CARLETON UNIVERSITY

SCHOOL OF
MATHEMATICS AND STATISTICS

HONOURS PROJECT

TITLE: The Circle Method Applied to Goldbach's Weak Conjecture

AUTHOR: Michael Cayanan

SUPERVISOR: Brandon Fodden

DATE: August 27, 2019

Acknowledgements

First and foremost, I would like to thank my supervisor Brandon Fodden not only for all of his advice and guidance throughout the writing of this paper, but also for being such a welcoming and understanding instructor during my first year of undergraduate study. I would also like to thank the second reader Kenneth Williams for graciously providing his time and personal insight into this paper. Of course, I am also very grateful for the countless professors and instructors who have offered me outlets to explore my curiosity and to cultivate my appreciation of mathematical abstraction through the courses that they have organized.

I am eternally indebted to the love and continual support provided by my family and my dearest friends throughout my undergraduate journey. They were always able to provide assurance and uplift me during my worst and darkest moments, and words cannot describe how grateful I am to be surrounded with so much compassion and inspiration. This work is as much yours as it is mine, and I love you all 3000.

THE CIRCLE METHOD APPLIED TO GOLDBACH'S WEAK CONJECTURE

Michael Cayanan

August 27, 2019

Abstract. Additive number theory is the branch of number theory that is devoted to the study of representations of integers subject to various arithmetic constraints. One of, if not, the most popular question that additive number theory asks arose in a letter written by Christian Goldbach to Leonhard Euler, famously dubbed *the Goldbach conjecture*. Dated June 7th, 1742, the translated conjecture states that "Every number n which is a sum of two primes is a sum of as many primes including unity as one wishes (up to n), and that every number > 2 is a sum of three primes". The primary goal of this paper is to tackle a much easier problem concerning the representation of every odd integer greater than 5 as a sum of three primes (appropriately named the *weak Goldbach conjecture*, or the *ternary Goldbach conjecture*), under the framework of the famed "circle method", mainly attributed to Hardy, Littlewood, Ramanujan and Vinogradov. The circle method will allow us to translate the weak conjecture into a complex-analytic problem relying on the estimation of integrals performed over a constructed partition of $[0, 1]$. We will also explore the extent and the limitations of the circle method when applied to Goldbach's binary conjecture, as well as its an application concerning Waring's problem.

1. Notation and Number Theory Preliminaries

Before we introduce the circle method, some preliminary definitions and results from elementary number theory will be briefly recalled. As usual, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{N}$ the set of all positive integers. In what follows, p will almost exclusively denote a prime number and $\mathcal{P}$ will denote the set of all prime numbers. We say that a nonzero integer d divides an integer n if $n = cd$ for some $c \in \mathbb{Z}$, and this will be denoted by $d \mid n$. The greatest common divisor of integers a and b (not both 0) will always be presented as (a, b). If $(a, b) = 1$, then we say that a and b are relatively prime, or co-prime.

Definition 1.1. An *arithmetic function* f is a complex valued function defined on the positive integers. If f is not identically zero and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, then f is a *multiplicative function*.

We now briefly describe two elementary arithmetic functions and some of their useful properties.

Definition 1.2. For $n > 1$ with prime factorization $n = p_1^{x_1} \dots p_k^{x_k}$, we define the *Möbius function* $\mu(n)$ as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } x_1 = \cdots = x_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 1$, then we set $\mu(1) = 1$.

Definition 1.3. For $n \geq 1$, the *Euler totient function* $\phi(n)$ is equal to the number of positive integers $k \leq n$ such that $(n,k) = 1$. That is,

$$\phi(n) = \sum_{\substack{k=1 \\ (n,k)=1}}^{n} 1.$$

It is well known that both μ and ϕ are multiplicative functions. Evaluating the totient function at prime powers yields $\phi(p^k) = p^k - p^{k-1} = p^k\big(1 - \frac{1}{p}\big)$, and more generally, $\phi(n) = n\prod_{p|n}\big(1 - \frac{1}{p}\big)$. Summing over all of the divisors d of n for $\phi(n)$ and $\mu(n)$, this gives

$$\sum_{d|n} \phi(d) = n, \text{ and } \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

All of these results may be found in any elementary number theory texts such as Apostol [1] (Chapters 2.2 and 2.3) or Nathanson [15] (Sections A.5 and A.6).

Proposition 1.4. *Given $\varepsilon > 0$, we have $n^{1-\varepsilon} < \phi(n) < n$ for all sufficiently large n.*

Proof. Clearly, $\phi(n) < n$, $\forall n > 1$. Now, we want to show that

$$\lim_{n\to\infty} \frac{n^{1-\varepsilon}}{\phi(n)} = 0.$$

Since ϕ is multiplicative, it suffices to show that the result holds for prime powers. Observe that for every prime p, we have that $\frac{p}{p-1} \leq 2$, hence

$$\frac{p^{m(1-\varepsilon)}}{\phi(p^m)} = \frac{p^{m(1-\varepsilon)}}{p^m - p^{m-1}} = \frac{p}{p-1}\frac{p^{m(1-\varepsilon)}}{p^m} \leq \frac{2}{p^{m\varepsilon}}.$$

Thus,

$$\lim_{p^m\to\infty} \frac{p^{m(1-\varepsilon)}}{\phi(p^m)} = 0.$$

□

For notational convenience, for $x \in \mathbb{R}$, we will set $e(x) := e^{2\pi i x}$. We will now define an exponential sum and prove an important reformulation using the Möbius function which we will make use of.

Definition 1.5. Let $q, n \in \mathbb{Z}$ and $q \geq 1$. We define the *Ramanujan sum* as the exponential sum

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e\Big(\frac{kn}{q}\Big).$$

Theorem 1.6. *The Ramanujan sum can be expressed in the form*

$$c_q(n) = \sum_{d|(q,n)} \mu\Big(\frac{q}{d}\Big)d.$$

Furthermore, if $(q,n) = 1$, we have $c_q(n) = \mu(q)$.

Proof. As noted above, we know that $\sum_{d|1} \mu(d) = 1$, hence

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e\Big(\frac{kn}{q}\Big) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e\Big(\frac{kn}{q}\Big) \cdot 1 = \sum_{k=1}^{q} e\Big(\frac{kn}{q}\Big) \sum_{d|(k,q)} \mu(d).$$

Therefore,

$$c_q(n) = \sum_{d|q} \mu(d) \sum_{\substack{k=1 \\ d|k}}^{q} e\Big(\frac{kn}{q}\Big) = \sum_{d|q} \mu(d) \sum_{\substack{l=1 \\ k=dl}}^{q/d} e\Big(\frac{dln}{q}\Big) = \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e\Big(\frac{ln}{q/d}\Big).$$

Define $f_d(n) := \sum_{l=1}^{d} e\big(\frac{ln}{d}\big)$, and note that

$$f_d(n) = \begin{cases} d & \text{if } d \mid n \\ 0 & \text{otherwise.} \end{cases}$$

From this, we then have that

$$\begin{aligned} c_q(n) &= \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e\Big(\frac{ln}{q/d}\Big) \\ &= \sum_{d|q} \mu(d) f_{q/d}(n) \\ &= \sum_{d|q} \mu\Big(\frac{q}{d}\Big) f_d(n) \\ &= \sum_{\substack{d|q \\ d|n}} \mu\Big(\frac{q}{d}\Big) d \\ &= \sum_{d|(q,n)} \mu\Big(\frac{q}{d}\Big) d. \end{aligned}$$

If $(q, n) = 1$, then the formula simplifies to $c_q(n) = \mu(q)$, as required. □

Using this result, we may now show that $c_q(n)$ is a multiplicative function in q. For if $q_1, q_2 \geq 1$ with $(q_1, q_2) = 1$, we have

$$c_{q_1}(n) \cdot c_{q_2}(n) = \sum_{d_1|(q_1,n)} \mu\Big(\frac{q_1}{d_1}\Big) d_1 \sum_{d_2|(q_2,n)} \mu\Big(\frac{q_2}{d_2}\Big) d_2 = \sum_{\substack{d_1|(q_1,n) \\ d_2|(q_2,n)}} \mu\Big(\frac{q_1}{d_1}\Big) \mu\Big(\frac{q_2}{d_2}\Big) d_1 d_2.$$

Writing $d = d_1 d_2$ and noting that μ is multiplicative, we rewrite the previous summation as

$$\sum_{d|(q_1,n)\cdot(q_2,n)} \mu\Big(\frac{q_1 q_2}{d}\Big) d = \sum_{d|(q_1 \cdot q_2, n)} \mu\Big(\frac{q_1 q_2}{d}\Big) d = c_{q_1 \cdot q_2}(n).$$

To conclude this introductory section, we introduce the last few number theoretic results and definitions which will greatly aid us in the proof of Vinogradov's Theorem. Excluding Theorem 1.11 and its following remark, the proofs of all of these results may be found in the appendix of Nathanson [15].

Theorem 1.7 (Euler Products)**.** *Let f be a multiplicative function. Suppose that the series $\sum_{n=1}^{\infty} f(n)$ converges absolutely. Then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p}(1 + f(p) + f(p^2) + \cdots) = \prod_{p}\big(1 + \sum_{k=1}^{\infty} f(p^k)\big).$$

If f is completely multiplicative, then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p}(1 - f(p))^{-1}.$$

Definition 1.8. Let f be any complex valued function, and let g be any positive function. Then $f = O(g)$ if and only if there exists $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x large enough. The constant C is called the *implied constant*. We may occasionally write $f \ll g$ (notation due to Vinogradov) in place of $f = O(g)$ as well.

Definition 1.9. We say that f is *asymptotic* to g, denoted $f \sim g$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Theorem 1.10 (Partial Summation)**.** *Let $a(n)$ be an arithmetic function, and suppose that f is a function with continuous derivative on an interval $[y, x]$ with $0 \leq y < x$. Define the sum function $A(x) = \sum_{n\leq x} a(n)$, and $A(x) = 0$ if $x < 1$. Then*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_{y}^{x} A(t)f'(t)dt.$$

In particular, taking $0 < y < 1$ yields

$$\sum_{n\leq x} a(n)f(n) = A(x)f(x) - \int_{1}^{x} A(t)f'(t)dt.$$

Theorem 1.11 (Prime Number Theorem)**.** *Define $\pi(x) = \sum_{p\leq x} 1$ as the prime-counting function. Then*

$$\pi(x) \sim \frac{x}{\log x}.$$

Remark 1.12. There are several equivalent forms for the Prime Number Theorem. Several equivalences involve weighted prime counting functions: Chebyshev's θ-function and Chebyshev's ψ-function, respectively given by the sums

$$\theta(x) = \sum_{p\leq x} \log p \quad\text{and}\quad \psi(x) = \sum_{p^k\leq x} \log p.$$

The Prime Number Theorem is equivalent to either of the asymptotic relations $\theta(x) \sim x$ and $\psi(x) \sim x$. Moreover, one can show that there exists $c_1, c_2 > 0$ such that

$$c_1 x \leq \theta(x) \leq \psi(x) \leq \pi(x)\log x \leq c_2 x.$$

This is known as Chebyshev's Theorem. A proof of the Prime Number Theorem is presented in great detail in Chapter 13 of Apostol [1], while Chebyshev's Theorem may be found in Section 6.2 of Nathanson [15].

2. The Circle Method

Consider a set A of nonnegative integers. For some positive integers n and s, we define $r_{s,A}(n)$ as the number of representations of n as the sum of s elements of A (the number of solutions to the equation $a_1 + \cdots + a_s = n$, where $a_1, \ldots a_s \in A$). That is,

$$r_{s,A}(n) = \sum_{\substack{a_1+\cdots+a_s=n \\ a_1,\ldots,a_s\in A}} 1.$$

Note that the order of the summands from A is taken into account, so for example, $2+3$ and $3+2$ both contribute to $r_{2,\mathbb{N}}(5)$. If the set A is clear from the context, we will simply denote this by $r_s(n)$.

We are interested in estimating $r_{s,A}(n)$ for specific values of s and sets A. For example, if we take $s = 2$ and $A = \mathcal{P}$ as the set of all prime numbers, then $r_{2,\mathcal{P}}(n)$ counts the number of ways n can be written as a sum of two primes. Indeed, showing that such a quantity is positive for all even $n \geq 4$ is equivalent to the verification of Goldbach's conjecture. Similarly, setting $s = 3$ allows one to study Goldbach's weak conjecture, which asserts that every odd number greater than 5 can be represented as the sum of three primes.

