-Preston Representation Theorem. By representing the elements of an inverse semigroup as partial bijections, we construct a relation between elements by thinking of functions and their extensions.

Definition 3.4. Natural Partial Order
Let S be an inverse semigroup and let $s, t \in S$. We define the relation $s \leq t$ if and only if $s = ts^{-1}s$. This is referred to as the *natural partial order.*

We now want to justify calling the relation a partial order.

Definition 3.5. Partial Order
A relation R is a *partial order* if:

(1) R is reflexive.
(2) R is antisymmetric
(3) R is transitive

Corollary 3.6. *The relation in Definition 3.4 is a partial order.*

Proof. Let S be an inverse semigroup, and let $s, t, u \in S$. By Definition 2.3,

$$s = ss^{-1}s$$

Thus, $s \leq s$, and so $\leq$ is reflexive.
Suppose $s \leq t$ and $t \leq s$. Then

$$\begin{aligned} &t = st^{-1}t \text{ and } s = ts^{-1}s \\ \implies &s = (st^{-1}t)s^{-1}s \\ &\quad = ss^{-1}st^{-1}t && \text{By Theorem 3.2} \\ &\quad = st^{-1}t \\ &\quad = t \end{aligned}$$

Thus, if $s \leq t$, and $s \neq t$, then $t \nleq s$. So $\leq$ is antisymmetric.
Now suppose $s \leq t$ and $t \leq u$. Then

$$\begin{aligned}
& s = ts^{-1}s \text{ and } t = ut^{-1}t \\
\implies & s = (ut^{-1}t)s^{-1}s \\
& = us^{-1}st^{-1}t && \text{By Theorem 3.2} \\
& = us^{-1}(ts^{-1}s)t^{-1}t \\
& = us^{-1}tt^{-1}ts^{-1}s && \text{By Theorem 3.2} \\
& = us^{-1}ts^{-1}s && \text{By Definition 2.3} \\
& = us^{-1}s
\end{aligned}$$

Thus, $s \leq u$, which implies $\leq$ is transitive. Therefore, the relation in Definition 3.4 is a partial order. □

Corollary 3.7. [2, Lemma 6, Pg 21] *The following statements are equivalent for inverse semigroup S and $s, t \in S$:*

(1) $s \leq t$
(2) $s = te$ *for some* $e \in E(S)$
(3) $s = ft$ *for some* $f \in E(S)$
(4) $s = ss^{-1}t$

Proof. (1) $\implies$ (2) is immediate from Lemma 3.1
(2) $\implies$ (3)
Let $e = t^{-1}ft$ for some $f \in E(S)$. Then

$$\begin{aligned}
\implies & s = t(t^{-1}ft) \\
& = (tt^{-1})ft \\
& = f(tt^{-1}t) \text{ by Theorem 3.2} \\
& = ft
\end{aligned}$$

(3) $\implies$ (4) is immediate from Lemma 3.1
(4) $\implies$ (1) Suppose $s = ss^{-1}t$. Then,

$$\begin{aligned}
s &= (ss^{-1}t)(t^{-1}ss^{-1})(ss^{-1}t) \\
&= (tt^{-1})(ss^{-1})(ss^{-1})(ss^{-1})t && \text{by Lemma 3.1 and Theorem 3.2} \\
&= t(t^{-1}(ss^{-1}s)(s^{-1}ss^{-1})t \\
&= t(t^{-1}ss^{-1}t)
\end{aligned}$$

Now, let $\eta = t^{-1}ss^{-1}t$. Then,

$$\begin{aligned}
s\eta &= s(t^{-1}ss^{-1}t) \\
&= (ss^{-1}t)(t^{-1}ss^{-1}t) && \text{by assumption} \\
&= ss^{-1}ss^{-1}tt^{-1}t \\
&= ss^{-1}t \\
&= s
\end{aligned}$$

Since $\eta = (s^{-1}t)^{-1}(s^{-1}t)$, by Lemma 3.1 $\eta \in E(S)$. So,

$$\begin{aligned} s^{-1}s &= s^{-1}s\eta \\ &= \eta s^{-1}s \\ &= (t^{-1}ss^{-1}t)s^{-1}s \\ \implies ts^{-1}s &= t(t^{-1}ss^{-1}t)s^{-1}s \\ &= ss^{-1}tt^{-1}ts^{-1}s \\ &= s^{-1}sts^{-1}s \\ &= ss^{-1}s \\ &= s \end{aligned}$$

Therefore, $s \leq t$ □

4. Examples

Example 4.1. Groups
Every group G is an inverse monoid. This is clear from Definition 2.1 and Definition 2.3

Example 4.2. Polycyclic Monoids[2, Pg 286]
Let $P_2 =< a_0, a_1, a_0^{-1}, a_1^{-1} | a_i^{-1}a_j = \delta_{ij} >$ be the monoid generated by the 4 generators satisfying the given relation, where δ_{ij} is the Kronecker Delta function:

$$\delta_{ij} = \begin{cases} 1_{id} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A monoid generated by a set of generators is the set of finite products of the generators the satisfy the given relation. Before we show that P_2 is an inverse semigroup, we first want to examine how the elements of P_2 look. Note, by 1_{id}, we mean the identity element in the semigroup, which we understand to be the empty word.

Claim 4.3. *All non-zero elements of P_2 can be written uniquely in the form $a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_1}^{-1}$ for $i_k, j_l \in \{0, 1\}$ for all $0 \leq k \leq n$ and $0 \leq l \leq m$*

Proof. We proceed by way of contradiction. Let $x \in P_2$ be non-zero. If x is not in the above form, then there must be at least one instance of an inverse element being multiplied on the left.

$$\begin{aligned} &\implies \exists a_j^{-1}a_i \text{ for some } i, j \\ &= \delta_{ij} \end{aligned}$$

If $i = j$, then $a_j^{-1}a_i = a_i^{-1}a_i = 1_{id}$. Since multiplying by the identity doesn't change the product, having $i = j$ removes $a_i^{-1}a_i$ from the product.
If $i \neq j$, then $a_j^{-1}a_i = 0$ by definition. This is a contradiction to $x \neq 0 \in P_2$.
Therefore, $\forall x \in P_2$, x has all non-inversed elements multiplied on the left, and all inversed elements multiplied on the right.
Now suppose we have $x = a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_1}^{-1}$ and $x = a_{s_1}...a_{s_k}a_{r_l}^{-1}...a_{r_1}^{-1}$, with $a_{j_{m'}}$ representing

the first instance of $a_j \neq a_r$. Then,

$$\begin{aligned}
x(a_{j_1}a_{j_2}...a_{j_{m'}}) &= a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_1}^{-1}(a_{j_1}a_{j_2}...a_{j_{m'}}) \\
&= a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_{m'+1}}^{-1}(1_{id}) \\
&= a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_{m'+1}}^{-1} \\
\text{And } x(a_{j_1}a_{j_2}...a_{j_{m'}}) &= a_{s_1}...a_{s_k}a_{r_l}^{-1}...a_{r_1}^{-1}(a_{j_1}a_{j_2}...a_{j_{m'}}) \\
&= a_{s_1}...a_{s_k}a_{r_l}^{-1}...a_{r_{m'}}^{-1}a_{j_{m'}} \\
&= a_{s_1}...a_{s_k}a_{r_l}^{-1}...a_{r_{m'+1}}^{-1}(0) \\
&= 0
\end{aligned}$$

Which is a contradiction. A similar argument can be made if each of the inversed elements are the same, but the non-inversed elements are different.
Now suppose that $x = a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_1}^{-1}$ and $x = a_{i_1}...a_{i_n}a_{i_{n+1}}a_{j_{m+1}}^{-1}a_{j_m}^{-1}...a_{j_1}^{-1}$. Then

$$\begin{aligned}
(a_{i_n}^{-1}...a_{i_1}^{-1})x(a_{j_1}...a_{j_m}) &= (a_{i_n}^{-1}...a_{i_1}^{-1})(a_{i_1}...a_{i_n}a_{j_m}^{-1}...a_{j_1}^{-1})(a_{j_1}...a_{j_m}) \\
&= (a_{i_n}^{-1}...a_{i_1}^{-1}a_{i_1}...a_{i_n})(a_{j_m}^{-1}...a_{j_1}^{-1}a_{j_1}...a_{j_m}) \\
&= 1_{id}1_{id} \\
&= 1_{id} \\
\text{And } (a_{i_n}^{-1}...a_{i_1}^{-1})x(a_{j_1}...a_{j_m}) &= (a_{i_n}^{-1}...a_{i_1}^{-1})(a_{i_1}...a_{i_n}a_{i_{n+1}}a_{j_{m+1}}^{-1}a_{j_m}^{-1}...a_{j_1}^{-1})(a_{j_1}...a_{j_m}) \\
&= (a_{i_n}^{-1}...a_{i_1}^{-1}a_{i_1}...a_{i_n}a_{i_{n+1}})(a_{j_{m+1}}^{-1}a_{j_m}^{-1}...a_{j_1}^{-1}a_{j_1}...a_{j_m}) \\
&= 1_{id}a_{i_{n+1}}a_{j_{m+1}}^{-1}1_{id} \\
&= a_{i_{n+1}}a_{j_{m+1}}^{-1}
\end{aligned}$$

So, we have $a_{i_{n+1}}a_{j_{m+1}}^{-1} = 1_{id}$, and thus does not contribute to the product. Thus, the form of x is unique. □

Now, we want to show that P_2 is a regular semigroup so that we can apply Theorem 3.2.

