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1 Introduction

1.1 Functional Calculus

A functional calculus is a theory or process which makes sense of a class of functions (often of a real or complex variable) taking instead an operator-valued input. A common first example of such an idea is of course the matrix exponential, but we are more interested in how we can apply general classes of functions to particular operator structures. For instance, it's easy to imagine we could define a polynomial function taking any operator input, and we might call this a polynomial functional calculus.

We will study principally the Riesz-Dunford or holomorphic functional calculus and subsequently define an analogous functional calculus for unbounded operators of half-plane type and functions with certain decay properties on a half plane. Finally, we show how once can extend this functional calculus to the exponential function with an unbounded input. Using Laplace transform methods one can connect this extension to the general theory of so called strongly continuous or C_0 semigroups. This gives an alternative approach to the Hille-Yosida theorem characterizing the generators of these semigroups, the main object of study which we will now motivate.

1.2 Semigroups in Quantum Mechanics

The state of a closed quantum system is described by a time dependent family of wave functions $\{\psi(t)\}_{t\geq 0}$ in a Hilbert space $\mathcal{H}$, where $\mathcal{H}$ is $L^2(\mathbb{R}^3)$ or its discrete analogue $\ell^2(\mathbb{Z}^3)$. The evolution of the system state is described by an abstract Cauchy problem generated by the self adjoint operator $-iH$, the Schrodinger equation;

$$i\partial_t \psi(t) = H\psi(t) \tag{1}$$
$$\psi(0) = \psi_0 \in \mathcal{H}. \tag{2}$$

Which admits the solution $\psi(t) = e^{-itH}\psi_0$. The family of operators $\{e^{-itH}\}_{t\in\mathbb{R}^+}$ forms a strongly continuous semigroup generated by $-iH$. For unitary operators on $\mathcal{H}$ as presently described, Stone's Theorem characterizes C_0 semigroups by their generator A, in which case the semigroup is precisely e^{itA}. Such a group gives the solutions to the differential equation analogously determined by A to the above. The formal statement of Stone's Theorem is as follows.

Theorem 1.1 (Stone's Theorem)**.** *Let $U(t) : \mathbb{R}^+ \to \mathcal{L}(\mathcal{H})$ be a strongly continuous one parameter unitary semigroup. That is, a C_0 semigroup of unitary operators. Then there is a unique self-adjoint operator A such that $U(t) = e^{itA}$. Where the domain of A, $\mathcal{D}(A)$ is given by*

$$\mathcal{D}(A) := \{\psi \in \mathcal{H} \ ; \ \lim_{h\to 0} \big(U(h)\psi - \psi)h^{-1}\big) \ \textit{exists}\}$$

Conversely, if A is an operator self-adjoint on its domain over $\mathcal{H}$. Then $U(t) := e^{itA}$ is a C_0 semigroup.

On the other hand we can consider the state of the system to be fixed and consider the evolution of the observables on the system. The evolution of said observable is described by the differential equation;

$$\partial_t A_t = SA_t. \tag{3}$$

Here A_t corresponds to a family in the Banach space $X := \mathcal{L}(\mathcal{H})$ and S is an operator acting on X. This system admits solutions through a C_0 semigroup of operators on X, $T(t)$ generated by S. More generally than Stone's Theorem, the Hille-Yosida theorem characterizes generators of C_0 semigroups on Banach spaces.

2 Preliminaries

The reader should have familiarity with real and complex analysis. In particular we will be taking results such as Cauchy's Theorem and Cauchy's Integral Formula in their complex variable form as known. An awareness of some of the basic theorems of functional analysis would also be helpful, but we will state them in this section.

Banach Spaces

Though the following may be familiar to most readers, these concepts are not always presented in an undergraduate analysis course so just to be safe;

Definition 2.1 (Banach Space/Algebra)**.** *We say that a normed vector space $\mathcal{A}$ is a Banach space if it is a complete metric space with respect to the metric induced by the norm. If in addition, $\mathcal{A}$ is an algebra such that for any $a, b \in \mathcal{A}$, $||ab|| \leq ||a|| \cdot ||b||$ we say that $\mathcal{A}$ is a Banach algebra. If the algebra has a multiplcative unit, we say it is unital.*

We will typically denote, in plain text, a Banach space by X while we will denote a Banach algebra by $\mathcal{A}$.

Definition 2.2. *Let $\mathcal{A}$ be a unital Banach Algebra. The spectrum of $a \in \mathcal{A}$ is;*

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} \ ; \ \lambda - a \text{ has no inverse in } \mathcal{A}\}$$

$\rho_A(a) = \sigma_A(a)^c$ is the called the resolvent set of a in A.

When the ambient algebra is evident, we may drop the subscript in $\sigma_{\mathcal{A}}(a)$ and $\rho_{\mathcal{A}}(a)$.

Definition 2.3. *Let $a \in \mathcal{A}$, a unital Banach algebra. Denote by $R_a : \rho(a) \to \mathcal{A}$ the resolvent function of a, given by $R_a(z) = (z - a)^{-1}$.*

Definition 2.4. *Let X be a Banach space and $D \in \mathbb{C}$ be open. We say a function $f : D \to X$ is holomorphic if for all $z \in D$, the limit*

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in X. Moreover, we say that f is weakly holomorphic if for any $x^* \in X^*$, $x^* f$ is holomorphic on D.

One can show quite easily that if f is holomorphic then $(x^* f)' = x^* f'$. That is, if f is holomorphic then it is weakly holomorphic. It is also true, but harder to show that these properties are equivalent; see section 1.7 in [4].

Definition 2.5 (Hilbert Space)**.** *An inner product space for which the norm induced by the inner product is a complete metric is a Hilbert space.*

We take a couple properties of Banach algebras without proof. The proofs can be found in any functional analysis text, including [2].

Proposition 2.1. *Let $\mathcal{A}$ be a unital Banach Algebra. If $a \in A$, $\lambda \in \mathbb{C}$ with $|\lambda| > ||a||$. Then $1\lambda - a$ is invertible and $(1\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$.*

Proposition 2.2. *Let $\mathcal{A}$ be a unital Banach Algebra. Let G be the set of invertible elements in $\mathcal{A}$. G is open and $a \mapsto a^{-1}$ is continuous.*

Definition 2.6. *(Bounded Operators) Let X, Y be Banach spaces. We say that an operator $A \in \mathcal{L}(X, Y)$ is bounded if there exists $M > 0$ such that for all $x \in X$*

$$\|Ax\| \leq M \|x\|$$

In this case, the operator norm $\|A\| := \sup\{\|Ax\| \ ; \ \|x\| \leq 1\}$ is finite and the above holds for $M = \|A\|$.

For Banach spaces X and Y we will write $\mathcal{L}(X, Y)$ and X^* to be set of bounded operators from X to Y and the space of bounded linear functionals on X (bounded operators from X to $\mathbb{C}$), respectively. In the case where $X = Y$, we will shorten to just $\mathcal{L}(X)$.

It is a well known fact that the above definition is equivalent to A being continuous, uniformly continuous and Lipschitz continuous. In this project we will also consider unbounded operators, in particular closed operators.

Definition 2.7. *(Closed Operators) Let A be an operator from $\mathcal{D}(A) \to Y$ where $\mathcal{D}(A) \in X$ is the domain of A in a Banach space X and Y also a Banach space. We say that A is closed if whenever $x_n \to x \in X$ in $\mathcal{D}(A)$ such that $Ax_n \to y \in Y$, then $x \in \mathcal{D}(A)$ and $Ax = y$.*

A consequence of Proposition 2.1 is the following power series representation for the resolvent.

Proposition 2.3. *Let A be a closed operator on a Banach space X. Then for $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$ satisfying $|\mu - \lambda| < 1/\|R_A(\mu)\|$, then*

$$R_A(\lambda) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_A(\mu)^{n+1}$$

Theorems of Functional Analysis

We will also borrow some of the keystone results of functional analysis. In particular, many of the proofs in this paper will make use of this particular formulation, a corollary really, of Hahn-Banach. Most of these can be found in any textbook on functional analysis and for our purposes we would refer the reader to [2].

Theorem 2.1 (Hahn-Banach)**.** *Let X be a Banach Space, then for $x, y \in X$, $x \neq y$ there exists $f \in X^*$ such that $f(x) \neq f(y)$.*

Theorem 2.2 (Uniform Boundedness)**.** *Let X be a Banach Space and Y a normed space. Let $\{F_i\}_{i\in\mathbb{N}} \subset \mathcal{L}(X,Y)$ be a family of bounded operators. If for every $x \in X$;*

$$\sup\{||F_i(x)|| \ ; \ F_i \in F\} < \infty$$

Then, F is uniformly bounded in operator norm. i.e;

$$\sup\{||F_i|| \ ; \ F_i \in F\} < \infty$$

Definition 2.8. *Let X be a complex vector space and $E \subseteq X$. Then, E is said to be weakly bounded if $\forall x^* \in X^*$, $x^*(E)$ is bounded.*

Corollary 2.1 (Weak Boundedness)**.** *In the setting of the above definition, E is bounded if and only if E is weakly bounded.*

Theorem 2.3 (Fubini's Theorem)**.** *Let A and B be complete measure spaces. If a function $f(x,y)$ is $A \times B$ measurable and*

$$\int_{A\times B} |f(x,y)| d(x,y) < \infty$$

Then the order of integration doesn't matter, that is;

$$\int_{AxB} f(x,y)d(x,y) = \int_A \int_B (f(x,y)dy)\, dx = \int_B \int_A (f(x,y)dx)\, dy$$

Theorem 2.4 (Vitali's Theorem)**.** *Let $U \in \mathbb{C}$ be open and connected and let (f_α) be a locally bounded net of functions holomorphic on U. If $\{z \in U \ ; \ f_\alpha(z) \text{ converges}\}$ has a limit point in U, then (f_α) converges to a holomorphic function f on compacts in U.*

We will use Vitali's Theorem once towards the end of the project, and we will consider a sequence instead of a net and in a setting where the requirements are more easily met. But for this section we will give the statement of the theorem as in [6]. This is to say, the reader doesn't need to be familiar with what a net is.

