Things to know for the final exam

Functions and Graphs

1. Know how to tell if an equation or a graph represents a function.
   • A function is a rule that assigns to any input a single output.
   • A function passes the vertical line test
   • The left figure represents a function, while the right figure does not.

2. Know how to recognize the parent function of an equation
   • A parent function is a function from a family of functions with the simplest equation that still maintains the shape.
     ◦ Example: From the family of parabolic functions, the function \( f(x) = x^2 \) is the parent function.
   • Required Parent Functions:
     ◦ Constant
       ▪ \( f(x) = a \)
     ◦ Linear
       ▪ \( f(x) = x \)
     ◦ Absolute Value:
       ▪ \( f(x) = |x| \)
     ◦ Quadratic
       ▪ \( f(x) = x^2 \)
     ◦ Cubic
       ▪ \( f(x) = x^3 \)
     ◦ Square Root
       ▪ \( f(x) = \sqrt{x} \)
     ◦ Cube Root
       ▪ \( f(x) = x^{\frac{1}{3}} \)
     ◦ Exponential
       ▪ \( f(x) = a^x \)
     ◦ Logarithmic
       ▪ \( f(x) = \log_a(x) \)
     ◦ Reciprocal
       ▪ \( f(x) = \frac{1}{x} \)
     ◦ Trigonometric Functions
       ▪ \( f(x) = \sin(x) \)
       ▪ \( f(x) = \cos(x) \)
       ▪ \( f(x) = \tan(x) \)
3. Know how to graph using transformations

- Strategy: Graphing Parent Functions using a Table of Values
  - How to use it: Start with a set of $x$ – values (think of ones that will be easy inputs for the function) and use the function to generate corresponding $y$ – values. You can then plot the points on the grid.
  - When To Use it: Although it can be used to graph any function, it is best when graphing parent functions (as it is possible to miss key information).

- Strategy: Graphing Functions using Transformations
  - How To Use it:
    - 1) Identify the parent function $f(x)$.
    - 2) Write the function in the form: $g(x) = a(f(b(x + c)) + d$
    - 3) Construct the table of values for the parent function.
    - 4) Apply the following transformations to the table of values to get the new table of values:
      - New $x$ values: $\frac{x}{b} - c$
      - New $y$ values: $ya + d$
    - 5) Graph the new function.

Example:

Graph the following equation: $g(x) = 3x^2 - 2$

**Solution:**

Parent function $f(x) = x^2$

$g(x) = 3[f(1[x + 0])] - 2$

$a = 3, b = 1, c = 0, d = -2$

We generate our table of values for the parent function:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$(-2)^2 = 4$</td>
</tr>
<tr>
<td>-1</td>
<td>$(-1)^2 = 1$</td>
</tr>
<tr>
<td>0</td>
<td>$(0)^2 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$1^2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$2^2 = 4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$3(4) - 2 = 10$</td>
</tr>
<tr>
<td>-1</td>
<td>$3(1) - 2 = 1$</td>
</tr>
<tr>
<td>0</td>
<td>$3(0) - 2 = -2$</td>
</tr>
<tr>
<td>1</td>
<td>$3(1) - 2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$3(4) - 2 = 10$</td>
</tr>
</tbody>
</table>

New x values: $\frac{x}{b} - c \rightarrow \frac{x}{1} - 0$

New y values: $(y \times a + d) \rightarrow 3y - 2$
4. Know how to find the domain from a graph or an equation

- Given a function, we call the domain the set of all allowable inputs (x-values) that the function can take.
- Similarly, we call the range the set of all allowable outputs (y-values) the function can map to.

- Strategy: Finding the domain and range by using a Scanner
  - How To Use it: To find the domain:
    - 1) Start the “scanning” process from the furthest left of the relation where you see points appear going towards the right.
    - 2) If you notice that there is an arrow on the left, this means that the points go all the way to $-\infty$. Similarly if there is an arrow pointing towards the right it will go all the way to $\infty$.
    - 3) If you notice there is a break in the domain (holes, vertical asymptotes, large gaps where there are no points) keep track of where the “scanner” will not pick up points.
    - 4) Put all of the information together using intervals where $)$ are used for “not including endpoints”, $[,$ are used for “including endpoints”, and $\cup$ is used to join to intervals.
  - How to find the Range: The same way as finding the domain except you scan bottom to top instead.

- Strategy: Finding the domain from an equation
  - How To Use it: To find the domain:
    - 1) Determine different issues that appear in placing numbers into the function. You should be looking for:
      - i) Division by zero
      - ii) Square roots of negatives
      - iii) Logs of 0 or negatives
      - iv) Trig functions that are undefined (Tan, Sec, Csc and Cot)
    - 2) Once you found all of the “problem points”, create an interval that excludes all of the problem areas. It helps to construct a number line to organize the information.
    - Note that finding the range from the equation is substantially more challenging. Typically, it is better to graph to find the range and this method only works for domain.

