CHECK LIST

□ Can I put a system of linear equations into an augmented matrix?
□ Can I put a system of linear equations into a coefficient matrix?
□ Can I row reduce? (Do I know the elementary row operations?)
□ Do I know how many solutions a system of linear equations has?
□ Do I know if a solution to a linear system is consistent or inconsistent?
□ Can I write a solution in standard form? Parametric form?
□ Do I know what a free variable is?
□ Can I write a vector as a linear combination of other vectors.
□ Do I know the definition of span?
□ Do I know what linear dependence / independence is?
□ Can I write down the dependence relation?
□ Do I know what an one-to-one linear transformation is?
□ Do I know what an onto linear transformation is?
□ Can I do the matrix operations? (sum, scalar multiple, matrix product)
□ Can I find the inverse of a square matrix?
□ Do I know when a transformation is invertible?
□ Do I know how to find the determinant of a matrix?
□ Can I do co-factor expansion?
□ Do I know the properties of upper (lower) triangular matrices?
□ Do I know how row operations change the determinant?
□ Do I know the properties of the determinant?
□ Do I know Cramer’s Rule?
□ Do I know what a vector space is?
☐ Can I show that a subset of a vector space is a subspace?
☐ Do I know what the Null space is?
☐ Do I know what the column space is?
☐ Do I know what the eigenvector / eigenvalue are? Can I find them?
☐ Do I know what diagonalizable means?
☐ Do I know what the properties of the inner product are?
☐ Do I know how to find the length of a vector in \( \mathbb{R}^n \)
☐ Do I know what a unit vector is?
☐ Do I know what it means for two vectors to be orthogonal?
☐ Do I know Pythagorean Theorem for the length of two vectors?
☐ Do I understand orthogonality? Can I find an orthogonal basis?
☐ Can I do the Gram-Schmidt process? Do I know what it gives me?
☐ Do I know how to do arithmetic with complex numbers?
Row reduction

There are three valid elementary row operations:

1. Adding a multiple of one row to another row
2. Interchanging two rows
3. Multiplying a row by a nonzero number

Step 1: Locate a nonzero entry appearing in the leftmost column. If necessary, use a row interchange to bring it to the top of the matrix. If necessary, multiply the row by a nonzero scalar so that the first nonzero entry is 1 (called the leading 1, or pivot).

Step 2: Add multiples of the row with the leading 1 to the other rows so that all other entries in the pivot column are zero.

Step 3: Repeat the above process, leaving any previous rows with leading 1s in their place.

Alternative forms of linear systems

There are different ways of writing systems of linear equations. You should be able to take a system of linear equations and rewrite it as a matrix-vector equation of the form $A\mathbf{x} = \mathbf{b}$

The homogeneous system associated with the system $A\mathbf{x} = \mathbf{b}$ is $A\mathbf{x} = \mathbf{0}$. If you used row operations to solve $A\mathbf{x} = \mathbf{b}$, then you should be able to solve $A\mathbf{x} = \mathbf{0}$. The RREF of $A\mathbf{x} = \mathbf{0}$ is almost identical to the RREF of $A\mathbf{x} = \mathbf{b}$. The only difference is in the last column. The last column of the RREF of $A\mathbf{x} = \mathbf{0}$ is all zeros.

RREF Examples

For each of the following system of equations do questions (a) to (d)

a) Rewrite this system as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.
b) Find the reduced row echelon form (RREF) of the augmented matrix of the system.
c) Use the RREF from part (b) to solve the system.
d) Write the solution in parametric form.

**Example 1**: Consider the following system of linear equations.

$$
\begin{align*}
  x_1 + 5x_2 &= 7 \\
  x_1 - 2x_2 &= -2
\end{align*}
$$

**Example 2**: Consider the following system of linear equations.

$$
\begin{align*}
  x_1 &- 2x_2 &- x_3 &+ 3x_4 &= 1 \\
  2x_1 &- 4x_2 &+ x_3 &= 5 \\
  x_1 &- 2x_2 &+ 2x_3 &- 3x_4 &= 4
\end{align*}
$$

**Example 3**: Consider the following system of linear equations.

$$
\begin{align*}
  x_1 &- 7x_2 &+ 6x_4 &= 5 \\
  x_3 &- 2x_4 &= -3 \\
  -x_1 &+ 7x_2 &- 4x_3 &+ 2x_4 &= 7
\end{align*}
$$
Determining whether a system is consistent

A system of linear equations has either no solution, exactly one solution, or infinitely many solutions. If a system has no solutions, we say it is inconsistent. If it has either exactly one solution or infinitely many solutions, then we say it is consistent.