Claim 4.4. *P_2 is a regular semigroup.*

Proof. Let $x \in P_2$. Then by Claim 4.3, we have,

$$x = a_{i_1}a_{i_2}...a_{i_n}a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}$$

Define an element $u \in P_2$ such that

$$u = a_{j_1}a_{j_2}...a_{j_m}a_{i_n}^{-1}a_{i_{n-1}}^{-1}...a_{i_1}^{-1}$$

Now consider the product

$$\begin{aligned} xux &= x(a_{j_1}a_{j_2}...a_{j_m}a_{i_n}^{-1}a_{i_{n-1}}^{-1}...a_{i_1}^{-1})(a_{i_1}a_{i_2}...a_{i_n}a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}) \\ &= x(a_{j_1}a_{j_2}...a_{j_m})(a_{i_n}^{-1}a_{i_{n-1}}^{-1}...a_{i_1}^{-1}a_{i_1}a_{i_2}...a_{i_n})(a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}) \\ &= x(a_{j_1}a_{j_2}...a_{j_m})(1_{id})(a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}) \\ &= (a_{i_1}a_{i_2}...a_{i_n}a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1})(a_{j_1}a_{j_2}...a_{j_m})(a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}) \\ &= (a_{i_1}a_{i_2}...a_{i_n})(a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}a_{j_1}a_{j_2}...a_{j_m})(a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}) \\ &= a_{i_1}a_{i_2}...a_{i_n}(1_{id})a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1} \\ &= x \end{aligned}$$

We can follow the same logic to show that $u = uxu$. Thus, u is an inverse of x. Therefore, P_2 is a regular semigroup. □

Now, by Theorem 3.2 we just need to show that idempotents commute.
From the previous claim, and Lemma 3.1, we can see that

$$e = xu \in P_2 \implies e = a_{j_1}a_{j_2}...a_{j_m}a_{j_m}^{-1}a_{j_{m-1}}^{-1}...a_{j_1}^{-1}$$

Or more compactly, $E(P_2) = \{xx^{-1} \in P_2 | x \in \{a_0, a_1\}^*\} \cup \{0\}$, where $\{a_0, a_1\}^*$ is the set of finite products of a_0, and a_1.

Claim 4.5. *Idempotents in P_2 commute.*

Proof. Let $xx^{-1}, yy^{-1} \in E(P_2)$. Then we could have 1 of 3 possibilities:

(1) $y = xv$ for some $v \in \{a_0, a_1\}^*$
(2) $x = yw$ for some $w \in \{a_0, a_1\}^*$
(3) $x = ux'$ and $y = uy'$ for some $x', y', u \in \{a_0, a_1\}^*$

Suppose (1). Then

$$\begin{aligned} xx^{-1}yy^{-1} &= xx^{-1}(xv)(xv)^{-1} \\ &= x(x^{-1}x)vv^{-1}x^{-1} \\ &= xvv^{-1}x^{-1} \\ &= yy^{-1} \end{aligned}$$

And,

$$\begin{aligned} yy^{-1}xx^{-1} &= (xv)(xv)^{-1}xx^{-1} \\ &= xvv^{-1}(x^{-1}x)x^{-1} \\ &= xvv^{-1}x^{-1} \\ &= yy^{-1} \end{aligned}$$

Thus, $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Supposing (2) yields the same argument. Now, suppose (3). Then

$$\begin{aligned} xx^{-1}yy^{-1} &= (ux')(ux')^{-1}(uy')(uy')^{-1} \\ &= (ux')x'^{-1}(u^{-1}u)y'y'^{-1}u^{-1} \\ &= (ux')(x'^{-1}y')y'^{-1}u^{-1} \\ &= x0y^{-1} \\ &= 0 \end{aligned}$$

And,

$$\begin{aligned} yy^{-1}xx^{-1} &= (uy')(uy')^{-1}(ux')(ux')^{-1} \\ &= (uy')y'^{-1}(u^{-1}u)x'x'^{-1}u^{-1} \\ &= (uy')(y'^{-1}x')x'^{-1}u^{-1} \\ &= y0x^{-1} \\ &= 0 \end{aligned}$$

Thus, $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Therefore, idempotents in P_2 commute, and by Theorem 3.2, P_2 is an inverse semigroup. □

There is a natural generalization to the monoid $P_n =< a_i, a_i^{-1} | 0 \leq i \leq n-1, a_i^{-1}a_j = \delta_{ij} >$ for $n \geq 2$

Example 4.6. Left-Inverse Hull
Let S be a semigroup and recall $I(S)$ as the inverse semigroup of partial bijections outlined in Example 2.5. We say S is a *left-cancellative* semigroup if, for all $p, q, r \in S$, we have

$$pq = pr \implies q = r$$

Let S be left-cancellative. For each $s \in S$, we can define a map $\lambda_s : S \to S$ by $\lambda_s(p) = sp$, for $p \in S$. Since S is left-cancellative, λ_s is a partial bijection between S and sS.

Definition 4.7. Left-Inverse Hull[4]
The *left-inverse hull* of S is the smallest inverse semigroup of $I(S)$, which contains each λ_s, which we denote $I_l(S)$.

Now, we let A be a finite set. We define A^* as the set of all finite sequences, or words, in A, together with the empty word ϕ. If $\alpha, \beta \in A^*$, then the word $\alpha\beta \in A^*$ is formed under concatenation. With this operation, ϕ acts as the identity in A^* with

$$\alpha\phi = \phi\alpha = \alpha \ \forall \alpha \in A^*$$

So that A^* is a monoid. We now want to provide an explicit description of $I_l(A^*)$.

Claim 4.8. *Let A be a finite set such that $|A| \geq 2$. Then $I_l(A^*) = \{\lambda_\alpha \lambda_\beta^{-1} : \alpha, \beta \in A^*\} \cup \{0\}$, where 0 is the empty map in $I(A^*)$*

Proof. First, let $S = \{\lambda_\alpha \lambda_\beta^{-1} : \alpha, \beta \in A^*\} \cup \{0\}$. We wish to show that $I_l(A^*) = S$, so we proceed via a set inclusion argument.
We note that for every α, β in A^*, their associated maps $\lambda_\alpha, \lambda_\beta$ are in $I_l(A^*)$ by Definition 4.7. And since $I_l(A^*)$ is an inverse semigroup, Definition 2.3 implies that λ_β^{-1} is in $I_l(A^*)$, hence

$\lambda_\alpha\lambda_\beta^{-1}$ is also in $I_l(A^*)$.
Since $|A| \geq 2$, we can find elements $a \neq b \in A$. Then,

$$\lambda_b^{-1} : bA^* \to A^*$$
$$\lambda_a : A^* \to aA^*$$
$$\implies \lambda_b^{-1}\lambda_a = 0$$

Since $dom(\lambda_b^{-1}) \cap ran(\lambda_a) = \{\}$. Thus, $0 \in I_l(A^*)$, and therefore, $S \subseteq I_l(A^*)$.
Now, we want to show that $I_l(A^*) \subseteq S$. Let $\alpha, \beta \in A^*$. Then we have

$$(\lambda_\alpha\lambda_\beta^{-1})^{-1} = \lambda_\beta\lambda_\alpha^{-1} \in S \tag{($\star\star$)}$$

So we conclude that S is closed under inverses.
Let $\alpha, \beta, \gamma, \delta \in A^*$. We wish to show that $(\lambda_\alpha\lambda_\beta^{-1})(\lambda_\gamma\lambda_\delta^{-1})$ is in S, as to prove that S is closed under multiplication. We claim:

$$\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1} = \begin{cases} \lambda_\alpha\lambda_{\delta\mu}^{-1} & \text{if } \beta = \gamma\mu \\ \lambda_{\alpha\kappa}\lambda_\delta^{-1} & \text{if } \gamma = \beta\kappa \\ 0 & \text{otherwise} \end{cases}$$

First, let $\beta = \gamma\mu$. Then:

$$\begin{aligned} \lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1} &= \lambda_\alpha\lambda_{(\gamma\mu)}^{-1}\lambda_\gamma\lambda_\delta^{-1} \\ &= \lambda_\alpha(\lambda_\gamma\lambda_\mu)^{-1}\lambda_\gamma\lambda_\delta^{-1} \\ &= \lambda_\alpha\lambda_\mu^{-1}\lambda_\gamma^{-1}\lambda_\gamma\lambda_\delta^{-1} \\ &= \lambda_\alpha\lambda_\mu^{-1}\lambda_\delta^{-1} \\ &= \lambda_\alpha\lambda_{\delta\mu}^{-1} \end{aligned}$$