In order to motivate the overall idea, we will define the generating function for A as the power series $f(z) = \sum_{a\in A} z^a$. Let $a(n) = 1$ if $n \in A$, and $a(n) = 0$ otherwise. Note that squaring $f(z)$ yields

$$f(z)^2 = \left(\sum_{a\in A} z^a\right)^2 = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \sum_{\substack{0\leq h,k\leq n \\ h+k=n}}\sum a(h)a(k).$$

Note that $a(h)a(k) = 1$ if $h, k \in A$, and $a(h)a(k) = 0$ otherwise. Thus, c_n counts the number of ways n may be expressed as a sum of two elements belonging in A, and so $c_n = r_{2,A}(n)$. Using the same argument as above, one may deduce that for $s \in \mathbb{N}$, the sth power of $f(z)$ is given by

$$f(z)^s = \left(\sum_{a\in A} z^a\right)^s = \sum_{n=0}^{\infty} r_{s,A}(n) z^n.$$

From this, we see that $f(z)^s$ is actually the generating function for $r_{s,A}(n)$. Explicitly determining the value of $r_{s,A}(n)$ for specific values of s and n can be difficult, especially when these parameters are rather large. However, certain choices for the set A can lead to easily obtainable formulas for $r_{s,A}(n)$.

Example 2.1. Let $s \geq 1$ and $\mathcal{K} = \{0, 1^k, 2^k, \ldots\}$ be the set of k powers, where k is a positive integer. Waring's problem asks whether each nonnegative integer is the sum of a bounded number of kth powers.

For $k = 1$ and $n \geq s$, we may obtain an explicit formula for $r_s(n)$. We are now studying the number of representations for the expression $n = a_1 + \cdots + a_s$. Subtracting s on both sides, we obtain $n - s = (a_1 - 1) + \cdots + (a_s - 1)$. If we denote by $\mathfrak{r}_s(n)$ the number of representations of n as the sum of s nonnegative integers, then the above argument tells us that $r_s(n) = \mathfrak{r}_s(n-s)$. We will show that

$$r_s(n) = \mathfrak{r}_s(n-s) = \binom{n-1}{s-1}.$$

Proof. Let $|z| < 1$. Then the geometric series $f(z) = \sum_{n=0}^{\infty} z^n$ converges to the sum $\frac{1}{1-z}$. Taking s powers, we obtain the generating function for $\mathfrak{r}_s(n)$: $f(z)^s = \sum_{n=0}^{\infty} \mathfrak{r}_s(n) z^n$. On the other hand, we have

$$\begin{aligned} f(z)^s &= \frac{1}{(1-z)^s} \\ &= \frac{1}{(s-1)!} \frac{d^{s-1}}{dz^{s-1}} \left(\frac{1}{1-z} \right) \\ &= \frac{1}{(s-1)!} \frac{d^{s-1}}{dz^{s-1}} \left(\sum_{k=0}^{\infty} z^k \right) \\ &= \frac{1}{(s-1)!} \sum_{k=s-1}^{\infty} k(k-1)\cdots(k-s+2) z^{k-s+1} \\ &= \sum_{k=s-1}^{\infty} \frac{k(k-1)\cdots(k-s+2)}{(s-1)!} \cdot \frac{(k-s+1)!}{(k-s+1)!} z^{k-s+1} \\ &= \sum_{k=s-1}^{\infty} \binom{k}{s-1} z^{k-s+1}. \end{aligned}$$

Re-indexing the last sum with $n = k - s + 1$, we obtain

$$f(z)^s = \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} z^n.$$

Thus,

$$\mathfrak{r}_s(n) = \binom{n+s-1}{s-1},$$

and by our previous observation we have that

$$r_s(n) = \mathfrak{r}_s(n-s) = \binom{n-1}{s-1}.$$

□

Now continuing from our original generating function, let $C_\rho = \{z \in \mathbb{C} \ : \ |z| = \rho\}$ denote the circle centred at the origin with radius ρ. Using Cauchy's Theorem, we can recover the coefficients of the generating function by considering a contour integral. We have

$$\frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)^s}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{C_\rho} \frac{\sum_{k=0}^{\infty} r_{s,A}(k) z^k}{z^{n+1}} dz = \frac{1}{2\pi i} \sum_{k=0}^{\infty} r_{s,A}(k) \int_{C_\rho} z^{k-n-1} = r_{s,A}(n)$$

since $\frac{1}{2\pi i}\int_{C_\rho} z^{k-n-1} = 1$, if $k-n-1=-1$ (i.e. $k=n$), and is 0 otherwise.

This was the original precursor to the circle method developed by Hardy, Littlewood, and Ramanujan in 1918-1920. Assuming the Generalized Riemann Hypothesis, Hardy and Littlewood were able to conditionally show that every large odd number is a sum of three primes, and that almost every even number is a sum of two primes. In layman terms, they established bounds for the contour integral by showing that there exists a constant N such that for all odd $n > N$, the integral is at least 1 (i.e. $r_{3,\mathcal{P}}(n) \geq 1$ for all such n). Establishing such bounds was accomplished by performing the integration over a specific partition of the unit interval. We shall elaborate more on this principle when dealing with Vinogradov's refinement of the circle method.

3. The Singular Series, Major Arcs, and Minor Arcs

Before we proceed further, a new arithmetic function needs to be formally introduced.

Definition 3.1. The function

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \frac{\mu(q)c_q(n)}{\phi(q)^3}$$

is called the *singular series* for the ternary Goldbach problem.

Under the assumption of the Generalized Riemann Hypothesis, Hardy and Littlewood's original efforts resulted in the asymptotic relation

$$r_{3,\mathcal{P}}(n) \sim \mathfrak{S}(n)\frac{n^2}{2(\log n)^3},$$

where n is any sufficiently large odd integer. Therefore, the quantity $\mathfrak{S}(n)$ can be bounded away from 0, implying that there are many ways to write n as the sum of three distinct prime numbers and proving Goldbach's weak conjecture for large enough odd n. In 1937, Vinogradov was able to refine the work of Hardy and Littlewood by showing that the bounds are not conditional on the assumption of the Generalized Riemann Hypothesis.

Theorem 3.2 (Vinogradov's Theorem)**.** *There exists positive constants c_1 and c_2 such that*

$$c_1 < \mathfrak{S}(n) < c_2$$

for all sufficiently large odd integers n. Furthermore, for any sufficiently large odd integer n, we have

$$r_{3,\mathcal{P}}(n) = \mathfrak{S}(n)\frac{n^2}{2(\log n)^3}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right).$$

A large amount of time and effort will be spent towards proving the latter statement, while the former is much easier in comparison. We first observe the following.

Theorem 3.3. *For odd n, the singular series $\mathfrak{S}(n)$ converges absolutely and uniformly with Euler product*

$$\mathfrak{S}(n) = \prod_{p}\Big(1+\frac{1}{(p-1)^3}\Big)\prod_{p\mid n}\Big(1-\frac{1}{p^2-3p+3}\Big).$$

Furthermore, for any $\varepsilon > 0$,

$$\mathfrak{S}(n,Q) := \sum_{q\leq Q}\frac{\mu(q)c_q(n)}{\phi(q)^3} = \mathfrak{S}(n) + O(Q^{-(1-\varepsilon)}),$$

where the implied constant depends only on ε.

Observe that proving this theorem for odd n gives

$$\prod_{p\mid n}\left(1-\frac{1}{p^2-3p+3}\right) < \mathfrak{S}(n) < \prod_{p}\left(1+\frac{1}{(p-1)^3}\right).$$

Note that $1-\frac{1}{p^2-3p+3} < 1$, and so the resulting product over all the prime divisors of n will provide a lower bound for c_1. Setting $\prod_p\big(1+\frac{1}{(p-1)^3}\big)$ as c_2 finishes our argument. One may also interpret this inequality as $\mathfrak{S}(n) \gg 1$ for odd n.

Proof. Let $\varepsilon > 0$ be given. Proposition 1.4 gives us $q^{1-\varepsilon} < \phi(q)$ for sufficiently large q. By definition of $c_q(n)$ and $\phi(q)$, we also have $c_q(n) = O(\phi(q))$ since

$$|c_q(n)| = \Bigg|\sum_{\substack{k=1\\(k,q)=1}}^{q} e\Big(\frac{kn}{q}\Big)\Bigg| \leq \sum_{\substack{k=1\\(k,q)=1}}^{q} 1 = \phi(q).$$

Noting that $|\mu(q)| \leq 1$, the summands of the singular series can then be estimated, as

$$\frac{\mu(q)c_q(n)}{\phi(q)^3} = O\bigg(\frac{1}{\phi(q)^2}\bigg) = O\bigg(\frac{1}{q^{2-\varepsilon}}\bigg).$$

That is, the singular series converges absolutely and uniformly. From this, we also obtain

$$\mathfrak{S}(n) - \mathfrak{S}(n,Q) = \sum_{q>Q}\frac{\mu(q)c_q(n)}{\phi(q)^3} = O\Bigg(\sum_{q>Q}\frac{1}{q^{2-\varepsilon}}\Bigg) = O\bigg(\frac{1}{Q^{1-\varepsilon}}\bigg).$$

Now observe that if $p \mid n$, then

$$c_p(n) = \sum_{\substack{k=1\\(k,p)=1}}^{p} e\Big(\frac{kn}{p}\Big) = \sum_{k=1}^{p-1} 1 = p-1.$$

On the other hand, $p \nmid n$ gives $c_p(n) = \mu(p) = -1$ by Theorem 1.6. The expression $\frac{\mu(q)c_q(n)}{\phi(q)^3}$ is multiplicative with respect to q (as it is a product and quotient of multiplicative functions), and together with absolute convergence, we may utilize Theorem 1.7. Since $\mu(p^k) = 0$ for $k \geq 2$ we can now obtain

$$\mathfrak{S}(n) = \prod_{p}\left(1+\sum_{k=1}^{\infty}\frac{\mu(p^k)c_{p^k}(n)}{\phi(p^k)^3}\right)$$

$$
\begin{aligned}
&= \prod_p \left(1 - \frac{c_p(n)}{\phi(p)^3}\right) \\
&= \prod_{p\nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right) \\
&= \frac{\prod_p \left(1 + \frac{1}{(p-1)^3}\right)}{\prod_{p|n} \left(1 + \frac{1}{(p-1)^3}\right)} \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right) \\
&= \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \frac{1 - \frac{1}{(p-1)^2}}{1 + \frac{1}{(p-1)^3}}.
\end{aligned}
$$

Simplifying the second product yields

$$
\frac{1 - \frac{1}{(p-1)^2}}{1 + \frac{1}{(p-1)^3}} = \frac{p^2 - 2p}{p^3 - 3p^2 + 3p} \cdot \frac{(p-1)^3}{(p-1)^2} = \frac{(p-2)(p-1)}{p^2 - 3p + 3} = \frac{p^2 - 3p + 3 - 1}{p^2 - 3p + 3} = 1 - \frac{1}{p^2 + 3p + 3}.
$$

Thus, we have

$$
\mathfrak{S}(n) = \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \left(1 - \frac{1}{p^2 - 3p + 3}\right),
$$

as required. □

Remark 3.4. The notion of a singular series $\mathfrak{S}(n)$ extends to other applications of the circle method, and is not exclusive to Goldbach's weak conjecture. Typically, one requires $\mathfrak{S}(n)$ to be some arithmetic function uniformly bounded above and below by positive constants c_1 and c_2. Other examples of singular series can be found in Sections 7 and 8.

Remark 3.5. The condition of n being odd in Theorem 3.3 cannot be stressed enough. For even n, we see that

$$
\begin{aligned}
\mathfrak{S}(n) &= \prod_{p\nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right) \\
&= \left(1 - \frac{1}{(2-1)^2}\right) \prod_{p\nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{\substack{p|n \\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) \\
&= 0.
\end{aligned}
$$

Note that if Vinogradov's result held true for even n, then one of p_1, p_2, or p_3 would have to be even. Without loss of generality, let $p_1 = 2$. Then we may write $n - 2 = p_2 + p_3$, and we have seemingly proved that every even number can be written as a sum of two primes. That is, if we were able to take n to be even in Theorem 3.3, Goldbach's strong conjecture would follow.

We now reformulate the definition of $r_{s,A}(n)$ slightly and consider a weighted sum over the number of representations of n as a sum of s elements of A:

$$
R_{s,A}(n) = \sum_{\substack{a_1 + \cdots + a_s = n \\ a_1, \ldots, a_s \in A}} \log a_1 \log a_2 \cdots \log a_s.
$$

In what follows, we will fix $s = 3$ and $A = \mathcal{P}$, so that $r_{3,\mathcal{P}}(n) = r(n)$ and $R_{3,\mathcal{P}}(n) = R(n)$. Working analogously as before, we will define a specific generating function for $R(n)$ from which we can retrieve its coefficients through integration. The main difference in the approach here is that the generating function is now an exponential sum, rather than a traditional power series. Define

$$F(x) := \sum_{p \leq n} (\log p) e(px).$$

Then, the orthogonality relation

$$\int_0^1 e(mx)e(-nx)dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

gives us

$$\begin{aligned} \int_0^1 F(x)^3 e(-nx)dx &= \int_0^1 \sum_{p_1,p_2,p_3 \leq n} (\log p_1)(\log p_2)(\log p_3) e((p_1+p_2+p_3)x)e(-nx)dx \\ &= \sum_{p_1,p_2,p_3 \leq n} (\log p_1)(\log p_2)(\log p_3) \int_0^1 e((p_1+p_2+p_3)x)e(-nx)dx \\ &= \sum_{p_1+p_2+p_3 = n} (\log p_1)(\log p_2)(\log p_3) \\ &= R(n). \end{aligned}$$

The logarithmic weights are introduced for analytic reasons, and will simplify future computations. It was mentioned before that Hardy and Littlewood's original result involves a partition for $[0, 1]$. These disjoint partitions are called major and minor arcs, and the above integral was then evaluated over both sets separately.

