Question 1

Find a vector equation and the parametric equations for the line segment from \((2, 2, 6)\) to \((4, 6, 2)\).

*Solution:* We have: \(\overrightarrow{OP_0} = (2, 2, 6)\) and \(\overrightarrow{OP_1} = (4, 6, 2)\). Hence,

\[ r(t) = (1 - t)(2, 2, 6) + t(4, 6, 2) = (2 + 2t, 2 + 4t, 6 - 4t) \text{ for } 0 \leq t \leq 1. \]

Note that, for \(0 \leq t \leq 1\), the above equation is equivalent to:

\[ r(t) = (1 - t)(2i + 2j + 6k) + t(4i + 6j + 2k) = (2i + 2j + 6k) + t(2i + 4j - 4k). \]

The corresponding parametric equations are \(x(t) = 2 + 2t, y(t) = 2 + 4t,\) and \(z(t) = 6 - 4t,\) for \(0 \leq t \leq 1.\)

Question 2

Find parametric equations and symmetric equations for the line through \((5, 1, 0)\) and perpendicular to both \(i+j\) and \(j+k\).

*Solution:* We have \(P = (5, 1, 0)\) and

\[v = (i+j) \times (j+k) = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i - j + k \]

where \(v\) is the direction of the line perpendicular to both \(i+j\) and \(j+k\). Then, the parametric equations are: \(x(t) = 5 + t, y(t) = 1 - t, z(t) = t,\)

and the symmetric equations are: \(x - 5 = \frac{y - 1}{-1} = z\) or \(x - 5 = 1 - y = z.\)
Question 3

Find an equation of the plane through the point $(-2, 8, 10)$ and perpendicular to the line $x(t) = 1 + t$, $y(t) = 2t$, $z(t) = 4 - 3t$.

Solution: Since the line is perpendicular to the plane, its direction vector $\mathbf{v} = (1,2,-3)$ is a normal vector to the plane. An equation of the plane is:

$$1(x - (-2)) + 2(y - 8) - 3(z - 0) = 0 \iff x + 2y - 3z = -16.$$ 

Question 4

Find an equation of the plane through the origin and parallel to the plane $2x - y + 3z = 1$.

Solution: Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = (2, -1, 3)$ and an equation of the plane is:

$$2(x - 0) - 1(y - 0) + 3(y - 0) = 0 \iff 2x - y + 3z = 0.$$ 

Question 5

Determine whether the following sets of planes are parallel, perpendicular, or neither. If neither, find the angle between them:

(a) $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$  
(b) $-x + 4y = 2z$, $3x - 12y + 6z = 1$  
(c) $x + y + z = 1$, $x - y + z = 1$ 

Solution: (a) Normal vectors for the planes are $\mathbf{n}_1 = (1, 4, -3)$ and $\mathbf{n}_2 = (-3, 6, 7)$, respectively. Clearly the normals are not parallel, and thus the planes aren’t parallel. However, $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals, and thus the planes, are perpendicular.

(b) Normal vectors for the planes are $\mathbf{n}_1 = (-1, 4, -2)$ and $\mathbf{n}_2 = (3, -12, 6)$, respectively. Since $\mathbf{n}_2 = -3\mathbf{n}_1$, the normals, and thus the planes, are parallel.

(b) Normal vectors for the planes are $\mathbf{n}_1 = (1, 1, 1)$ and $\mathbf{n}_2 = (1, -1, 1)$, respectively. Since the normals aren’t parallel, the planes are not parallel.
Moreover, the planes are not perpendicular because \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 1 - 1 + 1 \neq 0 \). Then the angle between the planes is given by:

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{1 - 1 + 1}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{1^2 + (-1)^2 + 1^2}} = \frac{1}{3}.
\]

Since \( \frac{1}{3} > 0 \), we take the angle to be \( \theta = \cos^{-1}(\frac{1}{3}) \).

**Question 6**

Let \( x = 4 \cos \theta \) and \( y = 5 \sin \theta \), \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). (i) Eliminate the parameter to find a Cartesian equation of the curve. (ii) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

**Solution:**

(i) \( \left( \frac{x}{4} \right)^2 + \left( \frac{y}{5} \right)^2 = \cos^2 \theta + \sin^2 \theta = 1. \)

The above is the equation of an ellipse with \( x \)-intercepts \((\pm 4, 0)\) and \( y \)-intercepts \((0, \pm 5)\).

(ii) We obtain the portion of the ellipse with \( x \geq 0 \) since \( 4 \cos \theta \geq 0 \) for \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).
Question 7

For $x = t + \ln t$, $y = t - \ln t$ find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, and determine for which values of $t$ is the curve concave up.

Solution:

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - \frac{1}{t}}{t + \frac{1}{t}} = \frac{t - 1}{t - 2} = 1 - \frac{2}{t + 1}.
\]

This implies that:

\[
\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left( 1 - \frac{2}{t + 1} \right)}{t + \frac{1}{t}} = \frac{2}{(t + 1)^2} = \frac{2t}{(t + 1)^3}.
\]

$\Rightarrow$ the curve is CU (concave up) for all $t > 0$.

Question 8

Find the points on the curve $x(t) = 2t^3 + 3t^2 - 12t$, $y(t) = 2t^3 + 3t^2 + 1$ where the tangent is horizontal or vertical.

Solution: We have:

\[
\frac{dx}{dt} = 6t^2 + 6t - 12 = 6(t + 1)(t - 1),
\]

so

\[
\frac{dx}{dt} = 0 \iff t = -2 \text{ or } 1 \iff (x, y) = (20, -3) \text{ or } (-7, 6).
\]

Similarly,

\[
\frac{dy}{dt} = 6t^2 + 6t = 6t(t + 1),
\]

so

\[
\frac{dy}{dt} = 0 \iff t = 0 \text{ or } -1 \iff (x, y) = (0, -1) \text{ or } (13, 2).
\]

Therefore, the curve has horizontal tangents at $(0, 1)$ and $(13, 2)$ and vertical tangents at $(20, -3)$ and $(-7, 6)$. 

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Question 9

Show that the curve $x(t) = \cos t$, $y(t) = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations.

Solution: We have:

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t.$$ 

Then $(x, y) = (0, 0) \iff \cos t = 0 \iff t$ is an odd multiple of $\frac{\pi}{2}$.

