CARLETON UNIVERSITY

SCHOOL OF
MATHEMATICS AND STATISTICS

HONOURS PROJECT

TITLE: Profit Testing of Equity-Linked Insurance Policies

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DATE: April 13, 2020

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1 Introduction

When insurance companies offer life insurance to potential clients, they are offering an opportunity for the policyholder to reduce their own risk by paying the insurer to take it for them. The insurer must then evaluate this risk and price their products accordingly. Traditionally, actuarial methods would involve the use of the expected present value approach to assess a contract and determine premiums and reserves. This approach comes with many problems as it is not flexible with variable interest rates or profit measures, and other more complex insurance contracts. Every change in the variables requires a manual recalculation of all other variables. Thus, we introduce cash flow analysis, also called profit testing when used for life insurance, as a method to determine profit, set reserves, and set premiums, while allowing for variability in interest rates and payment methods. This technique forecasts the income and expenses of an insurance company which relate to the product of interest and finding the net present value of these cash flows. Furthermore, profit testing can still be used to value a product even when interest rates are uncertain, simulating interest rates will generate a distribution of net present values, with which one can determine profitability of an insurance product.

2 Notation

Mortality Model

$_tp_x$ Probability of a person age x surviving to age $x + t$

$_tq_x$ Probability of a person age x dying before age $x + t$

K_x Random variable for the curtate future lifetime of a person at age x

T_x Random variable for the future lifetime of a person at age x

Insurance Policy

ν Discount rate

Π_t Profit signature at time t

$_tV$ Reserves held at time t for a policy

CV_t Cash for surrender payable at time t if the policyholder surrenders the contract in the period $[t-1, t)$

E_tV Expected cost of setting the reserves at time $t-1$ for a policy

E_t Expenses incurred at time t

EDB_t Expected cost of death benefits in the period $[t-1, t)$, paid at time t

ESB_t Expected cost of surrender benefits in the period $[t-1, t)$, paid at time t

F_t Amount invested in the policyholder's fund

I_t Interest earned on insurer's assets from time $t-1$ to t

NPV Net present value

P_t Premium paid at time $t-1$

Pr_t Emerging surplus for the period $[t-1, t)$ at time t

S_t Benefit to be paid at time t if the policyholder dies in the period $[t-1, t)$

3 Life Insurance Policies

This section will provide a basic understanding of how life insurance policies work and how the insurer might assess such policies in order to make them profitable. Life insurance policies are an agreement between the policyholder and the insurer. The policyholder pays the insurer an amount to the insurer so that in the future, should the policyholder die, the insurer will pay a benefit, denoted by S or S_t to the beneficiary. The payments are called premiums, denoted by P or P_t, and can either be paid as a lump sum, or as an annuity to the insurer. Typically the premiums are paid as an annuity and the benefit is a lump sum paid at the end of the year of death, however these can be reversed where the premium is a lump sum paid at the start of the contract and the benefits paid either as a lump sum or an annuity. The insurer uses the premiums first to pay its expenses, then uses the remaining amount to either invest or hold in reserve in case of a claim. To set the premiums and reserves, the insurer makes assumptions about the mortality of the policyholder, the expenses, and the interest rates.

3.1 Traditional

The most common types of policies are the whole life and term insurance policies. In the whole life case, the policyholder will pay premiums regularly until the year of death, when the insurer pays the benefit amount to the beneficiary. A term life insurance policy is similar except upon maturity, the policy ends and the premiums and benefits will not be paid. When the insurer evaluates these policies, they must predict the expected present value, also called the actuarial present value, of both the premiums and the benefits. We denote Z as a random variable for the present value of a benefit payable at the end of the year of death, defined by

$$Z = e^{-\delta K_x + 1}$$

where K_x is a random variable for the curtate future lifetime of a person age x and δ is the continuously compounding interest rate. Then the APV of a whole life insurance policy with a benefit of \$1 is denoted by A_x and is defined

$$A_x = E[Z] = E[e^{-\delta(K_x+1)}]. \tag{3.1}$$

The APV of the premiums is also evaluated, the present value random variable of an annuity due is denoted Y and is defined

$$Y = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - e^{-\delta(K_x+1)}}{d}. \tag{3.2}$$

Then the APV of level premiums of \$1 paid annually is denoted $\ddot{a}_x$ and is defined

$$\ddot{a}_x = E[Y] = E[\frac{1 - e^{-\delta(K_x+1)}}{d}] = \frac{1 - E[e^{-\delta(K_x+1)}]}{d} \tag{3.3}$$

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where d is the discount rate. Then by equation (3.1), we can see that

$$\ddot{a}_x = \frac{1 - A_x}{d} \qquad (3.4)$$

Similarly, the APV of an n-year term life insurance policy with a benefit of \$1 is denoted by $A^1_{x:\overline{n}|}$ and is defined

$$A^1_{x:\overline{n}|} = E[Z] \qquad (3.5)$$

where Z is now defined as

$$Z = \begin{cases} e^{-\delta T_x} & \text{if } K_x + 1 \leq n, \\ 0 & \text{if } K_x + 1 > n. \end{cases}$$

The term life APV can also be expressed as

$$A_{x:\overline{n}|} = A_x - A_{x+n}.$$

The PV of the annuity of premiums paid until death or time n is denoted Y, and defined

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & if K_x + 1 \leq n, \\ \ddot{a}_{\overline{n}|} & if K_x + 1 > n. \end{cases} \qquad (3.6)$$

Then the APV of the term life annuity due with payments of \$1 is denoted $\ddot{a}_{x:\overline{n}|}$ and is defined

$$\ddot{a}_{x:\overline{n}|} = E[Y] \qquad (3.7)$$

where Y is defined as in equation (3.6).

The insurer is then able to set premiums using the net present value of the loss. We define the future loss at time t as the amount the insurer will pay in an insurance policy, and denote it as L_t. The random loss at time 0 is defined as

$$L_0 = SZ - PY \qquad (3.8)$$

where S is the benefit amount, Z is the present value of a benefit of \$1, P is the premium amount, and Y is the present value of an annuity of \$1. Then taking the expected value and setting $E[L_0] = 0$ we can solve for the premium amount, where this amount gives a net present value of 0 for the policy.

$$E[L_0] = SE[Z] - PE[Y].$$

Setting $E[L_0]$ equal to zero and rearranging for P, we find

$$P = S\frac{E[Z]}{E[Y]}.$$

The future loss random variable is also used to calculate the reserves. We define L_t as

$$L_t = \text{PV(Future Benefits at time } t\text{)} - \text{PV(Future Premiums at time } t\text{)}$$

The expected future loss at time t gives the amount the insurer must hold in order to be able to pay for the expected costs. This number is called the reserve or the policy value. We denote the policy value at time t for a policyholder at age x as $_tV$ and it is defined as

$$_tV = \text{APV(Future Benefits at time } t\text{)} - \text{APV(Future Premiums at time } t\text{)}.$$

In the case of a whole life insurance policy, in actuarial notation this is written

$$_tV = SA_{x+t} - P\ddot{a}_{x+t}$$

This is the method that an insurer would use to set premiums and reserves and with some modification this allows for expenses, investments, and profit. The problem with this process is that it does not allow for flexible premiums or interest rates. A change would mean a manual recalculation of all parts of the policy.

3.2   Modern

Insurance companies offer alternative forms of insurance which provide the policyholder the opportunity to make an investment with the insurance company. Participating insurance is similar to term and whole life insurance except the insurer shares some of the investment income made on the premiums with the policyholders. Another common life insurance policy is Universal Life insurance, which is similar to Participating insurance, but with the added flexibility of premiums. The policyholder can change the premium amount that they wish to pay, as long as it is still enough to cover the insurance benefit set. The insurer pools the premiums with its own investments and pays a portion of the returns back to the policyholder in regular intervals.