Definition 3.6. For odd $n > 1$, fix $B > 0$ and let $Q = (\log n)^B$. For $1 \leq q \leq Q$, $1 \leq a \leq q$, and $(a, q) = 1$, we define the *major arc* $\mathfrak{M}(q, a)$ as the set

$$\mathfrak{M}(q,a) = \left\{ x \in [0,1] \ : \ \left| x - \frac{a}{q} \right| \leq \frac{Q}{n} \right\}.$$

The set of all major arcs will be denoted and given by

$$\mathfrak{M} = \bigcup_{q=1}^{Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q,a).$$

The set of *minor arcs* is correspondingly given by $\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$.

So the major arcs consist of real numbers $x \in [0, 1]$ that are well approximated by rationals with sufficiently small denominators that are within a distance of $\frac{Q}{n}$ from x.

Remark 3.7. For large values of n, the major arcs $\mathfrak{M}(q,a)$ are mutually disjoint for all a and q. Indeed, suppose $x \in \mathfrak{M}(q,a) \cap \mathfrak{M}(q',a')$ and $\frac{a}{q} \neq \frac{a'}{q'}$. Hence, $|aq' - a'q| \geq 1$, and with the triangle inequality, one obtains

$$\begin{aligned} \frac{1}{Q^2} \leq \frac{1}{qq'} \leq \frac{|aq'-a'q|}{qq'} &= \left|\frac{a}{q} - \frac{a'}{q'}\right| = \left|\frac{a}{q} - \frac{a'}{q'} + x - x\right| \\ &\leq \left|x - \frac{a}{q}\right| + \left|x - \frac{a'}{q'}\right| \leq \frac{2Q}{n} = \frac{2(\log n)^B}{n}. \end{aligned}$$

We may rearrange this to obtain $n \leq 2Q^3 = 2(\log n)^{3B}$, which is a contradiction for sufficiently large n.

Remark 3.8. Let μ be any measure on the σ-algebra of measurable sets on $\mathbb{R}$. We remark that the measure of the set $\mathfrak{M}$ tends to 0 as n goes to infinity. Since $\mu([a,b]) = b-a$ and μ is additive on disjoint sets, we have

$$\begin{aligned} \mu(\mathfrak{M}) = \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \mu(\mathfrak{M}(q,a)) = \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{2Q}{n} = \sum_{q=1}^{Q} \frac{2Q}{n} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} 1 = \sum_{q=1}^{Q} \frac{2Q}{n}\phi(q) \leq \frac{2Q}{n}\sum_{q=1}^{Q} q \\ \leq \frac{2Q}{n} \cdot \frac{Q(Q+1)}{2} \leq \frac{2Q^3}{n} = \frac{2(\log n)^{3B}}{n}, \end{aligned}$$

and this upper bound approaches 0 as n gets large.

This last remark also tells us that the measure of $\mathfrak{m}$ tends to 1 for increasing values of n, and so our minor arcs are actually considerably larger than our major arcs. This is a curious fact as we will end up showing that the main term in Vinogradov's theorem will come from the integral over the major arcs, while the contribution from the minor arcs will be negligible in comparison:

$$R(n) = \int_0^1 F(x)^3 e(-nx)dx = \int_{\mathfrak{M}} F(x)^3 e(-nx)dx + \int_{\mathfrak{m}} F(x)^3 e(-nx)dx.$$

4. Estimating the Major Arcs

We now introduce the singular integral

$$J(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta)^3 e(-n\beta)d\beta,$$

where $u(\beta)$ is an exponential sum given by $u(\beta) = \sum_{m=1}^{n} e(m\beta)$. The aforementioned contribution from the major arcs in Vinogradov's theorem will actually be the product of the singular series and the singular integral, with the latter being easy to deal with. Analogous to the argument utilizing the orthogonality relations preceding Definition 3.6, $J(n)$ can be seen to be nothing more than the number of ways to express n as a sum of three positive integers. Hence, $J(n) = r_s(n)$ from Example 2.1. An asymptotic formula for $J(n)$ is then given by

$$J(n) = \binom{n-1}{2} = \frac{(n-1)(n-2)(n-3)!}{2(n-3)!} = \frac{n^2-3n+2}{2} = \frac{n^2}{2} + O(n).$$

The next few results will rely on the following theorem.

Theorem 4.1 (Siegel-Walfisz)**.** *Let $C, B > 0$ and $(q, a) = 1$. Then for any $q \leq (\log x)^B$, we have*

$$\sum_{\substack{p \leq x \\ p \equiv a \mod q}} \log p = \frac{x}{\phi(q)} + O\left(\frac{x}{(\log x)^C}\right).$$

Here, the implied constant is dependent on B and C.

The Siegel-Walfisz theorem provides a uniform error term for primes in arithmetic progressions up to $q \leq (\log x)^B$. As $\phi(q)$ is at most q, which in turn is at most $(\log x)^B$, taking $C > B$ allows for the main term to be significantly larger then the error term. So it is in our best interest to keep C far away from B. A formal proof and the historical development of the Siegel-Walfisz theorem can be found in Chapters 21 and 22 of Davenport [4].

Lemma 4.2. *Let*

$$F_x(\alpha) := \sum_{p \leq x} (\log p) e(p\alpha).$$

Let $B, C > 0$. Suppose that $1 \leq q \leq Q = (\log n)^B$, and $(q, a) = 1$. For $1 \leq x \leq n$, we then have

$$F_x\left(\frac{a}{q}\right) = \frac{\mu(q)}{\phi(q)} x + O\left(\frac{Qn}{(\log n)^C}\right).$$

Here the implied constant depends only on B and C.

Proof. Let $p \equiv r \mod q$. Note that $p \mid q \iff (r, q) > 1$, and so

$$\sum_{\substack{r=1 \\ (r,q)>1}}^{q} \sum_{\substack{p \leq x \\ p \equiv r \mod q}} (\log p) e\left(\frac{pa}{q}\right) = \sum_{\substack{p \leq x \\ p \mid q}} (\log p) e\left(\frac{pa}{q}\right) = O\left(\sum_{p \mid q} \log p\right) = O(\log q) = O(\log Q).$$

We can then break $F_x\left(\frac{a}{q}\right)$ as sums over all $r = 1, \ldots, q$ separately for $(r, q) = 1$, and $(r, q) > 1$ as follows

$$\begin{aligned} F_x\left(\frac{a}{q}\right) &= \sum_{r=1}^{q} \sum_{\substack{p \leq x \\ p \equiv r \mod q}} (\log p) e\left(\frac{pa}{q}\right) \\ &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{\substack{p \leq x \\ p \equiv r \mod q}} (\log p) e\left(\frac{ra}{q}\right) + \sum_{\substack{r=1 \\ (r,q)>1}}^{q} \sum_{\substack{p \leq x \\ p \equiv r \mod q}} (\log p) e\left(\frac{pa}{q}\right). \end{aligned}$$

The Siegel-Walfisz Theorem then gives us

$$F_x\left(\frac{a}{q}\right) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{\substack{p \leq x \\ p \equiv r \mod q}} (\log p) e\left(\frac{ra}{q}\right) \; + \; O(\log Q)$$

$$= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ra}{q}\right) \sum_{\substack{p \le x \\ p \equiv r \mod q}} \log p \ + \ O(\log Q)$$
$$= c_q(a)\left(\frac{x}{\phi(q)} + O\left(\frac{x}{(\log x)^C}\right)\right) + O(\log Q)$$
$$= \frac{c_q(a)}{\phi(q)}x + O\left(\frac{|c_q(a)|x}{(\log x)^C}\right) + O(\log Q)$$
$$= \frac{\mu(q)}{\phi(q)}x + O\left(\frac{Qn}{(\log n)^C}\right).$$

Here we used the fact that $|c_q(n)| \le q \le Q$, and $c_q(a) = \mu(q)$ for $(q,a) = 1$ by Theorem 1.6. □

Given $x \in \mathbb{R}$, we define the integer part of x, denoted $\lfloor x \rfloor$, as the greatest integer $\le x$. That is, $\lfloor x \rfloor$ is the unique integer defined by the inequality $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. The fractional part of x is given by the quantity $\{x\} = x - \lfloor x \rfloor$, and so $\{x\} \in [0,1)$.

Lemma 4.3. *Suppose $B, C > 0$ with $C > 2B$. Let $x \in \mathfrak{M}(q,a)$ and $\beta = x - \frac{a}{q}$. Then*

$$F(x) = \frac{\mu(q)}{\phi(q)}u(\beta) + O\left(\frac{Q^2 n}{(\log n)^C}\right),$$

where the implied constant is dependent only on B and C.

Proof. Let $x \in \mathfrak{M}(q,a)$ and $\beta = x - \frac{a}{q}$. In particular, $x = \frac{a}{q} + \beta$, where $|\beta| \le \frac{Q}{n}$. Define

$$\lambda(m) := \begin{cases} \log p & \text{if } m = p \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \le x \le n$, we have

$$F(x) - \frac{\mu(q)}{\phi(q)}u(\beta) = \sum_{p \le n} (\log p) e(px) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{n} e(m\beta)$$
$$= \sum_{m=1}^{n} \lambda(m) e(mx) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{n} e(m\beta)$$
$$= \sum_{m=1}^{n} \lambda(m) e\left(\frac{ma}{q} + m\beta\right) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{n} e(m\beta)$$
$$= \sum_{m=1}^{n} \lambda(m) e\left(\frac{ma}{q}\right) e(m\beta) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{n} e(m\beta)$$
$$= \sum_{m=1}^{n} \left(\lambda(m) e\left(\frac{ma}{q}\right) - \frac{\mu(q)}{\phi(q)}\right) e(m\beta).$$

Let $a(m) = \lambda(m)e\left(\frac{ma}{q}\right) - \frac{\mu(q)}{\phi(q)}$. We wish to make use of partial summation (Theorem 1.10), but first we compute $A(x)$ using the previous lemma:

$$\begin{aligned}
A(x) &= \sum_{1\le m\le x}\left(\lambda(m)e\left(\frac{ma}{q}\right)-\frac{\mu(q)}{\phi(q)}\right)\\
&= \sum_{1\le m\le x}\lambda(m)e\left(\frac{ma}{q}\right)-\frac{\mu(q)}{\phi(q)}\lfloor x\rfloor\\
&= F_x\left(\frac{a}{q}\right)-\frac{\mu(q)}{\phi(q)}(x-\{x\})\\
&= F_x\left(\frac{a}{q}\right)-\frac{\mu(q)}{\phi(q)}x+\frac{\mu(q)}{\phi(q)}\{x\}\\
&= O\left(\frac{Qn}{(\log n)^C}\right)+O(1)\\
&= O\left(\frac{Qn}{(\log n)^C}\right).
\end{aligned}$$

Setting $f(m)=e(m\beta)$, partial summation yields

$$\begin{aligned}
\left|F(x)-\frac{\mu(q)}{\phi(q)}u(\beta)\right| &= \left|\sum_{m=1}^{n}a(m)f(m)\right|\\
&= \left|A(n)e(n\beta)-2\pi i\beta\int_1^n A(t)e(t\beta)dt\right|\\
&\le \left|A(n)e(n\beta)\right|+\left|2\pi i\beta\int_1^n A(t)e(t\beta)dt\right|\\
&\le |A(n)|+2\pi|\beta|\max_{1\le t\le n}|A(t)|\int_1^n 1dt\\
&= O\left(\frac{Qn}{(\log n)^C}\right)+O\left(\frac{Q^2n}{(\log n)^C}\right)\\
&= O\left(\frac{Q^2n}{(\log n)^C}\right),
\end{aligned}$$

as required. □

Moreover, by our choice of C, we have

$$\frac{Q^2n}{(\log n)^C}=\frac{(\log n)^{2B}n}{(\log n)^C}=\frac{n}{(\log n)^{C-2B}}.$$

Noting that $|u(\beta)|\le n$ and $\frac{1}{\phi(q)}\le 1$, cubing the result of Lemma 4.3 then yields

$$\begin{aligned}
F(x)^3 &= \left(\frac{\mu(q)}{\phi(q)}u(\beta)+O\left(\frac{Q^2n}{(\log n)^C}\right)\right)^3\\
&= \frac{\mu(q)}{\phi(q)^3}u(\beta)^3+\frac{3}{\phi(q)^2}u(\beta)^2\cdot O\left(\frac{Q^2n}{(\log n)^C}\right)+3\frac{\mu(q)}{\phi(q)}u(\beta)\cdot O\left(\frac{Q^4n^2}{(\log n)^{2C}}\right)+O\left(\frac{Q^6n^3}{(\log n)^{3C}}\right)\\
&= \frac{\mu(q)}{\phi(q)^3}u(\beta)^3+O\left(\frac{Q^2n^3}{(\log n)^C}\right)+O\left(\frac{Q^4n^3}{(\log n)^{2C}}\right)+O\left(\frac{Q^6n^3}{(\log n)^{3C}}\right)\\
&= \frac{\mu(q)}{\phi(q)^3}u(\beta)^3+O\left(\frac{Q^2n^3}{(\log n)^C}\right)+O\left(\frac{n^3}{((\log n)^{C-2B})^2}\right)+O\left(\frac{n^3}{((\log n)^{C-2B})^3}\right).
\end{aligned}$$

Therefore, we have,

$$F(x)^3 = \frac{\mu(q)}{\phi(q)^3} u(\beta)^3 + O\left(\frac{Q^2 n^3}{(\log n)^C}\right).$$

Before we definitively compute the integral $\int_{\mathfrak{M}} F(x)^3 e(-nx)dx$, we need one last lemma. We will define the distance from $x \in \mathbb{R}$ to its nearest integer n by

$$\|x\| = \min\{|n - x| : n \in \mathbb{Z}\} = \inf\{\{x\}, 1 - \{x\}\}.$$

Clearly, we have $\|x\| \in [0, \frac{1}{2}]$, and $x = n \pm \|x\|$ for some $n \in \mathbb{Z}$. From this, we obtain $|\sin(\pi x)| = \sin(\pi\|x\|)$. $\|\cdot\|$ also satisfies the triangle inequality.