When $t = \frac{\pi}{2}$ \Rightarrow $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1 \Rightarrow \frac{dy}{dx} = 1$.

When $t = \frac{3\pi}{2}$ \Rightarrow $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1 \Rightarrow \frac{dy}{dx} = -1$.

Therefore, the equations $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.

Question 10

Find the length of the curve $x = e^t \cos t$ and $y = e^t \sin t$ for $0 \leq t \leq \pi$.

Solution: We have $dx/dt = e^t(\cos t - \sin t)$ and $dy/dt = e^t(\sin t + \cos t)$, so that:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2$$

$$= (e^t)^2(\cos^2 t - 2 \cos t \sin t + \sin^2 t)$$

$$+ (e^t)^2(\sin^2 t + 2 \cos t \sin t + \cos^2 t)$$

$$= e^{2t}(2\cos^2 t + \sin^2 t) = 2e^{2t}.$$
Therefore,

\[ L = \int_0^\pi \sqrt{2\cos^2 \theta} \, d\theta = \int_0^\pi \sqrt{2\cos \theta} \, d\theta = \sqrt{2} \left[ \sin \theta \right]_0^\pi = \sqrt{2} (\sin \pi - \sin 0) = \sqrt{2} (0 - 0) = 0. \]

**Question 11**

Sketch the curve of \( r = \cos 2\theta \) and find the area enclosed by one loop of the curve.

*Solution:* we first sketch \( r = \cos 2\theta, \theta \in [0, 2\pi] \) in Cartesian coordinates. Then we see that as \( \theta \) increases from 0 to \( \pi \), \( r \) decreases from 0 to 1. As \( \theta \) increases from \( \pi \) to \( 2\pi \), \( r \) goes from 0 to \(-1\) (the distance from the origin is still 1, but the portion of the curve is now in the third quadrant). The remainder of the curve is drawn in a similar fashion.

To find the area of the region indicated above, we notice that the right loop is between the rays \( \theta = -\frac{\pi}{4} \) and \( \theta = \frac{\pi}{4} \).
Therefore, the area is given by:

\[
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta = \int_{0}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta
\]

\[
= \int_{0}^{\frac{\pi}{4}} (1 + \cos 4\theta) \, d\theta = \frac{\pi}{8}.
\]

**Question 12**

If \( f(x, y) = \frac{2x^2y}{x^4 + y^4} \) then does \( \lim_{(x, y) \to (0, 0)} \frac{2x^2y}{x^4 + y^4} \) exist?

**Solution:** We see that \( f(x, 0) = 0 \) for \( x \neq 0 \) so \( f(x, y) \to (0, 0) \) as \( (x, y) \to (0, 0) \) along the \( x \)-axis. However,

\[
f(x, x^2) = \frac{2x^4}{2x^4} = 1 \text{ for } x \neq 0 \text{ so } f(x, y) \to 1 \text{ as } (x, y) \to (0, 0) \text{ along } y = x^2.
\]

Since \( f(x, y) \) has two different limits along two different paths, the limit does not exist.

**Question 13**

If \( f(x, y) = \frac{x^4 - y^4}{x^2 + y^2} \) then does \( \lim_{(x, y) \to (0, 0)} \frac{x^4 - y^4}{x^2 + y^2} \) exist?

**Solution:**

\[
f(x, y) = \frac{x^4 - y^4}{x^2 + y^2} = \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = x^2 - y^2 \text{ for all } (x, y) \neq (0, 0).
\]

Since \( f(x, y) = x^2 - y^2 \) is continuous everywhere, we can find the limit now by direct substitution. Therefore, \( \lim_{(x, y) \to (0, 0)} \frac{x^4 - y^4}{x^2 + y^2} = 0. \)
Question 14

Let $f(x, y, z) = z \ln(xy^2z^4)$. Find $f_x$, $f_{xy}$, $f_{xxy}$, and $f_{xyz}$.

Solution: 

$f_x = x \cdot \frac{1}{xy^2z^3} (y^2z^3) + \ln(xy^2z^3) \cdot (1) = 1 + \ln(xy^2z^3)$,

$f_{xy} = \frac{1}{xy^2z^3} (2xyz^3) = \frac{2}{y}$, $f_{xxy} = -\frac{2}{y^2}$, and $f_{xyz} = 0$.

Question 15

If $z = \sin \alpha \tan \beta$, $\alpha = 3s + t$ and $\beta = s - t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Solution: By the Chain Rule, we have:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial s} = \cos \alpha \tan \beta \cdot 3 + \sin \alpha \sec^2 \beta \cdot 1$$

$$= 3 \cos \alpha \tan \beta + \sin \alpha \sec^2 \beta,$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial t} = \cos \alpha \tan \beta \cdot 1 + \sin \alpha \sec^2 \beta \cdot (-1)$$

$$= 3 \cos \alpha \tan \beta - \sin \alpha \sec^2 \beta.$$

Question 16

Let $\cos(x - y) = xe^y$. Find $dy/dx$.

Solution: Using implicit differentiation on both sides of the equation, we get:

$$-\sin(x - y)(1) - \sin(x - y)(-1)\frac{dy}{dx} = (1)e^y + xe^y\frac{dy}{dx}$$

$$\Rightarrow -xe^y\frac{dy}{dx} \sin(x - y)\frac{dy}{dx} - xe^y\frac{dy}{dx} = \sin(x - y) + e^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}$$
Question 17

Find the directional derivative of \( f(x, y) = \ln(x^2 + y^2) \) at the point (2, 1) in the direction of \( v = -i + 2j \).

Solution: A unit vector in the direction of \( v \) is:

\[
u = \frac{1}{\sqrt{1 + 4}}(-i + 2j) = -\frac{1}{\sqrt{5}}i + \frac{2}{\sqrt{5}}j,
\]

and,

\[
\nabla f(x, y) = \frac{2x}{x^2 + y^2}i + \frac{2y}{x^2 + y^2}j \quad \Rightarrow \quad \nabla f(2, 1) = \frac{4}{5}i + \frac{2}{5}j.
\]

Therefore,

\[
D_u f(2, 1) = \nabla f(2, 1) \cdot u = -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0.
\]

Question 18

Find the maximum rate of change of \( f(x, y) = \sin(xy) \) at the point (0, 1) and find the direction in which it occurs.