The most investment focused product that a life insurance company offers is the equity-linked insurance policy. Unlike the previously mentioned policies, equity-linked insurance is primarily an investment, with the added benefit of life insurance. The insurer offers many different investment funds to the policyholders that have different expected returns and riskiness, and the policyholder chooses which fund their premiums will be paid into. The policyholder also has the option to choose the amount and number of premiums paid into this fund, with a minimum amount set by the insurer to make sure the expenses are covered. The insurer also offers the option of guaranteed minimum benefits upon maturity or upon death of the policyholder.

A key difference between equity-linked insurance and other life insurance types is that these premiums must be held separately from the insurance company's assets.

For the insurer this means there is less need to hold reserves for these contracts, as the policyholder fund acts as a reserve itself. The risk of the investment is taken on by the policyholder, however, risk for the insurer is added when the guaranteed minimum benefits are added to the policy.

4 Profit Testing

4.1 Profit Testing Model

To begin profit testing, a basis of assumptions and details must be made by the insurer. First of these is the time interval. The time interval is set by an insurer to determine at what time the cash flows occur. The insurer can choose any interval they like, such as monthly or annually, as long as they use the appropriate discount rate for calculation. It is most convenient for the insurer to use the same interval that the premiums are paid in, unless the premium payments do not follow any defined structure, in which case monthly or annual cash flows would be the most convenient.

The insurer must also determine what is called the profit test basis. This is formed by assumptions made about the profit test model such as expenses, the mortality model, and interest rates.

4.1.1 Expenses

Usually when a contract is written by an insurer there are initial expenses, also called acquisition costs, and recurring expenses that continue for the duration of the contract. The initial expenses must always be allocated to time 0, as any expenses must be allocated as early as possible. The later expenses are costs arising from the maintenance of the policy and collection of premiums are much smaller than the acquisition expenses and are allocated to the beginning of the period they are paid. After defining the expenses, the insurer can set the amount to charge from the premiums paid.

4.1.2 Mortality model

An insurer must also make assumptions about the mortality model for policyholders as well as the probability of the policyholder withdrawing from the contract. Often life tables can be used to estimate mortality of the policyholder as shown in Appendix A, as they are very easy to derive probabilities of survival for each year. An alternative would be to use a distribution based on the force of mortality μ_x demonstrated here.

Let F_x be the distribution function of T_x, then

$$F_x(t) = Pr[T_x \leq t]. \tag{4.1}$$

F_x represents the probability of a person aged x dies before age $x + t$. We can also define the survival probability, S_x.

$$S_x(t) = Pr[T_x > t] = 1 - F_x(t). \tag{4.2}$$

We denote the probability density function of the random variable T_x, as f_x and its relationship to F_x and S_x is shown by

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t). \tag{4.3}$$

An important property for the random variable T_x is that the probability a person aged x dies within t more years is equal to the probability a person aged 0 dies within $x + t$ years given that they survive x years.

$$Pr[T_x \leq t] = Pr[T_0 \leq x + t | T_0 \leq x]. \tag{4.4}$$

Now by definiton

$$Pr[T_x \leq t] = Pr[T_0 \leq x + t | T_0 \leq x] = \frac{Pr[x < T_0 \leq x + t]}{Pr[T_0 > x]},$$

which is equivalent to

$$F_x(t) = \frac{F_0(x + t) - F_0(x)}{S_0(x)},$$

and using equation (4.2)

$$S_x(t) = \frac{S_0(x + t)}{S_0(x)}. \tag{4.5}$$

The force of mortality for a life at age x is denoted μ_x and is defined as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_0 \leq x + dx | T_0 > x].$$

or, by equation (4.4),

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_x \leq dx].$$

Then we relate μ_x to S_x by equations (4.1) and (4.2)

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx}(1 - S_x(dx)). \tag{4.6}$$

Recall that by equation (4.5), we can rewrite this as

$$\begin{aligned}
\mu_x &= \lim_{dx \to 0^+} \frac{1}{dx}\left(1 - \frac{S_0(x + dx)}{S_0(x)}\right) \\
&= \lim_{dx \to 0^+} \frac{1}{dx} \frac{S_0(x) - S_0(x + dx)}{S_0(x)} \\
&= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x + dx)}{dx} \\
&= \frac{1}{S_0(x)}\left(-\frac{d}{dx} S_0(x)\right)
\end{aligned}$$

and finally,

$$\mu_x = -\frac{1}{S_0(x)} \frac{d}{dx} S_0(x). \tag{4.7}$$

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which can also be written
$$\mu_x = \frac{f_0(x)}{S_0(x)}.$$

By the property stated in equation 4.5, we can define μ_{x+t} similarly. We assume x constant and t variable, then $d(x+t) = dt$.

$$\mu_{x+t} = -\frac{1}{S_0(x+t)} \frac{d}{d(x+t)} S_0(x+t)$$
$$= -\frac{1}{S_0(x)S_x(t)} \frac{d}{dt} S_0(x)S_x(t)$$
$$= -\frac{1}{S_x(t)} \frac{d}{dt} S_x(t)$$

resulting in
$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}. \tag{4.8}$$

Now we want to write $S_x(t)$ in terms of μ_{x+t}. We rewrite equation (4.7) as

$$\mu_{x+t} = -\frac{d}{dx} \log S_x(t)$$

Then integrating both sides over 0 to y gives

$$\int_0^y \mu_{x+t} = \log S_x(0) - \log S_x(y)$$

and since $S_x(0) = 1$,
$$S_x(y) = \exp\left\{ -\int_0^y \mu_{x+t} dt \right\} \tag{4.9}$$

From this equation, we are able to define a force of mortality and determine a mortality model. In 1825, Benjamin Gompertz proposed that the force of mortality increases exponentially with age, that is
$$\mu_x = Bc^x.$$

where B and c are constants such that $0 < \beta < 1$ and $c > 1$. Later William Makeham added to this theory, saying that there should also be a fixed term in the force of mortality, independent of age.
$$\mu_x = \alpha + Bc^x.$$

So from these examples we see ways in which an insurer could model mortality using the force of mortality.

4.1.3 Probability of Surrender

Depending on the type of policy, the insurer may also have to determine the probability of the policyholder withdrawing from the contract in any given period. This can be done based on historical results and forecasting. In such cases the insurer must use stochastic modelling for multiple states of the policyholder. In a 3-state model we denote state 0 as the survival with policy in force state, state 1 as survive and withdraw from policy, and state 2 as death. In this case the insurer is only concerned with transitions from state 0, so we use a probability vector for these probabilities. The single step probability vector for a person age x using this model is defined as

$$\begin{bmatrix} p_x^{00} & p_x^{01} & p_x^{02} \end{bmatrix}. \tag{4.10}$$

Under the assumption of independent and identically distributed lives, we can use this probability vector to predict how many policies are still in force in any given year. Then these probabilities can be used to define many terms in the profit test.