5. Know how to compose two functions

- For functions $f(x), g(x)$ where the range of g is contained in the domain of f, the composition $g \circ f(x) = g(f(x))$ is the function g evaluated at the output of the function f which is evaluated at the point $x$. 
Trigonometric Functions

6. **Know how to solve trigonometric equations using the unit circle?**

   - Given the unit circle (the circle of radius 1), a point (x,y) on the circle, and an angle starting from the point (1,0) moving the a counterclockwise direction, we can define the following trigonometric ratios:
     - \( \sin(\theta) = y \)
     - \( \cos(\theta) = x \)
     - \( \tan(\theta) = \frac{x}{y} \)
     - \( \csc(\theta) = \frac{1}{y} \)
     - \( \sec \theta = \frac{1}{x} \)
     - \( \cot(\theta) = \frac{x}{y} \)

   - **Strategy: Solving Trigonometric Equations using the Unit Circle (when solving for the ratio)**
     - **How To Use it:**
       - 1) Identify the angle \( \theta \)
       - 2) Add or subtract \( 2\pi \) to get an angle between 0 and \( 2\pi \)
       - 3) Plot the angle on the unit circle.
       - 4) Identify the point.
       - 5) Use the ratio to evaluate the expression

   - **Strategy: Solving Trigonometric Equations using the Unit Circle (when solving for the angle)**
     - **How To Use it:**
       - 1) Isolate the trig function on one side and have the ratio reduced on the other side.
       - 2) Use the ratio to determine the location(s) of the point(s) on the unit circle
       - 3) Determine the angle and solve for initial \( \theta \) at the point(s).
       - 4) Identify the period of the trig function:
         - i) Sin, Cos, Sec, and Csc: period = \( \frac{2\pi}{b} \)
         - ii) Tan and Cot: period = \( \frac{\pi}{b} \)
       - 5) The solution will be \( \theta = \text{initial theta} + \text{period}(k), \text{for } k \in \mathbb{Z} \)
Logarithms

7. Know the log laws.

\[
\begin{align*}
\log_a (a^x) &= x \\
a^{\log_a(x)} &= x \\
\log_a (bc) &= \log_a(b) + \log_a(c) \\
\log_a (b/c) &= \log_a(b) - \log_a(c) \\
\log_a (b^c) &= c(\log_a(b)) \\
\log_a(b) &= \frac{\log_e(b)}{\log_e(a)}
\end{align*}
\]

8. Know how to simplify and solve log equations.

- We can use the above log laws to greatly simplify most logarithmic equations. This trick can be very useful in computing limits and derivatives.
- Examples:

Solve the following equation: \(\log_2(-10 + x) + \log_2(-x) = 4\)

**Solution:**
Using exponent laws, we simplify the expression, then we work to solve by getting rid of log:
\[
\begin{align*}
\log_2(-10 + x) + \log_2(-x) &= 4 \\
\log_2(10x - x^2) &= 4 \\
2^{\log_2(10x - x^2)} &= 2^4 \\
10x - x^2 &= 16 \\
x^2 - 10x + 16 &= 0 \\
(x - 8)(x - 2) &= 0
\end{align*}
\]

Solve the following equation: \(4e^{3k} - 6 = 10\)

**Solution:**
We work to isolate \(k\) one step at a time:
\[
\begin{align*}
4e^{3k} - 6 &= 10 \\
4e^{3k} &= 16 \\
e^{3k} &= 4 \\
\ln(e^{3k}) &= \ln(4) \\
3k &= \ln(4) \\
k &= \frac{\ln(4)}{3}
\end{align*}
\]
9. **Know how to find the inverse of a function.**

- **Strategy: Finding Inverses Algebraically by switching coordinates**
  - 1) Rewrite \( f(x) \) as \( y \).
  - 2) Switch \( x \) for \( y \) (and vice versa) in the equation.
  - 3) Solve for \( y \).
  - 4) Replace \( y \) with \( f^{-1}(x) \)
  - 5) If the relation is not a function, impose a domain restriction on the original function (not the inverse function) so that the inverse will be a function. Relations that are not functions usually are things that include \( \pm \sqrt{ \) 

10. **Know how to change an absolute value function into a piecewise function.**

- The absolute value function \( |x| \) can be thought of as having two parts
  - \( f(x) = x \) if \( x \) is greater or equal to 0
  - \( f(x) = -x \) if \( x \) is less than zero
- This can also be written as \( f(x) = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases} \)
- Using this, we can deal with the absolute value function directly, or break the function up into intervals on which it is simply two linear functions.

11. **Know how to show a function is even or odd.**

- A function is even if when the function is reflected over the y axis it remains the same,
  - For an even function, \( f(x) = f(-x) \)
  - Examples: \( f(x) = x^2, f(x) = |x| \)
- A function is odd if when the function is reflected over both the x and y axis simultaneously it remains the same,
  - For an odd function, \( f(x) = -f(-x) \)
  - Example: \( f(x) = x^3 \)
- A function can be odd, even, or neither.
Limits

12. Know the difference between left hand, right hand, and both side limits and how to use them in an equation or find them on a graph.