Solution: For this question, we have:

\[
\nabla f(x, y) = y \cos(xy)i + x \cos(xy)j \quad \Rightarrow \quad \nabla f(1, 0) = 0 \cdot i + 1 \cdot j = j.
\]

Then the maximum rate of change is \( |\nabla f(0, 1)| = \sqrt{(1)^2} = 1 \) in the direction \( j \).

Question 19

Let \( C \) be defined by \( r(t) = (t, t^2, t^3) \) where \( t \in [0, 1] \). Evaluate \( \int_C (2x + 9z) \, ds \).

Solution: The tangent vector \( r'(t) = (1, 2t, 3t^2) \) so that

\[
|r'(t)| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}
\]
and \( f(r(t)) = (2t + 9t^3) \).

Hence,

\[
\int_C (2x + 9z) \, ds = \int_a^b f(r(t)) |r'(t)| \, dt = \int_0^1 (2t + 9t^3) \sqrt{1 + 4t^2 + 9t^4} \, dt
\]

\[
= \frac{1}{4} \int_1^{14} \sqrt{u} \, du = \frac{1}{6} u^{3/2} \bigg|_1^{14} = \frac{1}{6} (14^{3/2} - 1),
\]

where we used the substitution \( u = 1 + 4t^2 + 9t^4 \).

**Question 20**

Let \( C \) consists of the line segments from \((0, 0, 0)\) to \((1, 2, -1)\) and from \((1, 2, -1)\) to \((3, 2, 0)\). Evaluate \( \int_C (x^2 \, dx + y^2 \, dy + z^2 \, dz) \).

**Solution:**

\( C_1 \) is the line segment from \((0, 0, 0)\) to \((1, 2, -1)\) so \( r(t) = (1 - t)(0, 0, 0) + t(1, 2, -1) = (t, 2t, -t) \) for \( t \in [0, 1] \). Then parametric equations for \( A \) are

\( x = t, y = 2t, z = -t \) \( \implies dx = dt, dy = 2dt, dz = -dt \), for \( t \in [0, 1] \).

\( C_2 \) is the line segment from \((1, 2, -1)\) to \((3, 2, 0)\) so \( r(t) = (1 - t)(1, 2, -1) + t(3, 2, 0) = (2t, 0, t) + (1, 2, -1) \) for \( t \in [0, 1] \). Then parametric equations for \( B \) are \( x = 2t + 1, y = 2, z = t - 1 \) \( \implies dx = 2dt, dy = 0, dt = dz \), for \( t \in [0, 1] \).

Then, \( \int_C (x^2 \, dx + y^2 \, dy + z^2 \, dz) \)
\[
\begin{align*}
&= \int_{C_1} x^2 dx + y^2 dy + z^2 dz + \int_{C_2} x^2 dx + y^2 dy + z^2 dz \\
&= \int_0^1 t^2 dt + (2t)^2 \cdot 2dt + (-t)^2(-dt) \\
&\quad + \int_0^1 (1 + 2t)^2 \cdot 2dt + (2t)^2 \cdot 2dt + 0 + (-1 + t)^2(-dt) \\
&= \int_0^1 8t^2 dt + \int_0^1 (9t^2 + 6t + 3)dt \\
&= \frac{8}{3} \left[ t^3 \right]_0^1 + (3t^3 + 3t^2 + 3t) \bigg|_0^1 = \frac{35}{3}.
\end{align*}
\]

**Question 21**

Evaluate \(\int_C xy^4 ds\) where \(C\) is the right half of the circle \(x^2 + y^2 = 16\).

**Solution:** Parametric equations for \(C\) are \(x = 4\cos t\) and \(y = 4\sin t\), for \(t \in [-\frac{\pi}{2}, \frac{\pi}{2}]\). Then,

\[
\begin{align*}
\int_C xy^4 ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \sqrt{(4\sin t)^2 + (4\cos t)^2}dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \sqrt{16(\sin^2 t + \cos^2 t)}dt \\
&= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^4 t \ dt \quad \text{(use the substitution } u = \sin t) \\
&= \frac{4^6}{5} \sin^5 t \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2 \cdot 4^6}{5}.
\end{align*}
\]
Question 22

Evaluate \( \int_C \frac{xdy - ydx}{x^2 + y^2} \) counterclockwise around the circle \( C \) centered at \((0, 0)\) of radius \( R \).

Solution: Parametric equations for \( C \) are \( x = R \cos t \) and \( y = R \sin t \), for \( t \in [0, 2\pi] \). Then \( dx = -R \sin t \, dt, \, dy = R \cos t \, dt \), and:

\[
\int_C \frac{xdy - ydx}{x^2 + y^2} = \int_0^{2\pi} \frac{R^2 \cos^2 t \, dt + R^2 \sin^2 t \, dt}{R^2 \sin^2 t + R^2 \cos^2 t} \\
= \int_0^{2\pi} \frac{R^2 (\cos^2 t + \sin^2 t)}{R^2 (\cos^2 t + \sin^2 t)} \, dt \\
= \int_0^{2\pi} \frac{R^2}{R^2} \cdot (1) \, dt \\
= \int_0^{2\pi} dt = 2\pi.
\]

Question 23

Find the work done by the force field \( \mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j} \) on a particle that moves once around the circle \( x^2 + y^2 = 4 \) oriented in the counterclockwise direction.

Solution: We parametrize the circle \( C \) as:

\[
\mathbf{r}(t) = 2 \cos t \, \mathbf{i} + 2 \sin t \, \mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = -2 \sin t \, \mathbf{i} + 2 \cos t \, \mathbf{j}, \quad \text{for} \quad t \in [0, 2\pi],
\]

and

\[
\mathbf{F}(\mathbf{r}(t)) = 4 \cos^2 t \, \mathbf{i} + 4 \cos t \sin t \, \mathbf{j}. \quad \text{Therefore,}
\]

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \int_0^{2\pi} (4 \cos^2 t \sin t + 8 \cos^2 t \sin t) \, dt = 0.
\]
Question 24

Determine whether or not $F(x, y) = xe^y\mathbf{i} + ye^x\mathbf{j}$ is a conservative field, and if so, find a function $f$ (scalar field) such that $F = \nabla f$.

