MATH 1107 REVIEW

- Can I do computations with complex numbers? 2-3
- Am I comfortable with polar form and Demoivre’s theorem? 3
- Can I row reduce an augmented matrix to RREF and find a solution to a system of equations? 3-6
  - Can I add two tuples? Can I take the dot product of two tuples? 7
- Can I multiply matrices? Do I understand the basic matrix operations? 7-9
- Can I find the inverse of a matrix? 9-10
- Do I remember the properties of inverses? 10
- Can I compute the determinant of a matrix? 10-11
- Do I remember the properties of determinants? 11-12
- Can I determine whether a subset is a subspace? 12
- Can I determine whether a set of vectors is linearly dependent/independent? 13
- Do I understand bases and dimension? 13-14
- Can I find column space, row space, and null space? 14-16
- Can I find a tuple representation of a vector? 16
- Can I find eigenvalues and eigenvectors? Can I diagonalize a matrix? 16-19
- Do I understand linear transformations? 19
- Can I determine if a linear transformation is injective or surjective? 19-20
The basics

- We define the number $i$ to be the square root of $-1$. Note that no real number is a square root of $-1$. So, $i$ is a new number, different than the real numbers we’re used to working with.

- A complex number is a number of the form $a + bi$, where $a, b \in \mathbb{R}$. For instance, $2 + 3i$ is a complex number. We say that $a$ is the real part of $a + bi$ and $b$ is the imaginary part.

- It’s pretty straightforward to add and multiply complex numbers.

  $$(2 + 3i) + (4 + 7i) = (2 + 4) + (3 + 7)i = 6 + 10i.$$  

  $$(2 + 3i)(4 + 7i) = 2 \cdot 4 + 2 \cdot 7i + 3 \cdot 4i + 3 \cdot 7i^2 = 8 + 14i + 12i + 21(-1) = (8 - 21) + (14 + 12)i = -13 + 26i.$$  

- The conjugate of the complex number $a + bi$ is the complex number $a - bi$. We denote the conjugate $\overline{a + bi}$. Notice that

  $$(a + bi)(a - bi) = (a + bi)(a - bi) = a^2 - abi + abi - b^2(i^2) = a^2 - b^2(-1) = a^2 + b^2.$$  

So, when you multiply a complex number by its conjugate, you get a real number.

- We use the conjugate to define the absolute value of a complex number. Let $w = a + bi$. Then $|w| = |a + bi| = \sqrt{\overline{w}w} = \sqrt{a^2 + b^2}$.

- If you’re really being asked to do is to rewrite this expression in the form $e + fi$. To do so, all you need to do is to multiply $(a + bi)/(c + di)$ by $(c - di)/(c - di)$ (which is just the same as multiplying by 1) and then simplify. So, you multiply both the numerator and the denominator by the conjugate of the denominator.

- If you’re given a complex number $z = a + bi$ and you’re asked to find its inverse, then you just need to simplify the expression $1/z = 1/(a + bi)$ using the procedure outlined in the previous bullet point.

  (Easy) Compute $(2 + 5i)(4 + 4i)$.

  (Easy) Identify the real and imaginary parts of $2 + 5i$. Compute $|2 + 5i|$.

  (Medium) Simplify $(2 + 4i)/(1 + i)$. 


(Medium) Let \( z = -2 - 3i \). Find \( z^{-1} \).

**Polar form and Demoivre’s theorem**

- Let \( z = a + bi \). Let \( \theta \) be the angle that \( z \) makes with the positive real axis. Let \( r = |z| \). Then \( z \) has the polar form \( r(\cos \theta + i \sin \theta) \), which is sometimes abbreviated as \( rcis\theta \).

- Let \( z = r_1\text{cis}\theta_1 \) and \( w = r_2\text{cis}\theta_2 \).

\[
zw = r_1r_2\text{cis}(\theta_1 + \theta_2).
\]

\[
z/w = (r_1/r_2)\text{cis}(\theta_1 - \theta_2).
\]

- Let \( z = a + bi \). To convert \( z \) to polar form, we first compute its modulus to obtain \( r = |z| \). Next, we compute \( \alpha = \tan^{-1}(|b|/|a|) \). If \( z \) is in the first quadrant, \( \theta = \alpha \). If \( z \) is in the second quadrant, \( \theta = \pi - \alpha \). If \( z \) is in the third quadrant, \( \theta = \pi + \alpha \). Finally, if \( z \) is in the fourth quadrant, \( \theta = 2\pi - \alpha \).