3 Holomorphic Functional Calculus

Our goal here is to make sense of certain classes of holomorphic functions taking input $a \in \mathcal{A}$ where $\mathcal{A}$ is a Banach Algebra, (such as $\mathcal{L}(\mathcal{H})$). The Riesz-Dunford or holomorphic functional calculus achieves this by extending Cauchy's integral formula for functions holomorphic on a neighbourhood of the spectrum of a as in the definition;

$$f(a) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}$$

We will see that, for fixed $a \in \mathcal{A}$ the mapping $f \to f(a)$ gives an algebra homomorphism between functions holomorphic on a given neighbourhood of the spectrum of an element in $\mathcal{A}$, and $\mathcal{A}$ itself.

Throughout the construction of the holomorphic functional calculus we will extend many of well-known results of Complex Analysis.

We will also cover Banach space-valued integration here, after which we may define the holomorphic functional calculus. In this final subsection, we will also cover some interesting results that are of independent interest such as how power series of holomorphic functions behave under functional calculus and the Spectral Mapping Theorem.

Before we can say anything about functions holomorphic on the spectrum of $a \in \mathcal{A}$ we must cover some basic spectral theory.

3.1 Basic Spectral Theory

In the preliminaries we made reference to the spectrum, the resolvent and the resolvent function. Recall that the spectrum of $a \in \mathcal{A}$ is the set of complex numbers λ such that the operator $1_{\mathcal{A}}\lambda - a$ has no inverse (here $\mathcal{A}$ is unital). The resolvent is exactly the complement of the spectrum, and the resolvent function is the map from said set to the inverse of operators of the form $1_{\mathcal{A}}\lambda - a$. In addition to basic facts of the spectrum we will touch on holomorphicity for Banach space-valued functions, such as the resolvent function. As the name would suggest, this too will be crucial for defining the holomorphic functional calculus.

Proposition 3.1. *Let $\mathcal{A}$ be a unital Banach Algebra. For any $a \in \mathcal{A}$, $\sigma(a)$ is a compact subset in $\mathbb{C}$.*

Proof. Let $f : \mathbb{C} \to \mathcal{A}$ be given by $f(z) = z - a$. Let G be the set of invertible elements. Since f is continuous and G is open $f^{-1}(G) = \rho(a)$ is open, so $\sigma(a)$ is closed. On the other hand, find $\lambda \in \mathbb{C}$ such that $|\lambda| > ||a||$. For such λ, $a - \lambda 1$ is invertible, so $\sigma(a) \subset B_{||a||}(0)$. Hence, $\sigma(a)$ is compact.

□

Proposition 3.2. *Let $D \in \mathbb{C}$ be open and $f : D \to X$ weakly holomorphic, then f is continuous on D.*

Proof. For $\alpha \in D$, find $\bar{B}_r(\alpha) \in D$. For $x^* \in X^*$, write

$$g(z) = \frac{x^* f(z) - x^* f(\alpha)}{z - \alpha}$$

Since g is a quotient of a function analytic on D and a function with a zero of order 1 on D, g has a removable singularity. Consider g now as its analytic extension on D. The set

$$\left\{ \frac{f(z) - f(\alpha)}{z - \alpha} \;\; ; \; 0 < |z - x| \leq r \right\}$$

is weakly bounded, thus bounded in X. So, $||f(z) - f(\alpha)|| \leq M|z - x|$ for some M. Hence f is continuous. □

Theorem 3.1 (Liouville's Theorem)**.** *Let X be a Banach Space and $f : \mathbb{C} \to X$ be entire and bounded. Then f is constant.*

Proof. For every $x^* \in X^*$, we have that $x^* f$ is bounded and entire, hence constant by the usual Liouville's Theorem. If f is not constant, then by Hahn-Banach there is a linear functional which seperates any two distinct points in the range of f, which is a contradiction. Hence, f is constant. $\square$

Proposition 3.3. *Let $\mathcal{A}$ be a unital Banach Algebra. Then, $\forall a \in \mathcal{A}$, R_a is holomorphic on $\rho(a)$.*

Proof. Note first that for any invertible elements $x, y \in \mathcal{A}$ we have that $x^{-1} + y^{-1} = x^{-1}(x+y)y^{-1}$. Now, since $\rho(a)$ is open, for any $\alpha \in \mathbb{C}$ we may find $h \in \mathbb{C}$ of sufficiently small norm so that $\alpha + h \in \rho(a)$. Consider;

$$\begin{aligned}
\frac{R_a(\alpha+h) - R_a(\alpha)}{h} &= \frac{((\alpha+h)1_{\mathcal{A}} - a)^{-1} - (\alpha 1_{\mathcal{A}} - a)^{-1}}{h} \\
&= -\frac{((\alpha+h)1_{\mathcal{A}} - a)^{-1}((\alpha+h)1_{\mathcal{A}} - a - (\alpha 1_{\mathcal{A}} - a))(\alpha 1_{\mathcal{A}} - a)^{-1}}{h} \\
&= -((\alpha+h)1_{\mathcal{A}} - a)^{-1}(\alpha 1_{\mathcal{A}} - a)^{-1}
\end{aligned}$$

Which goes to $-(\alpha 1_{\mathcal{A}} - a)^{-2} = -R_a(z)^2$ as h tends to 0 by the continuity of inversion. $\square$

Theorem 3.2. *Let $\mathcal{A}$ be a unital Banach algebra, then for any $a \in \mathcal{A}$, $\sigma(a)$ is non empty.*

Proof. Assume that $\sigma(a)$ is empty. Then, $\rho(a) = \mathbb{C}$, so R_a is entire. Let $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 2||a||$. Then,

$$\begin{aligned}
||(1\lambda - a)^{-1}|| &= ||(\lambda(1 - \lambda^{-1}a))^{-1}|| \\
&= ||\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} a^n|| \\
&= ||\sum_{n=0}^{\infty} \lambda^{-n-1} a^n|| \\
&\leq \sum_{n=0}^{\infty} \frac{||a||^n}{|\lambda|^{n+1}} \\
&\leq \sum_{n=0}^{\infty} \frac{||a||^n}{2^{n+1}||a||^{n+1}} = \frac{1}{||a||}
\end{aligned}$$

Hence, $R_a(z)$ is bounded for $|z| \geq 2||a||$. Moreover, since R_a is continuous on $\mathbb{C}$, it is bounded for $|z| < 2||a||$. By Liouville's Theorem, R_a is constant, which cannot be true since $\rho(a)$ is the whole complex plane. $\square$

Definition 3.1. *Let $\mathcal{A}$ be a unital Banach Algebra. For $a \in \mathcal{A}$ define the spectral radius of a as;*

$$r(a) := \sup\{|\lambda| \ ; \ \lambda \in \sigma(a)\}$$

Lemma 3.1 (Spectral Mapping Theorem for Polynomials)**.** *Let $\mathcal{A}$ be a unital Banach Algebra and $p \in \mathbb{C}[z]$ a complex polynomial. For any $a \in \mathcal{A}$;*

$$\sigma(p(a)) = p(\sigma(a))$$

Proof. Let $\lambda \in \mathbb{C}$, and write $p(z) - \lambda = \beta(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)$ where $\beta, \alpha_i \in \mathbb{C}$. Hence, $p(a) - \lambda = \beta(a - \alpha_1)(a - \alpha_2) \dots (a - \alpha_n)$. If $\lambda \in \sigma(p(a))$, then there must exist an some $(a - \alpha_i)$ that is not invertible; i.e; $\alpha_i \in \sigma(a)$. $p(\alpha_i) - \lambda = 0$, hence $\lambda \in p(\sigma(a))$. On the other hand, if $\lambda \in p(\sigma(a))$, then there exists an element $\alpha \in \sigma(a)$ such that $p(\alpha) - \lambda = 0$. Then $\alpha_i = \alpha$ for some i, so $a - \alpha_i$ is not invertible, and so neither is $p(a) - \lambda$

□

Theorem 3.3 (Spectral Radius Formula)**.** *Let $\mathcal{A}$ be a unital Banach Algebra, then for $a \in \mathcal{A}$;*

$$r(a) = \lim_{n\to\infty} ||a^n||^{1/n}$$

Proof. By the lemma we have that $\sigma(a^n) = \sigma(a)^n$ where $\sigma(a)^n := \{\alpha^n \ ; \ \alpha \in \sigma(a)\}$. Hence, $r(a^n) = r(a)^n$. Note also that for $b \in \mathcal{A}$, $\sigma(b) \in B_{||b||}(0)$, that is $r(b) \leq ||b||$. Then, $r(a) = r(a^n)^{1/n} \leq ||a^n||^{1/n}$. Since this is true for all $n \in \mathbb{N}$, we have that $r(a) \leq \liminf ||a^n||^{1/n}$.