*Solution:* We let $P(x, y) = xe^y$ and $Q(x, y) = ye^x$. Then

$$\frac{\partial P}{\partial y} = xe^y \text{ and } \frac{\partial Q}{\partial x} = ye^x.$$ Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $F$ is not conservative.

Question 25

Determine whether or not $F(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$ is a conservative field, and if so, find a function $f$ (scalar field) such that $F = \nabla f$.

*Solution:* We let $P(x, y) = e^y$ and $Q(x, y) = xe^y$. Then

$$\frac{\partial P}{\partial y} = e^y = \frac{\partial Q}{\partial x} \Rightarrow F \text{ is conservative and there exists a function such that } F = \nabla f.$$ Thus,

$$f_x(x, y) = e^y \Rightarrow f(x, y) = \int e^y\,dx + g(y) = xe^y + g(y)$$

$$\Rightarrow f_y(x, y) = xe^y + g'(y).$$

But $f_y(x, y) = xe^y \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$, where $C = \text{constant}$. Therefore, $f(x, y) = xe^y + C$ is a potential function for $F$.

Question 26:

Let $\mathcal{R}$ be the region in $\mathbb{R}^2$ enclosed by $y = \frac{1}{2}x^2$ and $y = 2$. Sketch the region $\mathcal{R}$ and find $\iint_{\mathcal{R}} \sqrt{y}\,dA$. 

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Solution: The region \( \mathcal{R} \) is:

\[
\iint_{\mathcal{R}} \sqrt{y} \, dA = \int_{-2}^{2} \int_{0}^{\frac{1}{2}x^2} \sqrt{y} \, dy \, dx
\]

\[
= \int_{-2}^{2} \left( \frac{2}{3} y^{\frac{3}{2}} \bigg|_{0}^{\frac{1}{2}x^2} \right) \, dx
\]

\[
= \int_{-2}^{2} \left( \frac{2\sqrt{8}}{3} - \frac{2}{3\sqrt{8}} x^3 \right) \, dx
\]

\[
= \left( \frac{2\sqrt{8}}{3} x - \frac{2}{3\sqrt{8}} \cdot \frac{1}{4} x^4 \right) \bigg|_{-2}^{2} = \frac{8\sqrt{8}}{3}
\]

Question 27

Evaluate \( \int_{0}^{3} \int_{0}^{\frac{1}{3}y} e^{x^2} \, dx \, dy \) by reversing the order of integration.

Solution: The region of integration is:
Therefore, by reversing the order of integration, we have:

\[
\int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy = \int_0^3 \int_0^{x/3} e^{x^2} \, dx \, dy
\]

\[
= \int_0^3 e^{x^2} \left[ \frac{x}{3} \right]_0^3 \, dx
\]

\[
= \int_0^3 \frac{x e^{x^2}}{3} \, dx = \frac{1}{6} e^9 
\]

\[
= \frac{e^9 - 1}{6}.
\]

**Question 28**

Evaluate \( \iint_R (x + 2y) \, dA \) where \( R \) is the region between \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 9 \) above the x-axis.

**Solution:** Using polar coordinates is the easiest way to solve this problem. The region \( R \) of integration is:

Letting \( x = r \cos \theta, \ y = r \sin \theta, \ r^2 = x^2 + y^2, \) and \( dx \, dy = rdr \, d\theta, \) we get:
\[
\int\int_{R} (x + 2y) \, dA = \int_{0}^{\pi} \int_{1}^{3} r \cos \theta + 2r \sin \theta \, r \, dr \, d\theta
\]
\[
= \int_{0}^{\pi} \int_{1}^{3} r^2 \cos \theta + 2r^2 \sin \theta \, dr \, d\theta
\]
\[
= \int_{0}^{\pi} \left( \frac{r^3 \cos \theta}{3} + \frac{2r^3 \sin \theta}{3} \right)_{1}^{3} \, d\theta
\]
\[
= \int_{0}^{\pi} \frac{26 \cos \theta}{3} \, d\theta + \int_{0}^{\pi} \frac{52 \sin \theta}{3} \, d\theta = \frac{104}{3}.
\]

**Question 29**

Evaluate \( \iiint_{A} z \, dV \) where \( A \) is bounded by the cylinder \( y^2 + z^2 = 9 \) and the planes \( x = 0, y = 3x, \) and \( z = 0 \) in the first octant.

**Solution:** The region of integration is:

\[
\iiint_{A} z \, dV = \int_{0}^{1} \int_{3x}^{3} \int_{0}^{\sqrt{9-y^2}} z \, dz \, dy \, dx
\]
\[
= \int_{0}^{1} \int_{3x}^{3} \left( \frac{9 - y^2}{2} \right) dy \, dx
\]
\[
= \int_{0}^{1} \left( \frac{9}{2} - \frac{1}{6} y^3 \right)_{3x}^{3} \, dx = \int_{0}^{1} 9 - \frac{27}{2} x + \frac{9}{2} x^3 \, dx = \frac{27}{8}.
\]
Question 30

Evaluate \( \iiint_{\mathcal{E}} (x^2 + y^2) \, dz \, dy \, dx \), where \( \mathcal{E} = \{(x, y, z) : -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \sqrt{x^2 + y^2} \leq z \leq 2\} \).

Solution: The projection of \( \mathcal{E} \) onto the \( xy \)-plane is the disk \( x^2 + y^2 \leq 4 \). The lower surface of \( \mathcal{E} \) is the cone \( z = \sqrt{x^2 + y^2} \) and its upper surface is the plane \( z = 2 \). This region is much easier to describe using cylindrical coordinates: \( \mathcal{E} = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 2\} \). Therefore,

\[
\iiint_{\mathcal{E}} (x^2 + y^2) \, dz \, dy \, dx = \int_{0}^{2\pi} \int_{0}^{2} \int_{r}^{2} r \, dz \, dr \, d\theta
\]

\[
= \int_{0}^{2\pi} d\theta \int_{0}^{2} r^3 (2 - r) \, dr
\]

\[
= 2\pi \left( \frac{1}{2} r^4 - \frac{1}{3} r^5 \right) \bigg|_{0}^{2}
\]

\[
= \frac{16}{5} \pi.
\]

Question 31

Evaluate \( \iint_{\mathcal{R}} (x + y)^2 \, dx \, dy \), where \( \mathcal{R} \) is the parallelogram bounded by the lines \( x + y = 0 \), \( x + y = 1 \), \( 2x - y = 0 \) and \( 2x - y = 3 \).