If a system is inconsistent, then when you row reduce it, you will eventually run into a row in which every entry is zero, save for the last entry, which is nonzero.

Consistency Examples

Example 1:
For the augmented matrix A, determine whether A is consistent or inconsistent.

\[
A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 8 & 9 \end{pmatrix}
\]

Example 2:
For the augmented matrix B, determine what values of \( h \) make the system consistent.

\[
B = \begin{pmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{pmatrix}
\]

Example 3:
For the augmented matrix C, determine what values of \( h \) and \( k \) make the system consistent.

\[
C = \begin{pmatrix} 1 & h & 3 \\ 5 & -10 & k \end{pmatrix}
\]

Linear dependence

A set of vectors is linearly independent if and only if it is not linearly dependent. You should review the definition of linear dependence. But, you can still do linear dependence/independence problems, even if you don’t really have a clear understanding of the definitions of these terms.

A set of two vectors is linearly dependent if and only if one is a multiple of the other. Suppose you are given a set of three vectors \( \{a_1, a_2, a_3\} \) and you want to determine whether or not this set is linearly dependent. Stack the vectors up as the columns of a matrix \( A = (a_1, a_2, a_3) \). The set is linearly dependent if and only if the homogeneous system \( Ax = 0 \) has nontrivial solutions (i.e. if and only if it has a solution other than \( x_1 = x_2 = x_3 = 0 \)).

Recall that a consistent system has a free variable when there is a column (other than the rightmost column in the augmented matrix) that does not contain any pivots (i.e. a column that does not contain any leading ones). So if the RREF of the augmented matrix of \( Ax = 0 \) has a column (other than the right most column in the augmented matrix) that does not contain a pivot (leading one) then the system has nontrivial solutions, and so the set of vectors \( \{a_1, a_2, a_3\} \) is linearly dependent. If the system does not have any free variables, then the set of vectors is linearly independent.
Span

If \( v_1, \ldots, v_p \) are in \( \mathbb{R}^n \), then the set of all linear combinations of \( v_1, \ldots, v_p \) is denoted by \( \text{Span}\{v_1, \ldots, v_p\} \) and is called the subset of \( \mathbb{R}^n \) spanned by \( v_1, \ldots, v_p \). That is, \( \text{Span}\{v_1, \ldots, v_p\} \) is the collection of all vectors that can be written in the form \( c_1 v_1 + c_2 v_2 + \ldots + c_p v_p \) where \( c_1, \ldots, c_p \) are scalars. So asking whether a vector \( b \) is in \( \text{Span}\{v_1, \ldots, v_p\} \) amounts to asking whether the vector equation

\[
x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = b
\]

has a solution.

Linear Dependence and Span Examples

Example 1:
For
\[
a_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 3 \\ -3 \end{pmatrix}
\]

\( a) \) Determine whether \( \{a_1, a_2\} \) is linearly dependent or independent.

\( b) \) Do \( \{a_1, a_2\} \) span \( \mathbb{R}^2 \)

Example 2:
Let
\[
a_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 9 \\ 4 \\ -8 \end{pmatrix}
\]

Determine whether \( \{a_1, a_2, a_3\} \) is linearly dependent.

\( b) \) Do \( \{a_1, a_2, a_3\} \) span \( \mathbb{R}^3 \)

Example 3:
Let
\[
a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad a_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ h \end{pmatrix}
\]

For which values of \( h \) is the set \( \{a_1, a_2, a_3\} \) linearly dependent?

\( b) \) Do \( \{a_1, a_2, a_3\} \) span \( \mathbb{R}^4 \)

Example 4:
Are the vectors \((1, 2), (3, 7), (23, 1)\) linearly dependent or independent?

Example 5:
For
\[
A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 2 & 2 & -4 \end{pmatrix}
\]

\( a) \) Do the columns of \( A \) span \( \mathbb{R}^3 \)?

\( b) \) Are the columns of \( A \) linearly dependent or independent?

\( c) \) If the columns of \( A \) are dependent, write down the dependence relation among the columns of \( A \).
Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix $A$. We saw that $T$ is **one-to-one** if and only if the columns of $A$ are linearly independent. We saw that $T$ is **onto** if and only if the columns of $A$ span $\mathbb{R}^m$.