4.1.4 Interest rates

The insurer must also determine multiple interest rates to use in the profit test. First is the discount rate used to calculate the net present value of all of the cash flows in each period. This rate can be determined using the weighted average cost of capital, r_{WACC}. To calculate r_{WACC} you must use the cost of equity, r_E, the cost of debt, r_D, and the cost of preferred shares, r_{Pfd}, as well as the weights associated with these rates, w_E, w_D, w_{Pfd}. The cost of equity can be found using the capital asset pricing model, which says that

$$r_E = risk\ free\ rate + \beta_{firm\ equity} \times market\ risk\ premium.$$

The cost of debt is found simply by taking the interest rate associated to the outstanding debt for the firm. Then the effective cost of debt is found using the cost of debt and the corporate tax rate for the firm, T_c.

$$effective\ cost\ of\ debt = r_D \times (1 - T_c).$$

The cost of preferred equity is calculated using the dividends paid per share of preferred stock, Div_1, the price per share of preferred stock, P_0, and the growth rate of the dividends, g.

$$r_{Pfd} = \frac{Div_1}{P_0} + g.$$

The weights are calculated as the percentage of the firm value financed by equity, preferred stock, and debt. Now we can calculate r_{WACC} then we use this rate to find the discount rate ν.

$$r_{WACC} = w_D \times r_D(1 - T_c) + w_E \times r_E + w_{Pfd} \times r_{Pfd}.$$

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The discount rate is defined as

$$\nu = \frac{1}{1 + r_{WACC}}.$$

The insurer must also determine what interest is earned on its assets. We denote the amount of interest earned in the period $[t-1, t)$ at time t as I_t. If we say the interest rate earned on insurer assets over period $[t-1, t)$ is i_t, then the interest earned is

$$I_t = Amount\ invested \times i_t. \tag{4.11}$$

Finally, an insurer may also choose to invest the reserves held in short term, low risk assets. The rates on these assets must also be accounted for in the profit test.

4.1.5 Expected costs from benefits paid

Once the profit test assumptions are made, the insurer must then find the expected costs in each period in order to determine the emerging surplus. These costs are from the multiple states in which a policyholder could be in each period. Using the probability vector from equation (4.10), we define the following: The expected death benefit in year t, EDB_t

$$EDB_t = p^{02}_{x+t-1} \times S_t, \tag{4.12}$$

the expected cost of surrender (or withdraw) benefits, ESB_t

$$ESB_t = p^{01}_{x+t-1} \times CV_t, \tag{4.13}$$

and the expected cost of survivor benefits (also called endowment) EEB_t

$$EEB_t = p^{00}_{x+t-1} \times Endowment. \tag{4.14}$$

Usually an endowment occurs at the end of the contract, in the case of equity-linked life insurance this could be a percentage of the policyholder's fund, F_t. This method can be extended to any number of states where the transition probabilities between other states may become necessary, and a probability matrix must be formed to predict future proportions of policyholders in each state.

4.1.6 Reserves

The expected cost of reserves is also calculated using the survival model, which is considered a cost at the end of the period, then the actual reserve is carried forward as income at the start of the next period. To find the reserves we want to make the expected value of the policy equal to 0 in each period. The net present value of a policy in the period $[t-1, t)$ is given by

$$NPV_t = E[Benefits_t] + E[Expenses_t] - E[Premiums_t] - {}_tV. \tag{4.15}$$

where $Benefits_t$, $Expenses_t$, and $Premiums_t$ each represent the future amounts, starting from time t to the end of the term. We set $NPV_t = 0$ for every t to find the reserve in each period. Note that reserves can never be negative, in such cases we set the reserve equal to zero. Then the expected reserve for the period t is defined as

$$E_t V = p^{00}_{x+t-1} {}_t V. \tag{4.16}$$

Note that in the final policy period of whole life and term insurance policies, the reserve and expected reserve should be zero, since there is no benefit payment to be made in the future.

4.2 Deterministic Approach

In this section we look at the deterministic approach to cash flow analysis to show the basic ideas behind the method. We assume that interest rates are constant over the duration of the policy term.

4.2.1 Emerging Surplus

Using the cash flows defined in the previous section, the emerging surplus for the insurer in each year is defined as

$$Pr_t = {}_{t-1}V + P_t - E_t + I_t - EDB_t - ESB_t - EEB_t - E_t V. \tag{4.17}$$

We define the profit signature in period t, Π_t, as the expected surplus emerging at the end of period t for a contract in force at the issue date, time $t = 0$ and is calculated as follows,

$$\Pi_t = p^{00}_{x+t-1} Pr_t. \tag{4.18}$$

Then the net present value of a policy in force up to time t can be found.

$$NPV = \sum_{k=0}^{t} \Pi_k \nu^k. \tag{4.19}$$

With the net present value computed, it can be compared to the desired profit measure to determine if the product if profitable. Often this profit measure is the net present value itself, but alternatives include the internal rate of return, discounted payback period, and the profit margin.

4.2.2 Setting Reserves

The profit test can be used to set the reserves through a method called zeroization. We do this by first performing a profit test without reserves, then starting with the final period of the contract, set the reserve in the previous period such that the emerging

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surplus of the period is equal to 0. If we assume an interest rate of j on the amount held in reserves, then the amount needed at the end of the period to set the emerging surplus equal to zero is

$$_{n-1}V = max\left\{-\frac{Pr_t}{1+j}, 0\right\}.$$ (4.20)

For every period before the final period, the expected reserve must also be accounted for. We find the reserve of the periods before the final one using the following formula.

$$_tV = max\left\{-\frac{Pr_t}{1+j} + E_tV, 0\right\}$$ (4.21)

where E_tV is defined as in Equation (4.16). This pattern is repeated for every period of the contract working backwards from the final period to the first period. Once the zeroized reserves have been set, the net present value can be recalculated. This is done because the emerging surplus will come at an earlier point in time, and is therefore more valuable due to discounting.

4.2.3 Equity-Linked Life Insurance Profit Testing

Equity-linked life insurance policies add multiple factors to the calculation of emerging surplus. Since the insurer often takes a percentage of the policyholder's fund to cover expenses, called the management charge, or MC_t (this is also often called the management expense ratio, MER), the first step must be to determine what the policyholder fund will be in each period. To begin, the premiums paid are split into what is allocated to the policyholder's fund, AP_t, and what is allocated to the insurer's fund (or unallocated to the policyholder's fund), UAP_t. The policyholder fund starts with the first allocated premium at the beginning of the period, accrues interest, then the management charge is subtracted at the end of the period, which results in the policyholder's fund amount.

$$F_1 = AP_1 \times (1+i) - MC_1.$$

Then the future fund values will add the previous value to the allocated premium before interest and management charges are accounted for.

$$F_t = (F_{t-1} + AP_t) \times (1+i) - MC_t.$$ (4.22)

Once the policyholder fund is determined, the insurer can then calculate the emerging surplus from it's own cash flows. This is done similarly to a traditional life insurance policy except instead of collecting the premiums, the insurer collects the unallocated premiums and management charges. Equation (4.17) becomes

$$Pr_t = {}_{t-1}V + UAP_t + MC_t - E_t + I_t - EDB_t - ESB_t - EEB_t - E_tV.$$

Since the management charge is usually a percentage of the policyholder fund, we can denote this as MER,

$$MC_t = MER(F_{t-1} + AP_t)(1+i),$$

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and then we change the equation (4.22)

$$F_t = (F_{t-1} + AP_t) \times (1+i) \times (1 - MER).$$

Often are no reserves needed for an equity-linked life insurance policy, due to the policyholder fund being the benefit being paid. Also there may be no difference between surrender benefits and endowment benefits, if the insurer chooses to do so. In such cases the calculation of emerging surplus simplifies to

$$Pr_t = UAP_t + MC_t - E_t + I_t - EDB_t - ESB_t.$$

The guaranteed minimum maturity benefit (or death benefit) can increase risk for the insurer. In a profit test this is included after the net present value of the cash flows is calculated. In the case the guaranteed minimum maturity benefit (or death benefit) exceeds the policyholder's fund at the time the benefit must be paid, then this difference must be subtracted from the emerging surplus of the final period of the contract. If we suppose the policyholder survives until the end of the policy term, period n, then

$$EEB_t = {}_tp_x \; max \left\{ F_t, GMMB \times \sum_{k=1}^{t} P_k \right\}. \tag{4.23}$$

Similarly for the guaranteed minimum death benefit, if the policyholder dies in period t,

$$EDB_t = q_{x+t} \; max \left\{ F_t, GMDB \times \sum_{k=1}^{t} P_k \right\} \tag{4.24}$$

4.3 Stochastic Approach

Due to the variable nature of interest rates, it is not enough to simply take the expected interest rate and use it to perform a profit test on equity-linked life insurance type policies. Instead an insurer must account for this variability by making an assumption about the distribution of the interest rates on the policyholder's assets, and use it to generate a large sample. This sample can then be used to perform a large number of profit tests, and will give a sample of net present values, from which sample statistics can be drawn.

