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PART I: Multiple Choice Questions.

1. Consider the following augmented matrix of a system of linear equations:

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 3 \\ 1 & 2 & -2 & 2 & 3 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 & -3 \end{array}\right]. \text{ The system has}$$

a) a unique solution **b)** no solutions
c) infinitely many solutions with one free variable
d) infinitely many solutions with two variables
e) infinitely many solutions with three variables

2. Suppose that A is a 5×5 matrix, $\det A = 0$, and b is a vector in R^5.

The matrix equation $Ax = b$ has

a) exactly one solution **b)** no solutions
c) infinitely many solutions with one parameter
d) either no solution or infinitely many solutions
e) infinitely many solutions with two parameters

3. Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 \\ 2 \\ k \\ -3 \end{bmatrix}$.

For what value of k is the set $\{u_1, u_2, u_3, u_4\}$ linearly dependent?

a) -1 **b)** -2 **c)** -3 **d)** 0 **e)** 1

4. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 3 & -6 & k \end{bmatrix}$. For what value of k are the columns of A linearly dependent?

a) 0 **b)** -6 **c)** 6 **d)** -9 **e)** 9

5. Let $A = \begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}$. What is $\det A$?

a) -15 **b)** -8 **c)** 8 **d)** -7 **e)** 7

6. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. What is $\det A$?

a) 32 **b)** -32 **c)** 16 **d)** -16 **e)** 0

7. Let A be a 4×4 matrix such that $\det A = 9$. What is $\det(3A)$?

   a) 3^2 **b)** 3^4 **c)** 3^6 **d)** 3^8

8. Let A and B be 3×3 matrices such that $\det A = 3$, and $\det B = -4$.
   Find $\det\left(BA^T B^{-1}\right)$.

   a) 3 **b)** -3 **c)** 4 **d)** -4 **e)** $1/3$

9. Let $A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What is the dimension of the null space of A?

   a) 1 **b)** 2 **c)** 3 **d)** 4 **e)** 0

10. Let A be a 7×9 matrix. If the dimension of the null space of A is 4, what is the rank of A?

    a) 4 **b)** 7 **c)** 9 **d)** 3 **e)** 5

11. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. What are the eigenvalues of A?

    a) 3 and 1 **b)** 3 and -1 **c)** -3 and 1 **d)** -3 and -1 **e)** 0 and 2

12. $\lambda = 3$ is an eigenvalue for the matrix $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 5 & -10 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$.

    What is the dimension of the eigenspace for $\lambda = 3$?

    a) 1 **b)** 2 **c)** 3 **d)** 4 **e)** 0

13. Let $z = 2 - 4i$. What is the modulus of z?

    a) 2 **b)** 4 **c)** $\sqrt{5}$ **d)** $\dfrac{1}{\sqrt{5}}$ **e)** $2\sqrt{5}$

14. Let $z = 3 + 4i$ and $w = 1 - 2i$. What is $\left|\dfrac{z}{w}\right|$?

    a) 5 **b)** $5\sqrt{5}$ **c)** $\sqrt{5}$ **d)** $\dfrac{1}{\sqrt{5}}$ **e)** 1

15. Let $u = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -1 \end{bmatrix}$. What is the dot product of $\mathbf{u}$ and $\mathbf{v}$?

    a) -1 **b)** -2 **c)** 0 **d)** 1 **e)** 2

16. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. What is the inner product of **u** and **v**?

    a) 1 **b)** 2 **c)** 3 **d)** -1 **e)** 6

17. What is the angle between the vectors $u = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$?

    a) 0 **b)** $\pi/6$ **c)** $\pi/4$ **d)** $\pi/3$ **e)** $\pi/2$

18. What is the angle between the vectors $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

    a) 0 **b)** $\pi/6$ **c)** $\pi/4$ **d)** $\pi/3$ **e)** $\pi/2$

19. Which of the following matrices is invertible?

    a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **e)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

20. Let $A = \begin{bmatrix} a & 4 \\ 3 & 2 \end{bmatrix}$. For what value of a is the matrix A not invertible?

    a) 0 **b)** 2 **c)** 3 **d)** 4 **e)** 6

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PART II: Short Answer Questions.

1. Let $T : R^3 \longrightarrow R^2$ be a linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + z \\ 2x + y - z \end{bmatrix}$.

   Find a nonzero vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Is T onto?

2. Let $T : R^3 \longrightarrow R^2$ be a linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y - z \end{bmatrix}$.

   Find a basis for $\ker(T)$. Is T one-to-one?