- (Demoivre’s Theorem) Let \( z = rcis\theta \), and let \( n \in \mathbb{N} \). Then \( z^n = r^n\text{cis}n\theta \).

- Let \( n \in \mathbb{N} \). A complex number \( rcis\theta \) has \( n \) distinct \( n \)th roots:

\[
r^{1/n}\text{cis}\frac{\theta}{n}, r^{1/n}\text{cis}\left(\frac{2\pi}{n} + \frac{\theta}{n}\right), ..., r^{1/n}\text{cis}\left(\frac{2\pi(n-1)}{n}\right).
\]

(Easy) Let \( A \) denote the set \( \{x \in \mathbb{C} : x^4 = 16\} \). How many elements does \( A \) have?

(Medium) Let \( z \) denote the complex number \( 3 - i \). Let the polar form of \( z \) be given by \( rcis\theta \) where \( r = |z| \) and \( 0 \leq \theta < 2\pi \). Find \( \theta \).

**Row reduction**

- There are three valid elementary row operations: multiplying a row by a nonzero number, adding a multiple of one row to another row, and interchanging two rows.

- Start with the leftmost (nonzero) column. If the entry at the top of the column is zero, use a row interchange to bring a nonzero entry to the top of the column. If necessary, multiply the top row by a nonzero constant to change the entry at the top of this column into a leading one. Then, use row ops to make every other entry in that column equal to zero.
• The next step is to cover (or, in other words, ignore) the row containing the leading one, and apply the procedure outlined above to the sub-matrix that remains.

• Keep applying this procedure until there are no more nonzero rows left to modify. Once you’ve reached that point, the matrix will be in reduced row echelon form (i.e. RREF).

### Solution sets

• A system of linear equations has either no solution, exactly one solution, or infinitely many solutions. If a system has either one solution or infinitely many solutions, then we say that the system is consistent. If a system has no solutions, then we say that the system is inconsistent.

• If a system is inconsistent, then at some point while you’re row reducing its augmented matrix, you will run into a row in which every entry is zero, save for the rightmost entry, which is nonzero. Something like 0 0 0 0 4.

• If a system is consistent, then it has a unique solution if and only if in the RREF of the augmented matrix of the system, every column, save for the rightmost column, contains a leading 1.

• If a system is consistent, then it has infinitely many solutions if and only if the RREF of the augmented matrix of the system has a column (besides the rightmost column) in which there are no leading ones.

• You can find the solution set of a linear system by translating the rows of the RREF of the augmented matrix corresponding to the system back into equations. If the system has a unique solution, then when you make this translation, it will be pretty obvious what the solution is.

• For a system with infinitely many solutions, the variables that index columns not containing leading ones are called free variables. The variables that index columns containing leading ones are sometimes called leading variables.

• To write out the solution set of a system that has infinitely many solutions, set the free variables equal to some real numbers (with names like s and t). We call these real numbers parameters. Then translate the rows of the RREF back into equations and solve for the leading variables in terms of the free variables.
Alternative forms of linear systems

- Remember that there are different ways of writing systems of linear equations. You should be able to take a system of linear equations and rewrite it as a matrix-vector equation of the form $Ax = b$.

- The homogeneous system associated with the system $Ax = b$ is $Ax = 0$. If you’ve used row operations to solve $Ax = b$, then you should be able to solve $Ax = 0$. The RREF of $Ax = 0$ is almost identical to the RREF of $Ax = b$. The only difference is in the last column. The last column of the RREF of $Ax = 0$ is all zeros.

RREF examples

(Easy) Consider the following system of linear equations.

\[
\begin{align*}
  x_1 + 5x_2 &= 7 \\
  x_1 - 2x_2 &= -2
\end{align*}
\]

\(a\) Rewrite this system as a matrix equation of the form $Ax = b$.

\(b\) Find the reduced row echelon form (RREF) of the augmented matrix of the above system.

\(c\) Use the RREF obtained in \(b\) to solve the above system.

\(d\) Find the general solution of the associated homogeneous system.

(Medium) Consider the following system of linear equations.

\[
\begin{align*}
  x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\
  2x_1 - 4x_2 + x_3 &= 5 \\
  x_1 - 2x_2 + 2x_3 - 3x_4 &= 4
\end{align*}
\]

\(a\) Rewrite this system as a matrix equation of the form $Ax = b$.

\(b\) Find the reduced row echelon form (RREF) of the augmented matrix of the above system.

\(c\) Use the RREF obtained in \(b\) to solve the above system.