Now, let $\gamma = \beta\kappa$. Then:

$$\begin{aligned} \lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1} &= \lambda_\alpha\lambda_\beta^{-1}\lambda_{(\beta\kappa)}\lambda_\delta^{-1} \\ &= \lambda_\alpha\lambda_\beta^{-1}\lambda_\beta\lambda_\kappa\lambda_\delta^{-1} \\ &= \lambda_\alpha\lambda_\kappa\lambda_\delta^{-1} \\ &= \lambda_{\alpha\kappa}\lambda_\delta^{-1} \end{aligned}$$

Finally, let $\beta \neq \gamma\mu$ and $\gamma \neq \beta\kappa$. Then $dom(\lambda_\beta^{-1}) \cap ran(\lambda_\gamma) = 0$, and so:

$$\begin{aligned} \lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1} &= \lambda_\alpha 0 \lambda_\delta^{-1} \\ &= 0 \end{aligned}$$

Thus, the set S is closed under multiplication, and is therefore a semigroup. By construction of S, each λ_s is contained in S. By Definition 4.7 we have $I_l(A^*) \subseteq S$, which implies $I_l(A^*) = \{\lambda_\alpha\lambda_\beta^{-1} : \alpha, \beta \in A^*\} \cup \{0\}$. □

Proposition 4.9. *Let P_2 be the polycylic monoid with two generators as in Example 4.2*

$$P_2 =< a_0, a_1, a_0^{-1}, a_1^{-1} | a_i^{-1}a_j = \delta_{ij} >$$

Then the map $\Phi : I_l(\{0,1\}^*) \to P_2$ defined by $\lambda_\alpha\lambda_\beta^{-1} \mapsto a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1}$ where n_α is the length of the sequence $\alpha \in \{0,1\}^*$, together with $\phi \mapsto 1_{id}$, is an isomorphism of inverse semigroups.

Proof. We need to verify Φ is injective, surjective, multiplicative, and preserves inverses. Let $\lambda_\alpha\lambda_\beta^{-1} \in I_l(\{0,1\}^*)$. Then:

$$\begin{aligned}(\Phi(\lambda_\alpha\lambda_\beta^{-1}))^{-1} &= (a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1})^{-1}\\ &= (a_{\beta_1}...a_{\beta_{n_\beta}}a_{\alpha_{n_\alpha}}^{-1}...a_{\alpha_1}^{-1})\\ &= \Phi(\lambda_\beta\lambda_\alpha^{-1})\\ &= \Phi((\lambda_\alpha\lambda_\beta^{-1})^{-1}) && \text{By } (\star\star)\end{aligned}$$

Thus, Φ preserves inverses.
Now suppose $\Phi(\lambda_\alpha\lambda_\beta^{-1}) = \Phi(\lambda_\gamma\lambda_\delta^{-1})$ for $\lambda_\alpha\lambda_\beta^{-1}, \lambda_\gamma\lambda_\delta^{-1} \in I_l(\{0,1\}^*)$. Then:

$$\begin{aligned}a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1} &= a_{\gamma_1}...a_{\gamma_{n_\gamma}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ \implies n_\alpha &= n_\gamma \text{ and } n_\beta = n_\delta\\ &\implies \begin{cases}\alpha_i = \gamma_i & \text{for } 1 \le i \le n_\alpha\\ \beta_j = \delta_j & \text{for } 1 \le j \le n_\beta\end{cases}\end{aligned}$$

Since equal words in P_2 must have the same elements, or 'letters', see Claim 4.3. Thus, $\lambda_\alpha = \lambda_\gamma$ and $\lambda_\beta = \lambda_\delta$, and therefore $\lambda_\alpha\lambda_\beta^{-1} = \lambda_\gamma\lambda_\delta^{-1}$. Thus, Φ is injective.
Now, consider $\lambda_\alpha\lambda_\beta^{-1}, \lambda_\gamma\lambda_\delta^{-1} \in I_l(\{0,1\}^*)$. Suppose $\beta = \gamma\mu$. Then:

$$\begin{aligned}\Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1}) &= \Phi(\lambda_\alpha\lambda_{\gamma\mu}^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1})\\ &= \Phi(\lambda_\alpha\lambda_\mu^{-1}\lambda_\gamma^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1})\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\mu_{n_\mu}}^{-1}...a_{\mu_1}^{-1}a_{\gamma_{n_\gamma}}^{-1}...a_{\gamma_1}^{-1}a_{\gamma_1}...a_{\gamma_{n_\gamma}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\mu_{n_\mu}}^{-1}...a_{\mu_1}^{-1}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}(a_{\delta_1}...a_{\delta_{n_\delta}}a_{\mu_1}...a_{\mu_{n_\mu}})^{-1}\\ &= \Phi(\lambda_\alpha\lambda_{\delta\mu}^{-1})\end{aligned}$$

As we shown in Claim 4.8, $\Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1}) = \Phi(\lambda_\alpha\lambda_{\delta\mu}^{-1})$, and thus $\Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1}) = \Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1})$
Now suppose $\gamma = \beta\kappa$. Then:

$$\begin{aligned}\Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1}) &= \Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_{\beta\kappa}\lambda_\delta^{-1})\\ &= \Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\beta\lambda_\kappa\lambda_\delta^{-1})\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1}a_{\beta_1}...a_{\beta_{n_\beta}}a_{\kappa_1}...a_{\kappa_{n_\kappa}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\kappa_1}...a_{\kappa_{n_\kappa}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= \Phi(\lambda_{\alpha\kappa}\lambda_\delta^{-1})\end{aligned}$$

As we have shown in Claim 4.8, $\Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1}) = \Phi(\lambda_{\alpha\kappa}\lambda_\delta^{-1})$, and thus $\Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1}) = \Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1})$
Now, suppose that $\beta \neq \gamma\mu$ and $\gamma \neq \beta\kappa$. Then there exists some $1 \le j \le min\{n_\beta, n_\gamma\}$ such

that $\beta_j \neq \gamma_j$. Then:

$$\begin{aligned}\Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1}) &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1}a_{\gamma_1}...a_{\gamma_{n_\gamma}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_j}^{-1}a_{\gamma_j}...a_{\gamma_{n_\gamma}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_{j+1}}^{-1}(0)a_{\gamma_{j+1}}...a_{\gamma_{n_\gamma}}a_{\delta_{n_\delta}}^{-1}...a_{\delta_1}^{-1}\\ &= 0\end{aligned}$$

As we have shown in Claim 4.8, $\Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1}) = 0$, and thus $\Phi(\lambda_\alpha\lambda_\beta^{-1}\lambda_\gamma\lambda_\delta^{-1}) = \Phi(\lambda_\alpha\lambda_\beta^{-1})\Phi(\lambda_\gamma\lambda_\delta^{-1})$. Therefore, we can conclude that Φ is multiplicative.

Now, let $\alpha\beta^{-1} \in P_2$, such that $\alpha\beta^{-1} = a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1}$, where $a_{\alpha_i} = a_0, a_1$ and $a_{\beta_j}^{-1} = a_0^{-1}, a_1^{-1}$. Then by the definition of Φ, we have that $\Phi(\lambda_i) = a_i$, and $\Phi(\lambda_j^{-1}) = a_j^{-1}$ for $i, j = 0, 1$. Then:

$$\begin{aligned}\alpha\beta^{-1} &= a_{\alpha_1}...a_{\alpha_{n_\alpha}}a_{\beta_{n_\beta}}^{-1}...a_{\beta_1}^{-1}\\ &= \Phi(\lambda_{\alpha_1})...\Phi(\lambda_{\alpha_{n_\alpha}})\Phi(\lambda_{\beta_{n_\beta}}^{-1})...\Phi(\lambda_{\beta_1}^{-1})\\ &= \Phi(\lambda_{\alpha_1}...\lambda_{\alpha_{n_\alpha}}\lambda_{\beta_{n_\beta}}^{-1}...\lambda_{\beta_1}^{-1}) && \text{Since } \Phi \text{ is multiplicative}\\ &= \Phi(\lambda_{\alpha_1...\alpha_{n_\alpha}}\lambda_{\beta_{n_\beta}...\beta_1}^{-1})\\ &= \Phi(\lambda_\alpha\lambda_\beta^{-1})\end{aligned}$$

Thus, there is an element, $\lambda_\alpha\lambda_\beta^{-1}$ in $I_l(\{0,1\}^*)$ such that $\Phi(\lambda_\alpha\lambda_\beta^{-1}) = \alpha\beta^{-1}$, and therefore, Φ is surjective. □

Example 4.10. Matrix Units

Let $n \in \mathbb{N}$ and define the set of $n \times n$ matrix units to be:

$$U_n = \{A \in M_n(\{0,1\}) : A_{ij} = 1 \text{ for at most 1 index}\}$$

Then, for $1 \leq i, j, k, l \leq n$, we can represent U_n by:

$$U_n = \{(E_{ij})_{kl} = \delta_{i,k}\delta_{j,l}\} \cup \{0_n\}$$

where 0_n is the $n \times n$ zero matrix.