Proposition 4.4. *For all $x \in \mathbb{R}$ and all integers N_1, N_2 such that $N_1 < N_2$, we have*

$$\left|\sum_{n=N_1+1}^{N_2} e(nx)\right| \leq \min\left\{N_2 - N_1, \frac{1}{2\|x\|}\right\}.$$

Proof. First observe that

$$\left|\sum_{n=N_1+1}^{N_2} e(nx)\right| \leq \sum_{n=N_1+1}^{N_2} |e(nx)| = \sum_{n=N_1+1}^{N_2} 1 = N_2 - N_1,$$

with equality if $x \in \mathbb{Z}$. Next, for $x \notin \mathbb{Z}$, we must have $\|x\| > 0$ and $e(x) \neq 1$. Thus, we may apply the formula for the sum of a geometric series to obtain

$$\left|\sum_{n=N_1+1}^{N_2} e(nx)\right| = \left|e(x(N_1+1)) \sum_{n=0}^{N_2-N_1-1} e(x)^n\right| = |e(x(N_1+1))| \cdot \left|\frac{1 - e(x)^{N_2-N_1}}{1 - e(x)}\right| = \left|\frac{e(x(N_2 - N_1)) - 1}{e(x) - 1}\right|.$$

Noting that $2x \leq \sin(\pi x)$ on $\left[0, \frac{1}{2}\right]$, we conclude with

$$\left|\sum_{n=N_1+1}^{N_2} e(nx)\right| \leq \frac{2}{|e(x) - 1|} = \frac{2}{|e(\frac{x}{2}) - e(-\frac{x}{2})|} = \frac{2}{|2i\sin(\pi x)|} = \frac{1}{|\sin(\pi x)|} = \frac{1}{\sin(\pi\|x\|)} \leq \frac{1}{2\|x\|}.$$

□

Theorem 4.5. *Let $\varepsilon > 0$ and n be odd. Given $B, C > 0$ with $C > 5B$, the integral over the major arcs is*

$$\int_{\mathfrak{M}} F(x)^3 e(-nx)dx = \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{(\log n)^{(1-\varepsilon)B}}\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right),$$

where the implied constants are dependent only on B, C, and ε.

Proof. We will compute two separate integrals

$$\int_{\mathfrak{M}} \left(F(x)^3 - \frac{\mu(q)}{\phi(q)^3} u\left(x - \frac{a}{q}\right)^3\right) e(-nx)dx \quad \text{and} \quad \int_{\mathfrak{M}} \frac{\mu(q)}{\phi(q)^3} u\left(x - \frac{a}{q}\right)^3 e(-nx)dx,$$

and summing their estimates will yield our desired result.

Using Lemma 4.3 and the subsequent comments following it, we first have

$$\begin{aligned}
\left|\int_{\mathfrak{M}}\left(F(x)^3-\frac{\mu(q)}{\phi(q)^3}u\left(x-\frac{a}{q}\right)^3\right)e(-nx)dx\right| &= \left|\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)}\left(F(x)^3-\frac{\mu(q)}{\phi(q)^3}u\left(x-\frac{a}{q}\right)^3\right)e(-nx)dx\right| \\
&\ll \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)}\frac{Q^2n^3}{(\log n)^C}|e(-nx)|dx \\
&\ll \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{Q^2n^3}{(\log n)^C}\int_{\mathfrak{M}(q,a)}1dx \\
&\ll \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{Q^3n^2}{(\log n)^C} \\
&\le \frac{Q^5n^2}{(\log n)^C} \\
&\le \frac{n^2}{(\log n)^{C-5B}}.
\end{aligned}$$

For the second integral, recall that $x\in\mathfrak{M}(q,a)$ implies $x=\frac{a}{q}+\beta$ for some $|\beta|\le\frac{Q}{n}$. The definition of $\mathfrak{S}(n,Q)$ then gives us

$$\begin{aligned}
\int_{\mathfrak{M}}\frac{\mu(q)}{\phi(q)^3}u\left(x-\frac{a}{q}\right)^3e(-nx)dx &= \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{\mu(q)}{\phi(q)^3}\int_{\mathfrak{M}(q,a)}u\left(x-\frac{a}{q}\right)^3e(-nx)dx \\
&= \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{\mu(q)}{\phi(q)^3}\int_{a/q-Q/n}^{a/q+Q/n}u\left(x-\frac{a}{q}\right)^3e(-nx)dx \\
&= \sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{\mu(q)}{\phi(q)^3}\int_{-Q/n}^{Q/n}u(\beta)^3e\left(-n\left(\beta+\frac{a}{q}\right)\right)d\beta \\
&= \sum_{q\le Q}\frac{\mu(q)}{\phi(q)^3}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{na}{q}\right)\int_{-Q/n}^{Q/n}u(\beta)^3e(-n\beta)d\beta \\
&= \sum_{q\le Q}\frac{\mu(q)c_q(-n)}{\phi(q)^3}\int_{-Q/n}^{Q/n}u(\beta)^3e(-n\beta)d\beta \\
&= \mathfrak{S}(n,Q)\int_{-Q/n}^{Q/n}u(\beta)^3e(-n\beta)d\beta,
\end{aligned}$$

where we performed the substitution $\beta=x-\frac{a}{q}$ on the third equality, and used the fact that $c_q(n)=c_q(-n)$ for the last equality. We would like to integrate $u(\beta)^3e(-n\beta)$ from $-\frac{1}{2}$ to $\frac{1}{2}$ in order to make use of our estimate for the singular integral $J(n)$. Taking $|\beta|<\frac{1}{2}$ yields $u(\beta)\ll|\beta|^{-1}$ by Proposition 4.4. Thus,

$$\left|\int_{Q/n}^{1/2}u(\beta)^3e(-n\beta)d\beta\right|\ll\int_{Q/n}^{1/2}|u(\beta)|^3d\beta\ll\int_{Q/n}^{1/2}|\beta|^{-3}d\beta<\frac{n^2}{Q^2}.$$

A symmetric argument shows that

$$\int_{-1/2}^{-Q/n} u(\beta)^3 e(-n\beta)d\beta \ll \frac{n^2}{Q^2}$$

as well. Therefore, we see that

$$\begin{aligned}
\int_{-Q/n}^{Q/n} u(\beta)^3 e(-n\beta)d\beta &= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{Q/n}^{1/2} - \int_{-1/2}^{-Q/n}\right)\big(u(\beta)^3 e(-n\beta)\big)d\beta \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta)^3 e(-n\beta)d\beta + O\left(\frac{n^2}{Q^2}\right) \\
&= J(n) + O\left(\frac{n^2}{Q^2}\right) \\
&= \frac{n^2}{2} + O(n) + O\left(\frac{n^2}{Q^2}\right) \\
&= \frac{n^2}{2} + O\left(\frac{n^2}{Q^2}\right).
\end{aligned}$$

Recall that by Theorem 3.3, we have $\mathfrak{S}(n,Q) = \mathfrak{S}(n) + O(Q^{-(1-\varepsilon)})$. Now bringing all of our previous work together, the integral over the major arcs is then given by

$$\begin{aligned}
\int_{\mathfrak{M}} F(x)^3 e(-nx)dx &= \int_{\mathfrak{M}} \frac{\mu(q)}{\phi(q)^3} u\left(x - \frac{a}{q}\right)^3 e(-nx)dx + \int_{\mathfrak{M}} \left(F(x)^3 - \frac{\mu(q)}{\phi(q)^3} u\left(x-\frac{a}{q}\right)^3\right) e(-nx)dx \\
&= \mathfrak{S}(n,Q)\int_{-Q/n}^{Q/n} u(\beta)^3 e(-n\beta)d\beta + O\left(\frac{n^2}{(\log n)^{C-5B}}\right) \\
&= \left(\mathfrak{S}(n) + O\left(\frac{1}{Q^{1-\varepsilon}}\right)\right)\left(\frac{n^2}{2} + O\left(\frac{n^2}{Q^2}\right)\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right) \\
&= \mathfrak{S}(n)\frac{n^2}{2} + O\left(\mathfrak{S}(n)\frac{n^2}{Q^2}\right) + O\left(\frac{n^2}{Q^{1-\varepsilon}}\right) + O\left(\frac{n^2}{Q^{3-\varepsilon}}\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right) \\
&= \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{Q^{1-\varepsilon}}\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right) \\
&= \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{(\log n)^{(1-\varepsilon)B}}\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right),
\end{aligned}$$

which finishes the proof. □

5. Estimating the Minor Arcs

With the major arcs out of the way, our new goal is to estimate the integral over the remainder of the unit interval and show that the contribution of the minor arcs is of lower order than the contribution of the major arcs. More precisely, we will eventually show that for any $B > 0$,

$$\int_{\mathfrak{m}} F(x)^3 e(-nx)dx = O\left(\frac{n^2}{(\log n)^{(B/2)-5}}\right).$$

But first we must be able to make a more general statement for our exponential sum $F(x) = \sum_{p\le n}(\log p)e(px)$ regardless of whether or not x lies in the minor arcs.

Theorem 5.1 (Vaughan's Identity)**.** *For $u \geq 1$, we define*

$$M_u(k) := \sum_{\substack{d|k \\ d\leq u}} \mu(d).$$

If $\Phi(k,l)$ is any arithmetic function of two variables, we have

$$\sum_{u<l\leq n} \Phi(1,l) + \sum_{u<k\leq n} \sum_{u<l\leq \frac{n}{k}} M_u(k)\Phi(k,l) = \sum_{d\leq u} \sum_{u<l\leq \frac{n}{d}} \sum_{m\leq \frac{n}{ld}} \mu(d)\Phi(dm,l).$$

Proof. Consider the sum

$$S = \sum_{k=1}^{n} \sum_{u<l\leq \frac{n}{k}} M_u(k)\Phi(k,l).$$

We will show that S is equal to the left-hand side and right-hand side simultaneously. Recalling that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

one may easily deduce that

$$M_u(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } 1 < k \leq u. \end{cases}$$

With this we obtain

$$S = \sum_{u<l\leq n} \Phi(1,l) + \sum_{u<k\leq n} \sum_{u<l\leq \frac{n}{k}} M_u(k)\Phi(k,l).$$

On the other hand, we may also interchange the order of summation for S as follows

$$\begin{aligned} S &= \sum_{k=1}^{n} \sum_{u<l\leq \frac{n}{k}} \sum_{\substack{d|k \\ d\leq u}} \mu(d)\Phi(k,l) \\ &= \sum_{d\leq u} \sum_{\substack{k=1 \\ d|k}}^{n} \sum_{u\leq l\leq \frac{n}{k}} \mu(d)\Phi(k,l) \\ &= \sum_{d\leq u} \sum_{m\leq \frac{n}{d}} \sum_{u<l\leq \frac{n}{dm}} \mu(d)\Phi(dm,l) \\ &= \sum_{d\leq u} \sum_{u<l\leq \frac{n}{d}} \sum_{m\leq \frac{n}{ld}} \mu(d)\Phi(dm,l), \end{aligned}$$

where we let $k = dm$ in the third equality. □

Let

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k, \\ 0 & \text{otherwise} \end{cases}$$

be the von Mangoldt function. With the previous theorem in hand, it is desirable to find a new expression for $F(x)$ by setting u to a specific value and choosing a suitable product of arithmetic functions for $\Phi(k,l)$.

Corollary 5.2. *Define the sums*

$$S_1 := \sum_{d\le n^{2/5}} \sum_{l\le \frac{n}{d}} \sum_{m\le \frac{n}{ld}} \mu(d)\Lambda(l)e(xdlm)$$

$$S_2 := \sum_{d\le n^{2/5}} \sum_{l\le n^{2/5}} \sum_{m\le \frac{n}{ld}} \mu(d)\Lambda(l)e(xdlm)$$

$$S_3 := \sum_{k> n^{2/5}} \sum_{n^{2/5}<l\le \frac{n}{k}} M_{n^{2/5}}(k)\Lambda(l)e(xkl).$$

Then for all $x\in\mathbb{R}$, we have $F(x) = S_1 - S_2 - S_3 + O(n^{1/2})$.