- A limit expresses the behaviour of a function as we approach a point x from the left, right, or both directions.
  - $\lim_{x \to a^+} f(x) = L$
    - This says that as we approach the point a from the right side of the function f(x), the function goes towards the y value of L.
  - $\lim_{x \to a^-} f(x) = L$
    - This says that as we approach the point a from the left side of the function f(x), the function goes towards the y value of L.
  - $\lim_{x \to a} f(x) = L$
    - This says that as we approach the point a from both sides of the function f(x), the function goes towards the y value of L.

- Strategy: Finding Limits Using Graphs
  - How To Use it:
    - To find a limit from one side $a^+$ or $a^-$
    - 1) Pick a point close a on the right side of a (when doing $a^+$) or on the left side of a (when doing $a^-$).
    - 2) Trace the graph until you arrive at $x = a$ (note that you are not allowed to jump when arriving at a).
    - 3) This will be the value of the limit.
    - 4) Note: to find the limit when $x \to a$, you simply doing both $a^+$ and $a^-$ and see if the limits match. If they do, this is the value of the limit. If they do not, the limit does not exist.
  - To find a limit when $x \to \infty$ we choose a point on the far right of the graph and trace the graph until you see where the function is going (either it gets stuck at a value L, it shoots down to $-\infty$, shoots up to $\infty$, or bounces back and forth and does not exist). For $x \to -\infty$, it is the same idea except trace the graph to the left instead.

13. Know the limit strategy of direct substitution and when to use it.

- When asked to find $\lim_{x \to a} f(x)$ attempt to substitute the a value into the function. If the value can be calculated (ie, it turns out to be a number without issues like $\sqrt{-c}$, $\frac{0}{0}$, $\log(0)$, etc…), then this number is the value of the limit.
- This is the first strategy that should be attempted to calculate a limit.
- Note that this works when a function is continuous at the point a, if the function is not continuous at a, we have to try another method to calculate the limit.
14. **Know the limit strategy of factoring and simplifying and when to use it.**

- When you try direct substitution and the result is \( \frac{0}{0} \), we call this an indeterminate form, that is, more work is needed to find the limit.
- If the numerator and denominator are both zero, it is possible that a factor \((x-a)\) can appear in both numerator and denominator. Cancelling this factor will allow for the remaining expression to use direct substitution.
- We can try to:
  - 1) Factoring the numerator and denominator
  - 2) Cancelling expressions
  - 3) Try direct substitution on the new expression.
- Note some factoring techniques you will need to recall:
  - Difference of Squares: \( a^2 - b^2 = (a - b)(a + b) \)
  - Difference of Cubes: \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \)
  - Sum of Cubes: \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \)

15. **Know the limit strategy of “multiplying the conjugate” and when to use it.**

- A conjugate of a term of the form \((a + b)\) is the term \((a - b)\)
- If we come to an indeterminate form while trying direct substitution, we may be able to multiply the limit by a conjugate over a conjugate and then evaluate it.
  - 1) Take the conjugate of any root expressions that are present.
  - 2) Multiply the numerator and denominator by the conjugate.
  - 3) Only “Expand” the pieces that make a difference of squares.
  - 4) Try to simplify the expression. You may need to factor the expression if it is still indeterminate (of the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \))

16. **Know how to determine limits of piecewise functions by checking right and left side limits**

- When dealing with functions that end or change functions at the point of the limit, we can try:
  - 1) Take the left side limit.
  - 2) Take the right side limit.
  - 3) See if both limits go to the same value, this will be the value of the limit.
- Note that for piecewise functions, you will need to know the correct piece to substitute. For absolute value functions, you may need to change them to piecewise functions.
- This approach also works for functions that end at the limit value such as \( \sqrt{x} \)
17. **Know the statement of the Squeeze Theorem and know when to use it.**

- Let f, g, and h are functions defined on an interval [a, b], and let c be a point on the interval. If for every x in the interval not equal to c we have that \( g(x) \leq f(x) \leq h(x) \), and \( \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \), then \( \lim_{x \to c} f(x) = L \) as well.
- The squeeze theorem is very useful for limits involving trigonometric functions. Note that sine and cosine are bounded by 1 and -1, so if evaluating the limit directly yields \( \frac{\sin x}{\infty} \) or \( \frac{\cos x}{\infty} \) we may squeeze it with a limit of the form \( \frac{1}{\infty} \).
- How to use it:
  - Find an expression for that is higher than f(x). Call this h(x).
  - Find an expression that is lower than f(x). Call this g(x).
  - Show that the limits for g(x) and g(x) go to the same value L.
  - Conclude (by squeeze theorem) that f(x) also goes to value k.

18. **Know how to apply “l’hopital’s rule” and when to use it? (also how to change limits so you can use l’hopital’s rule on them)**

- **L’Hopital’s Rule**
  - Given a limit of the form \( \lim_{x \to a} \frac{N(x)}{D(x)} \) such that it approaches \( \frac{\infty}{\infty} \) or \( \frac{0}{0} \) we can find a new limit \( \lim_{x \to a} \frac{N'(x)}{D'(x)} \) with the same limit as the original.
- How To Use it:
  - Check if the limit is of the form \( \frac{\infty}{\infty} \) or \( \frac{0}{0} \).
  - Then derive the numerator and derive the denominator separately to get a new limit. Find the value of this limit to find the value of the original limit.
  - Note: It is possible that:
    - 1) You may need/want to use l’hopitals rule again on the new limit
    - 2) It is possible that the new limit is a lot more difficult to deal with than the first limit.
    In this case it is better to explore other limit strategies instead of l’hopital’s rule.
- If we have the indeterminant form \( 0 \times \infty \), then you can try:
  - Given AB, then rewrite the expression as \( \frac{A}{B} \) or \( \frac{B}{A} \).
  - This will change the limit to \( \frac{\infty}{\infty} \) or \( \frac{0}{0} \), so you can use l’hopital’s rule.