Now take $\lambda \in \mathbb{C}$ with $\lambda > \|a\|$. Then R_a is holomorphic at λ and $R_a(\lambda) = \sum\limits_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$. For any $a^* \in \mathcal{A}^*$ we have that

$$\sum_{n=0}^{\infty} \frac{|a^*(a^n)|}{|\lambda^{n+1}|} \leq \|a^*\| \sum_{n=0}^{\infty} \frac{||a^n||}{|\lambda^{n+1}|}$$

Where $\|a^*\|$ is the operator norm of a^*. Since $a^* R_a(\lambda)$ is analytic on $\rho(a)$, the above Laurent series converges for $|\lambda| > r(a)$ as well.

Now let $|\lambda| > r(a)$. The left hand side of the above series converges absolutely, so $\frac{a^*(a^n)}{\lambda^{n+1}}$ converges to zero. As $\frac{a^n}{\lambda^{n+1}}$ is weakly bounded, it is bounded uniformly, say by M. Then, $\limsup ||a^n|| \leq \limsup M|\lambda^{n+1}|$, or equivalently, $\limsup ||a^n||^{1/n} \leq \limsup M^{1/n}|\lambda|^{\frac{n+1}{n}} = |\lambda|$. It follows that $\limsup ||a^n||^{1/n} \leq r(a)$ which together with the previous work gives that $r(a) = \lim\limits_{n\to\infty} ||a^n||^{1/n}$. □

3.2 Vector-valued Integration

In this subsection we give the formalization of an integral for, in particular, a continuous function over a compact interval, and then a closed contour in the complex plane, into a Banach space X. Our main goal in this subsection is defining contour integrals and the extension of Cauchy's Integral Formula for a X-valued function, though the general construction here will be of use in later sections as well.

Definition 3.2 (Step-Valued Functions)**.** *We say that a function $f : [a, b] \to X$ is a step function if there exists a partition $P = \{a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n = b\}$ such that there is a unique $c_k \in X$ where for $\alpha_{k-1} \leq t_k \leq \alpha_k$*

$$f(t_k) = c_k$$

We denote $S([a, b], X)$ or S shorthand as the set of X-valued step functions over $[a, b]$, which is a linear subspace in $\ell^\infty([a, b], X)$. Indeed, for $f, g \in S$ with n and m distinct values on $[a, b]$ there are at most $nm + n + m$ distinct values for $f + g$ which it takes on compact subintervals of $[a, b]$, hence it is step-valued. It is also clear that for $\lambda \in \mathbb{C}$, λf is also step-valued and that $0 \in S$.

Definition 3.3. *Let $P = \{\alpha_i\}_{i=0,\cdots n}$ be a partition of the interval $[a, b]$. We say that P is an admissable partition for $f \in S([a, b], X)$ if for $k \in \{0, \cdots, n\}$, $f(t_k) = c_k$ for any $t_k \in (\alpha_{k-1}, \alpha_k)$.*

Definition 3.4 (Step-Valued Integration)**.** *Let $f \in S([a, b], X)$. Let $P = \{\alpha_i\}_{i=0,\ldots,n}$ be an admissable partition of f. Then, we define;*

$$\int_a^b f = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) c_i$$

It's easy to see that the above definition is independent of the chosen partition. Note also that $|| \int_a^b f|| \leq (b - a)||f||_\infty$. Hence the function $\psi : S \to X$ given by

$$\psi : f \to \int_a^b f$$

is in $\mathcal{L}(S, X)$. We can thus extend ψ to $\bar{S}$ in $\ell^\infty([a, b], X)$, which we now show contains $C([a, b], X)$.

Proposition 3.4. *$C([a, b], X)$ is approximated by $S([a, b], X)$.*

Proof. Take $f \in C([a, b], X)$. Since f is uniformly continuous on $[a, b]$, for $\epsilon > 0$ find $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. For a partition P let $||P|| = \max\{(\alpha_k - \alpha_{k-1}\}$. Set P to be a partition of $[a, b]$ such that $||P|| < \delta$. For each (α_{k-1}, α_k) find one value $f(t_k)$ where $t_k \in (\alpha_{k-1}, \alpha_k)$. Define a function $g \in S([a, b], X)$ by $g(x) = f(t_k)$ when $x \in (\alpha_{k-1}, \alpha_k)$. Then,

$$\begin{aligned} ||f - g||_\infty &= \sup |f(t) - g(t)| \\ &= \max_{k=1\ldots n} \{\sup\{|f(t) - f(t_k)| \ ; \ t \in [\alpha_{k-1}, \alpha_k]\}\} \\ &< \epsilon \end{aligned}$$

□

Thus we can make sense of the integration of any continuous functions $f : [a,b] \to X$. We will also make note that piecewise continuous functions may be approximated by S by approximating each piece and then concatenating them into one step-valued function.

Proposition 3.5. *Let $f : [a,b] \to X$ be continuous and Y a Banach Space and $A \in \mathcal{L}(X,Y)$. Then,*

$$A \int_a^b f(t)dt = \int_a^b Af(t)dt$$

Proof. Let $g_n \in S([a,b],X)$ be such that $g_n \to f$. By linearity of A it's easy to see that

$$A \int_a^b g_n(t)dt = \int_a^b A(g_n(t))dt$$

Note first that by our construction of integrating over f and the continuity of A we have;

$$\lim_{n\to\infty} A \int_a^b g_n(t)dt = A \lim_{n\to\infty} \int_a^b g_n(t)dt = A \int_a^b f(t)dt$$

Second, since $A(g_n(t))$ is a Y-valued step function with limit $A(f(t))$ we have by definition;

$$\lim_{n\to\infty} \int_a^b A(g_n(t))dt = \int_a^b A(f(t))dt$$

By uniqueness of limits we have the result.

$\square$

Theorem 3.4 (The Fundamental Theorem of Calculus)**.** *Let $f : [a,b] \to X$ be continuous. Define $F : [a,b] \to X$ by;*

$$F(x) = \int_a^x f(t)$$

Then F is contiuously differentiable with $F'(t) = f(t)$. Moreover, if f is contiuously differentiable then;

$$\int_a^b f'(t) = f(b) - f(a)$$

Proof. For the first result consider for any $x \in [a,b]$

$$\begin{aligned}
\left\|(F(x+h) - F(x))h^{-1} - f(x))\right\| &= \left\| \left(\int_x^{x+h} f(t)dt \right) h^{-1} - \left(\int_x^{x+h} f(x)dt \right) h^{-1} \right\| \\
&= h^{-1} \left\| \int_x^{x+h} f(t) - f(x)dt \right\| \\
&\leq h^{-1} \int_x^{x+h} \|f(t) - f(x)\| \, dt
\end{aligned}$$

By the continuity of $f(x) - f(t)$, $\|f(x) - f(t)\|$ is a continuous real-valued function on $[a, b]$. By the usual Fundamental Theorem of Calculus we have that the limit goes to $||f(x) - f(x)|| = 0$ as required. For the second result let $x^* \in X^*$. Then if f is continuously differentiable we have that $(x^* f)' = x^* f'$ is also continuously differentiable. Then;

$$x^* \int_a^b f' = \int_a^b x^* f' = x^*(f(b) - f(a))$$

By the usual Fundamental Theorem of Calculus. Hahn-Banach provides the desired equality. $\square$

Definition 3.5. *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve and X a Banach Space. Let $\gamma[a, b] \subset U \subset \mathbb{C}$ be open and take $f : U \to X$. We define the contour integral of f over γ to be;*

$$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)dt$$

Moreover, if $\Gamma = \bigcup_{i=1}^{n} \gamma_i$ where γ_i are contours, then we write;

$$\int_\Gamma f = \sum_{i=1}^{n} \int_{\gamma_i} f$$

We will quickly show that this definition is independent of parameterization. The reader may correctly expect this to be an application of Hahn-Banach. Let $x^* \in X^*$. If f is integrable we have that for contours γ and β parameterized over $[a, b]$ and $[c, d]$ respectively with $\mathrm{Im}(\gamma) = \mathrm{Im}(\beta)$, then

$$\begin{aligned} x^* \int_c^d f(\beta(t))\beta'(t)dt &= \int_c^d x^* f(\beta(t))\beta'(t)dt \\ &= \int_a^b x^* f(\gamma(t))\gamma'(t)dt \\ &= x^* \int_a^b f(\gamma(t))\gamma'(t)dt \end{aligned}$$

Indeed, $\int_\gamma f$ is independent of parametrization by Hahn-Banach.

We extended the Fundamental Theorem of Calculus to this context already and now that we have a notion of a contour integral we turn to the main goal of this subsection; the generalization of Cauchy's Integral Formula and Cauchy's Theorem.

Theorem 3.5 (Cauchy's Theorem)**.** *Let D be an open subset of $\mathbb{C}$ and $f : D \to X$ be holomorphic where X is a Banach Space. Then, for any simple closed contour $\gamma \in D$;*

$$\int_\gamma f = 0$$

In addition, for any z_0 in the interior of γ;

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

Proof. For $x^* \in X^*$, $x^* \circ f$ is holomorphic on D. By Cauchy's Theorem in $\mathbb{C}$,

$$\int_\gamma x^* \circ f(z) dz = x^* \int_\gamma f(z) dz = 0$$

Hence $\int_\gamma f(z)dz = 0$. Similarly for the second result,

$$x^* \int_\gamma \frac{f(z)}{z - z_0} dz = \int_\gamma x^* \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{x^* f(z)}{z - z_0} dz$$

Since $x^*(f(z))$ is holomorphic, the right hand side is equal to $2\pi i(x^*(f(z_0)))$ by Cauchy's Integral Formula. Hahn-Banach gives then that $f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$ as required. □

3.3 Functional Calculus

In the previous subsections we looked at the holomorphicity of the resolvent and now have a sensible definition of an integral for X-valued functions and so we have everything we need to insert an element of a Banach algebra into a holomorphic function.