4.3.1 Determining a Sample of Interest Rates

If we assume the interest rates are independent, identically distributed, and follow a lognormal distribution with mean μ and variance σ^2, $LN(\mu, \sigma^2)$, with n periods in the term of the policy, we generate a sample with size n from the standard normal distribution, $N(0,1)$. Then by definition of the lognormal distribution, the sample of interest rates, i_k, is found by

$$i_k = e^{\mu + \sigma z_k}, \tag{4.25}$$

where z_k is the kth element from the sample of standard normal random variables.

4.3.2 Changes to the Deterministic Approach

Once the interest rates have been determined the insurer can use them to perform a profit test. This is done very similarly to the deterministic approach, changing the interest rates in each period. When we examine the equity-linked life insurance policy, the effect of the variable interest rates on the policyholder's fund can be very large. This is especially important when there are guaranteed minimum maturity or death benefits offered in the policy. The formula for calculating the policyholder's fund becomes

$$F_t = (F_{t-1} + AP_t) \times (1 + i_t) \times (1 - MER) \tag{4.26}$$

and the rest of the calculation continues in the same way as demonstrated in section (4.2). This process is then repeated as many times as the insurer wishes to get a sufficiently large sample of independent net present values. Statistics can be drawn from this distribution to determine profit measures and assess if a product is worth accepting for an insurer. For example, an insurer may choose to accept a policy if, after 1000 simulations, the 5th percentile is positive. An insurer may also be interested in a confidence interval for the expected value of the NPV. For a sufficiently large sample size, N, we can apply the Central Limit Theorem, and say a $(1-\alpha\%)$ confidence interval for $\mathrm{E}[NPV]$ is given by

$$\left[\overline{NPV} - z_{\alpha/2} \frac{s}{\sqrt{N}}, \ \overline{NPV} - z_{\alpha/2} \frac{s}{\sqrt{N}} \right],$$

where s is the sample standard deviation of NPV.

4.3.3 Reserves

Insurers must use alternative methods to determine reserves when the risk associated with the policy is nondiversifiable, as is the case with equity-linked life insurance. The two most common methods are called quantile reserving and conditional tail expectation reserving (CTE), also commonly known as Value-at-Risk (VaR) and Tail Value-at-Risk (or TVaR) respectively.

The Value-at-Risk for a specified parameter α, where $0 \leq \alpha \leq 1$, is the amount such that the probability that a loss exceeds the reserve is equal to $1 - \alpha$. That is, for a random loss amount L with continuous distribution function F_L, we set the quantile reserve Q_α such that,

$$Pr[L \leq Q_\alpha] = \alpha. \tag{4.27}$$

Alternatively,

$$Q_\alpha = F_L^{-1}(\alpha). \tag{4.28}$$

The random loss, L, is defined by the future expected profits. Using the sets of interest rates simulated, each set will determine one element of the sample of loss random

variables, L_i. Let j denote the rate for interest earned on reserves, and let ${}_t p_x^{00}$ be the probability the policy is still in force at t and $Pr_{t,i}$ the profit emerging in period t of simulation i. Then

$$L_i = -\sum_{t=1}^{n} \frac{{}_{t-1}p_x^{00} Pr_{t,i}}{(1+j)^t} \tag{4.29}$$

This will give an empirical distribution for L from which the reserve ${}_0 V$ can be drawn. This method of reserving does not require any changes after setting the initial reserve provided that the assumptions made (see sections 3.2.1 to 3.2.4) are accurate. Then quantile reserving will give an amount such that the insurer will be able to cover the expected benefits with probability 0.95. This does not mean that the insurer should not adjust the reserve, in fact the insurer can adjust the reserve based on the result from the interested earned on the policyholder fund. If the fund does well, this means the insurer is less likely to have to pay the guaranteed minimum, as the fund will likely exceed the minimum. The insurer could then decrease the reserves held in order to profit more from other assets. Similarly if the fund does poorly, the insurer must increase the reserves in order to be able to pay the expected costs.

The conditional tail expectation reserving method, uses the conditional expectation of the random loss L given that the loss is greater than the quantile reserve, Q_α. That is, for a given level α, the CTE reserve is given by

$$CTE_\alpha = E[L | L > Q_\alpha] \tag{4.30}$$

CTE reserving looks at what happens if the loss is to actually exceed the quantile reserve, and set the reserves based on the tail distribution of the loss. This leads to a higher amount set for the reserves, and decreases risk for the insurer, at the cost of some potential profit earned on other assets.

5   Application

In this section we will see how an insurer would profit test for a term life insurance policy using a deterministic method, then compare the deterministic method to the stochastic method for an equity-linked insurance policy. First we must make our assumptions about the model. The mortality model used is the life table for all Canadians provided by Statistics Canada over the years 2016-2018. All other assumptions about the model are derived from the 2019 Annual Report of Canada Life Insurance. These assumptions are a rough estimate based on the total expenses, total investment income, total investments, and total premiums provided in the Annual Report. The information about the Core Conservative Growth segregated fund, such as historical returns, guaranteed minimums offered, and management expense ratio, can be found in the Canada Life segregated funds information folder.

The expenses depend on the policy, for term insurance we assume the initial expense is $98.85 per policy, with annual expenses of $107.10, and for the segregated fund policy, expenses depend on the fund chosen and premiums paid. We assume there is a 5% commision owed to the advisor as an initial expense. The annual expense estimated for segregated fund insurance is $1.70. We assume the insurer's assets earn interest at 7.7% annually, and reserves are held as cash with no interest earned. The discount rate used to calculate any net present values is 10.3%. The policyholder fund in the segregated fund policy is the Core Conservative Growth fund, with guaranteed minimum maturity/death benefits of 75%/100%, offers an expected return of 3.21% and standard deviation of 2.16%. The management expense ratio, or management charge, of this fund is 2.48%.

Summary of Model Assumptions and Data

Table 1: 10-year Term Life Insurance

Name	Amount
Initial Expense	$98.85
Annual Expenses	$107.10
Interest Rate	7.7%
Discount Rate	10.3%
Age of Policyholder	65
Amount Insured	$100000
Premium Amount	$1500

Table 2: 10-year Equity Linked Insurance

Name	Amount
Initial Expense	$0.05P_1$
Annual Expenses	$1.70
Initial Premium	$500
Annuity Premium	$100
Age of Policyholder	65
Expected Interest Rate of Fund	3.21%
Standard Deviation of Fund Interest Rate	2.161%
Guaranteed Minimum Maturity Benefit	75%
Guaranteed Minimum Death Benefit	100%
Management Expense Ratio	2.48%
Confidence Level	0.05
CTE Level	0.75

In the following examples year 5 will be shown as an example calculation, as well as any years where the calculation is different from the other years. The policies will be evaluated with the following profit measures: Net Present Value, Internal Rate of Return (IRR), Discounted Payback Period (DPP), and Profit Margin.