19. **Know the limit strategy of “ln and e method” and when to use it.**

- If we have a complicated limit f(x), it may be simpler to consider the limit \( f(x) = e^{\ln(f(x))} \)
- The two expressions are equal, but we can use our logarithm rules to simplify the exponent on the right hand side which may result in a limit that is easier to evaluate.
- Once we have simplified the limit, try other strategies like substitution.
20. **Know how to tell when/where a function is continuous.**

- Intuitively, a function is continuous if we can draw its graph without lifting our pencil from the paper or chalkboard. A function is discontinuous at a point if:
  - 1) There is a hole in the interval (replaceable discontinuity)
  - 2) There is a jump in the function somewhere in the interval (jump discontinuity)
  - 3) There is a vertical asymptote in the interval (asymptote discontinuity)
  - 4) The function ends abruptly (endpoint discontinuity)
- Continuity can also be tested algebraically:
  - To identify key points to test at a function we look at places where we think the function may be discontinuous:
    - 1) Divisions by zeros
    - 2) log(0)
    - 3) tan(x) that produce asymptotes
    - 4) Root functions that end.
    - 5) Piecewise functions where the function changes. Note you must also make sure that each piece is continuous on its domain as well!
  - To test a particular $x$-value we:
    - 1) Determine $f(a)$
    - 2) Determine $\lim_{x \to a} f(x)$ (note we may need to check both sides of the limit)
    - 3) Make sure that all values exist and are equal to each other.

21. **Know how to use the intermediate value theorem to find zeros/other $y$-values of a function.**

- The intermediate value theorem states:
  - Given: 1. $f(x)$ is a continuous on a closed interval $a,b$, and 2. $y$ is any number that is in between $f(a)$ and $f(b)$
  - Conclusion: We will always be able to find some $c$ in the interval $[a,b]$ such that $f(c) = y$
- We typically use the intermediate value theorem to determine if a function has roots in a given interval.
- How to use it:
  - 1) Ensure (state) that you have a continuous function on $[a,b]$
  - 2) Ensure that $y$ (usually $y = 0$) is in between $f(a)$ and $f(b)$.
  - 3) Conclude that we have a $c$ in the interval of $[a,b]$ such that $f(c) = y$.
- While we typically use it for roots, we can use it to say that we can always find an input $c$ that will produce $f(c) = y$ as long as $y$ is in between $f(a)$ and $f(c)$. 
Derivatives

22. Know how to find the average rate of change between two points.
   - The average rate of change between two points is simply the slope of the line between the two points. It can be calculated using the standard slope formula:
     \[ \frac{y_2 - y_1}{x_2 - x_1} \]

23. Know how to find the instantaneous rate of change at a point.
   - Given a function and a point on the graph, the instantaneous rate of change is the slope of the function at that point. We can’t do this directly, so we have to use the idea of the limit.
     - The slope of the tangent line is the limit \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \)

24. Know the definition of a derivative from first principles.
   - The derivative of a function encodes the slope of the tangent line, or the instantaneous rate of change at every point of the original function.
   - The derivative is defined as the difference quotient: \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) or \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \)

25. Know all of your derivative rules for standard functions.
   - Constant Rule:
     - If \( f(x) = c \) then \( f'(x) = 0 \)
   - Power Rule:
     - If \( f(x) = x^n \) then \( f'(x) = nx^{n-1} \)
   - Sum and Difference Rule:
     - If \( h(x) = f(x) \pm g(x) \) then \( h'(x) = f'(x) \pm g'(x) \)
   - Coefficient Rule:
     - If \( k \in \mathbb{R} \) and \( h(x) = kf(x) \) then \( h'(x) = kf'(x) \)
   - Product Rule:
     - If \( h(x) = f(x)g(x) \) then \( h'(x) = f'(x)g(x) + f(x)g'(x) \)
     - NOTE: This is the first rule where differentiation doesn’t behave exactly the same way limits do.
   - Quotient Rule:
     - If \( h(x) = \frac{f(x)}{g(x)} \) then \( h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \)
     - NOTE: You can also use the product and chain rule instead of the quotient rule, just by rewriting \( \frac{f(x)}{g(x)} = f(x)g(x)^{-1} \)
   - Chain Rule:
     - If \( h(x) = f \circ g(x) = f(g(x)) \) then \( h'(x) = f'(g(x))g'(x) \)
     - NOTE: The chain rule works for compositions of arbitrarily many functions, that is, it may be chained together multiple times.
   - Inverse Rule:
     - If we have an inverse function \( f^{-1}(x) \), then the derivative is \( [f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))} \)
     - That is the derivative of an inverse function is 1 divided by the derivative of the original function with an input of the inverse function.
Deriving piecewise functions

- To derive a piecewise function, we derive each part separately.
- For the points \( x = a \) where the function changes, we need to check whether the function is differentiable using the definition of the derivative. In general, you will need to check the limit on both sides of \( a \), since the function changes at \( a \).