Showing a Transformation is Linear

We can show that a transformation is linear two different ways.

1) By definition. That is, show that:
   - $T(u + v) = T(u) + T(v)$, and
   - $T(cu) = cT(u)$

2) We can use a result from class that says matrix mappings are linear. We can often rewrite a given transformation $T(x)$ in the form of $A x$ for a matrix $A$, and then conclude that $T$ is linear.
   The definition can be used to show a transformation is not linear.

Linear Transformation Examples

**Example 1:**
For $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 5z, y - 6z, x + y - z)$ determine if it is one-to-one, and if it maps onto its codomain.

**Example 2:**
For $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + 2y + 5z, 2x - y - z)$ determine if it is one-to-one, and if it maps onto its codomain.

**Example 3:**
For $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 2x + y, -3x + y)$ determine if it is one-to-one, and if it maps onto its codomain.

Matrix Arithmetic

Recall that we can add matrices, multiply them by a scalar, and do matrix multiplication. It is important to note that matrix division is not defined.

- We may add two matrices if they have the same $n \times m$ dimensions.
- To multiply by a constant we multiply each entry of the matrix by said constant.
- To multiply two matrices $AB$ the number of columns of $A$ must be the same as the number of rows of $B$.
- It is important to note that matrix multiplication is not commutative! (i.e. We might find that $AB \neq BA$ so we can **NOT** swap the order of matrices)
Matrix Arithmetic Examples

For

\[ A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 3 \\ 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 6 & 2 & 4 \end{pmatrix} \]

Example 1:
Find \(2A + 5AB\)

Example 2:
Find \(3D^2\)

Example 3:
Find \(BC\)

Example 4:
Find \(CB\)

Determinant

We saw the following regarding determinants.

- They can be evaluated with a cofactor expansion along any row or column.
- The determinant of an upper or lower triangular matrix is just the product of the diagonal entries.
- We saw how row operations change the determinant (Theorem 3 in section 3.2 or the textbook) and how we can use row operations to make zeros in a matrix in order to more easily evaluate the determinant.
  - If you multiply a row of a matrix \(A\) by \(c\) then \(\det A\) also gets multiplied by \(c\)
  - If you interchange two rows of \(A\), you negate the value of the determinant of \(A\).
  - Adding a multiple of one row to another does not change the value of a determinant.

- The determinant of a matrix is non-zero if and only if the matrix is invertible.
- \(\det A^T = \det (A)\)
- \(\det(AB) = (\det A)(\det B)\)
- \(\det(A^{-1}) = \frac{1}{\det(A)}\)

Finding Inverses

Suppose that we are given an \(n \times n\) matrix \(A\) and we are asked to find its inverse. Recall that the \(n \times n\) identity matrix \(I_n\) is the \(n \times n\) matrix with ones along its main diagonal and zeros everywhere else. So, for example

\[ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
We do the following to solve the inverse.

**Step 1:** Create the "double matrix" obtained by writing $A$ beside $I_n$. So we create the matrix $(A|I_n)$

**Step 2:** Preform the row operations needed to put $A$ into RREF. We do this for $(A|I_n)$ not just $A$. Once you have transformed $A$ into its RREF (which should be $I_n$), the matrix on the right will be $A^{-1}$. So $(A|I_n)$ will become $(I_n|A^{-1})$.

Recall that when we are dealing with a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have the inverse formula $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

We also saw that for a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be invertible, then $n = m$. From this we saw that $T$ is invertible if and only if it is both one-to-one and onto, which is equivalent to the standard matrix $A$ being invertible. If $T$ is inevitable then the standard matrix for $T^{-1}$ is $A^{-1}$

**Properties of Inverses**

For an $n \times n$ matrix $A$, $(A^T)^{-1} = (A^{-1})^T$

Shoes and Socks Property: for $n \times n$ matrices $A$ and $B$, $(AB)^{-1} = B^{-1}A^{-1}$

For an $n \times n$ matrix $A$, $(A^{-1})^{-1} = A$

**Finding Determinants and Inverse Examples**

**Example 1:**
Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 7 & 9 \\ 2 & 0 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

**Example 2:**
Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} c & d \\ 3a & 3b \end{pmatrix}$$

If $\det A = 4$, then what is $\det B$?