\(d\) Find the general solution of the associated homogeneous system.
(Medium) Consider the following system of linear equations.

\[
\begin{align*}
    x_1 - 7x_2 + 6x_4 &= 5 \\
    x_3 - 2x_4 &= -3 \\
    -x_1 + 7x_2 - 4x_3 + 2x_4 &= 7
\end{align*}
\]

\( a \) Rewrite this system as a matrix equation of the form \( Ax = b \).

\( b \) Find the reduced row echelon form (RREF) of the augmented matrix of the above system.

\( c \) Use the RREF obtained in \( b \) to solve the above system.

\( d \) Find the general solution of the associated homogeneous system.

**Systems over other fields**

- You might run into systems of equations defined over fields other than \( \mathbb{R} \). For instance, you might get a system defined over a finite field, like GF(2).

- You can still row reduce a system defined over another field. You just have to be careful to remember the rules of the field you’re working in.

- Alternatively, if you’re given a system of only a few equations in a few variables over a small field like GF(2), you can actually just test to see if there are any solutions by plugging in the elements of the field into the equations to see whether you get any solutions.

(Medium) How many solutions are there to the following system defined over GF(2)?

\[
\begin{align*}
    x + y + z &= 1 \\
    y + z &= 0
\end{align*}
\]

(Medium) Find the solutions of the following system defined over GF(2).

\[
\begin{align*}
    x + y + z &= 0 \\
    y + z &= 1
\end{align*}
\]
Tuples

- Addition and scalar multiplication of tuples is done component-wise.
  Eg,
  \[
  \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}.
  \]
  \[
  3 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 12 \end{pmatrix}.
  \]

- The dot product is a tool that helps us talk about what it means for two elements of \(\mathbb{R}^n\) to be orthogonal and to define the notion of an “angle” between two elements of \(\mathbb{R}^n\). To compute the dot product of two vectors in \(\mathbb{R}^n\), simply multiply together the corresponding elements of the two vectors and add up the resulting terms.
  Eg,
  \[
  \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 1 \cdot 2 + 3 \cdot 4 + 4 \cdot (-1) = 2 + 12 - 4 = 10.
  \]

- Two elements \(u\) and \(v\) of \(\mathbb{R}^n\) are orthogonal if their dot product is zero.

  (Easy) Let
  \[
  t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
  \]
  Compute \(t_1 + t_2\).

  (Easy) Compute the dot product
  \[
  \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \end{pmatrix}.
  \]

  (Medium) Compute the dot product
  \[
  \begin{pmatrix} -3 \\ 2 \\ 7 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 2 \\ 1 \end{pmatrix}.
  \]

Matrix multiplication

- Let \(A\) and \(B\) be matrices. Then so long as the number of columns of \(A\) equals the number of rows of \(B\), we can form the matrix product \(AB\). Be careful, though. If the
number of columns of $A$ does not equal the number of rows of $B$, then the matrix product $AB$ is not defined.

- The definition of matrix multiplication is a bit weird, but it is well-motivated by the theory of linear transformations (see the text). If $A$ is an $m \times n$ matrix and $B$ is a $n \times p$ matrix (so that $A$ has the same number of columns as $B$ has rows) then $AB$ is the $m \times p$ matrix such that the entry in its $i$th row and $j$th column is simply the dot product of the $i$th row of $A$ with the $j$th column of $B$.

- Be careful, even if the matrix products $AB$ and $BA$ are both defined, it is not necessarily the case that $AB = BA$.

- The transpose of a matrix $A$ is the matrix obtained by writing the rows of $A$ as columns. We denote the transpose of $A$ as $A^T$.

  Eg,\
  \[
  \begin{pmatrix}
  1 & 3 & 2 \\
  2 & 1 & 7
  \end{pmatrix}^T =
  \begin{pmatrix}
  1 & 2 \\
  3 & 1 \\
  2 & 7
  \end{pmatrix}
  \]

  Note that $A$ has the same number of columns as $A^T$ has rows, so $AA^T$ is defined. Likewise, $A^TA$ is defined.

- A matrix $A$ is symmetric if $A = A^T$.

- If $A$ is an $n \times n$ (i.e. square) matrix, we can define what it means to raise $A$ to a power: $A^2 = AA$, $A^3 = AAA$, etc.

- It’s often tedious to compute $A^n$. But, for certain matrices $A$, there are easy ways to compute $A^n$.