Strategy: Combining Multiple Derivative Rules

- 1) Identify the last operation you would need to perform, then this is the first differentiation rule you need to apply.
- 2) It helps to use shapes (for chain rules) and/or letters (for combinations of functions) to keep together the pieces you will need to derive next.
- 3) Repeat the above process one at a time until you have identified all derivatives.

Derivatives of Standard Functions

- \( \frac{d}{dx} x^n = nx^{n-1} \)
- \( \frac{d}{dx} a^x = a^x \ln(a) \)
- \( \frac{d}{dx} e^x = e^x \ln(e) = e^x \)
- \( \frac{d}{dx} \ln|x| = \frac{1}{x} \)
- \( \frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)} \)
- \( \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \)
- \( \frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}} \)
- \( \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} \)

26. Know how to use “implicit differentiation” and when you should use it.

- If we have a relation where \( y \) appears on both sides and we wish to differentiate, we can:
  - 1) Derive both sides using standard differentiation rules. Be careful to include \( y' \), but you may use the fact that \( x' = 1 \).
  - 2) Gather all \( y' \) terms to one side of the equation, and all other terms (terms only containing \( x \) and \( y \)) onto the other side.
  - 3) Factor the side with \( y' \), then you can isolate \( y' \) by dividing the expression over to the other side.

27. Know how to use “logarithmic differentiation” and when you should use it.

- If we have functions of the form \( f(x)^{g(x)} \) or if we have a function with many products/quotients, we can do the following to derive:
  - 1) Take the natural logarithm \( \ln \) of both sides of the equation
  - 2) Simplify the result using log laws.
  - 3) Use implicit differentiation to derive and find \( y' \)
  - 4) If we were given an expression for \( y \), don’t forget to sub in \( y \) so the derivative is only in terms of \( x \).
- This is useful because applying \( \ln \) to both sides gives us an opportunity to change exponentiation into products and products/quotients into sums/differences. This can simplify the expression to allow us to differentiate.
28. **Know the linearization formula, and know how to find the equation of the tangent line.**

- If we have a function $f(x)$ and we want to find the equation of the tangent line (linearization) at $x = a$, we can use the formula:
  - $L(x) = f(a) + f'(a)(x - a)$

29. **Know how to use linearization and/or differentials to approximate.**

- If we want to approximate an $x$-value on a function (call it $c$) without a calculator, we would:
  - 1) Determine an "$a$" value that we can evaluate exactly that is closest to “$c$”
  - 2) Find the linearization at $x = a$.
    - Find $f(x)$, $f'(x)$, $f(a)$, and $f'(a)$ and sub all of this into $L(x) = f(a) + f'(a)(x - a)$
  - 3) Sub in “$c$” into the linearization.
- This works because the tangent line’s $y$ value is close to the $y$ value of the function, as long as the $x$-value is close to the point that the line is tangent to.

30. **Know the extreme value theorem and how/when to use it to find absolute max/min values.**

- The Extreme Value Theorem
  - Let $f(x)$ be a continuous function on a closed interval $[a, b]$. Then $f(x)$ attains an absolute maximum and absolute minimum on $[a, b]$, and these occur at $f(a)$, $f(b)$ or at a critical point.
- How we can use it
  - Determine $f''(x)$ and find:
    - 1) All critical points when $f'(x) = 0$
    - 2) All critical points when $f''(x)$ is undefined.
    - 3) Evaluate $f(a)$, $f(b)$, and $f$(critical points).
  - The largest number in step 3 is the absolute maximum, and the smallest number in step 3 is the absolute minimum in the interval $[a, b]$.

31. **Know how to find critical points.**

- A critical point are the points that occur/exist on $f(c)$ and where $f'(x) = 0$ or when $f'(x)$ is undefined

- Classifying Critical Points using the Second Derivative
  - If we have found the critical point at $x = a$, and we have the second derivative of the function $f''(x)$, Then if we sub in $f''(a)$ we get:
    - 1) $f''(a)$ is positive means the critical point is a local minimum.
    - 2) $f''(a)$ is negative means the critical point is a local maximum.
    - 3) $f''(a)$ is undefined or 0 means the test fails.
32. **Know how to find intervals of increase and decrease and use this to classify critical points.**

- To find the intervals of increase/decrease we can:
  1. Identify the domain of the function.
  2. Identify all x values for the points of discontinuity.
  3. Determine $f'(x)$.
  4. Factor the expression fully and find all critical x-values.
  5. Create an interval table for $f'(x)$ (not $f(x)$) where we plot the domain, points of discontinuity, and critical points as the column separators, and the factors of the derivative as the rows. The product will tell us if the value is positive or negative which determines if the function is increasing (+) or decreasing (-).
  6. If a critical point comes from increasing then goes to decreasing, it is a local maximum. If it comes from decreasing and goes to increasing it is a local minimum. If it is anything else, it is neither a local maximum or minimum.