The IRR is defined the solution to the following equation,

$$0 = \sum_{t=0}^{n} \Pi_t \nu_{IRR}^t$$

where

$$\nu_{IRR} = \frac{1}{1 + IRR}.$$

The solution for IRR can be computed in Excel using the command IRR(). It is important to note that in some cases it is impossible to find a solution to this equation.

The DPP is the amount of time required for an investment to break even (ie. $NPV \geq 0$). To find this we simply look at the NPV column and set the DPP equal to the time at which the NPV first becomes nonnegative.

The profit margin is ratio of the net present value and the actuarial present value of the premiums.

$$\text{Profit margin} = \frac{NPV_n}{\sum_{t=0}^{n} P_t \; {}_t p_x \nu^t}$$

5.1 Term Life Insurance

In the first example we are an insurer offering a 10-year term life insurance policy for a person age 65 with a death benefit of $100000 payable at the end of the year of death. Level premiums of $1500 are paid annually with the first payment at the beginning of the first year. To begin we set up a table with columns: Time, Premiums, Expenses, Interest, Probability of Death, Expected Death Benefit, Profit, Profit Signature, and Net Present Value. The first 5 columns are all in the assumptions and can be entered immediately as seen in Table (3).

Table 3: Assumptions of Term Life Policy

Time	Premium	Expenses	Interest	q_x
0	$0	$98.85	$0	0
1	$1,500.00	$107.10	$115.49	0.00918
2	$1,500.00	$107.10	$115.49	0.01009
3	$1,500.00	$107.10	$115.49	0.0111
4	$1,500.00	$107.10	$115.49	0.01222
5	$1,500.00	$107.10	$115.49	0.01346
6	$1,500.00	$107.10	$115.49	0.01484
7	$1,500.00	$107.10	$115.49	0.01637
8	$1,500.00	$107.10	$115.49	0.01808
9	$1,500.00	$107.10	$115.49	0.01997
10	$1,500.00	$107.10	$115.49	0.02208

Table 4: Calculations

Time	EDB	Profit	Profit Signature	NPV
0	$0.00	-$98.85	-$98.85	-$98.85
1	$918.00	$590.39	$584.97	$642.00
2	$1,009.00	$499.39	$494.35	$1,068.20
3	$1,110.00	$398.39	$393.97	$1,383.57
4	$1,222.00	$286.39	$282.89	$1,593.83
5	$1,346.00	$162.39	$160.20	$1,704.39
6	$1,484.00	$24.39	$24.03	$1,719.79
7	$1,637.00	-$128.61	-$126.50	$1,644.52
8	$1,808.00	-$299.61	-$294.19	$1,482.00
9	$1,997.00	-$488.61	-$478.85	$1,236.37
10	$2,208.00	-$699.61	-$684.16	$910.51

The expected death benefit in each year is determined by Equation (4.12). In this case we assume there is no probability of surrender, so we have $p^{02}_{x+t-1} = q_x$.

$$EDB_5 = Sq_{x+5} = 100000(0.01346) = 1346.$$

Once the expected death benefit is calculated, the emerging surplus is then calculated using Equation (4.17).

$$Pr_5 = P_5 - E_5 + I_5 - EDB_5.$$
$$= 1500 - 107.1 + 115.49 - 1346$$
$$= 162.39$$

Then the profit signature is calculated using Equation (4.18).

$$\Pi_5 = p_{x+5}Pr_5$$
$$= (1 - q_{x+5})Pr_5$$
$$= (1 - 0.01346)(162.39)$$
$$= 160.2$$

Finally the NPV is calculated using Equation (4.19).

$$NPV_5 = \sum_{k=0}^{5} \Pi_k \nu^k$$
$$= -98.85 + (584.97)\left(\frac{1}{1.103}\right)^1 + ... + (160.2)\left(\frac{1}{1.103}\right)^5$$
$$= 1704.39$$

We note in Table (4) that the net present value of the policy is always positive after the first period, but in the later years of the policy there is negative emerging surplus, which must be avoided for proper risk management. We use the profit test to calculate the reserves using zeroization as shown in Section (4.2.2). Since there is no interest earned on reserves, we set $_9V = -Pr_{10} = \$699.61$. We then calculate backwards and find the values shown in Table (5).

$$_4V = max\{-Pr_5 + E_5V, 0\}$$
$$= -162.39 + 1489.63$$
$$= 1327.24.$$

Note that holding reserves decreases the NPV, but the emerging surplus is nonnegative for every year after the initial expenses which will decrease the risk for the insurer in the future years.

Summary of Profit Measures

$NPV = \$542.40$
$IRR = 383\%$
$DPP = 1$
Profit margin $= \dfrac{\$542.40}{\$1500 a_{65:\overline{10}|}} = 4.99\%$

Table 5: Reserve Calculations

Time	Pr_t Before	$_{t-1}V$	E_tV	Pr_t After	Π_t	NPV
0	-$98.85	$0.00	$0.00	-$98.85	-$98.85	-$98.85
1	$590.39	$0.00	$108.27	$482.12	$477.70	$542.40
2	$499.39	$109.27	$608.66	$0.00	$0.00	$542.40
3	$398.39	$614.87	$1,013.26	$0.00	$0.00	$542.40
4	$286.39	$1,024.63	$1,311.02	$0.00	$0.00	$542.40
5	$162.39	$1,327.24	$1,489.63	$0.00	$0.00	$542.40
6	$24.39	$1,509.95	$1,534.34	$0.00	$0.00	$542.40
7	-$128.61	$1,557.46	$1,428.85	$0.00	$0.00	$542.40
8	-$299.61	$1,452.63	$1,153.02	$0.00	$0.00	$542.40
9	-$488.61	$1,174.25	$685.64	$0.00	$0.00	$542.40
10	-$699.61	$699.61	$0.00	$0.00	$0.00	$542.40

5.2 Deterministic Testing of an Equity-Linked Insurance Policy

Now that a basic understanding of the profit test is demonstrated, we look at equity-linked insurance to see how an insurer would analyze the cash flows of this policy and set reserves if necessary to minimize risk. We use the information provided from the Canada Life segregated fund information folder about the Core Conservative Growth segregated fund to build the model of the policy. Suppose the policyholder is 65 years old and they wish to pay the minimum premiums as an annuity. The minimum amount set by Canada Life is an initial investment of $500.00, and any future premiums must be at least $100.00. When the policyholder sets up an annuity account, there is no initial fee charged, and the management expense ratio is 2.48%. Note that the entire premium is allocated to the policyholder's fund, thus there is no unallocated premium amount added to the insurer's fund ($UAP_t = 0$ for all t).

We begin the profit test by first predicting the policyholder fund, since the MER depends on the amount in the fund, then we calculate the emerging surplus using the MER, expenses and expected death benefit. The profit signature and net present value are calculated in the same way as before. We set up a table with columns: Time, Premium, Fund Before MC, MC, Fund After MC, Expenses, Probability of Death, Expected Death Benefits, Reserves at Time $t-1$, Expected Reserves Emerging Surplus, Profit Signature, and Net Present Value. First we find the fund value before subtracting the management charges, then we calculate the charges based on this value, and the final fund value at the end of the year is what remains after subtracting management charges.

$$\begin{aligned} F_1 \ Before &= P_1(1+i) \\ &= 500(1.0321) \\ &= 516.05. \end{aligned}$$

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We then find the management charge in the first year,

$$MC_1 = F_1 \ Before \times MER$$
$$= 516.05(0.0248)$$
$$= 12.8$$

The final value is then

$$F_1 = F_1 \ Before - MC_1$$
$$= 516.05 - 12.8$$
$$= 503.25$$

After finding the fund value after the first year, it must be accounted for in the future fund values.

$$F_5 \ Before = (F_4 + P_5)(1 + i)$$
$$= (817.05 + 100)(1.0321)$$
$$= 946.49.$$

$$MC_5 = F_5 \ Before \times MER$$
$$= 946.49(0.0248)$$
$$= 23.47$$

$$F_5 = F_5 \ Before - MC_5$$
$$= 946.49 - 23.47$$
$$= 923.02$$

Table (6) shows the result of the calculations of the policyholder's fund.