Proof. We appeal to Vaughan's identity by setting $u = n^{2/5}$ and $\Phi(k,l) = \Lambda(l)e(xkl)$. Working out each of the terms in the identity separately, the first term evaluates to

$$\begin{aligned}
\sum_{u<l\le n} \Phi(1,l) &= \sum_{n^{2/5}<l<n} \Lambda(l)e(xl) \\
&= \sum_{l=1}^{n} \Lambda(l)e(xl) - \sum_{l\le n^{2/5}} \Lambda(l)e(xl) \\
&= \sum_{p^k\le n} (\log p)e(xp^k) + O(n^{2/5}\log n) \\
&= \sum_{p\le n} (\log p)e(xp) + \sum_{\substack{p^k\le n\\ k\ge 2}} (\log p)e(xp^k) + O(n^{2/5}\log n) \\
&= F(x) + O\Bigg(\sum_{\substack{p^k\le n\\ k\ge 2}} \log p\Bigg) + O(n^{2/5}\log n).
\end{aligned}$$

Note that $p^k \le n \iff k\log p \le \log n \iff k \le \frac{\log n}{\log p}$. That is, $k \le \lfloor \frac{\log x}{\log p} \rfloor = \lfloor \log_p(x) \rfloor$. Hence, by Chebyshev's theorem (see Remark 1.12) and the Prime Number Theorem, we have

$$\begin{aligned}
\sum_{u<l\le n} \Phi(1,l) &= F(x) + O\Bigg(\sum_{p^2\le n} \lfloor \log_p(x)\rfloor \log p\Bigg) + O(n^{2/5}\log n) \\
&= F(x) + O\Bigg(\sum_{p\le n^{1/2}} \lfloor \log_p(x)\rfloor \log p\Bigg) + O(n^{2/5}\log n) \\
&= F(x) + O\big(\pi(n^{1/2})\cdot \log n\big) + O(n^{2/5}\log n) \\
&= F(x) + O(n^{1/2}).
\end{aligned}$$

The estimation of the second and third terms in Vaughan's identity is straightforward in comparison. Direct substitution of our choice of $\Phi(k,l)$ applied to the second term gives

$$\sum_{u<k\le n} \sum_{u<l\le \frac{n}{k}} M_u(k)\Phi(k,l) = \sum_{k>n^{2/5}} \sum_{n^{2/5}<l\le \frac{n}{k}} M_{n^{2/5}}(k)\Lambda(l)e(xkl) = S_3,$$

while the third term evaluates to

$$\begin{aligned}\sum_{d\le u}\sum_{u<l\le\frac{n}{d}}\sum_{m\le\frac{n}{ld}}\mu(d)\Phi(dm,l) &= \sum_{d\le n^{2/5}}\sum_{n^{2/5}<l\le\frac{n}{d}}\sum_{m\le\frac{n}{ld}}\mu(d)\Lambda(l)e(xdml)\\ &= \sum_{d\le n^{2/5}}\sum_{l\le\frac{n}{d}}\sum_{m\le\frac{n}{ld}}\mu(d)\Lambda(l)e(xdml) - \sum_{d\le n^{2/5}}\sum_{l\le n^{2/5}}\sum_{m\le\frac{n}{ld}}\mu(d)\Lambda(l)e(xdml)\\ &= S_1 - S_2.\end{aligned}$$

Therefore, $F(x) + O(n^{1/2}) + S_3 = S_1 - S_2$, and rearranging gives the desired result. □

Theorem 5.3. *Suppose a and q are integers such that $1 \le q \le n$ and $(a,q) = 1$. If $|x - \frac{a}{q}| \le \frac{1}{q^2}$, then*

$$F(x) \ll \left(\frac{n}{q^{1/2}} + n^{4/5} + n^{1/2}q^{1/2}\right)(\log n)^4.$$

The proof of the above theorem is very heavy in computations and technicalities, and so will be omitted from our development. These technicalities include relating our exponential sums with our integer norm $\|\cdot\|$, and approximations of both $\mu(d)\Lambda(l)e(xdlm)$ and $M_{n^{2/5}}(k)\Lambda(l)e(xkl)$, where $x \in \left[\frac{a}{q} - \frac{1}{q^2}, \frac{a}{q} + \frac{1}{q^2}\right]$. The interested reader will be referred to Section 3.1 of Vaughan [18] and Section 8.5 of Nathanson [15], as they hold proofs for the following claims:

$$S_1 \ll \left(\frac{n}{q} + n^{2/5} + q\right)(\log n)^2$$

$$S_2 \ll \left(\frac{n}{q} + n^{4/5} + q\right)(\log n)^2$$

$$S_3 \ll \left(\frac{n}{q^{1/2}} + n^{4/5} + q^{1/2}n^{1/2}\right)(\log n)^4.$$

The proofs utilize Proposition 4.4 as well as further estimates for exponential sums, properties regarding the divisor function $d(n) = \sum_{d|n} 1$, and an application of the Cauchy-Schwarz inequality. One may then easily deduce Theorem 5.3 by inserting the above approximations into Corollary 5.2. A different proof of Theorem 5.3 using a similar approach to Vaughan can also be found in Section 25 of Davenport [4]. One last lemma is required for the minor arc estimation, which will then directly lead to the proof of Vinogradov's theorem. Our desired result will depend on rational approximations for $x \in [0,1]$.

Lemma 5.4 (Dirichlet's Approximation Theorem)**.** *Let x and Q be real numbers where $Q \ge 1$. Then there exist integers a and q such that $1 \le q \le Q$, $(a,q) = 1$ and*

$$\left|x - \frac{a}{q}\right| < \frac{1}{qQ}.$$

Proof. Setting $n = \lfloor Q \rfloor$, we note that $n+1 > Q$. Now consider $\{qx\} \in [0,1)$ for some positive integer $q \leq n$. We consider three cases:

(1) there is some $q \leq n$ with $\{qx\} \in \left[0, \frac{1}{n+1}\right)$
(2) there is some $q \leq n$ with $\{qx\} \in \left[\frac{n}{n+1}, 1\right)$
(3) for all $q \leq n$, $\{qx\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right)$.

The proofs for the first two cases are straightforward. If $\{qx\} \in \left[0, \frac{1}{n+1}\right)$, setting $a = \lfloor qx \rfloor$ then yields

$$0 \leq \{qx\} = qx - \lfloor qx \rfloor = qx - a < \frac{1}{n+1},$$

and hence,

$$\left| x - \frac{a}{q} \right| < \frac{1}{q(n+1)} < \frac{1}{qQ}.$$

Similarly, if $\{qx\} \in \left[\frac{n}{n+1}, 1\right)$, choosing $a = \lfloor qx \rfloor + 1$ yields

$$\frac{n}{n+1} \leq \{qx\} = qx - \lfloor qx \rfloor = qx - a + 1 < 1,$$

from which we obtain $|qx - a| \leq \frac{1}{n+1}$, and so

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q(n+1)} < \frac{1}{qQ}.$$

Suppose now that for all $q = 1, \ldots, n$ we have

$$\{qx\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right).$$

Each of the n real numbers $\{qx\}$ must then lie in one of the $n-1$ intervals $\left[\frac{i}{n+1}, \frac{i+1}{n+1}\right)$, where $i = 1, \ldots, n-1$. By the pigeonhole principle, there then exist an integer $i \in \{1, \ldots, n-1\}$ and integers q_1, q_2 such that $1 \leq q_1 < q_2 \leq n$, and

$$\{q_1 x\}, \{q_2 x\} \in \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right).$$

Let $q = q_2 - q_1$ and $a = \lfloor q_2 x \rfloor - \lfloor q_1 x \rfloor$. We then obtain

$$|qx - a| = |(q_2 x - \lfloor q_2 x \rfloor) - (q_1 x - \lfloor q_1 x \rfloor)| = |\{q_2 x\} - \{q_1 x\}| < \frac{1}{n+1} < \frac{1}{Q}.$$

Dividing by q one last time completes the proof. □

Theorem 5.5. *Let $B > 0$. Then*

$$\int_{\mathfrak{m}} F(x)^3 e(-nx) dx = O\left(\frac{n^2}{(\log n)^{(B/2)-5}} \right),$$

where the implied constant is dependent only on B.

Proof. Let $x \in \mathfrak{m} = [0,1]\backslash\mathfrak{M}$. Dirichlet's approximation theorem ensures the existence of a rational $\frac{a}{q} \in [0,1]$ with $1 \leq q \leq \frac{n}{Q}$ and $(a,q) = 1$ such that

$$\left|x - \frac{a}{q}\right| \leq \frac{Q}{qn} \leq \min\left(\frac{Q}{n}, \frac{1}{q^2}\right).$$

Recall that we previously defined $Q = (\log n)^B$. Note that we must have $Q < q \leq \frac{n}{Q}$, otherwise, $q \leq Q$ would imply that $x \in \mathfrak{M}(\mathfrak{q},\mathfrak{a}) \subseteq \mathfrak{M}$, which is a contradiction. Appealing to Theorem 5.3 for our values of q and $Q = (\log n)^B$, we obtain

$$\begin{aligned} F(x) &\ll \left(\frac{n}{q^{1/2}} + n^{4/5} + n^{1/2}q^{1/2}\right)(\log n)^4 \\ &\ll \left(\frac{n}{(\log n)^{B/2}} + n^{4/5} + n^{1/2}\frac{n^{1/2}}{(\log n)^{B/2}}\right)(\log n)^4 \\ &\ll \frac{n}{(\log n)^{(B/2)-4}}. \end{aligned}$$

Next, we note that

$$\begin{aligned} \int_0^1 |F(x)|^2 dx &= \int_0^1 \left|\sum_{p\leq n} (\log p) e(px)\right|^2 dx \\ &= \int_0^1 \sum_{p_1,p_2\leq n} (\log p_1)(\log p_2) e((p_1-p_2)x) dx \\ &= \sum_{p_1,p_2\leq n} (\log p_1)(\log p_2) \int_0^1 e((p_1-p_2)x) dx \\ &= \sum_{p\leq n} (\log p)^2. \end{aligned}$$

Recall that by Chebyshev's theorem, $\theta(x) = \sum_{p\leq x} \log p = O(x)$. Hence,

$$\int_0^1 |F(x)|^2 dx = \sum_{p\leq n} (\log p)^2 \leq \log n \sum_{p\leq n} \log p = O(n \log n).$$

With this, we deduce that

$$\begin{aligned} \int_{\mathfrak{m}} F(x)^3 e(-nx) dx &\leq \int_{\mathfrak{m}} |F(x)|^3 dx \\ &\leq \sup\{|F(x)| : x \in \mathfrak{m}\} \int_{\mathfrak{m}} |F(x)|^2 dx \\ &\ll \frac{n}{(\log n)^{(B/2)-4}} \int_0^1 |F(x)|^2 dx \\ &\ll \frac{n^2}{(\log n)^{(B/2)-5}}. \end{aligned}$$

□

6. Proving Vinogradov's Theorem and Further Improvements

Combining our estimates for the major arcs and minor arcs, we may now finally evaluate the main term and the error term for $R(n) = \sum_{p_1+p_2+p_3=n} (\log p_1)(\log p_2)(\log p_3)$. For any $B, C, \varepsilon > 0$ with $C > 5B$ and $B > 10$, Theorems 4.5 and 5.5 yield

$$\begin{aligned} R(n) &= \int_0^1 F(x)^3 e(-nx)dx \\ &= \int_{\mathfrak{M}} F(x)^3 e(-nx)dx + \int_{\mathfrak{m}} F(x)^3 e(-nx)dx \\ &= \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{(\log n)^{(1-\varepsilon)B}}\right) + O\left(\frac{n^2}{(\log n)^{C-5B}}\right) + O\left(\frac{n^2}{(\log n)^{(B/2)-5}}\right), \end{aligned}$$

where the implied constants depend only on B, C, and ε.

Given $A > 0$, we can then set $B = 2A + 10$, $C = A + 5B$, and $\varepsilon < \frac{1}{2}$ to simplify the above estimate to

$$R(n) = \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{(\log n)^A}\right),$$

where n is any sufficiently large odd integer and our only implied constant depends on A. Note that this implies that $R(n) \ll n^2$, since $c_1 < \mathfrak{S}(n) < c_2$ for some $c_1, c_2 > 0$ by the first half of Theorem 3.2. We are now finally ready to derive Vinogradov's asymptotic formula and prove the second half of Theorem 3.2.