We now give the formal expression of functional calculus as an algebra homomorphism between $H(U)$ (functions holmorphic on U) and $\mathcal{A}$ and prove that it is well defined. To finish off the section, we will also give comforting results on the behaviour of the functional calculus in the sense of uniform convergence and power series as well as the Spectral Mapping Theorem.

Definition 3.6. *Take $a \in \mathcal{A}$, a unital Banach Algebra. Fix $U \subset \mathbb{C}$, an open set containing $\sigma(a)$. Let γ be a closed contour in U such that $\forall \alpha \in \sigma(a)$, $I(\gamma; \alpha) = 1$ (the winding number of γ). For $f : U \to \mathbb{C}$ holomorphic we define;*

$$f(a) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)}$$

In the subsection on Spectral Theory we showed that $R_a(z)$ is holomorphic and thus continuous on the resolvent set of a so this integral does indeed converge by what we covered in 3.2.

What's left to check is that this definition is equivalent for all contours satisfying the conditions in the definition. We show this in the next proposition, as well as the homomorphism properties of the functional calculus.

Proposition 3.6. *With the same definition as above, $f(a)$ is well-defined and the map $\phi : H(U) \to \mathcal{A}$ is an algebra homomorphism, where $H(U)$ is the algebra of complex functions holomorphic on U.*

Proof. To show that $f(a)$ is well-defined we must show that it is independent of the contour in U. Let γ_1 and γ_2 be two closed contours satisfying the requirements of the above definition. Write

$$b = \int_{\gamma_1} \frac{f(z)}{z-a}dz - \int_{\gamma_2} \frac{f(z)}{z-a}dz = \int\limits_{\Gamma} \frac{f(z)}{z-a}dz$$

Where $\Gamma = \gamma_1 - \gamma_2$. We aim to show that for any $a^* \in \mathcal{A}^*$, $a^*(b) = 0$, giving us the result by Hahn-Banach.

Note first that for $\lambda \notin U$, $I(\Gamma;\lambda) = 0$ as it is zero for both γ_1 and γ_2. Moreover, for $\lambda \in \sigma(a)$ we have that $I(\Gamma;\lambda) = I(\gamma_1;\lambda) - I(\gamma_2;\lambda) = 1 - 1 = 0$. Since $a^* \frac{f(z)}{(z-a)}$ is holomorphic on $\rho(a) \cap U$ and elements of $\sigma(a)$ do not contribute to integrals over Γ, it follows that $a^*(b) = 0$.

It remains to show that ϕ is an algebra homomorphism. The linearity of ϕ is clear based on the linearity of integration. To show that ϕ is multiplicative, let f, g be holomorphic on U. Construct two contours γ_1, and γ_2 in U such that γ_2 is in the interior of γ_1. Since fg is holmomorphic on U, $fg(a)$ is well defined. Consider;

$$\begin{aligned}
f(a)g(a) &= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \frac{f(z)}{z-a}dz \int_{\gamma_2} \frac{g(w)}{w-a}dw \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma_1}\int_{\gamma_2} f(z)g(w)(z-a)^{-1}(w-a)^{-1}dzdw \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma_1}\int_{\gamma_2} \frac{f(z)g(w)}{w-z}((z-a)^{-1} - (w-a)^{-1})dzdw \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \frac{f(z)}{z-a} \int_{\gamma_2} \frac{g(w)}{w-z}dwdz \quad - \quad \frac{1}{(2\pi i)^2} \int_{\gamma_2} \frac{g(w)}{w-a} \int_{\gamma_1} \frac{f(z)}{w-z}dzdw
\end{aligned}$$

In the last line, the first integral vanishes by Cauchy's Theorem because z is outside of γ_2 so $\frac{g(w)}{w-z}$ is holomorphhic on the inside of γ_2. On the other hand, for the integral on the right since w is inside of γ_1 Cauchy's Integral Formula gives;

$$\begin{aligned}
f(a)g(a) &= \frac{1}{(2\pi i)^2} \int_{\gamma_2} \frac{g(w)}{w-a} \int_{\gamma_1} \frac{f(z)}{z-w}dzdw \\
&= \frac{1}{2\pi i} \int_{\gamma_2} \frac{g(w)f(w)}{w-a}dw = fg(a)
\end{aligned}$$

□

Lemma 3.2. *Let f_n be a sequence of functions holomorphic on an open neighborhood U of $\sigma(a)$ converging uniformly to f on compact subsets of U. Then f is also holomorphic on U and*

$$\lim_{n\to\infty} ||f_n(a) - f(a)|| = 0$$

Proof. Let γ be a closed contour in U. Since $f_n \to f$ uniformly on compact subsets of U, f is holomorphic on U by the analytic convergence theorem [7]. By assumption, $f_n \to f$ uniformly on γ hence;

$$\begin{aligned}
||f_n(a) - f(a)|| &= \frac{1}{2\pi}|| \int_\gamma (f_n(z) - f(z))(z-a)^{-1}|| \\
&\leq \frac{1}{2\pi} \int_\gamma |f_n(z) - f(z)| \cdot ||R_a(z)|| \\
&\leq \frac{1}{2\pi} ||R_a||_\gamma \cdot ||f - f_n||_\gamma \cdot \ell(\gamma)
\end{aligned}$$

Where $||\cdot||_\gamma$ is the supremum norm on γ and $\ell(\gamma)$ is the length of γ. Everything here is uniform in n except for $||f - f_n||_\gamma$, which goes to zero. ☐

Theorem 3.6 (Power Series)**.** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. If $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ converges uniformly and is holomorphic on a neighbourhood of $\sigma(a)$, then $f(a) = \sum_{n=0}^{\infty} c_n a^n$.*

Proof. Find $r > 0$ and $\epsilon > 0$ such that $\sigma(a) \subset B_r(0)$ and $r(a) < \epsilon < r$. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = \epsilon e^{it}$. Then;

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i} \int_\gamma \left(\sum_{n=0}^{\infty} c_n z^n \right) (z-a)^{-1} \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} c_n \int_\gamma z^n (z-a)^{-1} \\
&= \sum_{n=0}^{\infty} c_n z^n(a)
\end{aligned}$$

Note that $z^n(a) = (z(a))^n$ by algebra homomorphism properties and that

$$
\begin{aligned}
z(a) &= \frac{1}{2\pi i}\int_\gamma z(z-a)^{-1} \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} a^n z^{-n} \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty} a^n \int_\gamma z^{-n} \\
&= \sum_{n=0}^{\infty} a^n \mathrm{Res}(z^{-n}, 0) \\
&= a
\end{aligned}
$$

Hence, $z^n(a) = a^n$ and so $f(a) = \sum_{n=0}^{\infty} c_n a^n$. $\square$

Theorem 3.7 (Spectral Mapping Theorem)**.** *For $a \in \mathcal{A}$ and f holomorphic on a neighbourhood of $\sigma(a)$*

$$\sigma(f(a)) = f(\sigma(a))$$

Proof. If $\lambda \notin f(\sigma(a))$, then $g(z) = (\lambda - f(z))^{-1}$ is holomorphic on any open neighbourhood of $\sigma(a)$. Thus we can write,

$$g(a)(\lambda - f(a)) = (g(z)(\lambda - f(z)))(a) = 1(a) = 1_{\mathcal{A}}$$

And symmetrically with the order of g and $\lambda - f$ reversed. Then $g(a)$ is an inverse for $\lambda - f(a)$, so $\lambda \notin \sigma(f(a))$. Now if $\lambda \in f(\sigma(a))$, then $\lambda - f(z)$ has a root on $\sigma(a)$ at z_0 and we can write $\lambda - f(z) = h(z)(z_0 - z)$ where h is holomorphic on the neighbourhood f is. By homomorphism properties we have that $\lambda - f(a) = h(a)(z_0 - a) = (z_0 - a)h(a)$. Since $(z_0 - a)$ is not invertible, neither then is $\lambda - f(a)$, so $\lambda \in \sigma(f(a))$. $\square$

4 C_0 Semigroups

So far we have only seen what C_0 semigroups are representative of from a physical perspective. Now we will define what they are and give their basic properties. In the previous section when we constructed the holomorphic functional calculus we did so in the context of a general Banach algebra. Now that we are fixed on semigroups, we will be working in the case where $\mathcal{A} = \mathcal{L}(X)$.

Moreover, we will move outside the Banach algebra when we talk about generators of semigroups that are possibly unbounded. After what is provided in this section, we will construct a functional calculus for such an operator which we use to prove Hille-Yosida. Towards this goal, the main result of this section is the Laplace transform representation of the generator of a C_0 semigroup.

But without further ado; what is a C_0 semigroup?

Definition 4.1 (C_0 semigroup)**.** *A strongly continuous one-parameter semigroup, also known as a C_0 semigroup, is a family $T(t)$ of bounded operators in $\mathcal{L}(X)$ satisfying, for all $t, s \geq 0$*

$$i)\ T(t+s) = T(t)T(s)$$

$$ii)\ T(0) = 1_X$$

As well as the strong continuity property which is the continuity of the orbit maps over X; $\xi_x : \mathbb{R}^+ \to X$ given by

$$\xi_x(t) = T(t)x$$

If all the above holds but over the whole of $\mathbb{R}$, then T is a C_0 group.