Eg:

$$U_2 = \left\{ \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \right\}$$

We want to show that under matrix multiplication, for $n \in \mathbb{N}$, U_n is an inverse semigroup. Let $A, B \neq 0 \in U_n$, then $A = E_{i_1j_1}$ and $B = E_{i_2j_2}$. Then,

$$\begin{aligned}(AB)_{ij} &= \sum_{r=1}^{n} A_{ir}B_{rj} \\ &= \sum_{r=1}^{n} \delta_{ii_1}\delta_{rj_1}\delta_{ri_2}\delta_{rj_2} \\ \implies (AB)_{ij} &= \begin{cases} \sum_{r=1}^{n} A_{i_1r}B_{rj_2} & \text{if } i = i_1 \text{and } j = j_2 \\ 0 & \text{otherwise} \end{cases} \\ \implies (AB)_{ij} &= \begin{cases} \delta_{j_1i_2} & \text{if } i = i_1 \text{and } j = j_2 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Therefore,

$$(\star\star\star) \qquad E_{i_1j_1}E_{i_2j_2} = \begin{cases} E_{i_1j_2} & \text{if } j_1 = i_2 \\ 0 & \text{otherwise} \end{cases}$$

Now consider $E_{ij} \neq 0 \in U_n$. Then E_{ji} is also in U_n. So we have:

$$\begin{aligned} E_{ij}E_{ji} &= E_{ii} && \text{By } (\star\star\star) \\ \implies E_{ij}E_{ji}E_{ij} &= E_{ii}E_{ij} \\ &= E_{ij} && \text{By } (\star\star\star) \end{aligned}$$

By a symmetric argument, we also have $E_{ji}E_{ij}E_{ji} = E_{ji}$, and thus, E_{ji} is an inverse of E_{ij}. To conclude that U_n is indeed an inverse semigroup, we just need to show that this inverse is unique.

Let $0 \neq E_{ij} \in U_n$ and assume E_{ji} and E_{kl} are both inverses of E_{ij}. Then by Definition 2.3, we have $E_{ij} = E_{ij}E_{ji}E_{ij}$ and $E_{ij} = E_{ij}E_{kl}E_{ij}$. Since $E_{ij} \neq 0$, it must be the case that $E_{ij}E_{kl} \neq 0$, and by $(\star\star\star)$, this implies that $k = j$. Similarly, $E_{kl}E_{ij} \neq 0$ implies $l = i$, and thus, $E_{kl} = E_{ji}$. Therefore, the inverses are unique, which implies U_n is an inverse semigroup with zero.

Example 4.11. Rook Matricies

Let $n \in \mathbb{N}$. Define the set of $n \times n$ *rook matricies*, denoted R_n, to be all matricies in $M_n(\{0, 1\})$ such that each row and each column has at most one non-zero entry.

Eg:

$$R_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Proposition 4.12. *The rook matrices R_n form a semigroup under standard matrix multiplication.*

Proof. Let A, B be rook matrices. If $A = 0$, then $AB = 0 \times B = 0$. A symmetric argument can be made if $B = 0$. Thus, $AB \in R_n$. Suppose that $A \neq 0 \neq B$, and assume $AB \notin R_n$. Then either $(AB)_{i,j} \neq 0, 1$, or for some $1 \leq i \leq n$, we have that $(AB)_{i,j_1} = 1 = (AB)_{i,j_2}$ for $1 \leq j_1 \neq j_2 \leq n$.

First, assume $(AB)_{i,j} = 2$. Then:

$$\sum_{r=1}^{n} A_{i,r}B_{r,j} = 2$$

$\implies$ There exists $1 \leq r_1 \neq r_2 \leq n$ such that $A_{i,r_1}B_{r_1,j} + A_{i,r_2}B_{r_2,j} = 2$. However, since $A, B \in R_n$, by definition of the rook matrices, each row and column of A and B can only have at most one non-zero entry. Thus $r_1 = r_2$, a contradiction.
Now assume that for some $1 \leq k \leq n$, we have $(AB)_{k,j_1} = (AB)_{k,j_2} = 1$ with $1 \leq j_1 \neq j_2 \leq n$. Then we have:

$$\sum_{r=1}^{n} A_{k,r}B_{r,j_1} = \sum_{r=1}^{n} A_{k,r}B_{r,j_2} = 1$$

Let r_k represent the index of the non-zero entry in the kth row of A. Since $A \in R_n$, there is only one entry that is non-zero, so the sum collapses to

$$\begin{aligned} 1 &= A_{k,r_k}B_{r_k j_1} = A_{k,r_k}B_{r_k j_2} \\ &= (1)B_{r_k,j_1} = (1)B_{r_k,j_2} \\ &= B_{r_k,j_1} = B_{r_k,j_2} \end{aligned}$$

Since B is a rook matrix, each row must have at most one non-zero entry. Thus, $j_1 = j_2$, a contradiction. Hence, we conclude that R_n is closed under multiplication and is therefore a semigroup. □

We want to show that, under standard matrix multiplication, the semigroup R_n is isomorphic to $I(\{1, 2, ..., n\})$, which we understand to be the set of partial bijections on a finite subset on $\mathbb{N}$, as in Example 2.5.
First, we would like to have a way of representing the elements of R_n in a convenient way. Consider a matrix $A \in R_n$. Then we can say $A = (A_1, A_2, ...A_n)$, where $A_i \in \{0, 1, ..., n\}$ represents the location of the non-zero entries of A. So in the ith row, the non-zero entry is in the A_ith column, if there exists a non-zero entry, or is in the zeroth column if the row is all zeros.
Eg:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies A = (3, 1, 0)$$

So in this new notation, it is important to understand what multiplication and transpose look like. Let $A, B \in R_n$ with $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$. Then:

$$AB = (B_{A_1}, ..., B_{A_n})$$

Where if $A_j = 0$ then $B_0 = 0$ for $1 \leq j \leq n$.
Now, consider $A = (A_1, ..., A_n) \in R_n$. Then :

$$A^t = \left(\sum_{r=1}^{n} r\delta_{1,A_r}, \sum_{r=1}^{n} r\delta_{2,A_r}, ..., \sum_{r=1}^{n} r\delta_{n,A_r} \right)$$

Where A^t is the transpose of the matrix A, and δ_{j,A_r} is the Kronecker Delta function.
Eg:

Let $A = (3, 4, 2, 1)$ and $B = (0, 1, 3, 0)$. Then

$$\begin{aligned} AB &= (B_3, B_4, B_2, B_1) \\ &= (3, 0, 1, 0) \end{aligned}$$

In standard notation, we would have:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And, if we want to find the transpose of A, then:

$$\begin{aligned} A^t &= (\sum_{r=1}^{n} r\delta_{1,A_r}, \sum_{r=1}^{n} r\delta_{2,A_r}, \sum_{r=1}^{n} r\delta_{3,A_r}, \sum_{r=1}^{n} r\delta_{4,A_r}) \\ &= (0+0+0+4, 0+0+3+0, 1+0+0+0, 0+2+0+0) \\ &= (4, 3, 1, 2) \end{aligned}$$

And in the standard notation, we would have:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now, we define the map $\varphi : R_n \to I(\{1, 2, ..., n\})$ by $\varphi(A) : dom(\varphi(A)) \to ran(\varphi(A))$, where $dom(\varphi(A)) = \{A_i \neq 0\}$ so that:

$$\begin{aligned} \varphi(A)(A_i) &= A \cdot A_i \\ &= i \end{aligned}$$

Eg:
Let $A = (1, 0, 0, 4, 2, 5) \in R_6$, then $dom(\varphi(A)) = \{A_1, A_4, A_5, A_6\} = \{1, 4, 2, 5\}$. Then we have:

$$\begin{aligned} \varphi(A)(1) &= 1 \\ \varphi(A)(4) &= 4 \\ \varphi(A)(2) &= 5 \\ \varphi(A)(5) &= 6 \end{aligned}$$

It is clear by construction, that each $\varphi(A)$ is bijective. We now wish to show that φ itself it a bijective map.
Let $A, B \in R_n$. Assume $\varphi(A) = \varphi(B)$. Then for each $A_i \in dom(\varphi(A))$, we have that:

$$\begin{aligned} & \varphi(A)(A_i) = \varphi(B)(A_i) \\ \implies & \varphi(B)(A_i) = i \\ \implies & A_i = B_i && \text{Since } \varphi(B) \text{ is injective} \\ \implies & A = B \end{aligned}$$

Therefore, φ is injective.
Now suppose $\psi \neq 0 \in I(\{1,...,n\})$. Then by Example 2.5, we know that $dom(\psi) \subseteq \{1,...,n\}$ and $ran(\psi) \subseteq \{1,...,n\}$. Suppose that $dom(\psi) = \{i_1,...,i_k\}$ for some $1 \leq k \leq n$. Since ψ is a partial bijection, it is injective, and thus we say $ran(\psi) = \{\psi(i_1),...,\psi(i_k)\}$.
Define $A_\psi \in M_n(\{1,0\})$ by:

$$(A_\psi)_{i,j} = \begin{cases} 1 & \text{if } j \in dom(\psi) \text{ and } \psi(j) = i \\ 0 & \text{otherwise} \end{cases}$$