3. Suppose that $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}^{-1} \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find $\mathbf{x}$.

4. Suppose A is a 2×2 matrix such that $A \cdot \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ -1 & 3 \end{bmatrix}$. Find A.

5. Let A and B be two 4×4 matrices such that $\det A = 2$ and $\det B = 9$.
   Find $\det(3AB^{-1}A^T)$.

6. Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$. Find the eigenvalues of A.

7. Let $A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$. Find the eigenvalues of A.

8. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & k \\ 0 & -4 & 4 \end{bmatrix}$. Find the value of k for which the rank of A is 2.

9. Write the complex number $\dfrac{-2 + 11i}{-3 + 4i}$ in the form $a + bi$, where a and b are real numbers.

10. Write down the polar form of the complex number $z = -2 + 2i$.

11. You are given that $\lambda = 2$ is an eigenvalue for the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

    Determine the eigenspace of A corresponding to $\lambda = 2$.

12. You are given that $\lambda = 3$ is an eigenvalue of the matrix $A = \begin{bmatrix} 5 & 2 & -4 \\ 2 & 5 & -4 \\ 2 & 2 & -1 \end{bmatrix}$.

    Determine a basis for E_3.

13. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.
Write down a vector $\mathbf{w}$ which is orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$.

14. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Write down a unit vector in the direction of $\mathbf{u}$.

15. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -5 \\ -1 \end{bmatrix}$.
Suppose that $x = c_1\mathbf{v_1} + c_2\mathbf{v_2}$, where c_1 and c_2 are real numbers. Find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

16. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $v_4 = \begin{bmatrix} 5 \\ -4 \\ h \end{bmatrix}$.
Find the value of h for which the vector v_4 is in Span$\{v_1, v_2, v_3\}$.

17. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -5 \\ 2 & 0 & 4 \end{bmatrix}$. Write down the transpose of A.

18. Let $A = \begin{bmatrix} 2 & k \\ 3 & 4 \end{bmatrix}$. Write down the value of k for which A is symmetric.

19. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 3 & 1 \\ 4 & 2 & -5 \end{bmatrix}$ and let $B = \begin{bmatrix} -1 & 2 & 5 & -3 \\ 4 & -2 & 1 & 6 \\ -2 & 1 & 3 & 2 \end{bmatrix}$.
Find the third column of AB.

20. Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $u_4 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$.
Find the dimension of the subspace $H =$Span$\{u_1, u_2, u_3, u_4\}$.

PART III: Long Answer Questions.

1. Find the general solution of the following system of linear equations.

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 1 \\ 2x_1 - 2x_2 + 3x_3 + 7x_4 &= 4 \\ 3x_1 - 3x_2 + 4x_3 + 9x_4 &= 5 \\ -2x_1 + 2x_2 - x_3 - x_4 &= 0 \end{aligned}$$

2. Find the general solution of the following matrix equation.

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 & -1 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$. Find the inverse of A and the inverse of B.

4. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. You are given that $\det(A - \lambda I) = -(\lambda + 1)^2(\lambda - 3)$.

   (a) Find all the eigenvalues of A.

   (b) For each eigenvalue, find a basis for the corresponding eigenspace.

   (c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

5. Let $z = \cos(\pi/16) + i\sin(\pi/16)$ and $w = 1 - i$.

   (a) Write z^8 in the form $a + bi$, where a and b are real numbers.

   (b) Write w^{12} in the form $a + bi$, where a and b are real numbers.

6. Find all complex numbers such that $z^3 = -64i$. Express your answers in the form $a + bi$, where a and b are real numbers.

7. Find all complex numbers such that $z^4 = 1$. Express your answers in the form $a + bi$, where a and b are real numbers.

8. Find all complex numbers such that $z^2 + 6z + 11 = 0$.

9. Let $A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 3 & 1 & 4 \end{bmatrix}$. Find the determinant of A.

10. Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 2 & 4 & 9 \\ 4 & 8 & 0 & -4 \end{bmatrix}$. Find a basis for Col$A$ and a basis for Nul$A$.

Verify that the Rank-Nullity Theorem holds for the matrix A.

11. Let $A = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 8 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 8 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Use Cramer's Rule to solve for x_4 (without solving for x_1, x_2 and x_3) in the matrix equation $Ax = b$.

12. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. You are given that $\det(A - \lambda I) = -(\lambda - 1)^2(\lambda - 4)$.

(a) Find all the eigenvalues of A.

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.