  Eg,\
  \[
  \begin{pmatrix}
  2 & 0 \\
  0 & 4
  \end{pmatrix}^n =
  \begin{pmatrix}
  2^n & 0 \\
  0 & 4^n
  \end{pmatrix}.
  \]

  (Easy) Compute

  \[
  \begin{pmatrix}
  2 & -3 \\
  4 & 1
  \end{pmatrix} \begin{pmatrix}
  2 & 2 \\
  3 & -4
  \end{pmatrix}.
  \]

  (Medium) Let $A$ be a $5 \times 8$ matrix. Let $C = A^TA$. How many rows are there in $C$?

  (Medium) Let $A$ be a $3 \times 4$ matrix and let $B$ be a $3 \times 5$ matrix. Is the matrix product $AB$ defined? What about the matrix product $BA$?
(Medium) Let

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Is \( A \) symmetric? Is it true that \( A^2 = A \)?

(Hard) Let

\[ A = \begin{pmatrix} 1 & 1 \\ -2 & k \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 \\ 0 & k \end{pmatrix}. \]

Let \( C = AB \). What must \( k \) be equal to if \( C \) is symmetric?

**Finding inverses**

- Recall that an \( n \times n \) matrix \( A \) is invertible (i.e. it has and inverse) if there exists an \( n \times n \) matrix \( A^{-1} \) such that \( AA^{-1} = A^{-1}A = I_n \).

- There is another term that is sometimes used interchangeably with the term “invertible” (at least for matrices whose entries come from fields). An invertible matrix is also called a nonsingular matrix. A noninvertible matrix is also called a singular matrix.

- Suppose we’re given an \( n \times n \) matrix \( A \) and we’re asked to find its inverse. Here’s how we would proceed.

- Recall that the \( n \times n \) identity matrix \( I_n \) is the \( n \times n \) matrix with ones along its main diagonal and zeros everywhere else. So, for example

\[ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

- Create the “double matrix” obtained by writing \( A \) beside \( I_n \). So, create the matrix \((A|I_n)\).

- Perform the row ops needed to put \( A \) into RREF. But don’t just perform these row ops on \( A \). Perform them on \( I_n \) as well. Once you’ve transformed \( A \) into its RREF (which should be \( I_n \)), the matrix on the right will be \( A^{-1} \).

Find the inverses of the following matrices.

(Easy)

\[ \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} \]
(Medium) \[
\begin{pmatrix}
1 & 2 & -11 \\
0 & -1 & 4 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

(Hard) \[
\begin{pmatrix}
1 & 0 & -2 \\
-3 & 1 & 4 \\
2 & -3 & 4 \\
\end{pmatrix}
\]

Properties of inverses

- Recall that for an \( n \times n \) matrix \( A \), \( (A^T)^{-1} = (A^{-1})^T \).
- Remember the socks-shoes property: for \( n \times n \) matrices \( A \) and \( B \), \( (AB)^{-1} = B^{-1}A^{-1} \).
- Remember that for an \( n \times n \) matrix \( A \), \( (A^{-1})^{-1} = A \).

(Easy) Rewrite \( (ABC)^{-1} \) in terms of \( A^{-1} \), \( B^{-1} \), and \( C^{-1} \).

(Medium) Rewrite \( (A^T B^{-1})^{-1} \) in terms of \( A^{-1} \), \( B^{-1} \), and \( C^{-1} \).

(Hard) Rewrite \( (A^{-1} B^T (C^{-1})^T)^{-1} \) in terms of \( A^{-1} \), \( B^{-1} \), and \( C^{-1} \).

Determinants

Calculating determinants

- You should be able to calculate determinants using cofactor expansion and using row ops.
- Remember how row ops affect the value of a determinant. If you multiply a row of a matrix \( A \) by \( c \), then det\( A \) also gets multiplied by \( c \). Interchanging two rows of \( A \) negates the value of the determinant of \( A \). Adding a multiple of one row to another does not change the value of a determinant.
- To compute a determinant using row ops, row reduce the matrix to a matrix that is in triangular form, keeping track of how the row ops you’re performing affect the value of the determinant as you go. Then, use the fact that the determinant of a triangular matrix is equal to the product of the entries on the main diagonal of the matrix.
Determinant examples

Compute the determinants of the following matrices.

(Easy) \[
\begin{pmatrix}
1 & 2 \\
2 & 7
\end{pmatrix}
\]

(Medium) \[
\begin{pmatrix}
1 & 0 & 3 \\
0 & 7 & 9 \\
2 & 0 & 7
\end{pmatrix}
\]

(Medium) \[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 & 2 \\
0 & 0 & 3 & 3 & 3 \\
0 & 0 & 0 & 4 & 4 \\
5 & 5 & 5 & 5 & 6
\end{pmatrix}
\]

(Medium) Let $A$ and $B$ be $3 \times 3$ matrices. Suppose that $B$ is obtained from $A$ by performing the following sequence of elementary row ops (in the order given): $R_1 \leftarrow 2R_1$ then $R_3 \leftarrow R_3 - 2R_2$. Let $\det B = 4$. Calculate $\det A$.