Solution: Letting \( u = x + y \) and \( v = 2x - y \), and using \( u + v = (x + y) + \) \( (2x - y) = 3x \) and \( 2u - v = (2x + 2y) - (2x - y) = 3y \), we have: \( x = \frac{u + v}{3} \) and \( y = \frac{2u - v}{3} \).

This transformation maps the rectangle \( \Gamma \) onto \( \mathcal{R} \):
The Jacobian is given by:

\[
J(u, v) = \begin{vmatrix}
\frac{\partial}{\partial u} \left( \frac{u + v}{3} \right) & \frac{\partial}{\partial u} \left( \frac{2u - v}{3} \right) \\
\frac{\partial}{\partial v} \left( \frac{u + v}{3} \right) & \frac{\partial}{\partial v} \left( \frac{2u - v}{3} \right)
\end{vmatrix} = \begin{vmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{vmatrix} = -\frac{1}{3}.
\]

Therefore,

\[
\iint_{\mathcal{R}} (x + y)^2 \, dx \, dy = \iint_{\Gamma} u^2 |J(u, v)| \, du \, dv
\]

\[
= \frac{1}{3} \int_0^3 \int_0^1 u^2 \, du \, dv
\]

\[
= \frac{1}{3} \int_0^3 dv \int_0^1 u^2 \, du
\]

\[
= \frac{1}{3}.
\]
Question 32

Let \( \mathcal{R} \) be the hemi-spherical shell which is between the two spheres \( x^2 + y^2 + z^2 = 36 \) and \( x^2 + y^2 + z^2 = 49 \), with \( z \geq 0 \). If the density at \((x, y, z)\) in \( \mathcal{R} \) is given by \( \delta(x, y, z) = 4(x^2 + y^2 + z^2)^{-\frac{1}{2}} \), calculate the mass of \( \mathcal{R} \).

Solution: We will use spherical coordinates to solve this problem. Here is the region of integration:

Hence,

\[
M = \iiint_{\mathcal{R}} 4(x^2 + y^2 + z^2)^{-\frac{1}{2}} dV
\]

\[
= 4 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_6^7 (\rho^2)^{-\frac{1}{2}} \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= 4 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_6^7 \rho \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= 4 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin \phi \left( \frac{\rho^2}{2} \right)^{\frac{1}{2}} \bigg|_6^7 \, d\phi \, d\theta
\]

\[
= 26 \int_0^{2\pi} (-\cos \theta) \bigg|_0^{\frac{\pi}{2}} \, d\theta
\]

\[
= 26 \int_0^{2\pi} d\theta = 52\pi.
\]
Question 33

Use Green’s Theorem to evaluate \( \oint_C (1 + 10xy + y^2)dx + (6xy + 5y^2)dy \)
where \( C \) is the square with vertices \((0, 0), (5, 0), (5, 5), (0, 5)\).

Solution: Let \( R \) be the square region enclosed by \( C \) (positively oriented):

\[
\begin{align*}
\text{With } P(x, y) &= 1 + 10xy + y^2 \Rightarrow \frac{\partial P}{\partial y} = 10x + 2y \\
\text{and } Q(x, y) &= 6xy + 5x^2 \Rightarrow \frac{\partial Q}{\partial x} = 6xy + 5x^2,
\end{align*}
\]

we get, by Green’s Theorem,

\[
\oint_C (1 + 10xy + y^2)dx + (6xy + 5y^2)dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy
\]

\[
= \int_0^5 \int_0^5 4y \, dx\, dy
\]

\[
= \int_0^5 x \left. 4y \right|_0^5 \, dy
\]

\[
= 20 \int_0^5 y \, dy
\]

\[
= 20 \cdot \frac{1}{2} y^2 \bigg|_0^5 = 250.
\]
Question 34

Use Green’s Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
$\mathbf{F}(x, y) = (e^x + x^2y)i + (e^y - xy^2)j$ and $C$ is the circle $x^2 + y^2 = 25$ oriented clockwise.

**Solution:** The region $\mathcal{R}$ enclosed by $C$ is the disk $x^2 + y^2 \leq 25$. Since $C$ is traversed clockwise, $-C$ gives the positive orientation. Moreover,

$P(x, y) = e^x + x^2y \Rightarrow \frac{\partial P}{\partial y} = x^2$

and $Q(x, y) = e^y - xy^2 \Rightarrow \frac{\partial Q}{\partial x} = -y^2$.

Then, by Green’s Theorem, we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} (e^x + x^2y)dx + (e^y - xy^2)dy$$

$$= -\iint_{\mathcal{R}} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx \; dy$$

$$= -\iint_{\mathcal{R}} (-y^2 - x^2)dx \; dy$$

$$= \iint_{\mathcal{R}} (y^2 + x^2)dx \; dy$$

$$= \int_0^{2\pi} \int_0^5 (r^2) \cdot r \; dr \; d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 r^3 \; dr$$

$$= 2\pi \left[ \frac{1}{4} r^4 \right]_0^5 = \frac{625}{2} \pi.$$
Question 35

Find an equation of the tangent plane to the surface $\mathbf{r}(u, v) = u^2 \mathbf{i} + 2u \sin v \mathbf{j} + u \cos v \mathbf{k}$ at the point $(1, 0)$.

Solution: We have $\mathbf{r}(1, 0) = (1, 0, 1)$, $\mathbf{r}_u(u, v) = 2u \mathbf{i} + 2 \sin v \mathbf{j} + \cos v \mathbf{k}$, and $\mathbf{r}_v(u, v) = 2u \cos v \mathbf{j} - u \sin v \mathbf{k}$.

So a normal vector to the surface at the point $(1, 0, 1)$ is:

$\mathbf{r}_u(1, 0) \times \mathbf{r}_v(1, 0) = (2 \mathbf{i} + \mathbf{k}) \times (2 \mathbf{j}) = -2 \mathbf{i} + 4 \mathbf{k}$.