The reader may think of this as a generalization of the exponential function. Indeed, now that we have functional calculus the operator e^{tA} forms a C_0 semigroup (or a C_0 group) where $A \in \mathcal{L}(\mathcal{H})$ as in Stone's Theorem. It can also be shown that so long as $T(t)$ is continuous in operator norm, then there is an operator $A \in \mathcal{L}(X)$ such that $T(t) = e^{tA}$. We are concerned with the case where the semigroup is merely strongly continuous and we collect some properties in this setting.

Proposition 4.1. *Let $T(t)$ be a C_0 semigroup. Then there exists $M, \alpha \in \mathbb{R}$ such that for any $t \geq 0$*

$$||T(t)|| \leq Me^{\alpha t}$$

Proof. For $s \in [0,1]$ note that $\xi_x(s) = T(s)x$ attains its supremum on $[0,1]$ by the extreme value theorem. Thus by uniform boundedness there exists an $M \geq 1$ such for any $s \in [0,1]$, $||T(s)|| \leq M$. For $t \geq 0$, find $s \in [0,1]$ and $n \in \mathbb{N}_{\not\vdash}$ such that $t = s + n$. Then,

$$\begin{aligned}
||T(t)|| &\leq ||T(s)|| \cdot ||T(n)|| \\
&\leq ||T(s)|| \cdot ||T(1)||^n \\
&\leq M^{n+1} \\
&= Me^{n \log(M)} \\
&\leq Me^{t \log(M)}
\end{aligned}$$

Thus we take $\alpha = \log(M)$. □

Definition 4.2 (Generators)**.** *We say that an operator A on X is the generator of a C_0 semigroup $T(t)$ where $Ax = \xi_x'(0)$ defined for $x \in X$ such that ξ_x is right differentiable at 0.*

We note that ξ_x is right differentiable at 0 if and only if ξ_x is differentiable on $\mathbb{R}^+$. Indeed;

$$\begin{aligned}\xi_x'(t) &= \lim_{h\to 0} \frac{T(t+h)x - T(t)x}{h} \\ &= T(t) \lim_{h\to 0} \frac{T(h)x - 1_X x}{h} \\ &= T(t)\xi_x'(0)\end{aligned}$$

Proposition 4.2. *(Generator Properties) Let A be the generator of a C_0 semigroup $T(t)$. Then;*

i) $A : \mathcal{D}(A) \to X$ is a linear operator

ii) If $x \in \mathcal{D}(A)$, then $T(t)x \in \mathcal{D}(A)$ for all $t \geq 0$ and

$$T'(t)x = T(t)Ax = AT(t)x$$

iii) For any $t \geq 0$ and $x \in X$,

$$\int_0^t T(s)x \; ds \in \mathcal{D}(A)$$

iv) For any $t \geq 0$

$$T(t)x - x = A \int_0^t T(s)x \; ds$$

and if $x \in \mathcal{D}(A)$ we have that

$$T(t)x - x = \int_0^t T(s)Ax \; ds$$

Proof. i): The linearity of A is clear in the definition of Ax given that $T(t)$ is linear and continuous.

ii): Observe that since $T(t)$ is continuous that

$$\begin{aligned}T(t)Ax &= T(t)\left(\lim_{h\to 0} \frac{T(h)x - x}{h}\right) \\ &= \lim_{h\to 0} \frac{T(t)T(h)x - T(t)x}{h} \\ &= AT(t)x\end{aligned}$$

Hence, $T(t)x \in \mathcal{D}(A)$ by the existence of the limit. Together with previous work we have that $T'(t)x = \xi_x'(t) = T(t)\xi_x'(0) = T(t)Ax = AT(t)x$.

iii): For any $t \geq 0$ and $x \in X$ we have;

$$\begin{aligned}
\lim_{h\to 0}\left(T(h)\int_0^t T(s)xds - \int_0^t T(s)xds\right)h^{-1} &= \\
\lim_{h\to 0}\frac{1}{h}\int_0^t T(s+h)xds - \frac{1}{h}\int_0^t T(s)xds &= \\
\lim_{h\to 0}\frac{1}{h}\int_h^{t+h} T(s)xds - \frac{1}{h}\int_0^t T(s)xds &= \\
\lim_{h\to 0}\frac{1}{h}\left(\int_t^{t+h} T(s)xds + \int_h^t T(s)xds - \int_0^t T(s)xds\right) &= \\
\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds &
\end{aligned}$$

And by the fundamental theorem of calculus, the first term converges to $T(t)x$ and the second to $T(0)x = x$. Thus, the integral is in $\mathcal{D}(A)$ by definition.

iv): In the previous part we saw that the limit converges to $T(t)x - x$, which is then A applied to the integral by definition. In a previous proof we showed that $T(s)$ is bounded uniformly on $[0,t]$, say by M. We have that for $x \in \mathcal{D}(A)$

$$\begin{aligned}
\left\|T(s)\frac{T(h)x - x}{h} - T(s)Ax\right\| &\leq ||T(s)|| \left\|\frac{T(h)x - x}{h} - Ax\right\| \\
&\leq M\left\|\frac{T(h)x - x}{h} - Ax\right\|
\end{aligned}$$

Hence we have convergence of the left limit uniform for $s \in [0,t]$. As such;

$$\begin{aligned}
\lim_{h\to 0}\frac{T(h) - 1_X}{h}\int_0^t T(s)xds &= \lim_{h\to 0}\int_0^t T(s)\left(\frac{T(h) - 1_X}{h}\right)xds \\
&= \int_0^t T(s)Axds
\end{aligned}$$

□

Theorem 4.1. *Let A be a generator of a C_0 semigroup $T(t)$. Then A is closed, densely defined and determines T uniquely.*

Proof. We first show that A is closed. Let $x_n \in \mathcal{D}(A)$ converge to $x \in X$ and Ax_n converge to y. From the previous proposition we have that

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds$$

Since $\xi_y(s)$ is continuous, it is integrable over $[0,t]$. As before, let M be a uniform bound on $\|T(s)\|$ for $s \in [0,t]$ and consider the following esimate;

$$\left\| \int_0^t T(s)Ax_n ds - \int_0^t T(s)y \right\| ds = \left\| \int_0^t T(s)(Ax_n - y) \right\| ds$$
$$\leq \int_0^t \|T(s)(Ax_n - y)\| \, ds$$
$$\leq \int_0^t \|T(s)\| \, \|Ax_n - y\| \, ds$$
$$\leq M||Ax_n - y||t$$

Which goes to zero by our assumption. Observe also that $\lim_{n\to\infty}(T(t)x_n - x_n) = T(t)x - x$ since $T(t)$ is bounded. By unique of limits we conclude that

$$T(t)x - x = \int_0^t T(s)y$$

And so for $t > 0$ we have

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)y$$

Taking the limit above as $t \to 0$ gives on the left Ax by defintion and on the right $T(0)y = y$ by the Fundamental Theorem of Calculus.

We will now show that A is densely defined. Let $x \in X$. As we showed before, for any $t \geq 0$ the integral

$$\int_0^t T(s)x \in \mathcal{D}(A)$$

The Fundamental Theorem of Calculus gives us the following approximation in n;

$$n\int_0^{1/n} T(s)x \to T(0)x = x$$

Since the sequence is in $\mathcal{D}(A)$, we are done.

For uniqueness, consider a second C_0 semigroup generated by A, $S(t)$. Fix $x \in \mathcal{D}(A)$ and $t \geq 0$. Write for fixed $x \in X$ and t the function $\psi_{x,t} : [0,t] \ni s \to T(t-s)S(s)x \in X$. Consider the derivative of the above function;

$$\frac{1}{h}(\psi_{t,x}(s+h) - \psi_{t,x}(s)) = \frac{1}{h}\left(T(t-s-h)S(s+h)x - T(t-s)S(s)x\right)$$
$$= [T(t-s-h)\frac{1}{h}(S(s+h)x - S(s)x)] + \frac{1}{h}[(T(t-s-h) - T(t-s))S(s)x]$$

By the previous theorem, $S(s)x \in \mathcal{D}(A)$. Since the generator of $T(t-s)$ is $-A$ the second part of the above goes to $-AT(t-s)S(s)x$. For the first part consider the following;

$$
\begin{aligned}
&\left\|T(t-s-h)\frac{1}{h}(S(s+h)x - S(s)x) - T(t-s)AS(s)x\right\| \\
&\leq \left\|T(t-s-h)\frac{1}{h}(S(s+h)x - S(s)x) - T(t-s-h)AS(s)x\right\| \\
&+ \|T(t-s-h)AS(s)x - T(t-s)AS(s)x\| \\
&\leq M\left\|\frac{1}{h}(S(s+h)x - S(s)x) - AS(s)x\right\| + \|(T(t-s-h) - T(t-s))(AS(s)x)\|
\end{aligned}
$$

Here the M comes the uniform boundedness of $T(t-s-h)$. Taking $h \to 0$ both parts go to zero by definition of $AS(s)x$ and strong continuity of T respectively. Thus, the first part of the original estimate converges to $T(t-s)AS(s)x$ as $h \to 0$. Since A commutes with T, the derivative of $\psi_{t,x}$ is constantly zero and $\psi_{t,x}$ is constant. Hence

$$\psi_{t,x}(0) = T(t)x = \psi_{t,x}(t) = S(t)x$$

as required. □

We end this section with a theorem connecting the resolvent of the generator of a C_0 semigroup with its Laplace transform. This result will allow us to connect the general semigroup theory outlined here with the exponential applied to an unbounded operator via the uniqueness of the Laplace transform. One can see [1] for more details on operator valued Laplace transforms. In particular see Theorem 1.7.3.