Proof of Theorem 3.2. Observe that if we can show that

$$0 \le (\log n)^3 r(n) - R(n) \ll \frac{n^2 \log\log n}{\log n}, \tag{1}$$

then using our newly obtained asymptotic formula for $R(n)$ (and provided that $A \ge 1$), we obtain

$$\begin{aligned} (\log n)^3 r(n) &= R(n) + O\left(\frac{n^2 \log\log n}{\log n}\right) \\ &= \mathfrak{S}(n)\frac{n^2}{2} + O\left(\frac{n^2}{(\log n)^A}\right) + O\left(\frac{n^2 \log\log n}{\log n}\right) \\ &= \mathfrak{S}(n)\frac{n^2}{2}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right). \end{aligned}$$

Dividing by $(\log n)^3$, we would then obtain the desired asymptotic formula for $r(n)$ and complete the proof. Therefore, it suffices to show that (1) holds. We must first bound $R(n)$ from above and below.

First note that

$$R(n) = \sum_{p_1+p_2+p_3=n} \log p_1 \log p_2 \log p_3 \le (\log n)^3 \sum_{p_1+p_2+p_3=n} 1 = (\log n)^3 r(n),$$

and this will serve as our upper bound. Now for $0 < \delta < \frac{1}{2}$, we define $r_\delta(n)$ as the number of representations of n in the form $p_1 + p_2 + p_3$ with $p_i \le n^{1-\delta}$ for some i. The Prime Number Theorem then gives us

$$r_\delta(n) \le 3 \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \le n^{1-\delta}}} 1 \ll \sum_{p_1 \le n^{1-\delta}} \Bigg(\sum_{p_2+p_3=n-p_1} 1 \Bigg) \le \sum_{p_1 \le n^{1-\delta}} \Bigg(\sum_{p_2<n} 1 \Bigg) \le \pi(n^{1-\delta})\pi(n) \ll \frac{n^{2-\delta}}{(\log n)^2}.$$

This then allows us to obtain a lower bound for $R(n)$. Observe that

$$\begin{aligned} R(n) &\ge \sum_{\substack{p_1+p_2+p_3=n \\ p_1,p_2,p_3>n^{1-\delta}}} \log p_1 \log p_2 \log p_3 \\ &\ge (1-\delta)^3 (\log n)^3 \sum_{\substack{p_1+p_2+p_3=n \\ p_1,p_2,p_3>n^{1-\delta}}} 1 \\ &= (1-\delta)^3(\log n)^3 (r(n) - r_\delta(n)). \end{aligned}$$

Rearranging for $(\log n)^3 r(n)$, we obtain

$$(\log n)^3 r(n) \le \frac{1}{(1-\delta)^3} R(n) + (\log n)^3 r_\delta(n) \ll \frac{1}{(1-\delta)^3} R(n) + (\log n) n^{2-\delta}.$$

Since $0 < \delta < \frac{1}{2}$, we have $\frac{1}{2} < 1-\delta < 1$, and so

$$0 < (1-\delta)^{-3} - 1 = \frac{1-(1-\delta)^3}{(1-\delta)^3} < 8(1-(1-\delta)^3) = 8(3\delta - 3\delta^2 + \delta^3) < 8 \cdot 3\delta = 24\delta,$$

since $\delta < 3$ implies $\delta^3 - 3\delta^2 < 0$. Thus,

$$\begin{aligned} (\log n)^3 r(n) - R(n) &\ll \big((1-\delta)^{-3} - 1\big) R(n) + (\log n) n^{2-\delta} \\ &\ll \delta R(n) + (\log n) n^{2-\delta} \\ &\ll \delta n^2 + (\log n) n^{2-\delta} \\ &= n^2 \left(\delta + \frac{\log n}{n^\delta} \right), \end{aligned}$$

where we used the fact that $R(n) \ll n^2$ for the third line. Note that this inequality holds for all $\delta \in (0, \frac{1}{2})$, and so our implied constant does not depend on δ. We can then set

$$\delta = \frac{2 \log\log n}{\log n},$$

so that

$$\delta + \frac{\log n}{n^\delta} = \frac{2\log\log n}{\log n} + \frac{\log n}{(\log n)^2} \ll \frac{\log\log n}{\log n}.$$

From this we finally obtain

$$0 \le (\log n)^3 r(n) - R(n) \ll \frac{n^2 \log\log n}{\log n},$$

which concludes the proof. □

With a valid asymptotic formula attained for $r(n)$, we deduce Goldbach's weak conjecture independent of the Generalized Riemann Hypothesis. Throughout our discussion, we required n to be "sufficiently large", ie. we require $n > N$ for some large value of N. In fact, the value of N was unspecified in Vinogradov's original 1937 paper. It wasn't until 1939 when Vinogradov's own student Konstantin Borozdkin proved that one may take

$$N = 3^{3^{15}} = 3^{14348907}.$$

It now remains to show that every $n < N$ can be written as a sum of three primes computationally. Of course, this is easier said than done due to the staggering magnitude of N (which has 6 846 169 digits!).

Further marginal improvements were made decades later with new bounds for N being given. In 1997, Deshouillers, Effinger, Riele, and Zinoviev re-implemented the use of the Generalized Riemann Hypothesis to improve the numerical constant down to $N = 10^{20}$. An improvement on Borozdkin's unconditional constant N was made 5 years later by Liu and Wang [12], bringing it down to

$$N = e^{3100} \approx 2 \cdot 10^{1346}.$$

Although being a significant improvement over the original constant, it was still unfeasible to work with and the computational power needed to computationally verify Goldbach's weak conjecture remained unfathomable. A decade later, Harald Helfgott published a series of papers in which he unconditionally proved Goldbach's weak conjecture with our best known value for N to date. Helfgott [8] proved that one could take

$$N = 10^{27},$$

and verification of the weak conjecture for odd integers up to $8.875 \cdot 10^{30}$ was then conducted shortly after by Helfgott and Platt [9]. Combining this with Helfgott's 2013 result, the weak conjecture has now been effectively resolved. Helgott's proof splits the integration over major and minor arcs as well, but his treatment over both sets of arcs are more reliant on Fourier analysis and the use of Dirichlet L-functions.

Remark 6.1. It immediately follows from Goldbach's weak conjecture that every number (odd or even) may also be written as a sum of four primes, where $n \geq 4$. Indeed, for any odd $n \geq 9$, $n-2$ is odd and $n-2 > 5$, hence $n-2$ can be written as a sum of three primes. That is, $n = p_1 + p_2 + p_3 + 2$ and n is a sum of four primes. Similarly for $n > 9$ even, $n-3$ is odd and $n-3 > 5$, hence $n-3$ can be also be written as a sum of three primes. Thus, $n = 3 + p_1 + p_2 + p_3$. A very straightforward inductive argument can be used to show that any sufficiently large number may be written as a sum of n primes for all $n \geq 4$.

7. Goldbach's Conjecture and the Exceptional Set

As viable and amazing as the circle method is, it is deficient in solving Goldbach's original conjecture. Proceeding analogously as before with redefined exponential sums

$$r_2(n) = \sum_{p_1+p_2=n} 1 \quad \text{and} \quad R_2(n) = \sum_{p_1+p_2=n} \log p_1 \log p_2,$$

we will see that the major contribution of the integral

$$R_2(n) = \int_0^1 F(x)^2 e(-nx)dx = \int_{\mathfrak{M}} F(x)^2 e(-nx)dx + \int_{\mathfrak{m}} F(x)^2 e(-nx)dx$$

will now come from the integral over the minor arcs. The majority of our above work remains unchanged, but do note that our new singular integral now satisfies

$$J(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\beta)^2 e(-n\beta)d\beta = \binom{n-1}{2-1} = n-1 = n + O(1),$$

by Example 2.1. We will also slightly alter our singular series for the ternary Goldbach problem and note some of its expected properties.

Definition 7.1. The function

$$\mathfrak{S}_2(n) = \sum_{q=1}^{\infty} \frac{\mu(q)^2 c_q(n)}{\phi(q)^2}$$

is called the *singular series* for the binary Goldbach problem.

Completely analogous to Theorem 3.3, for any $\varepsilon > 0$, we now have

$$\mathfrak{S}_2(n) - \mathfrak{S}_2(n, Q) = O\left(\frac{1}{Q^{1-\varepsilon}}\right),$$

where

$$\mathfrak{S}_2(n, Q) = \sum_{q \leq Q} \frac{\mu(q)^2 c_q(n)}{\phi(q)^2}$$

is similar to our ternary singular series before. $\mathfrak{S}_2(n)$ is also a multiplicative arithmetic function that converges absolutely and uniformly in n, and so its Euler product is given by

$$\begin{aligned}
\mathfrak{S}_2(n) &= \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\mu(p^k)^2 c_{p^k}(n)}{\phi(p^k)^2}\right) \\
&= \prod_p \left(1 + \frac{c_p(n)}{\phi(p)^2}\right) \\
&= \prod_{p \nmid n} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \mid n} \left(1 + \frac{1}{p-1}\right) \\
&= \prod_p \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \mid n} \left(\frac{p-1}{p-2}\right).
\end{aligned}$$

Lastly note that in stark contrast to $\mathfrak{S}(n)$, it follows from its Euler product that $\mathfrak{S}_2(n) \gg 1$ for even n, and $\mathfrak{S}_2(n) = 0$ for odd n.

Continuing along with our original treatment of the major arcs for the ternary Goldbach problem, squaring our result from Lemma 4.3 instead of cubing it now returns

$$F(x)^2 = \frac{\mu(q)}{\phi(q)^2} u(\beta)^2 + O\left(\frac{Q^2 n^2}{(\log n)^C}\right).$$

Also, the singular integral evaluated from $-\frac{Q}{n}$ to $\frac{Q}{n}$ is now

$$\int_{-Q/n}^{Q/n} u(\beta)^2 e(-n\beta)d\beta = n + O\left(\frac{n}{Q}\right).$$

Hence, our computation of the major arcs now yields

$$\begin{aligned}
\int_{\mathfrak{M}} F(x)^2 e(-nx)dx &= \int_{\mathfrak{M}} \frac{\mu(q)}{\phi(q)^2} u\left(x-\frac{a}{q}\right)^2 e(-nx)dx + \int_{\mathfrak{M}} \left(F(x)^2 - \frac{\mu(q)}{\phi(q)^2} u\left(x-\frac{a}{q}\right)^2\right) e(-nx)dx \\
&= \mathfrak{S}_2(n,Q)\int_{-Q/n}^{Q/n} u(\beta)^2 e(-n\beta)d\beta + O\left(\frac{n}{(\log n)^{C-5B}}\right) \\
&= \mathfrak{S}_2(n,Q)\cdot\left(n + O\left(\frac{n}{Q}\right)\right) + O\left(\frac{n}{(\log n)^{C-5B}}\right) \qquad (2)\\
&= n\mathfrak{S}_2(n,Q) + O\left(\frac{n}{(\log n)^{B}}\right) + O\left(\frac{n}{(\log n)^{C-5B}}\right).
\end{aligned}$$

Since $\mathfrak{S}_2(n,Q) = \mathfrak{S}_2(n) + O\left(\frac{1}{Q^{1-\varepsilon}}\right)$, we then obtain

$$\int_{\mathfrak{M}} F(x)^2 e(-nx)dx = n\mathfrak{S}_2(n) + O\left(\frac{n}{(\log n)^{(1-\varepsilon)B}}\right) + O\left(\frac{n}{(\log n)^{C-5B}}\right).$$

In comparison to our original derivation in Theorem 4.5, observe that the order of the main term and the error terms have all been reduced by 1. The same can not be said for our estimation of the minor arcs. Using the same approach by appealing to Chebyshev's Theorem and Theorem 5.3, we obtain

$$\int_0^1 |F(x)|dx \le \sum_{p\le n} \log p = O(n).$$

Hence,

$$\begin{aligned}
\int_{\mathfrak{m}} F(x)^2 e(-nx)dx &\le \int_{\mathfrak{m}} |F(x)|^2 dx \\
&\le \sup\{|F(x)| : x\in\mathfrak{m}\}\int_{\mathfrak{m}} |F(x)|dx \\
&\ll \frac{n}{(\log n)^{(B/2)-4}}\int_0^1 |F(x)|dx \\
&\ll \frac{n^2}{(\log n)^{(B/2)-4}}.
\end{aligned}$$

Therefore, our error term over the minor arcs remains of order $\frac{n^2}{(\log n)^A}$ for some $A>0$, which dominates the contribution made from our major arcs. One could cleverly employ the Cauchy-Schwarz inequality to reattempt the above computation by finding another upper bound for $\int_0^1 |F(x)|dx$:

$$\begin{aligned}
\int_{\mathfrak{m}} F(x)^2 e(-nx)dx &\le \sup\{|F(x)| : x\in\mathfrak{m}\}\int_0^1 |F(x)|dx \\
&\le \sup\{|F(x)| : x\in\mathfrak{m}\}\left(\int_0^1 |F(x)|^2\right)^{\frac{1}{2}}\left(\int_0^1 |1|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\ll \frac{n}{(\log n)^{(B/2)-4}} \cdot (n \log n)^{\frac{1}{2}}$$
$$= \frac{n^{3/2}}{(\log n)^{(B/2)-9/2}},$$

which is still unfortunately larger than the contribution from the major arcs. Though it is difficult to successfully apply the circle method to Goldbach's conjecture, it is possible to derive a formula concerning the density of even integers that are *not* representable as a sum of two odd prime numbers. It turns out that Vinogradov's theorem implies that "almost" all even natural numbers may be represented as a sum of two primes.