Thus an equation of the tangent plane at $(1, 0, 1)$ is:

$-2(x - 1) + 0(y - 0) + 4(z - 1) = 0 \quad \text{or} \quad -x + 2z = 1.$

Question 36

Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: We have $z = f(x, y) = y^2 - x^2 \Rightarrow \frac{\partial z}{\partial x} = -2x$ and $\frac{\partial z}{\partial y} = 2y$, and $1 \leq x^2 + y^2 = r^2 \leq 4$. Therefore, the surface area is:

$$\iint_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \iint_{\mathcal{R}} \sqrt{1 + 4x^2 + 4y^2} \, dA$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2} \, r \, dr$$

$$= \left[ \theta \right]_0^{2\pi} \cdot \frac{1}{12} (1 + r^2)^{3/2} \bigg|_1^2$$

$$= \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}).$$

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Question 37

Find the area of the surface of the helicoid (spiral ramp) with vector equation \( \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k} \) for \( 0 \leq u \leq 1, \ 0 \leq v \leq \pi \).

Solution: We have:

\[ \mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + 0 \mathbf{k} \quad \text{and} \quad \mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + 1 \mathbf{k} \]

\[ \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = |\sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k}| = \sqrt{\sin^2 v + (-\cos v)^2 + u^2} = \sqrt{1 + u^2}. \]

Therefore, the surface area is:

\[
\iint_{\mathcal{R}} |\mathbf{r}_u \times \mathbf{r}_v| dA = \int_0^\pi \int_0^1 \sqrt{1 + u^2} \, du \, dv
\]

\[
= \int_0^\pi dv \int_0^1 \sqrt{1 + u^2} \, du
\]

\[
= \pi \cdot \left( \frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right) \bigg|_0^1
\]

\[
= \frac{\pi}{2} \sqrt{2} + \frac{\pi}{2} \ln(1 + \sqrt{2}).
\]

Question 39

Find the flux of the vector field \( \mathbf{F}(x, y, z) = zi + yj + xk \) across the surface \( S \) where \( S \) is the unit sphere \( x^2 + y^2 + z^2 = 1 \).

Solution: We can find a vector function \( \mathbf{r}(u, v) \) for \( S \) via the parametric representation:

\[
\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}, \ \ 0 \leq \phi \leq \pi, \ \ 0 \leq \theta \leq 2\pi.
\]

Then, \( \mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos \phi \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \sin \phi \cos \theta \mathbf{k} \),
\( \mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k}, \) and

\[
\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot \mathbf{r}_\phi \times \mathbf{r}_\theta = \cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta.
\]

Therefore, the flux of \( \mathbf{F} \) across \( S \) is:

\[
\int_S \int \mathbf{F} \cdot \mathbf{S} = \int \int_{\mathcal{R}} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) \ dA
\]

\[
= \int_0^{2\pi} \int_0^\pi (2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta) \ d\phi \ d\theta
\]

\[
= 2 \int_0^\pi \sin^2 \phi \cos \phi \ d\phi \int_0^{2\pi} \cos \theta \ d\theta + \int_0^\pi \sin^3 \phi \ d\phi \int_0^{2\pi} \sin^2 \theta \ d\theta
\]

\[
= 0 + \int_0^\pi \sin^3 \phi \ d\phi \int_0^{2\pi} \sin^2 \theta \ d\theta
\]

\[
= \left( \frac{1}{3} \cos^3 \phi - \cos \phi \right) \bigg|_0^\pi \left( -\frac{\sin(2\theta)}{4} \right) \bigg|_0^{2\pi}
\]

\[
= \frac{4}{3} \pi.
\]

**Question 40**

Let \( \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zz \mathbf{k} \) and let \( S \) be the oriented surface that is the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and has upward orientation. Find the flux of \( \mathbf{F} \) across \( S \).

**Solution:** For this question, the surface \( S \) is given by the graph \( z = f(x, y) \), so we can think of \( x \) and \( y \) as the parameters. We let

\[
P(x, y, z) = xy, \ Q(x, y, z) = yz, \text{ and } R(x, y, z) = zz.
\]

Furthermore, \( \frac{\partial f}{\partial x} = -2x \) and \( \frac{\partial f}{\partial y} = -2y \).
Therefore,

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_\pi \left( -P \frac{\partial f}{\partial x} - Q \frac{\partial f}{\partial y} + R \right) d\mathbf{A} \]

\[ = \iint_\pi (-xy(-2x) - yz(-2y) + zx) d\mathbf{A} \]

\[ = \int_0^1 \int_0^1 2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2) dy dx \]

\[ = \int_0^1 \frac{1}{3} x^2 + \frac{11}{3} x - x^3 + \frac{34}{15} dx \]

\[ = \frac{713}{180}. \]

**Question 41**

Let \( \mathbf{F}(x, y, z) = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \) where \( S \) is the part of the paraboloid \( z = 9 - x^2 - y^2 \) that lies in the plane \( z = 5 \), oriented upward. Use Stokes' Theorem to evaluate \( \iint_S \text{curl} \ \mathbf{F} \cdot d\mathbf{S} \).

**Solution:** The plane \( z = 5 \) intersects the paraboloid \( z = 9 - x^2 - y^2 \) in the circle \( x^2 + y^2 = 4 \), \( z = 5 \). This boundary curve \( C \) is oriented in the counterclockwise direction, so the vector equation is:

\[ \mathbf{r}(t) = 2 \cos ti + 2 \sin tj + 5k \] for \( 0 \leq t \leq 2\pi \). Then,

\[ \mathbf{r}'(t) = -2 \sin ti + 2 \cos tj, \]

\[ \mathbf{F}(\mathbf{r}(t)) = 10 \sin ti + 10 \cos tj + 4 \cos t \sin tk, \]

and by Stokes' Theorem, we get:
\[
\int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} \\
= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
= \int_0^{2\pi} (-20 \sin^2 t + 20 \cos^2 t) dt \\
= 20 \int_0^{2\pi} \cos 2tdt = 0.
\]

**Question 42**

Verify Stokes’ Theorem by:

(a) directly computing the line integral \( \int_C (2yz)dx + (zx)dy + (xy)dz \) where the curve \( C \) is the intersection between the surfaces \( x^2 + y^2 = 1 \) and \( z = y^2 \),

(b) evaluating the line integral by applying Stokes’ Theorem.