We want to show that $A \in R_n$. Since ψ is injective, for each $i \in ran(\psi)$, there is a unique $j \in dom(\psi)$ such that $\psi(j) = i$. Thus each row of A_ψ has at most one non-zero entry. Therefore, $A_\psi \in R_n$. If $\psi = 0$, then $A_\psi = 0 \in R_n$
Now we just need to show that $\varphi(A_\psi) = \psi$. Let $x \in dom(\varphi(A_\psi))$. Then the xth column of A_ψ is not all zero. This implies that there exists a $y \in \{1,...,n\}$ such that $(A_\psi)_{y,x} \neq 0$. By the definition of A_ψ, this implies that $x \in dom(\psi)$. Thus, $dom(\varphi(A_\psi)) \subseteq dom(\psi)$.
Now suppose $x \in dom(\psi)$. Since ψ is a partial bijection, there exists some $y \in ran(\psi)$ such that $\psi(x) = y$. Thus, by the definition of A_ψ, we have $A_{y,x} \neq 0$. This implies that the xth column of A_ψ is not all zero. Therefore, $x \in dom(\varphi(A_\psi))$, and we conclude that $dom(\varphi(A_\psi) = dom(\psi)$.
Now, let x be in $dom(\varphi(A_\psi))$. Then as before, there exists some $y \in \{1,...,n\}$ such that (A_ψ) is non-zero. Since $A_\psi \in R_n$, there is only one such y that satisfies this, that being $y = \psi(x)$ by the definition of A_ψ. Therefore, we have $\varphi(A_\psi) = \psi$, which implies φ is surjective, and thus is bijective.
The final thing we would like to show is the preservation of the operation. Let $0 \neq A, B \in R_n$. Then

$$\varphi(AB)(AB)_{i_j} = j$$

By our definition of matrix multiplication, we have that $(AB)_{i_j} = B_{k_{A_{i_j}}}$, so that

$$\begin{aligned} \varphi(A)\varphi(B)(AB)_{i_j} &= \varphi(A)\varphi(B)(B_{k_{A_{i_j}}}) \\ &= \varphi(A)(A_{i_j}) \\ &= j \end{aligned}$$

So, we conclude that $\varphi(AB) = \varphi(A)\varphi(B)$, and thus, φ preserves multiplication.

5. Directed Graphs

Definition 5.1. Directed Graph
A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$, where E^0 represents the set of vertexes, E^1 represents the set of edges, $r : E^1 \to E^0$ represents the range function, and $s : E^1 \to E^0$ represents the source function. We can picture an element $\alpha \in E^1$ as an arrow pointing from the vertex $s(\alpha)$ to the vertex $r(\alpha)$.

$$s(\alpha) \xrightarrow{\alpha} r(\alpha)$$

Let $E^* = E^0 \cup \{\alpha = \alpha_1\alpha_2...\alpha_n : r(\alpha_{i+1}) = s(\alpha_i), i = 1,...,n-1)\}$ be the set of finite paths in E. Then $E^* \cup \{0\}$ is a semigroup under concatenation. That is,

$$\text{For } \alpha, \beta \in E^* \cup \{0\}, \alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } r(\beta) = s(\alpha) \\ 0 & otherwise \end{cases}$$

If α is in $E^* \setminus E^0 \cup \{0\}$ and β is a vertex, then if $\alpha\beta$ is non-zero, it must be the case that $\beta = s(\alpha)$. Similarly, if α is a vertex and β is a non-zero path in E^*, then $\alpha\beta$ being non-zero implies $\alpha = r(\beta)$. Finally, if both α and β are vertexes, then $\alpha\beta$ being non-zero implies $\alpha = \beta$.

Note, we will assume associativity of concatenation, as not much is gained by proving it here.

Remark 5.2. There are two conventions for dealing with the concatenation of the edges. We will use the the convention common in the Southern hemisphere, popularized by Raeburn[5].

Definition 5.3. Graph Inverse Semigroup
Let $S_E = \{(\alpha, \beta) \in E^* \times E^* : s(\alpha) = s(\beta)\} \cup \{0\}$ be the *graph inverse semigroup* associated to E under the operation:

$$(\alpha, \beta)(\gamma, \delta) = \begin{cases} (\alpha, \delta\mu) & \text{if } \beta = \gamma\mu \\ (\alpha\kappa, \delta) & \text{if } \gamma = \beta\kappa \\ 0 & \textit{otherwise} \end{cases}$$

Our first objective with the graph inverse semigroup is to prove that it is in fact an inverse semigroup. Once again, we will not prove associativity of the operation.
It is worthwhile to note that, another way to characterise the operation is by whether or not β and γ *agree*. By agree, we mean that one is a prefix of the other. So the product is 0 if and only if β and γ do not agree.

Proposition 5.4. *The graph inverse semigroup is an inverse semigroup.*

Proof. Let $(\alpha, \beta) \in S_E$. Then we have $s(\alpha) = s(\beta) \implies (\beta, \alpha) \in S_E$. Then,

$$(\alpha, \beta)(\beta, \alpha) = \begin{cases} (\alpha, \alpha\mu) & \text{if } \beta = \beta\mu \\ (\alpha\kappa, \alpha) & \text{if } \beta = \beta\kappa \end{cases}$$

In both cases, it must be the case that $\mu = s(\beta) = \kappa$, as $\beta s(\beta) = \beta$ by the preservation of lengths. And since $s(\alpha) = s(\beta)$, we have

$$\begin{aligned} (\alpha, \alpha\mu) &= (\alpha, \alpha s(\alpha)) \\ &= (\alpha, \alpha) \\ &= (\alpha s(\alpha), \alpha) \\ &= (\alpha\kappa, \alpha) \end{aligned}$$

Therefore,

$$(\alpha, \beta)(\beta, \alpha) = (\alpha, \alpha) \tag{1}$$

Now consider $(\alpha, \beta)(\beta, \alpha)(\alpha, \beta)$. By (1), we have:

$$\begin{aligned} (\alpha, \beta)(\beta, \alpha)(\alpha, \beta) &= (\alpha, \alpha)(\alpha, \beta) \\ &= \begin{cases} (\alpha, \beta\mu') & \text{if } \alpha = \alpha\mu' \\ (\alpha\kappa', \beta) & \text{if } \alpha = \alpha\kappa' \end{cases} \\ &= (\alpha, \beta) & & \text{by above} \end{aligned}$$

Now, consider $(\beta,\alpha)(\alpha,\beta)(\beta,\alpha)$. By above, we have:

$$\begin{aligned}(\beta,\alpha)(\alpha,\beta)(\beta,\alpha) &= (\beta,\alpha)(\alpha,\alpha) \\ &= \begin{cases}(\beta,\alpha\mu) & \text{if } \alpha=\alpha\mu \\ (\beta\kappa,\alpha) & \text{if } \alpha=\alpha\kappa\end{cases} \\ &= (\beta,\alpha)\end{aligned}$$

Thus, $(\beta,\alpha)(\alpha,\beta)(\beta,\alpha) = (\beta,\alpha)$. Therefore, (β,α) is an inverse of (α,β)
$\implies S_E$ is regular.
By Theorem 3.2, it is sufficient to show that the idempotents of S_E commute.
Let $(\alpha,\beta)\in E(S_E)$. Then $s(\alpha)=s(\beta)$ and $(\alpha,\beta)(\alpha,\beta)=(\alpha,\beta)$. Then,

$$(\alpha,\beta)(\alpha,\beta) = \begin{cases}(\alpha,\beta\mu) & \text{if } \beta=\alpha\mu \\ (\alpha\kappa,\beta) & \text{if } \alpha=\beta\kappa\end{cases}$$

$$\implies (\alpha,\beta) = \begin{cases}(\alpha,\beta\mu) & \text{if } \beta=\alpha\mu \\ (\alpha\kappa,\beta) & \text{if } \alpha=\beta\kappa\end{cases}$$

Again, by the preservation of lengths, either $\mu=s(\beta)$ or $\kappa=s(\alpha)$
$\implies \beta=\alpha s(\beta)=\alpha s(\alpha)=\alpha$ or $\alpha=\beta s(\alpha)=\beta s(\beta)=\beta$
$\implies \alpha=\beta$
Thus, $E(S_E)=\{(\alpha,\alpha)\in E^*\times E^*\}\cup\{0\}$. Now that we know what the idempotents look like, we can finally show commutativity.
Let $(\alpha,\alpha),(\beta,\beta)\in E(S_E)$.
If $\alpha=\beta\mu$, then

$$\begin{aligned}(\alpha,\alpha)(\beta,\beta) &= (\beta\mu,\beta\mu)(\beta,\beta) \\ &= (\beta\mu,\beta\mu)\end{aligned}$$

and

$$\begin{aligned}(\beta,\beta)(\alpha,\alpha) &= (\beta,\beta)(\beta\mu,\beta\mu) \\ &= (\beta\mu,\beta\mu)\end{aligned}$$

Thus, $(\alpha,\alpha)(\beta,\beta)=(\beta\mu,\beta\mu)=(\beta,\beta)(\alpha,\alpha)$
If $\beta=\alpha\kappa$, then

$$\begin{aligned}(\alpha,\alpha)(\beta,\beta) &= (\alpha,\alpha)(\alpha\kappa,\alpha\kappa) \\ &= (\alpha\kappa,\alpha\kappa)\end{aligned}$$

and

$$\begin{aligned}(\beta,\beta)(\alpha,\alpha) &= (\alpha\kappa,\alpha\kappa)(\alpha,\alpha) \\ &= (\alpha\kappa,\alpha\kappa)\end{aligned}$$

Then $(\alpha,\alpha)(\beta,\beta)=(\alpha\kappa,\alpha\kappa)=(\beta,\beta)(\alpha,\alpha)$
If $\alpha\neq\beta\mu$ and $\beta\neq\alpha\kappa$, then

$$(\alpha,\alpha)(\beta,\beta)=0=(\beta,\beta)(\alpha,\alpha)$$

Therefore, idempotents commute, and by Theorem 3.2, S_E is an inverse semigroup. □

Now that we have the inverse semigroup of interest, we set out to explore some properties that translate to properties of C^* algebras.