Definition 7.2. Let E denote the set of even integers greater than 2 that cannot be written as the sum of two primes. The set E is called the *exceptional set* for the binary Goldbach conjecture. The quantity $E(n)$ will denote the number of integers in E that do not exceed n.

Of course, proving Goldbach's conjecture to be true will imply that $E(n) = 0$ for all n. We will instead show that $\forall A > 0$,

$$E(n) \ll \frac{n}{(\log n)^A}. \tag{3}$$

Our preceding attempt in applying the circle method to the binary Goldbach conjecture will be of great use in what follows. For brevity's sake, we will denote by

$$\mathfrak{R}(n) = \int_{\mathfrak{M}} F(x)^2 e(-nx)dx \quad \text{and} \quad \mathfrak{r}(n) = \int_{\mathfrak{m}} F(x)^2 e(-nx)dx$$

as the integrals over the major and minor arcs respectively. Therefore, $R_2(n) = \mathfrak{R}(n) + \mathfrak{r}(n)$. In order to show that (3) holds, we will require an estimate for

$$\sum_{m=1}^{n} |R_2(m) - m\mathfrak{S}_2(m)|^2.$$

In dealing with $\mathfrak{R}(n)$, we shall briefly derive an alternate asymptotic relation for the difference $\mathfrak{S}(n) - \mathfrak{S}_2(n, Q)$. First note the following preliminary facts concerning the Ramanujan sum and the Euler totient function:

(i) The Ramanujan sum can be expressed in the form

$$c_q(n) = \frac{\mu\left(\frac{q}{(q,n)}\right)\phi(q)}{\phi\left(\frac{q}{(q,n)}\right)}.$$

(ii) We have

$$\sum_{n \le x} \frac{1}{\phi(n)} \ll \log x \quad \text{and} \quad \sum_{n > x} \frac{1}{\phi(n)^2} \ll \frac{1}{x}.$$

The proof of the former can be found in Section A.7 of Nathanson [15], while the latter can be found in Section A.5 of Nathanson as well.

Lemma 7.3. *Defining $\mathfrak{S}_2(n,Q)$ as before, we have*

$$\mathfrak{S}_2(n,Q) = \mathfrak{S}_2(n) + O(\log n).$$

Also, for any $A > 0$, we have

$$\sum_{m=1}^{n} |\mathfrak{S}_2(m) - \mathfrak{S}_2(m,Q)|^2 \ll \frac{n}{(\log n)^A},$$

where the implied constant depends only on A.

Proof. First note that if $d \mid n$ and $(r,n) = 1$, then $(r,d) = 1$ as well. Hence,

$$\begin{aligned}
\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q) &= \sum_{q>Q} \frac{\mu(q)^2}{\phi(q)^2} \frac{\mu\big(\frac{q}{(q,n)}\big)\phi(q)}{\phi\big(\frac{q}{(q,n)}\big)} \\
&= \sum_{d \mid n} \sum_{\substack{q>Q \\ (q,n)=d}} \frac{\mu(q)^2}{\phi(q)^2} \frac{\mu\big(\frac{q}{d}\big)\phi(q)}{\phi\big(\frac{q}{d}\big)} \\
&= \sum_{d \mid n} \sum_{\substack{r>\frac{Q}{d} \\ (r,\frac{n}{d})=1}} \frac{\mu(rd)^2}{\phi(rd)^2} \frac{\mu(r)\phi(rd)}{\phi(r)},
\end{aligned}$$

where we have set $r = \frac{q}{d}$. Also note that if $(r,d) > 1$, then $(r,d)^2 \mid rd$, and thus $\mu(rd) = 0$. On the other hand, if $(r,d) = 1$, then rd is square-free, and so $\mu(rd) = \pm 1$. Thus,

$$\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q) = \sum_{d \mid n} \sum_{\substack{r>\frac{Q}{d} \\ (r,\frac{n}{d})=1 \\ (r,d)=1}} \frac{\mu(r)}{\phi(rd)\phi(r)} = \sum_{d \mid n} \sum_{\substack{r>\frac{Q}{d} \\ (r,n)=1}} \frac{\mu(r)}{\phi(r)^2\phi(d)} = \sum_{d \mid n} \frac{1}{\phi(d)} \sum_{\substack{r>\frac{Q}{d} \\ (r,n)=1}} \frac{\mu(r)}{\phi(r)^2}.$$

From this, we then obtain

$$|\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q)| \le \sum_{d \mid n} \frac{1}{\phi(d)} \sum_{\substack{r>\frac{Q}{d} \\ (r,n)=1}} \frac{1}{\phi(r)^2} \le \sum_{d \mid n} \frac{1}{\phi(d)} \sum_{r=1}^{\infty} \frac{1}{\phi(r)^2}.$$

Note that for $r \ge 2$, we have $\phi(r) \ge \frac{\log 2}{2} \frac{r}{\log r}$. Then, $\frac{1}{\phi(r)^2} \le \frac{4}{(\log 2)^2} \frac{(\log r)^2}{r^2}$, and the infinite series $\sum_{r=2}^{\infty} \frac{1}{\phi(r)^2}$ converges (say, to some K) by the comparison test with $\sum_{r=2}^{\infty} \frac{(\log r)^2}{r^2}$. Therefore, we have

$$\begin{aligned}
|\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q)| &\le (1+K) \sum_{d \mid n} \frac{1}{\phi(d)} \\
&\le (1+K) \sum_{d \le n} \frac{1}{\phi(d)} \\
&= O(\log n).
\end{aligned}$$

Squaring and taking the sum over $m = 1, \ldots, n$ then yields

$$\begin{aligned}
\sum_{m=1}^{n} |\mathfrak{S}_2(m) - \mathfrak{S}_2(m,Q)|^2 &\ll (\log n) \sum_{m=1}^{n} |\mathfrak{S}_2(m) - \mathfrak{S}_2(m,Q)| \\
&\leq (\log n) \sum_{m=1}^{n} \sum_{d|m} \frac{\mu(d)^2}{\phi(d)} \left(\frac{d}{Q}\right) \\
&\leq (\log n) \sum_{d=1}^{n} \frac{m}{d} \frac{\mu(d)^2}{\phi(d)} \left(\frac{d}{Q}\right) \\
&\leq \frac{n}{(\log n)^{B-1}} \sum_{d=1}^{n} \frac{1}{\phi(d)} \\
&\ll \frac{n}{(\log n)^{B-2}}.
\end{aligned}$$

Setting $B = A + 2$ finishes the proof. □

Theorem 7.4. *Let $A > 0$ be given. Then*

$$\sum_{m=1}^{n} |\mathfrak{R}(m) - m\mathfrak{S}_2(m)|^2 \ll \frac{n^3}{(\log n)^A},$$

where the implied constant depends only on A.

Proof. Utilizing our work from (2) for $B \geq A$ and $C = A + 5B$ we have

$$\mathfrak{R}(n) - n\mathfrak{S}_2(n,Q) = \int_{\mathfrak{M}} F(x)^2 e(-nx)dx - n\mathfrak{S}_2(n,Q) \ll \frac{n}{(\log n)^A}.$$

Now, Lemma 7.3 along with the fact that $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ gives us

$$\begin{aligned}
\sum_{m=1}^{n} |\mathfrak{R}(m) - m\mathfrak{S}_2(m)|^2 &= \sum_{m=1}^{n} |\mathfrak{R}(m) - m\mathfrak{S}_2(m,Q) + m\mathfrak{S}_2(m,Q) - m\mathfrak{S}_2(m)|^2 \\
&\ll \sum_{m=1}^{n} |\mathfrak{R}(m) - m\mathfrak{S}_2(m,Q)|^2 + \sum_{m=1}^{n} |m|^2 |\mathfrak{S}_2(m,Q) - \mathfrak{S}_2(m)|^2 \\
&\ll \sum_{m=1}^{n} \left(\frac{n^2}{(\log n)^{2A}}\right) + \frac{n^3}{(\log n)^A} \\
&\ll \frac{n^3}{(\log n)^A},
\end{aligned}$$

as required. □

A similar result for the minor arcs can also be attained, but a slight detour towards a result from Fourier analysis is needed. Bessel's inequality states that if $e_1, \ldots, e_n$ are orthonormal members of some inner product space V, then for any $x \in V$, we have

$$\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \leq \|x\|^2,$$

where $\langle \cdot, \cdot \rangle$ is the underlying inner product. If we take our inner product to be the integral inner product of functions restricted to $\mathfrak{m}$ given by

$$\langle f, g \rangle = \int_{\mathfrak{m}} f(x)g(x)dx,$$

then its associated induced norm (the L^2-norm) is given by

$$\|f\| = \left(\int_{\mathfrak{m}} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Taking $g_m(x) = e(-mx)$ where $m = 1, \ldots, n$ to be our orthonormal sequence of functions, Bessel's inequality then gives us

$$\sum_{m=1}^{n} |\mathfrak{r}(m)|^2 = \sum_{m=1}^{n} \left| \int_{\mathfrak{m}} F(x)^2 e(-mx)dx \right|^2 = \sum_{m=1}^{n} |\langle F(x)^2, e(-mx)\rangle|^2 \leq \|F(x)^2\|^2 = \int_{\mathfrak{m}} |F(x)|^4 dx.$$

We may then estimate as usual, yielding

$$\int_{\mathfrak{m}} |F(x)|^4 dx \leq \sup\{|F(x)| : x \in \mathfrak{m}\}^2 \int_{\mathfrak{m}} |F(x)|^2 dx \ll \frac{n^2}{(\log n)^{B-8}} \int_0^1 |F(x)|^2 dx \ll \frac{n^3}{(\log n)^{B-9}}.$$

With a choice of $B = A + 9$, we finally obtain

$$\sum_{m=1}^{n} |\mathfrak{r}(m)|^2 = \int_{\mathfrak{m}} |F(x)|^4 dx \ll \frac{n^3}{(\log n)^A}.$$

We are now ready to prove the main results of this section.

Theorem 7.5. *Let $A > 0$ be given. Then*

$$\sum_{m=1}^{n} |R_2(m) - m\mathfrak{S}_2(m)|^2 \ll \frac{n^3}{(\log n)^A},$$

where the implied constant depends only on A.

Proof. The proof follows directly from Theorem 7.4 and our previous work regarding $\mathfrak{r}(m)$. For any $A > 0$, we have

$$\begin{aligned} \sum_{m=1}^{n} |R_2(m) - m\mathfrak{S}_2(m)|^2 &= \sum_{m=1}^{n} |(\mathfrak{R}(m) - m\mathfrak{S}_2(m)) + \mathfrak{r}(m)|^2 \\ &\ll \sum_{m=1}^{n} |\mathfrak{R}(m) - m\mathfrak{S}_2(m)|^2 + \sum_{m=1}^{n} |\mathfrak{r}(m)|^2 \\ &\ll \frac{n^3}{(\log n)^A}, \end{aligned}$$

where we used the fact that $R_2(n) = \mathfrak{R}(n) + \mathfrak{r}(n)$ in the first line. ☐

Corollary 7.6. *For all $A > 0$, the number $E(n)$ of even numbers m in the exceptional set E that do not exceed n satisfies*

$$E(n) \ll \frac{n}{(\log n)^A},$$

where the implied constant is dependent only on A.

Proof. Recall that for even m, $\mathfrak{S}_2(m) \gg 1$. Also, for each m that cannot be written as a sum of two primes, we have $R_2(m) = 0$, and so

$$m^{-1}|R_2(m) - m\mathfrak{S}_2(m)| = \mathfrak{S}_2(m) \gg 1.$$

Squaring the above result and summing over each m counted by $E(n)$, we have

$$E(n) \ll \sum_{m=1}^{n} m^{-2}|R_2(m) - m\mathfrak{S}_2(m)|^2.$$

If we set $a(m) = |R_2(m) - m\mathfrak{S}_2(m)|^2$ and $f(m) = m^{-2}$, then $A(n) = \sum_{m=1}^{n}|R_2(m) - m\mathfrak{S}_2(m)|^2$ and $f'(m) = -2m^{-3}$. Partial summation then allows us to deduce that

$$\begin{aligned} E(n) &\ll \sum_{m=1}^{n} m^{-2}|R_2(m) - m\mathfrak{S}_2(m)|^2 \\ &= A(n)f(n) - \int_1^n A(t)f'(t)dt \\ &= \sum_{m=1}^{n}|R_2(m) - m\mathfrak{S}_2(m)|^2 n^{-2} + 2\int_1^n t^{-3}\sum_{m=1}^{t}|R_2(m) - m\mathfrak{S}_2(m)|^2 dt \\ &\ll \frac{n^3}{(\log n)^A} n^{-2} + \int_1^n \frac{1}{(\log t)^A} dt \\ &\ll \frac{n}{(\log n)^A}, \end{aligned}$$

and this completes the proof. □

This result was observed independently by Van der Corput [17], Chudakov [3], and Estermann [5] shortly after Vinogradov's original series of papers. This rather weak error term on the exceptional set was not improved upon until 1975, where Vaughan and Montgomery [14] ensured the existence of a constant $\delta > 0$ such that

$$E(n) = O(n^{1-\delta}).$$

Their method significantly strengthened an estimate for $F(x)$ over the major arcs by considering the zero-free region of a particular L-function. An explicit numerical value for δ was first computed by Chen and Pan [2], where they showed that the method of Montgomery and Vaughan yielded $\delta = 0.01$. This value has been subsequently sharpened by several authors and is currently known to hold for $\delta = 0.086$. In 2004, Pintz [15] unveiled a refinement on Vaughan and Montgomery's work by establishing that one may take $\delta = \frac{1}{3}$. Moreover, Pintz was also able to prove that with the exception of $O(x^{3/5+\varepsilon})$ even integers $n \leq x$, it must be the case that either n or $n-2$ is a sum of two primes.