Table 6: Policyholder Fund

Time	Premium	F_t Before MC	MC	F_t
0	$0.00	$0.00	$0.00	$0.00
1	$500.00	$516.05	$12.80	$503.25
2	$100.00	$622.62	$15.44	$607.18
3	$100.00	$729.88	$18.10	$711.77
4	$100.00	$837.83	$20.78	$817.05
5	$100.00	$946.49	$23.47	$923.02
6	$100.00	$1,055.86	$26.19	$1,029.67
7	$100.00	$1,165.94	$28.92	$1,137.02
8	$100.00	$1,276.73	$31.66	$1,245.07
9	$100.00	$1,388.24	$34.43	$1,353.81
10	$100.00	$1,500.48	$37.21	$1,463.27

The calculations of the fund value at the end of the year can also be done in one step as follows.

$$F_1 = P_1(1+i)(1-MER)$$
$$= 500(1.0321)(1-0.0248)$$
$$= 503.25.$$

The fund values after time $t = 1$ are calculated similarly,

$$F_5 = (F_4 + P_5)(1+i)(1-MER)$$
$$= (817.05 + 100)(1.0321)(1-0.0248)$$
$$= 923.02$$

The test continues by calculating the emerging surplus as in Equation (**??**). In this case there is no negative emerging surplus in any year, so reserves do not need to be held. The interest earned on insurer assets is also zero since there is no amount invested in relation to this policy.

$$Pr_5 = MC_5 - E_5 - EDB_5$$
$$= 23.47 - 1.7 + 12.42$$
$$= 9.35$$

Notice the only instance where the emerging surplus is negative is at time $t = 0$, since

Table 7: Insurer Cash Flows From Equity-Linked Policy

Time	MC_t	E_t	q_{x+t}	EDB_t	Pr_t
0	$0.00	$25.00	0	$0.00	-$25.00
1	$12.80	$1.70	0.00918	$4.62	$6.48
2	$15.44	$1.70	0.01009	$6.13	$7.61
3	$18.10	$1.70	0.0111	$7.90	$8.50
4	$20.78	$1.70	0.01222	$9.98	$9.09
5	$23.47	$1.70	0.01346	$12.42	$9.35
6	$26.19	$1.70	0.01484	$15.28	$9.20
7	$28.92	$1.70	0.01637	$18.61	$8.60
8	$31.66	$1.70	0.01808	$22.51	$7.45
9	$34.43	$1.70	0.01997	$27.04	$5.69
10	$37.21	$1.70	0.02208	$32.31	$3.20

this expense is paid immediately there is no need to hold reserves for this policy, the policyholder fund itself acts as the reserves.

Next we calculate the profit signature and net present value as before in order

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to value the policy.

$$\Pi_5 = p_{x+5}Pr_5$$
$$= (1 - qx + 5)Pr_5$$
$$= (1 - 0.01346)(9.35)$$
$$= 9.22$$

Table 8: Net Present Value Of Equity-Linked Policy

Time	Pr_t	p_{x+t}	Π_t	NPV
0	-$25.00	1	-$25.00	-$25.00
1	$6.48	0.99082	$6.42	-$19.18
2	$7.61	0.98991	$7.54	-$12.99
3	$8.50	0.9889	$8.41	-$6.72
4	$9.09	0.98778	$8.98	-$0.65
5	$9.35	0.98654	$9.22	$5.00
6	$9.20	0.98516	$9.07	$10.03
7	$8.60	0.98363	$8.46	$14.29
8	$7.45	0.98192	$7.32	$17.63
9	$5.69	0.98003	$5.58	$19.94
10	$3.20	0.97792	$3.13	$21.11

Summary of Profit Measures

$NPV = \$21.11$

$IRR = 28\%$

$DPP = 5$

Profit margin $= \dfrac{\$21.11}{\$500 + \$100\nu a_{66:\overline{9|}}} = 1.99\%$

5.3 Stochastic Testing of an Equity-Linked Insurance Policy

The deterministic test is good to get an idea of what to expect from the product, but it is not realistic as interest rates tend to vary. With the stochastic test we follow the same procedure as before while changing the expected interest rates earned on the policyholder fund in each period. In this case we assume interest rates follow a lognormal distribution with mean 0.0321 and standard deviation 0.0216. First we do one simulation to calculate the NPV of the policy including an interest rate column in the table for calculations. The change in Equation (4.22) is with interest rate notation, where we use i_t instead of i to reflect the annual change in interest rates. We calculate

the policyholder fund as follows.

$$F_1 = P_1(1 + i_1)(1 - MER)$$
$$= 500(1.03597)(1 - 0.0248)$$
$$= 505.14.$$

$$F_5 = (F_4 + P_5)(1 + i_5)(1 - MER)$$
$$= (819.8 + 100)(1.03815)(1 - 0.0248)$$
$$= 931.21.$$

Table 9: Stochastic Test of Policyholder Fund

Time	P_t	$i_{1,t}$	F_t Before	MC	F_t
0	$0.00	0	$0.00	$0.00	$0.00
1	$500.00	0.03597	$517.98	$12.85	$505.14
2	$100.00	0.05327	$637.37	$15.81	$621.57
3	$100.00	0.03221	$744.81	$18.47	$726.34
4	$100.00	0.01732	$840.65	$20.85	$819.80
5	$100.00	0.03815	$954.89	$23.68	$931.21
6	$100.00	0.01635	$1,048.06	$25.99	$1,022.07
7	$100.00	0.05934	$1,188.66	$29.48	$1,159.18
8	$100.00	0.06573	$1,341.95	$33.28	$1,308.67
9	$100.00	0.02903	$1,449.55	$35.95	$1,413.60
10	$100.00	0.01898	$1,542.32	$38.25	$1,504.08

Once the expected policyholder fund has been calculated, we then look at the cash flows for the insurer. We take the management charges which have already been calculated in the first step, then subtract expenses and the expected death benefit payable.

$$EDB_5 = q_{x+5} \ max \left\{ F_5, GMDB \times \sum_{t=1}^{5} P_t \right\}$$
$$= 0.01346 \ max\{931.21, 500 + 4(100)\}$$
$$= 0.1345(931.21)$$
$$= 12.53$$

It is important to remember the guaranteed minimum death benefit as this is the source of risk for the insurer. In this case the minimum death benefit is 100% of the premiums paid, $GMDB = 1$. Note that if we set $GMDB = 1.5$, the expected death benefit will increase

$$EDB_5 = 0.01346 \ max\{931.21, 1.5(500 + 4(100))\}$$
$$= 0.01345 \ max\{931.21, 1350\}$$
$$= 18.16.$$

We then calculate the emerging surplus in each year as before, except we include the guaranteed minimum maturity benefit in the final year.

$$Pr_5 = MC_5 - E_5 - EDB_5$$
$$= 23.68 - 1.7 - 12.53$$
$$= 9.45$$

$$Pr_{10} = MC_{10} - E_5 - EDB_5 - {}_{10}p_{65}max\{F_{10} - GMMB\sum_{t=1}^{1}0P_t, 0\}$$

$$= MC_{10} - E_5 - EDB_5 - \left(\prod_{t=1}^{10}(1 - q_{65+t})\right)max\{F_{10} - GMMB\sum_{t=1}^{1}0P_t, 0\}$$

$$= 38.25 - 1.7 - 33.21 - 0.8619(max\{500 + 9(100) - 1504.08, 0\})$$

$$= 3.34$$

We can build Table (10) with these calculations.