**Solution:** (a) We can parametrize the curve \( C \) by seeing that since the \( x, y \) coordinates lie on the unit circle, we have:

\[ \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin^2 t\mathbf{k} \text{ for } 0 \leq t \leq 2\pi. \]

\[ \Rightarrow \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\sin t\cos t\mathbf{k}. \]

Taking the vector field

\[ \mathbf{F}(x, y, z) = 2yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \mathbf{F}(\mathbf{r}(t)) = 2\sin^3 t\mathbf{i} + \cos t\sin^2 t\mathbf{j} + \cos t\sin t\mathbf{k}, \]

\[ \Rightarrow \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 3\cos^2 t\sin^2 t - 2\sin^4 t. \]

Since \( \int_C Pdx + Qdy + Rdz = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \), we have:
\[
\int_C (2yz)dx + (xz)dy + (xy)dz = \int_0^{2\pi} 3 \cos^2 t \sin^2 t - 2 \sin^4 t \, dt
\]

\[
= \int_0^{2\pi} \left[ 3 \left( \frac{\sin 2t}{2} \right)^2 - 2 \left( \frac{1 - \cos t}{2} \right)^2 \right] \, dt
\]

\[
= \int_0^{2\pi} \left[ \frac{3 \sin^2 2t - 2 \cos^2 t + 4 \cos 2t - 2}{4} \right] \, dt
\]

\[
= \int_0^{2\pi} \left[ \frac{\sin^2 2t + 3 \cos 2t - 2}{4} \right] \, dt
\]

\[
= \int_0^{2\pi} \left[ \frac{1 - \cos 4t + 6 \cos 2t - 4}{8} \right] \, dt
\]

\[
= \int_0^{2\pi} \frac{1 - 4}{8} \, dt = -\frac{3\pi}{4}.
\]

where the second last line comes from the fact that \( \int_0^{2\pi} \frac{-\cos 4t + 6 \cos 2t}{8} \, dt = 0 \).

(b): We must choose a surface \( S \) that has \( C \) as its boundary. We can simply choose the part of the surface \( z = y^2 \) that is enclosed within \( x^2 + y^2 = 1 \), that is, \( S = \{(x, y, z) : x^2 + y^2 \leq 1, z = y^2\} \). Furthermore, we can parametrize the surface as:

\[
r(r, \theta) = r \cos \theta i + r \sin \theta j + r^2 \sin^2 \theta k \text{ for } 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi,
\]

\[
\Rightarrow \frac{\partial r}{\partial r} = r_r = \cos \theta i + \sin \theta j + 2r \sin^2 \theta k, \text{ and}
\]

\[
\frac{\partial r}{\partial \theta} = r_\theta = -r \sin \theta i + r \cos \theta j + 2r^2 \sin \theta \cos \theta k.
\]

We find \( n \) by calculating \( n = r_r \times r_\theta = -2r^2 \sin \theta j + r k \).

We note that the \( z \) component of the normal vector is positive, so as to point upward, conforming with the counterclockwise orientation of the line integral.
taken before. Furthermore, the curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = y\mathbf{j} - z\mathbf{k} \Rightarrow \nabla \times \mathbf{F}|_r = r \sin \theta \mathbf{j} - r^2 \sin^2 \theta \mathbf{k}.$$  

Then,

$$\nabla \times \mathbf{F}|_r \cdot \mathbf{n} = -3r^3 \sin^2 \theta,$$

and therefore,

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} d\mathbf{S}$$

$$= \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS$$

$$= \int_0^{2\pi} \int_0^1 -3r^3 \sin^2 \theta \, dr \, d\theta$$

$$= -\frac{3}{8} \int_0^{2\pi} \sin^2 \theta \, d\theta$$

$$= -\frac{3\pi}{4}.$$  

**Question 43**

Use the Divergence Theorem to evaluate $\iint_S (3\mathbf{i} + 2y\mathbf{j}) d\mathbf{S}$, where $S$ is the sphere $x^2 + y^2 + z^2 = 9$.

**Solution:** Note that we could parameterize the surface and evaluate the surface integral, but it is much faster to use the Divergence Theorem. Since

$$\text{div}(3\mathbf{i} + 2y\mathbf{j}) = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial x}0 = 5,$$

the divergence theorem gives:

$$\iint_S (3\mathbf{i} + 2y\mathbf{j}) d\mathbf{S} = \iiint_\xi \text{div} \mathbf{F} dV = \iiint_\xi 5dV = 5 \cdot \text{(volume of sphere)} = 180\pi.$$
Question 44

Verify that the Divergence Theorem is true for the vector field \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k} \) on the region \( \mathcal{E} \), where \( \mathcal{E} \) is the solid bounded by the paraboloid \( z = 4 - x^2 - y^2 \) and the \( xy \)-plane.

**Solution:** The following is a sketch of the region \( \mathcal{E} \) and the closed surfaces \( S_1 \) and \( S_2 \):

Now, \( \text{div}\mathbf{F} = 2x + x + 1 = 3x + 1 \), so, by converting to polar coordinates, we get:

\[
\iiint_{\mathcal{E}} \text{div}\mathbf{F} \, dV = \iiint_{\mathcal{E}} (3x + 1) dV
\]

\[
= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r + \cos \theta + 1) r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 r(3r + \cos \theta + 1)(4 - r^2) \, d\theta \, dr
\]

\[
= 2\pi \int_0^2 (4r - r^3) \, dr = 8\pi
\]

Now, on \( S_1 \) the surface is \( z = f(x, y) = 4 - x^2 - y^2 \), \( x^2 + y^2 \leq 4 \), with upward orientation and \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + (4 - x^2 - y^2) \mathbf{k} \). Then,
\[
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\pi} \left( -P \frac{\partial f}{\partial x} - Q \frac{\partial f}{\partial y} + R \right) dA
\]

\[= \iint_{\pi} -(-x^2)(2x) - (xy)(-2y) + (4 - x^2 - y^2) dA\]

\[= \iint_{\pi} [(2x(x^2 + y^2)) + 4 - (x^2 + y^2)] dA\]

\[= \int_{0}^{2\pi} \int_{2}^{0} (2r \cos \theta \cdot r^2 + 4 - r^2) r \, dr \, d\theta\]

\[= \int_{0}^{2\pi} (\frac{64}{5} \cos \theta + 4) d\theta\]

\[= 8\pi.\]