- This works because increasing means that the derivative is positive and decreasing means that the derivative is negative. This strategy just gives us a list of steps to identify all of these places by looking at the derivative.

- For classifying local maximums or minimums, we see that maximums are increasing on the left then decreasing on the right. Similarly, minimums are decreasing on the left and then increasing on the right.

33. **Know how to find intervals of concavity and use this to find inflection points.**

- We say that a function is concave up when the point appears to be on a valley
- We say that a function is concave down when the point appears to be on a hill.
- We call an inflection point a point that changes from concave up to concave down (or vice versa) on a continuous function.

- Finding intervals of Concavity and Inflection Points
  1. Identify the domain of the function.
  2. Identify all x values for the points of discontinuity.
  3. Determine $f''(x)$.
  4. Factor $f''(x)$ fully and find all possible inflection points x-values (where the second derivative is 0 or undefined).
  5. Create an interval table for $f''(x)$ where we plot the domain, points of discontinuity, and potential inflection points as the column separators, and the factors of the derivative as the rows. The product will tell us if the value is positive or negative which determines if the function is concave up (+) or concave down (-).
  6. If a point switches concavity (and is not a point of discontinuity) then those are the inflection points.
34. **Know how to find all asymptotes and holes of an equation using limits.**

- To find vertical asymptotes/holes we:
  1) Find all points of discontinuity (division by zero, logs, tan, change in piecewise, endpoints, etc…). Call any such point “a”
  2) Test the limit at “a”. Note that it may be needed to test the right hand and left hand limits for each point of discontinuity: \( \lim_{x \to a^+} f(x) = L_1 \) and \( \lim_{x \to a^-} f(x) = L_2 \)
  3) If one of the limits (or both) go to a number \( L_1 \) (or \( L_2 \)), this produces a hole at the point \( (a, L_1) \) or a hole at \( (a, L_2) \), or possibly both.
  4) If one of the limits (or both) go to infinity (or negative infinity), then this produces a vertical asymptote (on one or both sides of \( a \) depending on the limits).

- To find horizontal asymptotes we:
  1) Find the limits as \( x \to \infty \) and as \( x \to -\infty \)
  2) If the limit goes to a number \( L \), then we have a horizontal asymptote on that side (it is possible to have two different horizontal asymptotes one for each side). If it goes to \( \infty \) or \( -\infty \) then there is no horizontal asymptote.

35. **Know how to sketch a curve given the information (intercepts, asymptotes, critical points ,…)**

- To sketch the curve we should:
  1) Plot all of the key points given: endpoints, intercepts, local max/min, holes, inflection points, vertical and horizontal asymptotes.
  2) For each vertical asymptote, determine the behaviour on either side using increasing and decreasing and plot a “short arrow” do indicate what is happening on each side of vertical asymptote.
  3) For each horizontal asymptote, determine whether the function is coming in on top or below the asymptote by looking at increasing and decreasing.
  4) For each point/asymptote arrow to the next, determine the concavity increasing/decreasing nature in that interval, and connect the pieces together.
Integration

36. Know how to find an antiderivative of all of our standard functions and the value of the unknown constant.

- Let $f(x)$ be a function, then an Antiderivative is a new function $F(x)$ such that $F'(x) = f(x)$.
- That is, the antiderivative of a function is a new function whose derivative gives us our original $f(x)$ back.

- Finding Antiderivatives Using Try and Derive Tables
  - 1) Think of a new function $F(x)$ that you will feel will have a derivative that will result in the given function $f(x)$. Place the $F(x)$ on the “try side” and determine the derivative and place it on the “derive side”.
  - 2) Derive the function $F(x)$ to see if you get $f(x)$. If it is only of by a coefficient, you can fix your coefficient by multiplying the coefficient on both sides.
  - 3) Once you have the correct derivative equal to $f(x)$ then you have found your $F(x)$
  - 4) Do not forget to add the $+c$ at the end to account for the unknown constant.

- NOTE: We cannot multiply a variable on both sides to “fix” the antiderivative. Why? Deriving would result in a product rule (which odds are was not what you were considering)

Here is a useful list of antiderivatives to remember:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1} + c$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$\frac{1}{ax+b}$</td>
<td>$\frac{\ln</td>
</tr>
<tr>
<td>$\sin(ax)$</td>
<td>$-\frac{\cos(ax)}{a} + c$</td>
</tr>
<tr>
<td>$\cos(ax)$</td>
<td>$\frac{\sin(ax)}{a} + c$</td>
</tr>
<tr>
<td>$\sec^2(ax)$</td>
<td>$\frac{\tan(ax)}{a} + c$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$\frac{e^{ax}}{a} + c$</td>
</tr>
<tr>
<td>$b^{ax}$</td>
<td>$\frac{b^{ax}}{a \ln(b)} + c$</td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td>$\arctan(x) + c$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin(x) + c$</td>
</tr>
</tbody>
</table>

- Finding the value of the unknown constant is possible when we have an initial value, this is called solving an initial value problem. When solving for the specific antiderivative we can:
  - 1) Use antidifferentiation to find our general antiderivative (with the $+c$).
  - 2) Sub in the initial point into our antiderivative to find $c$. 