5.1. E^*-Unitary & F^*-Inverse.

The first properties we are interested in are E^*-unitary and F^*-inverse. These properties hold without limitations on the directed graph.

Definition 5.5. E-unitary

(1) Let S be an inverse semigroup. We say that S is *E-unitary* if, for $e \in E(S)$, $e \leq s$ implies $s \in E(S)$.
(2) If S is an inverse semigroup with zero, then we say S is *E^*-unitary* if $0 \neq e \leq s$ implies $s \in E(S)$

Remark 5.6. In our discussion, we are not concerned with E-unitary semigroups, as the inverse semigroups we are looking at are all inverse semigroups with zero, with the exception of Example 4.1. This does not imply that inverse semigroups with zero cannot be E-unitary. If we consider an inverse semigroup with zero, S, where every element is an idempotent, then S would in fact be E-unitary as well as E^*-unitary.

Theorem 5.7. *Let E be a directed graph. The graph inverse semigroup S_E is E^*-unitary.*

Proof. Let $(\alpha, \alpha) \in E(S_E)$, and $(\gamma, \delta) \in S_E$. Suppose $(\alpha, \alpha) \leq (\gamma, \delta)$ in the partial order. Then:

$$\begin{aligned}(\alpha, \alpha) &= (\gamma, \delta)(\alpha, \alpha)(\alpha, \alpha) \\ &= (\gamma, \delta)(\alpha, \alpha) \\ &= \begin{cases} (\gamma, \alpha\mu) & \text{if } \delta = \alpha\mu \\ (\gamma\kappa, \alpha) & \text{if } \alpha = \delta\kappa \end{cases}\end{aligned}$$

If $\delta = \alpha\mu$, then $(\alpha, \alpha) = (\gamma, \alpha\mu)$

$$\begin{aligned}&\implies \alpha = \alpha\mu \text{ and } \gamma = \alpha \\ &\implies \mu = s(\alpha) \\ &\implies \delta = \alpha\end{aligned}$$

Thus, $(\gamma, \delta) = (\alpha, \alpha) \in E(S_E)$
If $\alpha = \delta\kappa$, then $(\alpha, \alpha) = (\gamma\kappa, \alpha)$

$$\begin{aligned}&\implies \alpha = \gamma\kappa \\ &\implies \delta\kappa = \gamma\kappa \\ &\implies \delta = \gamma\end{aligned}$$

Thus, $(\gamma, \delta) = (\gamma, \gamma) \in E(S_E)$. Therefore, S_E is E^*-unitary in the partial order. □

Being E^*-unitary is useful, but we desire a stronger property. But we first need a few definitions.

Definition 5.8. Prehomomorphism[2, Pg 80]
Let S, T be inverse semigroups, we say $\varphi : S \to T$ is a *prehomomorphism* if $\varphi(st) = \varphi(s)\varphi(t)$ whenever $st \neq 0$.

Definition 5.9. Idempotent Pure
Let S, T be inverse semigroups. Then $\varphi : S \to T$ is *idempotent pure* if $\varphi^{-1}(E(T)) = E(S)$. That is, if only the idempotents of S are mapped to the idempotents of T.

Now that we have mapping conditions, we can explore the property stronger than being E^*-unitary.

Definition 5.10. Strongly E^*-unitary
Let S be an inverse semigroup with zero. We say that S is *strongly E^*-unitary* if there exists a group G, and an idempotent pure prehomomorphism $\varphi : S \setminus \{0\} \to G$

Definition 5.11. Free Groups
Let S be a set of elements and define $\mathbb{F}_S$ to be the *free group over the set* S, under the operation of concatenation. We impose that each element of S has an associated inverse and a *reducing relation*, so that if $a, b, b^{-1} \in S$, we have $red(abb^{-1}) = a$.

Theorem 5.12. *The inverse semigroup S_E is strongly E^*-unitary.*

Proof. Define $\varphi : S_E^* \to \mathbb{F}_{E^1}$ by $\varphi((\alpha, \beta)) = \alpha\beta^{-1}$. Then

$$\varphi((\alpha,\beta)(\gamma,\delta)) = \begin{cases} \varphi((\alpha,\delta\mu)) & \text{if } \beta = \gamma\mu \\ \varphi((\alpha\kappa,\delta)) & \text{if } \gamma = \beta\kappa \end{cases}$$
$$= \begin{cases} \alpha(\delta\mu)^{-1} = \alpha\mu^{-1}\delta^{-1} \\ \alpha\kappa\delta^{-1} \end{cases}$$

And, $\varphi((\alpha,\beta))\varphi((\gamma,\delta)) = \alpha\beta^{-1}\gamma\delta^{-1}$
Then if $\beta = \gamma\mu$,

$$\begin{aligned} \implies \varphi((\alpha,\beta))\varphi((\gamma,\delta)) &= \alpha(\gamma\mu)^{-1}\gamma\delta^{-1} \\ &= \alpha\mu^{-1}\gamma^{-1}\gamma\delta^{-1} \\ &= \alpha\mu^{-1}\delta^{-1} \end{aligned}$$

If $\gamma = \beta\kappa$,

$$\begin{aligned} \implies \varphi((\alpha,\beta))\varphi((\gamma,\delta)) &= \alpha\beta^{-1}\beta\kappa\delta^{-1} \\ &= \alpha\kappa\delta^{-1} \end{aligned}$$

Therefore, $\varphi((\alpha,\beta)(\gamma,\delta)) = \varphi((\alpha,\beta))\varphi((\gamma,\delta))$, so φ is a prehomomorphism.
Now we want to show that φ is idempotent pure. Let $\epsilon \in \mathbb{F}_{E^1}$ represent the identity in the free group. Then, if $\varphi((\alpha,\beta)) = \epsilon$

$$\implies \alpha\beta^{-1} = \epsilon$$

By the uniqueness of inverses in groups,

$$\begin{aligned} &\implies \beta^{-1} = \alpha^{-1} \\ &\implies \varphi(\alpha,\beta) = \varphi(\alpha,\alpha) \\ &\implies \text{if } \varphi(\alpha,\beta) = \epsilon, \text{ then } (\alpha,\beta) \in E(S_E) \end{aligned}$$

Therefore, φ is idempotent pure, and thus S_E is strongly E^*-unitary. ◻

Definition 5.13. F^*-inverse
Let S be an inverse semigroup. Then S is *F^*-inverse* if, for all $s \in S$, there exists a unique maximal element above s.

Theorem 5.14. *The inverse semigroup S_E is F^*-inverse.*

Proof. Let $(\alpha, \beta) \in S_E$. Then by definition, $s(\alpha) = s(\beta)$. Let $\alpha = \alpha'\mu$ and $\beta = \beta'\mu$ for some $\mu \in E^*$. We claim that $(\alpha, \beta) \leq (\alpha', \beta')$. Consider:

$$\begin{aligned}(\alpha', \beta')(\alpha, \beta)^{-1}(\alpha, \beta) &= (\alpha', \beta')(\beta, \alpha)(\alpha, \beta)\\ &= (\alpha', \beta')(\beta, \beta)\\ &= (\alpha', \beta')(\beta'\mu, \beta'\mu)\\ &= (\alpha'\mu, \beta'\mu)\\ &= (\alpha, \beta)\end{aligned}$$

$\implies (\alpha, \beta) \leq (\alpha', \beta')$ by Definition 3.4.
If $\mu = s(\alpha)$, then $\alpha = \alpha'$ and $\beta = \beta'$. Thus,

$$\begin{aligned}(\alpha, \beta) &= (\alpha, \beta)(\alpha, \beta)^{-1}(\alpha, \beta)\\ &= (\alpha', \beta')(\beta, \alpha)(\alpha, \beta)\end{aligned}$$

$\implies (\alpha, \beta) \leq (\alpha', \beta')$. Thus, $\forall (\alpha, \beta) \in S_E$, there exists a maximal element above it, constructed by removing all common suffixes of α and β.
Now suppose some other element (γ, δ) is also a maximal element above (α, β).
Then $(\alpha, \beta) \leq (\gamma, \delta)$

$$\begin{aligned}\implies (\alpha, \beta) &= (\gamma, \delta)(\beta, \beta)\\ &= \begin{cases}(\gamma, \beta\mu) & \text{if } \delta = \beta\mu\\ (\gamma\kappa, \beta) & \text{if } \beta = \delta\kappa\end{cases}\end{aligned}$$

If $\delta = \beta\mu$, then

$$\begin{aligned}&\implies (\alpha, \beta) = (\gamma, \beta\mu)\\ &\implies \mu = s(\beta) \text{ and } \alpha = \gamma\\ &\implies \delta = \beta\\ &\implies (\alpha, \beta) = (\gamma, \delta)\end{aligned}$$

If $\beta = \delta\kappa$, then

$$\begin{aligned}&\implies (\alpha, \beta) = (\gamma\kappa, \delta)\\ &\implies \alpha = \gamma\kappa\\ &\implies (\alpha, \beta) = (\gamma\kappa, \delta\kappa)\end{aligned}$$

But this is exactly how we defined α' and β'. So $(\gamma, \delta) = (\alpha', \beta')$. Therefore, (α', β') is a unique maximal element above (α, β), and thus, S_E is F^*-inverse. □

As before, we have a stronger property than being F^*-inverse.