Properties of determinants

- If $A$ and $B$ are $n \times n$ matrices, then $\det AB = \det A \det B$.
- If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.
- If $A$ is an $n \times n$ matrix, then $\det A^{-1} = 1/\det A$.
- If $A$ is an $n \times n$ matrix and $c \in \mathbb{R}$, then $\det cA = c^n \det A$.

- Determinants provide a way of determining whether or not a given matrix is singular. Let $A$ be an $n \times n$ matrix. Then $A$ is singular if and only if $\det A = 0$.

Let $A$ and $B$ be $3 \times 3$ matrices, and let $\det A = -2$ and $\det B = 4$.

(Easy) Compute $\det AB$.

(Medium) Compute $\det A^T BA$. 

(Hard) Compute $\det 2A^2A^TA^{-1}$.

(Medium) Let 

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}.$$ 

Let $B = A^TA$. Compute $\det B$.

(Medium) Let 

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 2i \end{pmatrix}.$$ 

Compute $\det A^6$.

(Medium) If the matrix 

$$\begin{pmatrix} 3 & -6 \\ k & 4 \end{pmatrix}$$ 

is singular, then what does $k$ equal?

(Hard) If the matrix 

$$\begin{pmatrix} k & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & k & 1 \\ 1 & -1 & 2 & -1 \end{pmatrix}$$

is singular, then what does $k$ equal?

**Vector spaces**

**Subspaces**

- Suppose you’re given a nonempty subset $H$ of some vector space $V$ and you’re asked to determine whether or not $H$ is a subspace of $V$. There is a simple test you can use to answer this question. It turns out that $H$ is a subspace of $V$ if and only if the following two conditions hold.

  (ii) For every $x, y \in H$, $x + y \in H$, also.

  (iii) For every $x \in H$ and $c \in \mathbb{R}$, $cx \in H$, also.

(Easy) Let $H = \{(a, a) : a \in \mathbb{R}\}$. Determine whether $H$ is a subspace of $\mathbb{R}^2$.

(Medium) Let $H = \{(a, b, c) : a - b = c\}$. Determine whether $H$ is a subspace of $\mathbb{R}^3$.

(Medium) Let $H = \{(a, b, c) : a \geq 0\}$. Determine whether $H$ is a subspace of $\mathbb{R}^3$. 

Linear dependence and linear independence

- A set of two vectors is linearly dependent if and only if one is a multiple of the other.

- Suppose you’re given a set of three vectors \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \) and you want to determine whether or not this set is linearly dependent. Stack the vectors up as the columns of a matrix \( \mathbf{A} = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3) \). The set is linearly dependent if and only if the homogeneous system \( \mathbf{A}\mathbf{x} = \mathbf{0} \) has nontrivial solutions (i.e. if and only if it has a solution other than \( x_1 = x_2 = x_3 = 0 \)).

- Thus, we need to check whether or not the augmented matrix of the system \( \mathbf{A}\mathbf{x} = \mathbf{0} \) has a column that doesn’t contain any leading ones. If it does, this means one of the variables is a free variable, which means the system has infinitely many solutions (and, so, the set of vectors is linearly dependent).

(Easy) Let

\[
\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 3 \\ -3 \end{pmatrix}
\]

Determine whether \( \{\mathbf{a}_1, \mathbf{a}_2\} \) is linearly dependent.

(Medium) Let

\[
\mathbf{a}_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_3 = \begin{pmatrix} 9 \\ 4 \\ -8 \end{pmatrix}
\]

Determine whether \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \) is linearly dependent.

Bases

- The span of a set \( S = \{\mathbf{v}_1, ..., \mathbf{v}_n\} \) of vectors is the set of all the vectors that can be written as linear combinations of vectors in \( S \). We denote the span of \( S \) by \( \text{span} S \).

- We say that the set \( S \) is a spanning set of a vector space \( V \) if \( \text{span} S = V \).

- A basis of a vector space \( V \) is a linearly independent spanning set of \( V \). The dimension of \( V \), denoted \( \dim V \), is the number of vectors in a basis of \( V \).