Theorem 4.2. *Let $T(t)$ be a strongly continuous semigroup with generator A. If $\lambda \in \mathbb{C}$ is such that*

$$R(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)xdt$$

exists for all $x \in X$, then $\lambda \in \rho(A)$ and $R(\lambda) = R_A(\lambda)$.

Proof. First we consider the case where $\lambda = 0$. Here we have;

$$
\begin{aligned}
\frac{T(h) - 1_X}{h}R(0)x &= \frac{T(h) - 1_X}{h}\int_0^\infty T(t)xdt \\
&= h^{-1}\int_0^\infty T(t+h)xdt - h^{-1}\int_0^\infty T(t)xdt \\
&= h^{-1}\int_h^\infty T(t)xdt - h^{-1}\int_0^\infty T(t)xdt \\
&= -h^{-1}\int_0^h T(t)xdt
\end{aligned}
$$

Taking $h \to 0$ we get $-x$. Hence $R(0)x \in \mathcal{D}(A)$ and $AR(0)x = -x$ by definition. We also of course have

$$\lim_{t\to\infty} \int_0^t T(s)xds = R(0)x$$

and for $x \in \mathcal{D}(A)$

$$\lim_{t\to\infty} A \int_0^t T(s)xds = \lim_{t\to\infty} \int_0^t T(s)Axds = R(0)Ax$$

Which, since A is closed gives us that $AR(0)x = R(0)Ax = -x$. That is, $R(0) = (-A)^{-1}$. Now consider the C_0 semigroup $S(t) = e^{-\lambda t}T(t)$ and x such that the following limit exists;

$$\begin{aligned}
\lim_{h\to 0}\left(\frac{S(h)x - x}{h}\right) &= \lim_{h\to 0}\left(\frac{S(h)x - T(h)x + T(h)x - x}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{e^{-\lambda h} - 1}{h}\right)\lim_{h\to 0}(T(h)x) + \lim_{h\to 0}\left(\frac{T(h)x - x}{h}\right) \\
&= -\lambda x + Ax
\end{aligned}$$

Hence, by definition the generator of S is $-\lambda 1_X + A$. Thus, if we apply the previous work for $\lambda = 0$ but for the Laplace transform of S we get;

$$R(\lambda)x = \int_0^\infty S(t)xdt = (\lambda 1_X - A)^{-1}x = R_A(\lambda)x$$

as required. $\square$

5 Functional Calculus for Unbounded Operators

In the previous section we outlined a procedure for taking bounded operator-valued arguments for holomorphic functions. However, generators of semigroups need not be bounded so for our approach we need a definition which relaxes the structure of the input. This requires us to be more specific with the types of functions we consider.

We consider a particular holomorphic functional falculus on an algebra of at least square decaying functions on a half plane, which we show allows for the input of so-called half plane type operators. As metioned in the introduction we seek a definition of e^{tA}, so we require some sort of extension of the functional calculus to bounded holomorphic functions. We touch on this procedure in the abstract for readability in the first subsection. Then. we introduce operators of half-plane type in the second subsection before constructing their functional calculus and giving the preliminaries for the proof of Hille-Yosida.

5.1 Abstract Framework

Setting the table, let X be a Banach Space, $\mathcal{M}$ a commutative, unital algebra and $\mathcal{E}$ a subalgebra of M, not necessarily containing the unit. Assume also that there is an algebra homomorphism $\Phi : \mathcal{E} \to \mathcal{L}(X)$. In this context we call the tuple $(\mathcal{E}, \mathcal{M}, \Phi)$ an abstract functional calculus.

Definition 5.1. *We say that $(\mathcal{E}, \mathcal{M}, \Phi)$ is a proper abstract functional calculus if*

$$Reg(\mathcal{E}) := \{e \in \mathcal{E} \ ; \ \Phi(e) \text{ is injective}\}$$

is non empty. We call elements of $Reg(\mathcal{E})$ regularisers. If $f \in \mathcal{M}$ has an element $e \in Reg(\mathcal{E})$ such that $ef \in \mathcal{E}$ we say that f is regularisable by $\mathcal{E}$ and e a regulariser of f. Last, we denote $\mathcal{M}_r$ to be the set of regularisable elements of $\mathcal{M}$.

Note that $\mathcal{M}_r$ is unital if and only if the abstract functional calculus is proper. Indeed, 1 being regularisable implies the non-emptiness of $Reg(\mathcal{E})$ and conversely if $Reg(\mathcal{E})$ is non empty then one element's product with the unit is just that element. Moreover, $\mathcal{M}_r$ is a subalgebra of $\mathcal{M}$ containing $\mathcal{E}$.

In the particular case to come in the next subsection, $\mathcal{E}$ will be a set of functions with a decay property required for the convergence of an integral on a vertical line, for which we will show we may insert an unbounded operator.

Holomorphic functions can then be regularized by taking products with decaying functions so that their product with f is in $\mathcal{E}$. We will then apply the functional calculus to the holomorphic function in the following way.

Proposition 5.1. *For $f \in \mathcal{M}_r$ with regulariser e the extension of Φ;*

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef)$$

is a well-defined, closed operator.

For this proof, any $e \in \mathcal{E}$ we alternatively write $\Phi(e) = e_\bullet$.

Proof. Note first that since $\Phi(ef)$ is bounded and $\Phi(e)^{-1}$ is the left inverse of a bounded operator, hence closed (inversion swaps the order of the graph's direct sum homeomorphically, so it remains closed). It follows that $\Phi(f)$ is closed as the composition of closed operators.

Let e, h be regularisers for $f \in \mathcal{M}_r$. Then by homomorphism properties and commutativity we have that

$$e_\bullet h_\bullet = h_\bullet e_\bullet$$

Which gives that $(e_\bullet h_\bullet)^{-1} = h_\bullet^{-1} e_\bullet^{-1} = e_\bullet^{-1} h_\bullet^{-1}$. Hence, $\Phi(f)$ is independent of the regulariser;

$$e_\bullet^{-1}(ef)_\bullet = e_\bullet^{-1} h_\bullet^{-1} h_\bullet (ef)_\bullet = e_\bullet^{-1} h_\bullet^{-1} e_\bullet (hf)_\bullet = h_\bullet^{-1}(hf)_\bullet$$

Since the abstract functional calculus is proper, $\mathcal{E} \subset \mathcal{M}_r$. Take $e \in Reg(\mathcal{E})$ and $f \in \mathcal{E}$. Then,

$$\Phi(f) = e_\bullet^{-1}(ef)_\bullet = e_\bullet^{-1} e_\bullet f_\bullet = f_\bullet$$

So Φ is indeed an extension of the original homomorphism over $\mathcal{E}$. □

5.2 Functional Calculus for Operators of Half Plane Type

The above can be used to define functional calculus for various classes of operators, but we will now focus on the type of operators which generate C_0 semigroups, so-called operators of half-plane type.

We introduce a functional calculus analogous to the holomorphic functional calculus which can be applied to functions with certain asymptotic properties. Then, we use the extension procedure to define e^{tA}.

As mentioned in section 4, we then connect the family of operators e^{tA} with a semigroup generated by A using the uniqueness of Laplace transforms. In the final section we will characterize when half-plane operators generate semigroups and then give a lemma for the final proof of Hille-Yosida.

Definition 5.2. *Let $\omega \in \mathbb{R}$ and $L_\omega := \{z \in \mathbb{C}\ ;\ Re(z) < \omega\}$, $R_\omega := \{z \in \mathbb{C}\ ;\ Re(z) > \omega\}$ denote the left and right half planes at ω. We say an operator A on X is half-plane type ω if $R_\omega \subset \rho(A)$ and*

$$M_\alpha(A) := \sup\{||R_A(z)||\ ;\ Re(z) \geq \alpha\} < \infty$$

For all $\alpha > \omega$. In that case we write $s_0(A) := \min\{\omega\ ;\ A$ is of half-plane type $\omega\}$.

For an operator of half-plane type ω A, $s_0(A)$ exists. Let $\beta := \min\{Re(z)\ ;\ z \in \sigma(A)\}$. Then, $R_\beta \subset \rho(A)$ but for $\beta - \epsilon$ this does not hold, thus $\beta \leq \omega$. For $\alpha > \beta$ the resolvant has a norm maximum on $[\alpha, \omega]$ and since $M_\alpha(A)$ exists for all $\alpha > \omega$, it then exists for all $\alpha > \beta$.

For $\omega \in \mathbb{R}$ we denote $\mathcal{E}(L_\omega) := \{f \in H(L_\omega) \quad | \ \exists M, n \in \mathbb{N}\ ;\ |f(z)| \leq \frac{M}{|z|^{1+n}}$ as $z \in L_\omega$ goes to $\infty\}$ where $H(L_\omega)$ is the holomorphic functions on L_ω. Our goal now is to use the asymptotic behaviour on this set to construct a functional calculus for an operator of half-plane type from $\mathcal{E}(L_\omega)$ to $\mathcal{L}(X)$.

For the remainder of this section we will work under the assumption that A is of half-plane type with $s_0(A) \leq 0$, as all results are invariant under translation.