Table 10: Stochastic Test of Insurer Cash Flows

Time	MC_t	E_t	q_{x+t}	EDB_t	Pr_t
0	$0.00	$25.00	0	$0.00	-$25.00
1	$12.85	$1.70	0.00918	$4.64	$6.51
2	$15.81	$1.70	0.01009	$6.27	$7.83
3	$18.47	$1.70	0.0111	$8.06	$8.71
4	$20.85	$1.70	0.01222	$10.02	$9.13
5	$23.68	$1.70	0.01346	$12.53	$9.45
6	$25.99	$1.70	0.01484	$15.17	$9.12
7	$29.48	$1.70	0.01637	$18.98	$8.80
8	$33.28	$1.70	0.01808	$23.66	$7.92
9	$35.95	$1.70	0.01997	$28.23	$6.02
10	$38.25	$1.70	0.02208	$33.21	$3.34

Then we finally calculate the profit signature and net present value as before.

$$\Pi_5 = p_{65+5}Pr_5$$
$$= 0.9865(9.45)$$
$$= 9.32$$

$$NPV = \sum_{t=0}^{5}\Pi_t\nu^t$$
$$= -25 + 6.45\frac{1}{1.103} + ... + 9.32(\frac{1}{1.103})^5$$
$$= 5.44$$

Table 11: NPV of Equity-Linked Policy

Time	Pr_t	p_{x+t}	Π_t	NPV
0	-$25.00	1	-$25.00	-$25.00
1	$6.51	0.99082	$6.45	-$19.15
2	$7.83	0.98991	$7.76	-$12.78
3	$8.71	0.9889	$8.61	-$6.36
4	$9.13	0.98778	$9.02	-$0.27
5	$9.45	0.98654	$9.32	$5.44
6	$9.12	0.98516	$8.99	$10.43
7	$8.80	0.98363	$8.66	$14.79
8	$7.92	0.98192	$7.78	$18.34
9	$6.02	0.98003	$5.90	$20.78
10	$3.34	0.97792	$3.27	$22.01

We can build the Table (11) using these equations.

Summary of Profit Measures

$NPV = \$22.01$

$IRR = 29\%$

$DPP = 5$

$$\text{Profit margin} = \frac{\$22.01}{\$500 + \$100\nu a_{66:\overline{9}|}} = 2.07\%$$

Now that it is determined how to do one instance of the profit test, we want to simulate this same process many times. In this example we will do 1000 simulations to determine a distribution for NPV and the loss L. From these distributions we want to find the maximum, minimum, average, standard deviation, and a 95% confidence interval for the expected net present value, as well as use the loss to determine the reserve using the CTE method. We set the level of the CTE reserve at 75%. The reserve value will then be equal to the average loss in the losses that exceed the 75th percentile. To see the code used to perform this simulation see Appendix B. We get the following results from the simulation:

Statistics

Average $= \$12.38$

Standard Deviation $= \$1.64$

Maximum $= \$19.87$

Minimum $= \$6.77$

95% CI $= [\$12.28, \$12.48]$

CTE75 reserve $= \$0.00$

From these statistics the insurer would be able to determine whether the product is profitable enough.

An important result of simulation is the ease of changing factors in the policy. Specifically, the insurer may want to change the guaranteed minimum benefits offered. We see that with the 75/100 maturity/death minimums, there is no need to hold reserves for this policy. Lets see how the policy looks with multiple different guaranteed minimum benefits.

100/100

Average = $11.18

Standard Deviation = $4.76

Maximum = $18.18

Minimum = −$20.30

95% CI = [$10.89, $11.48]

CTE75 reserve = $0.00

In this case we offer 100% minimum maturity and death benefits to the policyholder. We see the average drops slightly and a large increase in the standard deviation. Also there are now negative values appearing for the net present value when the fund does poorly. The reserves held are still zero as the risk of a large loss is still low for the insurer.

100/125

Average = −$7.13

Standard Deviation = $6.57

Maximum = $7.52

Minimum = −$41.78

95% CI = [−$7.54, −$6.73]

CTE75 reserve = $0.00

Here is the same policy with 100% guaranteed minimum maturity benefit and 125% death benefit. This is clearly not worth offering as the insurer would be losing money in most cases. The insurer might considering adjusting the premium so that a only a portion of the premium is allocated to the policyholder fund, with the rest being allocated to insurer assets. What is interesting here is that even though the policy is not profitable for the insurer, there is no need for reserves as the loss amounts remain negative. This is due to the initial expense being paid at the start of the contract, after which all emerging surplus amounts are positive, while not being sufficient to make up for the initial expense.

<u>75/75</u>

Average = $12.79

Standard Deviation = $1.30

Maximum = $19.53

Minimum = $10.23

95% CI = [$12.71, $12.87]

CTE75 reserve = $0.00

We look at the case of 75% guaranteed minimum benefits offered in both categories. We notice that the numbers are very similar to the 75/100 policy, but the insurer does a little bit better as there would be fewer cases where the fund falls below 75% of total premiums paid.

<u>110/100</u>

Average = −$13.84

Standard Deviation = $18.57

Maximum = $20.02

Minimum = −$62.16

95% CI = [−$14.99, −$12.68]

CTE75 reserve = $63.81

This policy offers a 110% guaranteed minimum maturity benefit, with 100% death benefit. This is clearly the most volatile of the policies we have looked at and it also would not be worthwhile for the insurer to offer, as most cases are negative NPV. Similar to the 100/125 a solution may be to allocate some of the premiums to the insurer fund to make up for the loss.

6   Conclusion

In this paper we have seen how an insurer would evaluate its products using cash flow analysis. Specifically the traditional 10-year term life insurance and 10-year equity-linked insurance policies were examined. Cash flow analysis using software offers added flexibility to the manual calculation method as it is very easy to change inputs provided the program is set up properly. An insurer is able to easily implement variable premiums, expenses, interest rates to test the product. We also looked at how an insurer can use the profit test to set premiums and reserves. Using the conditional tail expectation of a loss, the insurer can set the desired level of risk, and maximizes profit at this risk level. Finally we examined some types of profit measures an insurer

can use based on the profit test such as the net present value and profit margin in the case of the deterministic test, and statistics of the distribution of the net present value in the stochastic test.

When the equity-linked insurance policy was tested, we find that adjustments in the guaranteed minimum benefits can change the profitability for the insurer, yet even with reasonable changes to these amounts, the CTE75 reserve is still zero, as the risk is not so high with this policy to need reserves. When the guaranteed minimums offered are very large amounts, the insurer would then have to hold reserves.