- Sometimes you can find a basis for a vector space using just a bit of simple reasoning. If you’re given a set defined by a simple rule, you may be able to find vectors that span this set. This spanning set (or some subset of it) will form a basis for the vector space.
(Easy) Let $W$ be the subspace of the vector space of $2 \times 2$ matrices defined as follows:

$$W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Find $\dim W$.

(Medium) Let $W$ be the subspace of the vector space of $2 \times 2$ matrices defined as follows:

$$W = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a + 2b + 3c = 0 \right\}.$$

Find $\dim W$.

**Column space, row space, and null space**

- Suppose you’re given a matrix $A$ and you’re asked to find bases for the column space (Col$A$), row space (Row$A$), and null space (Null$A$) of $A$. The first step is to find the reduced row echelon form $R$ of $A$.

  - To find a basis for the column space, proceed as follows. Figure out which columns of $R$ contain leading ones. The corresponding columns of the original matrix $A$ form a basis for the column space of $A$. So, for instance, if the columns of $R$ that contain pivots are columns 1, 2, and 4, then columns 1, 2, and 4 of the matrix $A$ form a basis for the column space.

  - If you are given a set of tuples $v_1, \ldots, v_k$ in $\mathbb{R}^n$ and you are asked to find a basis for $\operatorname{span}\{v_1, \ldots, v_k\}$, then you can proceed as follows. Stack the vectors $v_1, \ldots, v_k$ as the columns of a matrix $A$. Then a basis for the column space of $A$ will also be a basis for $\operatorname{span}\{v_1, \ldots, v_k\}$.

  - To find a basis for the row space, proceed as follows. Figure out which rows of $R$ contain leading ones. Those same rows of $R$ form a basis for the row space of $A$.

- Note that the basis of the column space is made up of columns of the original matrix $A$, but the basis of the row space is made up of rows from the reduced row echelon form matrix $R$.

  - To find the null space, proceed as follows. Just use the reduced row echelon form $R$ of $A$ to find the solution set of the system $Ax = 0$. You will need to be able to extract a basis from this solution set.
• The rank of $A$ (denoted $\text{rank}(A)$) is the dimension of the column space of $A$ (i.e. the number of vectors in a basis of the column space of $A$).

• The nullity of $A$ (denoted $\text{nullity}(A)$) is the dimension of the null space of $A$ (i.e. the number of vectors in a basis for the null space of $A$).

• The Rank Theorem relates the dimensions of the null space and the column space of a $m \times n$ matrix $A$ (i.e. a matrix $A$ that has $n$ columns). It states that $\text{rank}(A) + \text{nullity}(A) = n$.

(Easy) Let $A$ be a $5 \times 7$ matrix such that $\text{rank}(A) = 4$. What is $\text{nullity}(A)$?

(Medium) Let

$$A = (v_1 \ v_2 \ v_3 \ v_4) = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Find a basis for $\text{Col}A$.

(Medium) Let $A \in \mathbb{R}^{2 \times 3}$ (i.e. let $A$ be a $2 \times 3$ matrix). What is the largest possible value for $\text{rank}(A)$?

(Medium) Find the dimension of the span of

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix} \right\}.$$ 

(Medium) Let

$$A = \begin{pmatrix} 1 & -2 & 2 & 1 & -3 \\ -3 & 1 & -2 & -4 & 0 \\ 4 & -1 & 4 & 7 & -1 \end{pmatrix}.$$ 

The reduced row echelon form of $A$ is

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}.$$ 

a) Find a basis for the column space of $A$.

b) Find a basis for the row space of $A$.

c) Find a basis for the null space of $A$.

d) Verify that the dimension of the column space of $A$ plus the dimension of the null space of $A$ is equal to the number of columns of $A$. 
(Hard) For how many values of $a$ is
\[
\begin{pmatrix}
0 \\
1 \\
a
\end{pmatrix}
\]
in the column space of
\[
\begin{pmatrix}
1 & -1 & 0 \\
-2 & 2 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

Tuple representations

- Let $S$ be a subspace of a vector space $V$. To be concrete, let’s assume $S$ is a subspace of $\mathbb{R}^3$. Suppose, for concreteness, that $B = \{v_1, v_2\}$ is an ordered basis for $S$. Suppose further that $x \in S$ equals $a_1v_1 + a_2v_2$. Then the tuple representation of $x$ with respect to $B$ is
\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}
\]
If $S$ had a basis containing three vectors, then any vector in $S$ would have a tuple representation containing three entries. And so forth.

(Easy) Let $B = \{v_1, v_2\}$ be a basis for a subspace $H$ of $\mathbb{R}^n$. Let $x = 3v_1 + 4v_2$. Find $[v]_B$.