Proposition 5.2. *Let $f \in \mathcal{E}(L_\omega)$ where $\omega \in (0,1)$. For $\delta \in (0, \omega)$ write $V_\delta := \{z \in \mathbb{C}\ ;\ Re(z) = \delta\}$, as the vertical line in $\mathbb{C}$ at δ. Then for $Re(z_0) \leq 0$*

$$f(z_0) = \frac{1}{2\pi i} \int_{V_\delta} \frac{f(z)}{z - z_0} dz.$$

Moreover, if A is an operator of half-plane type with $s_0(A) \leq 0$, then the following integral converges absolutely.

$$f(A) := \frac{1}{2\pi i} \int_{V_\delta} f(z) R_A(z) dz.$$

Proof. Let $R > |\delta - z_0|$ and $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ be given by $\gamma_1(t) := t(\delta + iR) + (1-t)(\delta - iR)$, $\gamma_2 := \delta + Re^{i(\pi/2 + t\pi)}$. Write $\Gamma_R := \gamma_1 + \gamma_2$. Then, Γ_R is a closed half circle centered at δ of radius R, which has z_0 in its interior. Consider the integral over γ_2

$$
\begin{aligned}
\left|\int_{\gamma_2} \frac{f(z)}{z-z_0} dz\right| &\leq M \int_{\gamma_2} \frac{1}{|z-a||z|^{1+s}} dz \\
&\leq \frac{M}{R^{1+s}} \int_{\gamma_2} \frac{1}{|z-a|} dz \\
&\leq \frac{M}{N_R R^{1+s}} (\pi R)
\end{aligned}
$$

We note note here that the R^{1+s} and M come from the fact that $f \in \mathcal{E}(L_\omega)$ and that $|z| = \sqrt{R^2+\delta^2} \geq R$ on γ_2, while N_R comes from the continuity of $1/|z-a|$ on γ_2 and the fact that it is compact. Lastly πR is the length of γ_2.

Given the above, if we take $R \to \infty$ we have that the integral over γ_2 goes to zero (N_R goes to zero as $R \to \infty$ as well). Since taking $R \to \infty$ takes the integral over γ_1 to the integral over V_δ, we have by Cauchy's Integral Formula;

$$
\begin{aligned}
\lim_{R\to\infty} f(z_0) &= \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\Gamma_R} \frac{f(z)}{z-z_0} dz \\
&= \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma_1} \frac{f(z)}{z-z_0} dz \\
&= \frac{1}{2\pi i} \int_{V_\delta} \frac{f(z)}{z-z_0} dz
\end{aligned}
$$

And so we have the first equality. The second follows from the bound on the resolvent, which we denote α. Thus

$$
||f(A)|| = \frac{1}{2\pi} \left\| \int_{V_\delta} f(z) R_A(z) dz \right\| \leq \frac{\alpha}{2\pi} \int_{V_\delta} |f(z)| dz
$$

The right hand side converges by the same argument as before.

□

Proposition 5.3. *Let A be an operator of half plane type with $s_0(A) \leq 0$. For $f \in \mathcal{E}(L_\omega)$ where $\omega \in (0,1)$. Then if we write $\Phi(f) := f(A)$ we have*

(i) $\Phi : \mathcal{E}(L_\omega) \to \mathcal{L}(X)$ is a well defined algebra homomorphism.

(ii) If $T \in \mathcal{L}(X)$ commutes with A, then it commutes with $f(A)$.

Proof. (i) In the proof of 5.2, the argument follows irrespective of the δ chosen. The rest of this is analogous to the holomorphic functional calculus proof.

(ii)

$$
\begin{aligned}
Tf(A) &= \frac{1}{2\pi} T \int_{V_\delta} f(z) R_A(z) dz \\
&= \frac{1}{2\pi} \int_{V_\delta} f(z) T R_A(z) dz \\
&= \frac{1}{2\pi} \int_{V_\delta} f(z) R_A(z) dz T
\end{aligned}
$$

□

Now that we have a functional calculus for $f \in \mathcal{E}(L_\omega)$, we want to extend this functional calculus to the exponential function by regularizing it. Indeed, for any bounded $f \in H^\infty(L_\omega)$, $f/(1-z)^2 \in \mathcal{E}(L_\omega)$. Since $R_A(1)^2$ is injective we may define $f(A)$ as outlined in section 5.1;

$$
f(A) := R_A(1)^2 \Phi\left(f(z)/(1-z)^2\right)
$$

In particular, since e^{tz} is bounded on left half-planes this gives us a definition of e^{tA} for an unbounded operator, A. With this in hand, we bridge what we have done here with the general C_0 semigroup theory via Laplace transforms.

Lemma 5.1 (Laplace Lemma)**.** *Let A be an operator of half-plane type with $s_0(A) \leq 0$. Then for $x \in \mathcal{D}(A^2)$ the function $\psi(t) : [0, \infty) \to X$ given by $\psi(t) = e^{-\omega t} e^{At} x$ is continuous and bounded. Moreover for $Re\lambda > \omega$ its Laplace transform is*

$$
\int_0^\infty e^{-\lambda t} e^{tA} x dt = R_A(\lambda) x
$$

Proof. Let $t_n \to t$ be a sequence converging in $[0, \infty)$. $e^{t_n z}$ converges to e^{tz} locally uniformly and since $\bigcup \{t_n\}_{n \in \mathbb{N}}$ is compact there exists a maximum of $\bigcup \{e^{t_n z}\}_{n \in \mathbb{N}}$, call it M. It follows that in norm;

$$
\begin{aligned}
||(e^{t_n A} - e^{tA})x|| &= \left\| \frac{e^{t_n z} - e^{tz}}{(1-z)^2}(A)(1-A)^2 x \right\| \\
&= \left\| \int_{V_\delta} \frac{e^{t_n z} - e^{tz}}{(1-z)^2} R_A(z)(1-A)^2 x dz \right\| \\
&\leq \int_{V_\delta} \frac{|e^{t_n z} - e^{tz}|}{|1-z|^2} \left\| R_A(z)(1-A)^2 x \right\| dz \\
&\leq \int_{V_\delta} \frac{2M}{|1-z|^2} \left\| R_A(z)(1-A)^2 x \right\| dz
\end{aligned}
$$

Since the integral at the end converges, by dominated convergence applied to the second last integral, the limit in n goes to zero and continuity over $\mathcal{D}(A^2)$ is satisfied. For the resolvent equality consider the following;

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t} e^{tA} x dt &= \int_0^\infty e^{-\lambda t} \int_{V_\delta} \frac{e^{tz}}{(1-z)^2} R_A(z)(1-A)^2 x dz dt \\
&= \int_{V_\delta} \int_0^\infty \frac{e^{-\lambda t} e^{tz}}{(1-z)^2} R_A(z)(1-A)^2 x dt dz \\
&= \int_{V_\delta} \frac{R_A(z)(1-A)^2}{(1-z)^2} \int_0^\infty e^{-\lambda t} e^{tz} dt x dz \\
&= \int_{V_\delta} \frac{R_A(z)(1-A)^2}{(1-z)^2(\lambda - z)} x dz \\
&= (1-A)^2 \int_{V_\delta} \frac{R_A(z)}{(1-z)^2(\lambda - z)} x dz \\
&= (1-A)^2 (1-A)^{-2} R_A(\lambda) x \\
&= R_A(\lambda) x
\end{aligned}
$$

Where the second line is by Fubini's Theorem and the second to last is by the functional calculus.

□

We will see in the final section how the Laplace lemma, together with the uniqueness of Laplace transforms allows us to connect this definition of e^{tA} with the semigroup theory outlined in section 4 . Note however that in this case the Laplace transform is equal to the resolvent for $x \in \mathcal{D}(A^2)$ as opposed to $\mathcal{D}(A)$. Crucially, $\mathcal{D}(A^2)$ is also dense in X.

Lemma 5.2. *Let A be a densely defined operator of half-plane type. Then $\mathcal{D}(A^2)$ is dense in X.*

Proof. Let $x \in \mathcal{D}(A)$ and $n \in \mathbb{N}$ large enough that $n \in \rho(A)$. We have that;

$$
\begin{aligned}
x + \frac{A}{n-A} x &= \frac{n-A}{n-A} x + \frac{A}{n-A} x \\
&= \left(\frac{n-A}{n-A} + \frac{A}{n-A} \right) x \\
&= R_A(n)(n - A + A) x \\
&= R_A(n) n x \\
&= n R_A(n) x
\end{aligned}
$$

That is, $n R_A(n) x = x + R_A(n) A x$. Taking $n \to \infty$ we have that $R_A(n) x \to 0$. For $x \in X$ we may find $x_0 \in \mathcal{D}(A)$ such that $\|R_A(n)(x - x_0)\| \leq \|x - x_0\| M < \epsilon/2$ where M is the uniform bound on the resolvent from the fact that A is half plane type. Then we have,

$$
\begin{aligned}
\|R_A(n)x\| &= \|R_A(n)x - R_A(n)x_0 + R_A(n)x_0\| \\
&\leq \|R_A(n)(x - x_0)\| + \|R_A(n)x_0\| \\
&< \epsilon/2 + \|R_A(n)x_0\|
\end{aligned}
$$

Note that the first norm being less that $\epsilon/2$ is independent of n. Taking $n \to \infty$ makes $\|R_A(n)x\| < \epsilon$.