**Example 3:**
Let $A$ and $B$ be $3 \times 3$ matrices, and let $\det A = -2$ and $\det B = 4$. Compute

(a) $\det AB$
(b) $\det A^TBA$
(c) $\det 2A^2A^{-1}$

**Example 4:**
Show that the linear transformation given by $T(x, y) = (x + 2y, 2x + 7y)$ is invertible and find an explicit formula for $T^{-1}(x, y)$
Example 5:

Find the determinant of \( B = \begin{pmatrix} 1 & 2 & -11 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \). Is \( B \) invertible? If \( B \) is invertible, find its inverse.

Example 6:

Find the determinant of \( C = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 4 & -6 & 8 \end{pmatrix} \). Is \( C \) is invertible. Find \( \det(C^3) \).

Example 7:

Find all \( k \) for which \( E = \begin{pmatrix} k & 1 & k \\ 0 & 2 & k \\ 0 & 1 & 3k \end{pmatrix} \) is invertible.

Cramer’s Rule

Cramer’s Rule is especially helpful when you are given a system of linear equations but you are only asked to solve for one of the variables.

When \( A \) is an invertible matrix, the unique solution to the system \( Ax = b \) has entries given by \( x_i = \frac{\det(A_{i,\cdot}b)}{\det(A)} \). Here, \( A_{i,\cdot} \) is the matrix obtained from \( A \) by replacing the \( i \)th column with \( b \).

Cramer’s Rule Examples

Example 1:

Let \( A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} \), \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), and, \( b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \). Use Cramer’s Rule to solve for \( x_1 \)

Example 2:

Let \( A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \), \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \), and, \( b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \). Use Cramer’s Rule to solve for \( x_1 \)

Example 3:

Let \( A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \), \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \), and, \( b = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} \). Use Cramer’s Rule to solve for \( x_3 \)

The Standard Matrix

Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Let \( \{e_1, e_2, \ldots, e_n\} \) be the standard basis of \( \mathbb{R}^n \). The standard matrix of the linear transformation \( T \) is the \( m \times n \) matrix whose first column is \( T(e_1) \), whose second column is \( T(e_2) \), and whose \( n^{th} \) column is \( T(e_n) \)

Vector Space

A vector space is any set with two operations that satisfies the list of axioms
1. The sum of \( u \) and \( v \), denoted by \( u + v \), is in \( V \).

2. \( u + v = v + u \).

3. \( (u + v) + w = u + (v + w) \).

4. There is a \textbf{zero} vector \( 0 \) in \( V \) such that \( u + 0 = u \).

5. For each \( u \) in \( V \), there is a vector \( -u \) in \( V \) such that \( u + (-u) = 0 \).

6. The scalar multiple of \( u \) by \( c \), denoted by \( cu \), is in \( V \).

7. \( c(u + v) = cu + cv \).

8. \( (c + d)u = cu + du \).

9. \( c(du) = (cd)u \).

10. \( 1u = u \).

We saw that \( \mathbb{R}^n \), \( \mathbb{P}_n \), and the set of \( n \times m \) matrices, and the set of all functions that map from \( D \) (a fixed subset of \( \mathbb{R} \)) to \( \mathbb{R} \) are vector spaces.

\section*{Subspace}

We saw that a subset \( H \) of a vector space is a subspace if and only if the following three axioms are satisfied

1. the zero vector, \( 0 \), is in \( H \).

2. for any \( u, v \) in \( H \), \( u + v \) is in \( H \)

3. for any constant \( c \) and vector \( u \), \( cu \) is in \( H \).

We also saw that the span of a given set of vectors forms a subspace. (This can be much easier to use to than the subspace theorem.) So we have two ways to show a set of vectors forms a subspace.

\section*{Vector Subspace Examples}

\textbf{Example 1:}
Let \( H = \{(a, a) : a \in \mathbb{R}\} \). Determine whether \( H \) is a subspace of \( \mathbb{R}^2 \).

\textbf{Example 2:}
Let \( S \) be the set of all vectors in \( \mathbb{R}^5 \) of the form \( \begin{pmatrix} 8r - 5s + 3t \\ 5r + 4s + t \\ r - s \\ r - 2s + 2t \\ r - t \end{pmatrix} \). Show that \( S \) is a subspace of \( \mathbb{R}^5 \).

\textbf{Example 3:}
Is \( \{3x, 4x - 1, x^2\} \) a subspace of \( \mathbb{R}^3 \)?
Null Space

The **null space** is the set of all solutions of the homogeneous system with coefficient matrix $A$.