(Easy) Let $B = \{v_1, v_2\}$ be a basis for a subspace $H$ of $\mathbb{R}^n$. Let $x = 2v_2 - v_1$. Find $[v]_B$.

(Medium) The set $B = \{(1, 1), (1, 0)\}$ forms a basis for $\mathbb{R}^2$. Find the coordinates of the vector $(1, -1)$ relative to $B$.

(Medium) Let
\[
B = \left( \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right)
\]
be an ordered basis for $\mathbb{R}^2$. Let
\[
u = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2.
\]
Suppose that
\[[\nu]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\]
Compute $a + b$.

Eigenvalues and eigenvectors

- Let $A$ be an $n \times n$ matrix. If there is a number $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$, then we say that $\lambda$ is an eigenvalue of $A$ and $x$ is an eigenvector corresponding
to \( \lambda \). Eigenvalues can be real or complex.

- The eigenvalues of a triangular matrix are the entries on its main diagonal.

- A scalar \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \) if and only if \( \lambda \) satisfies the characteristic equation \( \det(A - \lambda I_n) = 0 \). If you are asked to find the eigenvalues of a matrix \( A \), then you need to find the roots of the polynomial \( \det(A - \lambda I_n) \). Note that you should probably compute \( \det(A - \lambda I_n) \) using cofactor expansion rather than row operations. It just winds up being easier that way.

- Recall that the roots of \( ax^2 + bx + c \) are
  \[
  x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
  \]

- You can find the eigenvalues of any \( 2 \times 2 \) matrix using the quadratic formula since if \( A \) is \( 2 \times 2 \), then \( \det(A - \lambda I_2) \) is a quadratic polynomial.

- The set of all eigenvectors corresponding to an eigenvalue \( \lambda \) is called the eigenspace of \( \lambda \). Note that \( Ax = \lambda x \) if and only if \( Ax - \lambda x = 0 \) if and only if \( (A - \lambda I_n)x = 0 \). So, the eigenspace corresponding to \( \lambda \) is simply the null space of \( A - \lambda I_n \). If you are asked to find a basis for the eigenspace corresponding to \( \lambda \), you simply need to find a basis for the null space of \( A - \lambda I_n \).

- The algebraic multiplicity of an eigenvalue is the number of times it appear as a root in the characteristic polynomial \( \det(A - \lambda I_n) \). The geometric multiplicity of an eigenvalue is the dimension of its eigenspace.

- A diagonal matrix is a matrix that has zeros everywhere except (possibly) along its main diagonal. The identity matrices are examples of diagonal matrices. A matrix \( A \) is diagonalizable if there is an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

- An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors. To check whether \( A \) is diagonalizable, you just need to find its eigenvalues and find bases of the corresponding eigenspaces. If the sum of the dimensions of the eigenspaces equals \( n \), then \( A \) is diagonalizable. If the sum is less than \( n \), then \( A \) is not diagonalizable.

- If \( A \) is diagonalizable (with \( A = PDP^{-1} \)) then the columns of \( P \) are \( n \) linearly independent eigenvectors of \( A \). In this case, the diagonal entries of \( D \) are eigenvalues of \( A \) ordered in a way that they correspond, respectively, to the given eigenvectors in \( P \).
• The easiest situation to deal with (in terms of diagonalization) is the case in which you have an \( n \times n \) matrix \( A \) that has \( n \) distinct eigenvalues. In this case, there is a theorem that says that \( A \) is diagonalizable. Furthermore, we can find the matrices \( P \) and \( D \) such that \( A = PDP^{-1} \) as follows. Say \( A \) has eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \). Let \( x_1, x_2, ..., x_n \) be eigenvectors corresponding to \( \lambda_1, \lambda_2, ..., \lambda_n \), respectively. Let \( D \) be the diagonal matrix with the numbers \( \lambda_1, \lambda_2, ..., \lambda_n \) along its main diagonal. Let \( P = (x_1, x_2, \ldots, x_n) \) be the matrix whose columns are the vectors \( x_1, ..., x_n \). Then \( A = PDP^{-1} \).

• It’s easy to compute a power of a diagonal matrix. Indeed,
\[
\begin{pmatrix}
  d_1 & 0 \\
  0 & d_2
\end{pmatrix}^k = \begin{pmatrix}
  d_1^k & 0 \\
  0 & d_2^k
\end{pmatrix}.
\]
We can use a similar formula to calculate a power of a \( 3 \times 3 \) diagonal matrix (indeed, we can use a similar formula to calculate a power of an \( n \times n \) diagonal matrix).