Since A commutes with its resolvent we must have that $R_A(n)x \in \mathcal{D}(A)$. Then, we have that $n^2 R_A(n)x = nAR_A(n)x + nx$. We may apply A to this to get that $n^2 AR_A(n)x = nA^2 R_A(n)x + nAx$. Thus $nR_A(n)x \in \mathcal{D}(A^2)$. But we have that for any $x \in \mathcal{D}(A)$ that $nR_A(n)x = x + R_A(n)Ax$ which goes to x as n gets large since $R_A(n)Ax$ goes to 0 by the above. Hence, $\mathcal{D}(A)$ is in the closure of $\mathcal{D}(A^2)$ so by density of $\mathcal{D}(A)$, $\mathcal{D}(A^2)$ is dense. $\square$

6 Hille-Yosida Theorem

We now have all the background necessary to give necessary and sufficient conditions for an unbounded operator generating a C_0 semigroup via the Hille-Yosida Theorem.

Proposition 6.1. *Let A be an operator of half plane type with $s_0(A) \leq 0$. Then A is the generator of a C_0 semigroup if and only if A is densely defined and for $t \in [0, 1]$, e^{tA} is a bounded operator and uniformly bounded in norm. In this case, $T(t) = e^{tA}$.*

Proof. Let A generate a C_0 semigroup. By general properties, A is densely defined. Thus, $\mathcal{D}(A^2)$ is dense in X and $R_A(t)$ is the Laplace transform of e^{tA}. Hence by uniqueness of Laplace transform (see Theorem 1.7.3 in [1]) we have that $e^{tA} = T(t)$ on $\mathcal{D}(A^2)$. But, since e^{tA} is closed and $T(t)$ bounded we have for x_n in $\mathcal{D}(A^2)$ approximating x that $e^{tA}x_m = T(t)x_m \to T(t)x$ implies $e^{tA}x = T(t)x$. Thus, $e^{tA} = T(t) \in \mathcal{L}(X)$. General theory of C_0 semigroups gives that $T(t)$ or e^{tA} is uniformly bounded on compacts, particularly $[0, 1]$.

On the other hand, if A is densely defined and $T(t) := e^{tA}$, then $T(t)$ is a semigroup. Indeed we have that $e^{0A} = (1)(A) = (1-A)^2(1-A)^{-2} = I$ and by algebra homomorphism properties;

$$
\begin{aligned}
e^{(t+s)A} &= (1-A)^2(e^{tz}/(1-z)^2 e^{sz}/(1-z)^2)(A) \\
&= (1-A)^2(e^{tz}/(1-z)^2)(A)(1-A)^2(e^{sz}/(1-z)^2)(A) \\
&= e^{tA}e^{sA}
\end{aligned}
$$

Moreover, the Laplace lemma tells us that $T(t)$ is strongly continuous for $x \in \mathcal{D}(A^2)$. Thus by density it is strongly continuous on X and is uniformly bounded in operator norm on compact intervals (Proposition 4.1). Lastly, by the Laplace lemma, the transform of A agress with the resolvent of A on $\mathcal{D}(A^2)$ and thus on X by density so A generates T by general theory. $\square$

Lemma 6.1. *(Convergence Lemma) Let A be a densely defined operator of half plane type with $s_0(A) \leq 0$. Let $\omega \in (0,1)$ and f_n be a sequence in $H^\infty(L_\omega)$ satisfying;*

a) $\sup_n \|f_n\|_\infty < \infty$
b) $f_n(A) \in \mathcal{L}(X)$*, for all n and* $\sup_n ||f_n(A)|| < \infty$
c) $f(z) := \lim_n f_n(z)$ *exists for all* $z \in L_\omega$

Then $f \in H^\infty(L_\omega)$, $f(A) \in \mathcal{L}(X)$, $f_n(A) \to f(A)$ strongly and $||f(A)|| < \limsup_n ||f_n(A)||$.

Proof. First, Vitali's Theorem [6] tells us that f is holomorphic and that f_n converges to f locally uniformly. Condition i) also gives us that f is bounded

$$|f(z)| = \lim_{n\to\infty} |f_n(z)| \leq \sup_n \|f_n\|_\infty$$

We also have that from the functional calculus

$$\begin{aligned}\left\|(f_n(z)(1-z)^{-2})(A) - (f(z)(1-z)^{-2})(A)\right\| &= \left\|\int_{V_\delta} \frac{f_n(z)-f(z)}{(1-z)^2} R_A(z)dz\right\| \\ &\leq \int_{V_\delta} \frac{|f_n(z)-f(z)|}{(1-z)^2} \|R_A(z)\|\, dz\end{aligned}$$

Since $|f_n - f|$ is bounded and $(1-z)^{-2}(A)$ converges absolutely we have by dominated convergence that $(f_n(z)(1-z)^{-2})(A) \to (f(z)(1-z)^{-2})(A)$. It follows that for $x \in \mathcal{D}(A^2)$

$$f_n(A)x = (f_n(z)(1-z)^{-2})(A)(1-A)^2 x \to (f(z)(1-z)^{-2})(A)(1-A)^2 x = f(A)x$$

We also have that $\|f(A)x\| = \lim_{n\to\infty} \|f_n(A)x\| \leq \|x\| \limsup_n \|f_n(A)\| < \infty$. For $x \in X$ find a sequence (x_n) in $\mathcal{D}(A^2)$ converging to it. Then, by boundedness $f(A)x_n$ is Cauchy, and thus has a limit $y \in X$. But, since $f(A)$ is closed we have that $x \in \mathcal{D}(f(A))$ and $f(A)x = y$. The boundedness condition holds for such x as well. Lastly, we have that for $x \in X$;

$$\|f_n(A)x - f(A)x\| = \lim_{m\to\infty} \|f_n(A)x_m - f(A)x_m\|$$

Which goes to zero since the difference in the limit goes to zero as $n \to \infty$ for all x_m (by boundedness). Hence, $f_n(A)x \to f(A)x$ for all $x \in X$. ☐

Theorem 6.1 (Hille-Yosida)**.** *Let A be a densely defined operator with $(0,\infty) \subset \rho(A)$ and $M := \sup_{n\in\mathbb{N},\lambda>0} ||(\lambda R_A(\lambda))^n|| < \infty$. Then A is of half plane type with $s_0(A) \leq 0$ and $||e^{tA}|| \leq M$ for $t \in \mathbb{R}^+$.*

Proof. Fix $\mu \in \mathbb{C}$ such that $Re\mu > 0$ and let $\lambda \in \mathbb{R}$ be large enough so that $\lambda > |\mu|^2/(2Re\mu)$. In this case we have that $\lambda > |\mu - \lambda|$, hence by the power series representation of the resolvent we have;

$$
\begin{aligned}
\|R_A(\mu)\| &= \left\| \sum_{n=0}^{\infty} |\mu-\lambda|^n R_A(\lambda)^{n+1} \right\| \\
&\leq M \sum_{n=0}^{\infty} \frac{|\mu-\lambda|^n}{\lambda^{n+1}} \\
&= \frac{M}{\lambda - |\mu-\lambda|} \\
&= M \frac{1/\lambda}{1-|\mu/\lambda - 1|}
\end{aligned}
$$

L'Hopital's rule gives us that the limit as $\lambda \to \infty$ of the last expression converges to $\frac{M}{Re\mu}$. Thus, $\|R_A(\mu)\| \leq M/Re(\mu)$. It follows that A is of halfplane type with $s_0(A) \leq 0$.

Now denote $r_{n,t}(z) := (1 - tz/n)^{-n}$. Fix $\omega \in (0,1)$. Then for large $n \in \mathbb{N}$ we have

$$
\sup_{Re(z)\leq\omega} |r_{n,t}(z)| \leq \left(\inf_{Re(z)\leq\omega} |1 - \frac{tz}{n}| \right)^{-n} = \left(1 - \frac{\omega t}{n}\right)^{-n}
$$

Which converges to $e^{t\omega}$ taking $n \to \infty$. Hence, $\sup_n \|r_{n,t}\|_\infty < \infty$ over L_ω. We also have that $\|r_{n,t}(A)\| = \|(1-(t/n)A)^{-n}\| = \|((n/t)R_A(n/t))^{-n}\| \leq M$ by assumption. Lemma 6.1 yields then that $\left\|e^{tA}\right\| \leq M$.

□

We now give the statement of Hille-Yosida, characterizing generators of C_0 semigroups in the following corollary. We consider the special case where the uniform bound on $\|T(t)\| \leq M$. Our work here may be adapted more generally for when $\|T(t)\| \leq Me^{\omega t}$.

Corollary 6.1 (Hille-Yosida 2)**.** *Let A be a linear operator on X. The following conditions are equivalent;*

(i) A generates a C_0 semigroup $T(t)$ with $\|T(t)\| \leq M$.
(ii) A is closed, densely defined with $(0,\infty) \in \rho(A)$ and

$$
\sup_{n\in\mathbb{N},\lambda>0} \|\lambda^n R_A(\lambda)^n\| < \infty
$$

Proof. (ii) $\implies$ (i) follows from the previous theorem and Proposition 6.1. In the other case, we know that A is closed and densely defined by Theorem 4.1. We also know that $(0,\infty) \in \rho(A)$ by our uniform bound on $T(t)$ and Theorem 4.2. Differentiating the power series of the resolvent at λ one can show that;

$$
\partial_\lambda^n R_A(\lambda) = n!(-1)^n R_A(\lambda)^{n+1}
$$

Differentiating the Laplace transform of $T(t)$ with the above yields

$$
\begin{aligned}
\|R_A(\lambda)^n\| &= \left\| \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-\lambda s} T(s) ds \right\| \\
&\leq \frac{M}{(n-1)!} \int_0^\infty s^{n-1} e^{-\lambda s} ds
\end{aligned}
$$

Evaluating the above we get that $\|R_A(\lambda)^n\| \leq M/\lambda^n$ as required.

$\square$

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