Column Space

The **column space** is the set of all linear combinations of the columns. (i.e. the span of the columns).

Basis

A **basis** of a vector space is a linearly independent spanning set. We know that finite spanning set can always be reduced to a basis.

In the case of spans of vectors in $\mathbb{R}^n$, we saw how to row reduce to find a basis for a spanning set.

*To find a basis for Nul $A$ we:*

1. Find the RREF of the coefficient matrix $A$.
2. Write out basic variables in terms of the free variables.
3. Decompose the general solution into a linear combination of vectors.
4. Every linear combination of these vectors is in Nul $A$. So the span of these vectors forms a basis for Nul $A$.

*To find a basis for Col $A$ we:*

1. Remember that elementary row operations preserve dependence relations among the columns. So find the RREF of the coefficient matrix $A$.
2. Locate the columns that are independent in the RREF, since all other columns can be written as a linear combination of them.
3. Now locate the same columns in the original matrix, these columns form a basis of Col $A$.

**WARNING:** the pivot columns of a matrix $A$ are evident when $A$ has been reduced only to echelon form. Be careful to use the pivot columns of $A$ itself for the basis of Col $A$. This is because the row operations may change the column space of a matrix. The columns of the echelon form of $B$ of $A$ are often not in the column space. (as seen in the first example)
Basis, Null Space, and Column Space Examples

Example 1:
Given below is a matrix $A$ and its reduced row echelon form $B$.

$$
A = \begin{pmatrix}
1 & 3 & 2 & 5 & 10 & -4 & 4 & 9 \\
1 & 3 & 2 & 1 & 2 & -4 & 4 & 5 \\
5 & 15 & 1 & 2 & 5 & -2 & 20 & 22 \\
2 & 6 & -5 & -8 & 3 & 10 & 8 & 0 \\
10 & 30 & 3 & -5 & 6 & -6 & 40 & 35 \\
\end{pmatrix},
B = \begin{pmatrix}
1 & 3 & 0 & 0 & 0 & 0 & 4 & 4 \\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

Find a basis for $\text{Nul } A$ and a basis for $\text{Col } A$.

Example 2:
Show that the following vectors form a basis of $\mathbb{R}^3$

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \text{ and } a_3 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

Example 3:
Consider the following polynomials in $P_2$.

$$p(t) = 5t^2 + t + 1, q(t) = 15t^2 + 3t, r(t) = 10t^2 + 2t - 1$$

(a) Using the standard basis $B = \{1, t, t^2\}$, write down the coordinate vectors for each of these polynomials.

(b) Find the reduced row echelon form of the matrix with columns equal to the coordinate vectors you found in part (a).

(c) Using your answer from part (b), determine if the given polynomials are linearly independent or dependent. If they are dependent, write down a dependence relation among the polynomials.

Example 3: Let

$$A = \begin{pmatrix}
1 & -2 & 2 & 1 & -3 \\
-3 & 1 & -2 & -4 & 0 \\
4 & -1 & 4 & 7 & -1 \\
\end{pmatrix}, B = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 \\
\end{pmatrix}$$

Eigenvalues and Eigenvalues

Let $A$ be an $n \times n$ matrix. If there is a number $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$, for $x \neq 0$, then we say that $\lambda$ is an eigenvalue of $A$ and $x$ is an eigenvector corresponding to $\lambda$.

The eigenvalues of a triangular matrix are the entries on its main diagonal.

The scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic polynomial $\det(A - \lambda I_n) = 0$. If you are asked to find the eigenvalues of a matrix $A$, then you need to find the roots of the polynomial $\det(A - \lambda I_n)$. Note that you should probably compute $\det(A - \lambda I_n)$ using a combination of elementary row operations and cofactor expansions for ease.
The set of all eigenvectors corresponding to an eigenvalue $\lambda$ is called the **eigenspace** of $\lambda$. Note that the following are equivalent

- $Ax = \lambda x$
- $Ax - \lambda x = 0$
- $(A - \lambda I_n)x = 0$

So, the eigenspace corresponding to $\lambda$ is simply the null space of $A - \lambda I_n$. If you are asked to find a basis for the eigenspace corresponding to $\lambda$, you simply need to find a basis for the null space of $A = \lambda I_n$.