Definition 5.15. Strongly F^*-inverse
Let S be an inverse semigroup with zero. Then S is *strongly F^*-inverse* if there exists an idempotent pure map $\varphi : S_E \to G$ such that for all $g \in G, \varphi^{-1}(g)$ has a maximal element whenever its non-empty.

Theorem 5.16. *S_E is strongly F^*-inverse.*

Proof. Taking $G = \mathbb{F}_{E^1}$ and φ as is Theorem 5.12, we have an idempotent pure map to a group. Consider an element $\alpha\beta^{-1} \in \mathbb{F}_{E^1}$. Then $\varphi^{-1}(\alpha\beta^{-1}) = \{(\alpha\mu, \beta\mu) \in S_E : \mu \in E^*\}$. Since S_E is F^*-inverse, there exists a unique maximal element above any element in S_E. Namely, (α, β) if $\alpha\beta^{-1} = red(\alpha\beta^{-1}) \in \mathbb{F}_{E^1}$ as shown in Theorem 5.14 and Definition 5.11.
It is worthwhile to note that if we consider $\epsilon \in \mathbb{F}_{E^1}$, then $\varphi^{-1}(\epsilon) = E(S_E)$.
Then $\varphi((\alpha, \alpha)) = \epsilon$, and $(\alpha, \alpha) \leq (r(\alpha), r(\alpha))$.
We also have $\varphi((\beta, \beta)) = \epsilon$ and $(\beta, \beta) \leq (r(\beta), r(\beta))$, but

$$(r(\beta), r(\beta)) \nleq (r(\alpha), r(\alpha)) \nleq (r(\beta), r(\beta))$$

$\implies$ There is a maximal element, but it need not be unique. □

5.2. **Minimal.**

Although the previous properties did not depend on the directed graph, we now want to see how applying conditions to the graph yield different properties to the associated C^*-algebra.

Definition 5.17. Outer Cover[6, Definition 2.9, Pg 279]
Let S be an inverse semigroup. A set $C \subseteq E(S)$ is said to be an *outer cover* of $e \in E(S)$ if for all $0 \neq f \leq e$, there exists $c \in C$ such that $fc \neq 0$.

Definition 5.18. Minimal
Let S be an inverse semigroup. We say S is *minimal* if, for all $e, f \in E(S)^*$, there exists $s_1, ..., s_n \in S$ such that $\{s_i f s_i^{-1}\}_{1\leq i\leq n}$ is an outer cover of e.

Definition 5.19. Irreducible Graph
A graph E is said to be *irreducible* if, $\forall v, w \in E^0$, $\exists \mu \in E^*$ such that $s(\mu) = v$ and $r(\mu) = w$. In other words, there is a path between every pair of vertices.

Lemma 5.20. *If E is irreducible, then $\forall \alpha \in E^*, \forall v \in E^0, \exists s \in S_E$ such that $s(\alpha, \alpha)s^{-1} = (v, v)$*

Proof. Let $\alpha \in E^*$ and $v \in E^0$. Since E is irreducible, $\exists \mu$ such that $s(\mu) = v, r(\mu) = s(\alpha)$. Take $s = (v, \alpha\mu)$. Then

$$\begin{aligned} s(\alpha, \alpha)s^{-1} &= (v, \alpha\mu)(\alpha, \alpha)(\alpha\mu, v) \\ &= (v, \alpha\mu)(\alpha\mu, v) \\ &= (v, v) \end{aligned}$$

As required. □

Theorem 5.21. *If E is irreducible, then S_E is minimal. That is, $\forall e, f \in E(S_E)$, $\exists s_1, s_2, ..., s_n \in S_E$ such that $\{s_i f s_i^{-1}\}_{1\leq i\leq n}$ is a cover of e.*

Proof. Suppose E is irreducible.
Given $(\gamma, \gamma), (\delta, \delta) \in E(S_E)$, by Lemma 5.20, there exists $s \in S_E$ such that $s(\gamma, \gamma)s^{-1} = (r(\delta), r(\delta))$ is a cover of $(r(\delta), r(\delta))$. Let $\kappa, \eta \in E^*$ such that $s(\kappa) = s(\eta) = r(\delta)$. Then:

$$(r(\delta)\kappa, r(\delta)\kappa)s(\gamma, \gamma)s^{-1} \neq 0$$

$$\text{And } (\delta\eta, \delta\eta)s(\gamma, \gamma)s^{-1} \neq 0$$

$\implies$ $s(\gamma, \gamma)s^{-1}$ is a cover of (δ, δ). Therefore, S_E is minimal. □

5.3. **Transitive.**

As before, we need a few definitions before we start proving conditions for transitivity.

Definition 5.22. Fixed Idempotents[6, Definition 4.1, Pg 289]
Let S be an inverse semigroup. Let $s \in S$ and $e \in E(S)$. For $e \leq s^{-1}s$, we say that e is *fixed* if $se = e$. We say that e is *weakly fixed under* s if $sfs^{-1}f \neq 0$ for all non-zero idempotents $f \leq e$.

Definition 5.23. Transitive
Let S be an inverse semigroup. If, for all $s \in S$ and for all $e \in E(S)$ weakly fixed under s, there exists a cover of e consisting of fixed idempotents, then S is said to be *transitive.*

We note here that, if S is E^*-unitary, by Definition 5.5(2), if $s \in S \setminus E(S)$, then the only fixed idempotent under s is 0. Thus, by Theorem 5.7, an equivalent argument for $S(E)$ being transitive is for it to have no weakly fixed idempotents.

Definition 5.24. Cycle
Let E be a directed graph. An element $\eta \in E^* \setminus E^0$ with $s(\eta) = r(\eta)$ is called a *cycle.* For $\eta = \eta_1 ... \eta_n$ if there exists some $\alpha \in E^1$ such that, for some $1 \leq i \leq n$, $r(\eta_i) = r(\alpha)$ with $\alpha \neq \eta_i$, the α is called an *entry to the cycle* η.

Theorem 5.25. *Let E be a directed graph. Then $S(E)$ is transitive if and only if every cycle in E has an entry.*

Proof. Suppose every cycle in E has an entry. We want to show for all $s \in S_E \setminus E(S_E)$, s has no weakly fixed idempotents. Suppose $(\alpha, \beta) \in S_E \setminus E(S_E)$ and assume $e = (\gamma, \gamma)$ is weakly fixed under (α, β)
$\implies (\gamma, \gamma) \leq (\beta, \alpha)(\alpha, \beta) = (\beta, \beta)$
$\implies \gamma = \beta\mu$ for some $\mu \in E^*$
Since $e \leq e$, we have

$$(\alpha, \beta)(\gamma, \gamma)(\beta, \alpha)(\gamma, \gamma) \neq 0$$
$$(\alpha, \beta)(\beta\mu, \beta\mu)(\beta, \alpha)(\beta\mu, \beta\mu) \neq 0$$
$$(\alpha\mu, \beta\mu)(\beta, \alpha)(\beta\mu, \beta\mu) \neq 0$$
$$(\alpha\mu, \alpha\mu)(\beta\mu, \beta\mu) \neq 0$$

In particular, α and β agree. So, we must look at the relationship between the sizes of α and β. This leads to 3 cases:

(1) $|\alpha| = |\beta|$
(2) $|\alpha| > |\beta|$
(3) $|\beta| < |\alpha|$

Case (1) yields an obvious contradiction, since α and β agree, and $(\alpha, \beta) \notin E(S_E)$
Assume (2). Then we have that $\alpha = \beta\tau$ for some $\tau \in E^* \setminus \{0\}$ since the product is non-zero. Then

$$\implies \beta\tau\mu \text{ agrees with } \beta\mu$$
$$\implies \tau\mu \text{ agrees with } \mu$$

Then $r(\tau) = r(\mu)$ and $s(\tau) = s(\mu)$. Therefore, τ is a cycle.
$\implies s = (\beta\tau, \beta)$. Now, we compare the sizes of μ and τ.