37. **Know how to find definite integrals.**

- Let $f(x)$ be a function, we define the definite integral of $f$ from $a$ to $b$ as the area under the graph of $x$ but above the $x$ axis. We write this as $\int_a^b f(x)\,dx$.
- If the graph passes below the $x$-axis, then the area below the curve is represented as a negative number.
- Hence, the integral of a function on an interval is the ‘net’ area under the graph, adding the positive parts and subtracting the negative parts.

- We compute definite integrals by using the fundamental theorem of calculus:
  - Let $f(x)$ be continuous on $[a,b]$, and let $F(x) = \int_a^x f(t)\,dt$. then
  - $F(x)$ is a continuous function on $[a,b]$
  - $F(x)$ is differentiable on $(a,b)$
  - $F'(x) = f(x)$
  - $\int_a^b f(t)\,dt = F(b) - F(a)$

- With this we can easily compute definite integrals by finding and evaluating antiderivatives.
  - When calculating an integral $\int_a^b f(t)\,dt$
  - 1) Ensure the function is continuous on $[a,b]$
  - 2) Find the antiderivative $F(x)$
  - 3) Determine $F(b) - F(a)$ which is the value of our integral.

38. **Know how to find indefinite integrals.**

- An indefinite integral can also be thought of as the most general antiderivative
- We denote the indefinite integral $\int f(x)\,dx = F(x) + c$
- Finding indefinite integrals is equivalent to finding an antiderivative.

39. **Know how to find the total area (not net) made with a curve and the x-axis.**

- To find the total area we have to add the negative areas instead of subtracting them:
  - 1) Find all x intercepts of the function inside the interval. (For example, suppose we have 3 of them, call them $x_1, x_2, x_3$)
  - Subdivide the interval into subintervals with the boundaries given by the intercepts.
  - Add up the absolute values of the definite integral on each subinterval
  - 2) Our total area would be:
    - Total Area = $|\int_a^{x_1} f(x)\,dx| + |\int_{x_1}^{x_2} f(x)\,dx| + |\int_{x_2}^{x_3} f(x)\,dx| + |\int_{x_3}^b f(x)\,dx|$  
  - Note that if we have more x-intercepts, then we would continue to break up the integral over more pieces which expands the above formula.
- By splitting the interval up based on the x intercepts, we are guaranteeing that the pieces are either all positive area or all negative area. If we then take the absolute value of each piece, we then force all pieces to be positive and can add them together.
40. **Know how to find the area contained between two curves.**

- To find the area contained between two curves, we subtract the integral of the lower curve from the upper curve.
- If the curves cross, we need to deal with each part separately.
- How to find the area between two curves \( f(x), g(x) \) on an interval \([a,b] \):
  - Find all x-values for the points of intersection between the two curves (For example, suppose we have 4 points of intersection: \( x_1, x_2, x_3, x_4 \))
  - 2) Our contained area would be
  - Contained Area =
    \[
    \left| \int_{x_1}^{x_2} f(x) - \int_{x_1}^{x_2} g(x) \, dx \right| + \left| \int_{x_2}^{x_3} f(x) - \int_{x_2}^{x_3} g(x) \, dx \right| + \left| \int_{x_3}^{x_4} f(x) - \int_{x_3}^{x_4} g(x) \, dx \right|
    \]
  - Note that if we have more x-intercepts, then we would continue to break up the integral over more pieces which expands the above formula.
It is **most beneficial** to you to write this mock midterm **UNDER EXAM CONDITIONS**. This means:

- Complete the midterm in 3 hour(s).
- Work on your own.
- Keep your notes and textbook closed.
- Attempt every question.

After the time limit, go back over your work with a different colour or on a separate piece of paper and try to do the questions you are unsure of. Record your ideas in the margins to remind yourself of what you were thinking when you take it up at PASS.

The purpose of this mock exam is to give you practice answering questions in a timed setting and to help you to gauge which aspects of the course content you know well and which are in need of further development and review. Use this mock exam as a **learning tool** in preparing for the actual exam. In addition, there may be additional concepts covered on the exam not included in this mock.

Please note:

- **Come to the PASS workshop with your mock exam complete.** During the workshop you can work with other students to review your work.

- **Often, there is not enough time to review the entire exam in the PASS workshop.** Decide which questions you most want to review – the Facilitator may ask students to vote on which questions they want to discuss in detail.

- **Facilitators do not bring copies of the mock exam to the session.** Please print out and complete the exam before you attend.

- **Facilitators do not produce or distribute an answer key for mock exams.** Facilitators help students to work together to compare and assess the answers they have. If you are not able to attend the PASS workshop, you can work alone or with others in the class.

**Good Luck writing the Mock Exam!!**

**Dates and locations of mock exam take-up:** Thursday December 14th, 2PM-4PM ME3380. Monday December 18th, 4PM-6PM ME3380.