7 References

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A Life Table Excerpt - Population of Canada, Both Sexes, Age 55 to 75

Age	l_x	d_x	q_x	p_x	e_x
55	94814	352	0.00371	0.99629	30.3
56	94462	382	0.00405	0.99595	29.4
57	94080	416	0.00442	0.99558	28.6
58	93664	452	0.00483	0.99517	27.7
59	93212	492	0.00528	0.99472	26.8
60	92720	536	0.00578	0.99422	25.9
61	92184	584	0.00633	0.99367	25
62	91600	636	0.00694	0.99306	24.2
63	90964	692	0.00761	0.99239	23.3
64	90272	754	0.00836	0.99164	22.5
65	89517	822	0.00918	0.99082	21.7
66	88696	895	0.01009	0.98991	20.8
67	87801	975	0.0111	0.9889	19.2
68	86826	1061	0.01222	0.98778	18.4
69	85765	1155	0.01346	0.98654	17.7
70	84611	1256	0.01484	0.98516	16.9
71	83355	1365	0.01637	0.98363	16.1
72	81991	1482	0.01808	0.98192	15.4
73	80508	1608	0.01997	0.98003	14.7
74	78900	1742	0.02208	0.97792	13.3
75	77158	1885	0.02443	0.97557	12.6

l_x - Number of people alive at age x per 100000 alive at age 0

d_x - Number of deaths at age x per 100000 alive at age 0

q_x - Probability of death during the year at age x

p_x - Probability of surviving the year at age x

e_x - Expected future lifetime at age x

B Code used to Simulate the Stochastic Profit Test 1000 times

This function is for use in Microsoft Visual Basic for Applications, also known as VBA in Excel. See appendix C for the inputs of this function. The function returns the statistics shown in section (5.3).

```
[fontsize=\small]
Public Function EQUITYLINKED(i As Double, initial_premium _
+ As Double, annuity_premium As Double, age As Integer, _
+ expected_PH_fund_interest_rate As Double, _
+ PH_fund_interest_standard_deviation As Double, _
+ gmmb As Double, gmdb As Double, _
+ mer As Double, initial_expenses As Double, _
+ annual_expenses As Double, ci_level, _
+ cte_level As Double) As Variant


'Define variables as arrays since we may want to remember
'values in previous years
Dim NPV(1 To 1000) As Variant
Dim loss(1 To 1000) As Variant
Dim interest(1 To 10) As Double
Dim fund(1 To 10) As Double
Dim charge(1 To 10) As Double
Dim edb(1 To 10) As Double
Dim expenses(1 To 10) As Double
Dim profit(1 To 10) As Double
Dim sig(1 To 10) As Double
Dim kNPV(1 To 10) As Double
Dim px(1 To 10) As Double
Dim nu(1 To 10) As Double
Dim kloss(1 To 10) As Double
Dim important_statistics(1 To 7) As Variant
Dim large_loss As Variant

'Counting variables
Dim t As Integer
Dim k As Integer

'Variables for the Normal mean and standard deviation
```

```
Dim mu As Double
Dim sd As Double

'Calculate Normal mean and standard deviation
mu = Log((expected_PH_fund_interest_rate) ^ (2) / _
+ (Sqr((expected_PH_fund_interest_rate) ^ (2) + _
+ (PH_fund_interest_standard_deviation) ^ (2))))
sd = Sqr(Log(1 + (PH_fund_interest_standard_deviation) ^ (2) / _
+ (expected_PH_fund_interest_rate) ^ (2)))


'Loop to calculate discount rates and set survival probabilities
'based on the Excel Sheet "Life Table"
'This sheet is provided from Statistics Canada
For t = 1 To 10
nu(t) = (1 / (1 + i)) ^ t
px(t) = Worksheets("Life Table").Cells(age + 5 + t, 6).Value
Next t


'Initialize profit test variables
For k = 1 To 1000
interest(1) = Exp(mu + WorksheetFunction.Norm_Inv_
+(Rnd, 0, 1) * sd)
charge(1) = initial_premium * (1 + interest(1)) * mer
fund(1) = initial_premium * (1 + interest(1)) - charge(1)
edb(1) = WorksheetFunction.Max((1 - px(1)) * fund(1), _
+ (1 - px(1)) * initial_premium * gmdb)
expenses(1) = annual_expenses * annuity_premium
profit(1) = charge(1) - expenses(1) - edb(1)
sig(1) = profit(1) * px(1)
kloss(1) = -profit(1)
kNPV(1) = sig(1) * nu(1) - initial_expenses


'loop through profit test
For t = 2 To 10
interest(t) = Exp(mu + WorksheetFunction._
+Norm_Inv(Rnd, 0, 1) + * sd)
charge(t) = (fund(t - 1) + annuity_premium) _
+ * (1 + interest(t)) * mer
fund(t) = (fund(t - 1) + annuity_premium) * _
+ (1 + interest(t)) - charge(t)
```

```
edb(t) = WorksheetFunction.Max((1 - px(t)) * _
+ fund(t), _
+ (1 - px(t)) * (initial_premium + annuity_premium * _
+ (t - 1)) * gmdb)
expenses(t) = annual_expenses * annuity_premium


'want to check the gmmb if it is time 10
If t = 10 Then
profit(t) = charge(t) - expenses(t) - edb(t) - TPX(age, 10) * _
+ WorksheetFunction.Max(0, (initial_premium + 9 * annuity_premium) _
+ *gmmb - fund(10))

Else
profit(t) = charge(t) - expenses(t) - edb(t)
End If

sig(t) = profit(t) * px(t)
kloss(t) = kloss(t - 1) - profit(t) * TPX(age, t - 1)
kNPV(t) = kNPV(t - 1) + sig(t) * nu(t)
Next t

'set the loss and npv arrays as the final values
loss(k) = kloss(10)
NPV(k) = kNPV(10)
Next k


'define the output values
important_statistics(1) = WorksheetFunction.Average(NPV)
important_statistics(2) = WorksheetFunction.StDev_S(NPV)
important_statistics(3) = WorksheetFunction.Max(NPV)
important_statistics(4) = WorksheetFunction.Min(NPV)
important_statistics(5) = important_statistics(1) - _
+ WorksheetFunction.Confidence_Norm(0.05, _
+ important_statistics(2), 1000)
important_statistics(6) = important_statistics(1) + _
+WorksheetFunction.Confidence_Norm(0.05, _
+ important_statistics(2), 1000)
important_statistics(7) = CTE(cte_level, loss)

EQUITYLINKED = important_statistics
End Function
```

C Inputs for EQUITYLINKED Function

Name	Notation
Discount Rate	i
Initial Premium	P_1
Annuity Premium	P_t
Age of Insured	N/A
PH Fund E[Interest]	μ
PH Fund SD[Interest]	σ
GMMB	$gmmb$
GMDB	$gmdb$
Management Charge	mc_t
Initial Expense	E_0
Annual Expenses	E_t
CI alpha	α_{CI}
CTE alpha	α_{CTE}

D   CTE Function

This function is used in the EQUITYLINKED function to calculate the CTE reserve.

```
Public Function CTE(level As Double, losses As Variant)


Dim losses_above_quantile() As Variant
Dim quantile_loss As Double
Dim k As Integer
Dim count As Integer
Dim avg As Double


'Find the minimum loss of the new distribution
quantile_loss = WorksheetFunction.Percentile(losses, level)


'Build an array of losses that exceed the set percentile
For k = 1 To UBound(losses)
If losses(k) >= quantile_loss Then

count = count + 1
ReDim Preserve losses_above_quantile(1 To count)
losses_above_quantile(count) = losses(k)

End If
Next k

'Take the average of the large losses
avg = WorksheetFunction.Average(losses_above_quantile)

'Set the reserve to 0 if the loss is negative
If avg > 0 Then
CTE = avg
Else
CTE = 0
End If
End Function
```

E $_tp_x$ function

The TPX function is used in the EQUITYLINKED function to determine $_tp_x$ when needed.

```
Public Function TPX(age As Integer, time As Integer)


'Define variables and set their length equal to time input
Dim kpx() As Double
Dim px() As Double
Dim t As Integer
ReDim kpx(time)
ReDim px(time)

'take the survival probability from the life table
kpx(1) = Worksheets("Life Table").Cells(age + 5, 6).Value

If time >= 2 Then
For t = 2 To time
px(t) = Worksheets("Life Table").Cells(age + 5 + t - 1, 6).Value
kpx(t) = px(t) * kpx(t - 1)
Next t
End If

'Return the product of the single year survival probabilities
TPX = kpx(time)

End Function
```