• If \( A \) is an \( n \times n \) diagonalizable matrix, then it’s also easy to compute a power of \( A \). Suppose that \( A = PDP^{-1} \) (where \( D \) is a diagonal matrix). Then \( A^k = PD^kP^{-1} \). Since it’s easy to compute \( D^k \), it’s also easy to compute \( A^k \).

(Intermediate) How many distinct eigenvalues does the matrix
\[
\begin{pmatrix}
  2 & 0 & 0 \\
  2 & 1 & 2 \\
  2 & 2 & 1
\end{pmatrix}
\]
have?

(Intermediate) Let
\[
A = \begin{pmatrix}
  7 & 2 \\
-4 & 1
\end{pmatrix}.
\]

a) Find all of the eigenvalues and eigenvectors of \( A \).

b) Is \( A \) diagonalizable? If yes, then find invertible matrices \( P \) and \( P^{-1} \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

c) Compute \( A^5 \).

(Intermediate) Let
\[
A = \begin{pmatrix}
  2 & 1 & 0 \\
  0 & -2 & 3 \\
  0 & -3 & 4
\end{pmatrix}.
\]
It is known that the eigenvalues of \( A \) are 1 and 2.
a) Show that
\[
\begin{pmatrix}
1 \\
-1 \\
-1
\end{pmatrix}
\]
is an eigenvector of $A$.

b) Show that $A$ is not diagonalizable.

**Linear transformations**

**Basic definition**

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a map such that for each $x, y \in \mathbb{R}^n$ and for each $a, b \in \mathbb{R}$, $T(ax + by) = aT(x) + bT(y)$.

- Suppose that you’re told that $T$ is a linear transformation, but you aren’t given the rule that defines $T$. Suppose also that you’re told where $T$ maps a couple of vectors, $u_1$ and $u_2$, and you’re asked to determine where $T$ maps a third vector $u_3$. The way to proceed is to write $u_3$ as a linear combination of $u_1$ and $u_2$. So, find $a$ and $b$ such that $au_1 + bu_2 = u_3$. Then $T(u_3) = T(au_1 + bu_2) = aT(u_1) + bT(u_2)$. Then you can use the values of $T(u_1)$ and $T(u_2)$ to find $T(u_3)$.

(Easy) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that $T((1,0)) = (2,3)$ and $T((0,1)) = (3,4)$. Determine $T((4,5))$.

(Medium) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T((1,0,-1)) = (5,-2)$ and $T((0,1,-1)) = (1,2)$. Determine $T((2,-1,-1))$.

**Injectivity and surjectivity**

- Let $T : V \to W$ be a linear transformation. Then the kernel of $T$, denoted $\text{ker}(T)$, is the set of all vectors in $V$ that $T$ maps to the zero vector in $W$.

- A linear transformation $T : V \to W$ is injective if and only if $\text{ker}(T)$ consists only of the zero vector from $V$. In other words, $T$ is injective if and only if the only vector that $T$ maps to the zero vector in $W$ is the zero vector in $V$. Equivalently, $T$ is injective if and only if the dimension of $\text{ker}(T)$ is zero.

- A linear transformation $T : V \to W$ between finite dimensional vector spaces is surjective if and only if the dimension of $\text{range}(T)$ equals the dimension of $W$. 
So, if \( T \) is a matrix transformation defined by \( Tx = Ax \) (for some \( m \times n \) matrix \( A \)) then \( T \) is injective if and only if \( \text{nullity}(T) = 0 \) and \( T \) is surjective if and only if \( \text{rank}(A) = n \).

(Medium) Let \( P_2 \) denote the vector space of quadratic polynomials in \( x \) with real coefficients. Consider the linear transformation \( T : P_2 \to \mathbb{R}^2 \) given by

\[
T(ax^2 + bx + c) = \begin{pmatrix} 2a - b + c \\ b - 2c \end{pmatrix}.
\]

Compute the dimension of \( \ker(T) \). Is \( T \) injective?

(Hard) Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be a linear transformation given by \( T(x) = Ax \), for some matrix \( A \in \mathbb{R}^{2 \times 3} \). Explain why, if \( T \) is injective, the matrix

\[
\begin{pmatrix}
1 & -1 & 2 \\
-2 & 2 & -4
\end{pmatrix}
\]

cannot be the matrix \( A \).

(Hard) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation given by

\[
T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ -x_1 + 2x_2 \\ x_1 + x_2 + \beta x_3 \end{pmatrix}.
\]

For how many values of \( \beta \) is \( T \) not injective?