**EXTRA OFFICE HOUR:** 2PM Monday December 18th, 4th Floor ML
Part 1: Multiple Choice (30 marks = 15x2 marks each)

1. Is the function \( f(x) = 3x^3 + 2x \)
   a. Even  
b. Odd  
c. Neither

2. What is the inverse of \( y = x^{-2} + \frac{1}{2} \)
   a. \( y = \frac{1}{x} - 2 \)
   b. \( y = \frac{1}{x^2} - 2 \)
   c. \( y = \sqrt{2 - \frac{1}{x}} \)
   d. \( y = \sqrt{\frac{1}{x} - 2} \)

3. When is the function \( f(x) = \frac{x+1}{x^2-4x+3} \) discontinuous?
   a. -1  
b. -1, -3  
c. 1, 3  
d. Never

For question 4-7, evaluate the limit

4. \( \lim_{x \to \infty} e^{1/x} \)
   a. 0  
b. 1  
c. e  
d. DNE

5. \( \lim_{x \to 0} \frac{\sin(x)}{x} \)
   a. 0  
b. 1  
c. \( \pi \)  
d. DNE
6. \( \lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} \)
   a. 1
   b. \( \frac{1}{2} \)
   c. \( \frac{1}{4} \)
   d. DNE

7. \( \lim_{x \to 0^+} \frac{1}{x} \)
   a. \( \infty \)
   b. \(- \infty \)
   c. 0
   d. DNE

For question 8-12, find the derivative

8. \( f(x) = \sqrt{3x} + 1 \)
   a. \( f'(x) = \frac{3}{2\sqrt{3x+1}} \)
   b. \( f'(x) = \frac{1}{2\sqrt{3x+1}} \)
   c. \( f'(x) = \frac{3}{2}\sqrt{3x+1} \)
   d. \( f'(x) = \frac{1}{2}\sqrt{3x+1} \)

9. \( f(x) = x^{5/3} + x^{2/3} \)
   a. \( f'(x) = \frac{5}{3}x + \frac{2}{3} \)
   b. \( f'(x) = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{-1/3} \)
   c. \( f'(x) = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{1/3} \)
   d. \( f'(x) = \frac{5}{3}x + \frac{2}{3}x \)

10. \( f(x) = \sin(3x^2 + 4) \)
    a. \( f'(x) = 3\cos(6x) \)
    b. \( f'(x) = (3x)\cos(3x^2 + 4) \)
    c. \( f'(x) = 3\cos(3x^2 + 4) \)
    d. \( f'(x) = (6x)\cos(3x^2 + 4) \)
11. \( f(x) = \tan(\ln(x)) \)
   a. \( f'(x) = \csc^2(\ln(x)) \)
   b. \( f'(x) = \frac{\sec^2(\ln(x))}{x} \)
   c. \( f'(x) = \frac{\csc^2(\ln(x))}{x} \)
   d. \( f'(x) = \frac{1}{\tan(\ln(x))} \)

12. \( f(x) = e^{x^2+4} + 3e^{3x} \)
   a. \( f'(x) = e^{x^2+4} + 3e^{3x} \)
   b. \( f(x) = (2x)e^{x^2+4} + 9e^{3x} \)
   c. \( f(x) = 2e^{x^2+4} + 9e^{3x} \)
   d. \( f(x) = (2x)e^{2x} + 9e^3 \)

13. What is the implicit differentiation with regards to \( x \) of the following equation: \( y^2 + 2xy + x^2 = 25 \)
   a. \( 4y + 4x = 0 \)
   b. \( y' = \frac{-(y+x)}{(y+x)} \)
   c. \( y' = \frac{-2x-y}{y} \)
   d. \( y' = 4y + 4x \)

14. What is the general anti-derivative of \( f(x) = 3\sec^2 x + \frac{1}{\sqrt{1-x^2}} \)
   a. \( f'(x) = 3\tan(x) + \ln(\sqrt{1-x^2}) + C \)
   b. \( f'(x) = \tan(3x) + 2\sqrt{1-x^2} + C \)
   c. \( f'(x) = \tan(3x) + \arcsin(x) + C \)
   d. \( f'(x) = 3\tan(x) + \arcsin(x) + C \)

15. Evaluate the following definite integral: \( \int_1^3 (3x^2 + 4x + 5) \, dx \)
   a. 52
   b. 60
   c. -8
   d. -52
Part 2: Long answer (45 marks = 3x15 marks each + 2 extra practice)

1. Find the derivative of $f(x) = 3x^2 + 4x$ using first principles.
2. Linearize the function $f(x) = \sqrt[3]{x}$ at 27 and use it to approximate the value of $\sqrt[3]{27.27}$.

3. Find the equation of the tangent of $f(x) = x^4 + 7x^3 + 2x + 1$ at the point $x=1$. 

 DISCLAIMER: PASS handouts are designed as a study aid only for use in PASS workshops. Handouts may contain errors, intentional or otherwise. It is up to the student to verify the information contained within.
4. Find the following information for the curve $y = x^4 - 2x^2 + 3$.

a. Local Minimums and Maximums

b. Intervals of Increase and Decrease

c. Points of Inflection

d. Intervals of Concavity
5. Find the area enclosed by the curves $y = x + 1$ and $y = 4 + 3x - x^2$. 