8. Waring's Problem

We will now briefly describe another application of the circle method unrelated to Goldbach's problems. Among their first series of papers that formalized the circle method, Hardy and Littlewood explored Warings problem. As remarked in Example 2.1, if $\mathcal{K} = \{0, 1^k, 2^k, \dots\}$, then Waring's problem asserts that $r_{s,\mathcal{K}}(n) > 0$.

A combinatorial proof involving algebraic identities was given primarily by David Hilbert in 1909. A decade later, an asymptotic formula was derived using the circle method.

Theorem 8.1 (Hardy-Littlewood Asymptotic Formula). *For $s \geq s_0(k)$, there exists $\delta = \delta(s,k) > 0$ such that*

$$r_{s,\mathcal{K}}(n) = \mathfrak{S}(n)\Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} n^{(s/k)-1} + O(n^{(s/k)-1-\delta}),$$

where

$$\Gamma(s) = \int_0^\infty e^{-s}x^{s-1}dx$$

is the Gamma function, and $\mathfrak{S}(n)$ is the corresponding singular series dependent on our choice of s and k.

The proof of this result in its entirety is non-trivial and is beyond the scope of this paper. There is however, a great deal of overlap method-wise between the special case of $s_0(k) = 2^k + 1$ and our development of Vinogradov's theorem. For $n \geq 2^k$, let $P = \lfloor n^{1/k} \rfloor$ and $F(x) = \sum_{m=1}^{P} e(xm^k)$ be our unweighted trigonometric polynomial. This polynomial will serve as our generating function, and so

$$r_{s,\mathcal{K}}(n) = \int_0^1 F(x)^s e(-nx)dx,$$

as expected.

Construction of the major and minor arcs proceeds very similarly to Definition 3.6. Let $n \geq 2^k$, $P = \lfloor n^{1/k} \rfloor \geq 2$, and $0 < \nu < \frac{1}{5}$. For $1 \leq q \leq P^\nu$, $0 \leq a \leq q$, and $(a,q) = 1$, we define the major arc for Waring's problem, denoted $\mathfrak{W}(q,a)$, as the set

$$\mathfrak{W}(q,a) = \left\{x \in [0,1] \ : \ \left|x - \frac{a}{q}\right| \leq \frac{1}{P^{k-\nu}}\right\}.$$

The set of all major arcs is the union

$$\mathfrak{W} = \bigcup_{q=1}^{Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{W}(q,a),$$

and the set of minor arcs for Waring's problem is the complement $\mathfrak{w} = [0,1] \setminus \mathfrak{W}$. Next, the singular series and singular integral for Waring's problem in this framework are respectively defined as

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right)^s e\left(\frac{-na}{q}\right) \quad \text{and} \quad J(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(\beta)^s e(-n\beta)d\beta,$$

where

$$S(q,a) = \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right) \quad \text{and} \quad \nu(\beta) = \sum_{m=1}^{n} \frac{1}{k} m^{(1/k)-1} e(\beta m)$$

are auxiliary functions analogous to our expressions for $c_q(n)$ and $u(\beta)$. The proof of our special case can then broken down into 5 steps:

(1) Provide an estimate for the integral over the minor arcs by showing that there exists $\delta_1 > 0$ such that

$$\int_{\mathfrak{w}} F(x)^s e(-nx)dx = O(P^{s-k-\delta_1}).$$

The proof of this is fairly elementary, requiring Dirichlet's approximation theorem (Lemma 5.4), and asymptotic relations involving our function $F(x)$ provided by making use of two standard results found in number theory, namely Weyl's inequality and Hua's Lemma.

(2) In the same spirit as Theorem 4.5, the integral over the major arcs can be expressed as a product between a singular series and the singular integral, as well as an additional error term. Defining

$$\mathfrak{S}(n,Q) = \sum_{1\le q\le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right)^s e\left(\frac{-na}{q}\right) \quad \text{and} \quad J^*(n) = \int_{-P^{\nu-k}}^{P^{\nu-k}} \nu(\beta)^s e(-n\beta)d\beta,$$

one can show that

$$\int_{\mathfrak{W}} F(x)^s e(-nx)dx = \mathfrak{S}(n,P^\nu)J^*(n) + O(P^{s-k-\delta_2}),$$

where $\delta_2 = \frac{1}{k}(1-5\nu)$.

(3) Prove that there exists $\delta_3 > 0$ such that

$$J(n) \ll P^{s-k} \quad \text{and} \quad J^*(n) = J(n) + O(P^{s-k-\delta_3}).$$

Furthermore, one would also have to show that for $s \ge 2$, the singular integral may be rewritten as

$$J(n) = \Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} n^{(s/k)-1} + O(n^{(s-1)/(k-1)}).$$

This result, along with Step 2, would then yield the main term in Theorem 8.1.

(4) Deduce the existence of constants $c_1 = c_1(k,s)$ and $c_2 = c_2(k,s)$ that depend on k and s such that $c_1 < \mathfrak{S}(n) < c_2$, and the existence of $\delta_4 > 0$ such that

$$\mathfrak{S}(n,P^\nu) = \mathfrak{S}(n) + O(P^{-\nu\delta_4}).$$

These bounds require a multitude of results pertaining to the multiplicative nature of $\mathfrak{S}(n)$'s coefficients, and the solvability of congruence relations modulo prime powers.

(5) Conclude that suitably choosing $\delta_0 = \min(1,\delta_1,\delta_2,\delta_3,\nu\delta_4)$ and $\delta = \frac{\delta_0}{k}$ completes the proof by computing

$$r_{s,\mathcal{K}}(n) = \int_0^1 F(x)^s e(-nx)dx = \int_{\mathfrak{W}} F(x)^s e(-nx)dx + \int_{\mathfrak{w}} F(x)^s e(-nx)dx$$

directly and substituting the aforementioned results found in each step above.

Details on the above procedure and the series of lemmas required can be found in Section 5 of Nathanson [15].

Remark 8.2. When considering the range of validity for the Hardy-Littlewood asymptotic formula, we will define $\widetilde{G}(k)$ as the smallest integer s_0 such that Theorem 8.1 holds for all $s \geq s_0$. Then,

$$\widetilde{G}(k) \leq k^2(\log k + \log\log k + O(1)).$$

This was proved by Kevin Ford [6] almost 70 years after Hardy and Littlewood's first proof with the aid of Vinogradov's results on the circle method.

In the same vein as Remark 8.2, an interesting discussion arises when we start considering the least s such that every natural number n may be expressed as a sum of at most s kth powers. Denoting this quantity as $g(k)$, it is known that $g(3) = 9$ and $g(4) = 19$, but otherwise, known exact values are sparse. A trivial bound can be achieved by considering the integers of the form

$$n = 2^k \left\lfloor \frac{3^k}{2^k} \right\rfloor - 1.$$

Clearly $n < 3^k$, and so can only be a sum of kth powers of 1 and 2, and the simplest example of such is to take $\lfloor \left(\frac{3}{2}\right)^k \rfloor - 1$ kth powers of 2 and $2^k - 1$ kth powers of 1:

$$n = \left(\left\lfloor \frac{3^k}{2^k} \right\rfloor - 1 \right) \cdot 2^k + (2^k - 1) \cdot 1.$$

Hence, $g(k) \geq 2^k + \lfloor \left(\frac{3}{2}\right)^k \rfloor - 2$. A great wealth of asymptotic relations have been discovered for the related quantity $G(k)$, which is defined for $k \geq 2$ as the minimum value for s such that every sufficiently large integer n, n may expressed as a sum of s kth powers. Note that the only exact values known are $G(2) = 4$ and $G(4) = 16$. This function was introduced by Hardy and Littlewood, who were also able to obtain the bound

$$G(k) \leq (k-2)2^{k-1},$$

where $k \geq 3$. During the 1930's, a series of improvements on this bound were made by Vinogradov using his refinement of the circle method. Currently, the best known value for $G(k)$ was obtained via the circle method by Wooley [19], who in particular proved that

$$G(k) < k(\log k + \log\log k + O(1)).$$

9. Conclusion

The collaborative effort of mathematicians over the decades have provided significant progress towards the area of additive number theory, as well as the elusive Goldbach conjecture. Much of this progress would not have been possible without the conception of the groundbreaking circle method, with its utility and outreach being nothing short of remarkable. Though the Goldbach conjecture remains unresolved, the recent breakthrough by Helfgott on Goldbach's weak conjecture indicates that the circle method is still young and blossoming.

With further technological advancements on the horizon, the accelerated verification of future number theoretic bounds and estimates will be accepted with open arms. Of course, any future estimates achieved may rely heavily on further refinements to the methods of Hardy, Littlewood and Vinogradov. Alternative approaches to the Goldbach conjecture and other related problems neglected in our discussion include the area of sieve theory, and the continual work on the Generalized Riemann Hypothesis, both of which may potentially fill in the gaps left behind by the circle method. Helfgott's result also encourages further implementation of Fourier methods to strengthen the bounds on both the major and minor arcs; further emphasizing that there is still much to be done before we may reclassify Goldbach's conjecture as a bonafide theorem.

10. References

[1] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. *Springer-Verlag, New York-Heidelberg,* 1976.

[2] Chen, Jing Ren; Pan, Cheng Dong. The exceptional set of Goldbach numbers. I. *Sci. Sinica 23,* (1980), no. 4, 416-430.

[3] Chudakov Nikolai. On the density of the set of even integers which are not representable as a sum of two odd primes, *Izv. Akad. Nauk SSR*, 1938.

[4] Davenport, Harold. Multiplicative number theory. Third edition. Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000.

[5] Estermann, Theodor. On Goldbach's Problem : Proof that Almost all Even Positive Integers are Sums of Two Primes. *Proc. London Math. Soc. (2)* 44 (1938), no. 4, 307-314.

[6] Ford, Kevin B. New estimates for mean values of Weyl sums. *Internat. Math. Res. Notices* 1995, no. 3, 155171.

[7] Hardy, G. H.; Ramanujan, S. Asymptotic formulae in combinatory analysis. *Proc. London Math. Soc. (2)* 17 (1918), 75-115.

[8] Helfgott, Harald Andrs. The ternary Goldbach problem. *Proceedings of the International Congress of Mathematicians-Seoul 2014. Vol. II, 391-418, Kyung Moon Sa, Seoul,* 2014.

[9] Helfgott, H. A.; Platt, David J. Numerical verification of the ternary Goldbach conjecture up to $8.875 \cdot 10^{30}$. *Exp. Math* (2013), no. 4, 40-409.

[10] Kumchev, A. V.; Tolev, D. I. An invitation to additive prime number theory. *Serdica Math. J. 31* (2005), no. 1-2, 1-74.

[11] Li, Hong Ze. Goldbach numbers in short intervals. *Sci. China Ser. A* 38 (1995), no. 6, 641-652.

[12] Liu, Ming-Chit; Wang, Tianze. On the Vinogradov bound in the three primes Goldbach conjecture. *Acta Arith.* 105 (2002), no. 2, 133-175.

[13] Miller, Steven J.; Takloo-Bighash, Ramin. An invitation to modern number theory. With a foreword by Peter Sarnak. *Princeton University Press, Princeton, NJ,* 2006.

[14] Montgomery, H. L.; Vaughan, R. C. The exceptional set in Goldbach's problem. Collection of articles in memory of Juri Vladimirovi Linnik. *Acta Arith.* 27 (1975), 353-370.

[15] Nathanson, Melvyn B. Additive Number Theory The Classical Bases, *Springer-Verlag, New York-Heidelberg,* 1996.

[16] Pintz, Janos. Explicit formulas and the exceptional set in Goldbachs problem, *lecture at the CNTA VIII meeting in Toronto*, 2004.

[17] Van der Corput, J. G. Sur la dmonstration de l'hypothse de Goldbach pour les nombres impairs, donne par Vinogradow. (French) *Bull. Soc. Math. France 67* (1939), 87-99.

[18] Vaughan, R.C. The Hardy-Littlewood Method, *Cambridge University Press,* 1997.

[19] Wooley, Trevor D. Large improvements in Waring's problem. *Ann. of Math. (2)* 135 (1992), no. 1, 131-164.