On \(S_2\), the surface is \(z = 0\) with downward orientation, so \(\mathbf{F} = x^2 \mathbf{i} + xy \mathbf{j}\), \(\mathbf{n} = -\mathbf{k}\)

\[\Rightarrow \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} 0 \, dS = 0.\]

\[\Rightarrow \iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 8\pi = \iiint_{\varepsilon} \text{div} \mathbf{F} \, dV.\]

**Question 45**

Calculate the flux of \(\mathbf{F}(x, y, z) = xy \sin zi + \cos(xy)j + y \cos zk\) across the surface \(S\), where \(S\) is the ellipsoid \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.\)
Solution: The easiest way to calculate this is via the Divergence Theorem. Accordingly,

$$\text{div} \mathbf{F} = y \sin z + 0 - y \sin z = 0,$$
so we get:

$$\text{flux of } \mathbf{F} \text{ across } S = \iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_\varepsilon 0 \, dV = 0.$$

Question 46

Find the local maximum and minimum and saddle points, if they exist, for the following functions: (a) \( f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \) (b) \( f(x, y) = x^4 + y^4 - 4xy + 2 \) and (c) \( f(x, y) = e^x \cos y \).

Solution: (a) \( f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \)

\[ f_x = -2 - 2x, \quad f_y = 4 - 8y, \quad f_{xx} = -2, \quad f_{yy} = -8, \quad f_{xy} = 0. \]

Then, \( f_x = -2 - 2x = 0 \Rightarrow x = -1 \) and \( f_y = 4 - 8y = 0 \Rightarrow y = \frac{1}{2} \)

\[ \Rightarrow \text{the only critical point is } (-1, \frac{1}{2}). \text{ Then, } D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 16 \]
and since \( D(-1, \frac{1}{2}) = 16 > 0 \) and \( f_{xx}(-1, \frac{1}{2}) = -2 < 0, f(-1, \frac{1}{2}) = 11 \) is a local maximum, by the Second Derivative Test.

(b) \( f(x, y) = x^4 + y^4 - 4xy + 2 \)

\[ f_x = 4x^3 - 4y, \quad f_y = 4y^3 - 4x, \quad f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4. \]

Then, \( f_x = 0 \Rightarrow y = x^3 \) and substitution into \( f_y = 0 \Rightarrow x = y^3 \) gives

\[ x^9 - x = 0 \Rightarrow x = 0 \text{ or } x = \pm 1 \]

\[ \Rightarrow \text{the critical points are } (0, 0), \ (1, 1) \text{ and } (-1, -1). \text{ Now, by the Second Derivative Test,} \]

\[ D(0, 0) = 0 \cdot 0 - (-4)^2 = -16 < 0 \text{ so } (0, 0) \text{ is a saddle point.} \]

\[ D(1, 1) = (12)(12) - (-4)^2 > 0 \text{ and } f_{xx}(1, 1) = 12 > 0, \text{ so } f(1, 1) = 0 \text{ is a local minimum.} \]

\[ D(-1, -1) = (12)(12) - (-4)^2 > 0 \text{ and } f_{xx}(-1, -1) = 12 > 0, \text{ so } f(-1, -1) = 0 \text{ is also a local minimum.} \]
\( (c) \ f(x, y) = e^x \cos y \)

\[ \Rightarrow f_x = e^x \cos y, \quad f_y = -e^x \cos y. \]  
But, \( f_x = 0 \Rightarrow \cos y = 0 \) or \( y = \frac{\pi}{2} + n\pi \) for an integer \( n \). But \( \sin \left(\frac{\pi}{2} + n\pi\right) \neq 0 \), so there are no critical points.

**Question 47**

Use Lagrange multipliers to find the maximum and minimum values of the function \( f(x, y) = 4x + 6y \) subject to the constraint \( x^2 + y^2 = 13 \).

**Solution:** In this question, \( f(x, y) = 4x + 6y \) and \( g(x, y) = x^2 + y^2 = 13 \). Using Lagrange multipliers, we solve the equations \( \nabla f = \lambda \nabla g \) and \( g(x, y) = 13 \), which can be written as:

\[ f_x = \lambda g_x \Rightarrow 4 = \lambda 2x \Rightarrow x = \frac{2}{\lambda}, \]
\[ f_y = \lambda g_y \Rightarrow 4 = \lambda 2y \Rightarrow y = \frac{3}{\lambda}. \]

But \( 13 = x^2 + y^2 = \left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 \Rightarrow 13 = \frac{13}{\lambda^2} \Rightarrow \lambda = \pm 1 \)

\[ \Rightarrow f \] has possible extreme values at \((2, 3)\) and \((-2, -3)\).

Since \( f(2, 3) = 26 \) and \( f(-2, -3) = -26 \) the maximum value of \( f \) on \( x^2 + y^2 = 13 \) is \( 26 \) and the minimum value is \(-26\).

**Question 48**

Use Lagrange multipliers to find the maximum and minimum values of the function \( f(x, y, z) = 8x - 4z \) subject to the constraint \( x^2 + 10y^2 + z^2 = 5 \).

**Solution:** Again, we solve the equations \( \nabla f = \lambda \nabla g \) and \( g(x, y, z) = 5 \), which can be written as:

\[ f_x = \lambda g_x \Rightarrow 8 = \lambda 2x \Rightarrow x = \frac{4}{\lambda}, \]
\[ f_y = \lambda g_y \Rightarrow 0 = \lambda 20y \Rightarrow y = 0. \]
\[ f_z = \lambda g_z \Rightarrow -4 = \lambda 2z \Rightarrow x = -\frac{2}{\lambda}. \]

But \[ 5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10 \cdot 0 \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{40}{\lambda^2} \Rightarrow \lambda = \pm 2 \]

\[ \Rightarrow f \text{ has possible extreme values at } (2,0,-1) \text{ and } (-2,0,1). \text{ Then the maximum value of } f \text{ on } x^2 + 10y^2 + z^2 = 5 \text{ is } f(2,0,-1) = 20 \text{ and the minimum value is } f(-2,0,1) = -20. \]