If $|\mu| \leq |\tau|$, then we can find a δ such that $r(\delta) = s(\mu)$ so that $|\mu\delta| \geq |\tau|$ and $\mu\delta$ agrees with $\tau\tau$ except for the last entry of δ, which is the entrance of τ. Let $f = (\beta\mu\delta, \beta\mu\delta)$. Then

$$\begin{aligned}
&\implies f \leq e \\
&\implies sfs^{-1}f = (\beta\tau\mu\delta, \beta\tau\mu\delta)(\beta\mu\delta, \beta\mu\delta) \neq 0 \\
&\implies \beta\tau\mu\delta \text{ agrees with } \beta\mu\delta \\
&\implies \tau\mu\delta \text{ agrees with } \mu\delta
\end{aligned}$$

And since $\mu\delta$ agrees with τ, we have that $\tau\mu\delta$ begins with $\tau\tau$, but $\mu\delta$ does not by construction, a contradiction.
If $|\mu| > |\tau|$, then $\mu = \tau\eta$ for $\eta \in E^*$. Then

$$\begin{aligned}
&\implies \tau\eta \text{ agrees with } \tau\mu \\
&\implies \eta \text{ agrees with } \mu \\
&\implies \eta \text{ agrees with } \tau
\end{aligned}$$

This continues, so we can find η so that $|\eta| \leq |\tau|$ and $\mu = \tau^n\eta$. Finding δ as before so that $\mu\delta$ and τ^{n+1} agree, but $\mu\delta$ and τ^{n+2} do not. Let $f = (\beta\mu\delta, \beta\mu\delta)$ as before. Then

$$\begin{aligned}
&\implies \beta\tau\mu\delta \text{ agrees with } \beta\mu\delta \\
&\implies \tau\mu\delta \text{ agrees with } \mu\delta
\end{aligned}$$

But $\tau\mu\delta$ agrees with τ^{n+2} while $\mu\delta$ does not agree with τ^{n+2}, a contradiction.
Assuming (3) follows a similar procedure. Therefore, $(\alpha\mu, \alpha\mu)(\beta\mu, \beta\mu) = 0$, which implies e is not weakly fixed under (α, β), a contradiction. Thus, there are no weakly fixed idempotents, and by Definition 5.23, S_E is transitive.

Now suppose E has a cycle κ with no entry. Let $s = (\kappa, s(\kappa))$ and let $e = (s(\kappa), s(\kappa))$. We want to show that e is weakly fixed under s.
Let $f = (\gamma, \gamma) \leq (s(\kappa), s(\kappa))$, then $r(\gamma) = s(\kappa)$.
Since κ has no entry, $\kappa = \kappa_1\kappa_2...\kappa_n$ implies $r(\kappa_1) = s(\kappa_n)$. So if $r(\alpha) = r(\kappa_i)$ for some $1 \leq i \leq n$, then $\alpha = \kappa_i$ So,

$$\begin{aligned}
&r(\gamma) = r(\kappa) \\
&\quad \implies r(\gamma_1) = r(\kappa_1) \\
&\quad \implies \gamma_1 = \kappa_1 \text{ and so} \\
&r(\gamma_2) = s(\gamma_1) = s(\kappa_1) = r(\kappa_2) \\
&\quad \implies \gamma_2 = \kappa_2
\end{aligned}$$

And so on. Thus, γ agrees with κ. Moreover, $\gamma = \kappa^n\mu$, where μ is some initial segment of κ. Then

$$\begin{aligned}
sfs^{-1}f &= (\kappa\gamma, \kappa\gamma)(\gamma, \gamma) \\
&= (\kappa^{n+1}\mu, \kappa^{n+1}\mu)(\kappa^n\mu, \kappa^n\mu) \\
&= (\kappa^{n+1}\mu, \kappa^{n+1}\mu) \neq 0
\end{aligned}$$

$\implies$ e is weakly fixed by s. But e does not have a cover consisting of fixed idempotents since S_E has no fixed idempotents. Therefore, if a cycle in E has no entries, then S_E is not transitive. □

6. C^*-Algebras

It is useful to remind the reader what we mean by a C^* algebra before discussing the properties implied from the graph inverse semigroup.

Definition 6.1. C^*-Algebra
A Banach algebra A over $\mathbb{C}$ is said to be a *C^*-algebra* if, $\exists$ an involution $* : x \mapsto x^*$ on A such that $\forall x \in A$, $||x^*x|| = ||x||^2$

Although a deep subject of study, we are mainly interested in one type of C^*-algebra, aptly called the Graph C^*-algebra.

Definition 6.2. Graph C^*-Algebra[5, Pg 13]
Let E be a directed graph. Then $\mathcal{O}_E$ is the universal C^*-algebra generated by a set of projections $\{P_v\}_{v\in E^0}$ and a set of partial isometries $\{S_e\}_{e\in E^1}$ that satisfy the Cuntz-Krieger relations:

(1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$
(2) $P_v = \sum_{\{e:E^1:r(e)=v\}} S_e S_e^*$

For any inverse semigroup S, Exel associated a C^*-algebra, denoted $C^*_{tight}(S)$[9]. In [9], Exel argues that, when applied to the graph inverse semigroup, S_E, we obtain the universal C^*-algebra $\mathcal{O}_E$.
In subsequent works of Milan-Steinberg[8], Li[7], and Exel-Pardo[6], various properties of an inverse semigroup were shown to give rise to properties of the $C^*_{tight}(S)$. What follows is a summary of some of their results, and a note about what these properties imply about $\mathcal{O}_E$.

Remark 6.3. Being strongly E^*-unitary implies that the C^*-algebra, $C^*_{tight}(S_E)$ can be expressed as a partial crossed product.[7, Proposition 3.1, Pg 8]

Remark 6.4. The graph inverse semigroup being strongly F^*-inverse implies that the C^*-algebra, $C^*_{tight}(S_E)$ is *morita equivalent* to a crossed product C^*-algebra.[8, Corollary 6.17, Pg 510]

Remark 6.5. The graph inverse semigroup being minimal, transitive, and E^*-unitary implies the C^*-algebra, $C^*_{tight}(S_E)$ is *simple*.[6, Theorem 6.8, Pg 302]

It is of interest to know when $\mathcal{O}_E$ is *simple*, that is, when it has no 2-sided ideals. The universal C^*-algebra is simple if and only if the graph is *cofinal* and when every cycle has an entry (See Definition 5.24)[5, Theorem 4.14, Pg 39].

Definition 6.6. Cofinal
Let E be a directed graph and let $E^{\leq\infty} = E^\infty \cup E^{<\infty}$ represent all possible paths. We say that E is *cofinal* if, for all $v \in E^0$ and for all $\alpha \in E^{\leq\infty}$, there exists a vertex w in α, and a $\beta \in E^*$ such that $r(\beta) = v$ and $s(\beta) = w$. In other words, for every path and vertex in the graph, there is a vertex in the path with a path connecting the two vertices.

Remark 6.7. It is worth noting that if and only if E is cofinal, then S_E is minimal. One can go about proving this, but it requires product topology on the infinite path space, which is outside of the scope of this paper.

This, taken together with what was proven in Section 5 shows that when the directed graph E is finite, $\mathcal{O}_E$ can be expressed as a partial crossed product and is Morita equivalent to a crossed product. This result was already known, see [10] for details. Additionally, $\mathcal{O}_E$

is simple if and only if E is cofinal and every cycle in E has an entry[5, Theorem 4.14, Pg 39]. In particular, this occurs when E is irreducible and when every cycle has an entry.

References

[1] Gallian, Joseph, A. *Contemporary Abstract Algebra, Ninth edition* Cengage Learning (2017), 43

[2] Lawson, Mark V. *Inverse Semigroups: The Theory of Partial Symmetries* World Scientific Publishing (1998) ISBN: 981-02-3316-7

[3] Lawson, Mark, V. *Primer on Inverse Semigroups I* (arXiv:2006.01628)

[4] Norling, M.D. *Inverse Semigroup C^*-Algebras Associated with Left Cancellative Semigroups*, Proceedings of the Edinburgh Mathematical Society (2014)

[5] Raeburn, Iain, *Graph Algebras*, 1991 Mathematics Subject Classification. Primary 46L05 Secondary 46L08, 46L35, 46L55, 46L80, 22D35

[6] Exel, Ruy; Pardo, Enrique. *The Tight Groupoid of an Inverse Semigroup*, Semigroup Forum (2016)

[7] Li, Xin, *Partial Transformation Groupoids Attached to Graphs and Semigroups*, International Mathematics Research Notices, Vol. 2016, No.00

[8] Milan, David; Steinberg, Benjamin, *On Inverse Semigroup C^*-Algebras and Crossed Products*, Groups Geometries and Dynamics Vol. 8 (2014)

[9] Exel, Ruy; Inverse semigroups and combinatorial C^*-algebras. *Bull. Braz. Math. Soc. (N.S.)*, 39(2):191–313, 2008.

[10] Exel, Ruy; *Partial Dynamical Systems, Fell Bundles and Applications*, volume 224 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2017.