We saw that the eigenvectors from distinct eigenspaces are linearly independent. We also saw that the dimension of the eigenspace corresponding to $\lambda_i$ is bounded above by the multiplicity of the root $\lambda_i$ in the characteristic polynomial. This can help us save time! If we know that the dimension of an eigenspace is at most $n_i$, and can find $n_i$ linearly independent eigenvectors by inspection (given by the coefficients in a linear combination of the columns of $A - \lambda I$ that is equal to the zero vector), then we know that we have a basis for the eigenspace.

A **diagonal matrix** is a matrix that has zeros everywhere except (possibly) along its main diagonal. The identity matrices are examples of diagonal matrices. A matrix $A$ is **diagonalizable** if there is an invertible matrix $P$ and a diagonal matrix $D$ such that

$$A = PDP^{-1}$$

The diagonalization theorem states that an $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix if and only if the columns of $P$ and $n$ linearly independent eigenvectors of $A$. In this case the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the given eigenvectors is $P$.

The easiest situation to deal with (in terms of diagonalization) is the case in which you have an $n \times n$ matrix $A$ that has $n$ distinct eigenvalues. In this case, there is a theorem that says that $A$ is diagonalizable. Furthermore, we can find the matrices $P$ and $D$ such that $A = PDP^{-1}$ as follows. Say $A$ has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let $x_1, x_2, ..., x_n$ be eigenvectors respectively. Let $D$ be the diagonal matrix with the numbers $\lambda_1, \lambda_2, ..., \lambda_n$ along its main diagonal. Let $P = (x_1, x_2, ..., x_n)$ be the matrix whose columns are the vectors $x_1, x_2, ..., x_n$. Then $A = PDP^{-1}$.

It’s easy to compute a power of a diagonal matrix.

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^k = \begin{pmatrix} d_1^k & 0 \\ 0 & d_2^k \end{pmatrix}$$

We can use a similar formula to calculate a power of a $3 \times 3$ matrix (indeed, we can use a similar formula to calculate a power of an $n \times n$ matrix).

If $A$ is an $n \times n$ diagonalizable matrix, then it’s also easy to compute a power of $A$. Suppose that $A = PDP^{-1}$ (where $D$ is a diagonal matrix). Then $A^k = PD^kP^{-1}$. Since its easy to compute $D^k$ its also easy to compute $A^k$. 

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Eigenvalue / Eigenvector Examples

Example 1:
Find the formula for the entries of $A^m$ for

$$A = \begin{pmatrix} -1 & -1 \\ 4 & -6 \end{pmatrix}$$

Example 2:
(a) Find all the real eigenvalues of the given matrix $A$.
(b) Comment on the possible values for the dimension of each eigenspace.
(c) For each eigenvalue $\lambda$, find a basis for the corresponding eigenspace.
(d) Determine if $A$ is diagonalizable.
(e) If $A$ is diagonalizable, find the invertible $P$ and diagonal $D$ with $A = PDP^{-1}$

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Example 3:
Let $A$ be a $2 \times 2$ matrix with eigenvectors

$$v_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

corresponding to eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$, respectively. What is the matrix $A$?

Inner Product and Orthogonality

The inner product (or dot product) of $u = \{u_1, u_2, \ldots, u_n\}$ and $v = \{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$ is the scalar

$$u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

The properties of the inner product are:

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + V \cdot w$
3. $(cu) \cdot v = c(u \cdot v)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$

The length of $v \in \mathbb{R}^n$ is $||v|| = \sqrt{v \cdot v}$. This corresponds with the usual notion of length in $\mathbb{R}^2$ and $\mathbb{R}^3$.

A unit vector is a vector with length 1. If $v \neq 0$, then $v/||v||$ is a unit vector that is a scalar multiple of $v$ (that is, it is in the ”same direction” as $v$).

We saw $u$ and $v$ are orthogonal if $u \cdot v = 0$. In $\mathbb{R}^2$ and $\mathbb{R}^3$ two vectors are orthogonal if and only if they are perpendicular.
We saw the **Pythagorean Theorem**: two vectors in \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \) are orthogonal if and only if \( ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}|| + ||\mathbf{v}|| \).

An **orthogonal basis** of a subspace \( V \) of \( \mathbb{R}^n \) is a basis of \( V \) for which the vectors are pairwise orthogonal (that is, the inner product of any two distinct vectors in the basis is 0). If \( \mathbf{u}_1, ..., \mathbf{u}_p \) is an orthogonal basis of \( V \) and \( \mathbf{v} \in V \), then \( \mathbf{v} = c_1 \mathbf{u}_1 + ... + c_p \mathbf{u}_p \) for \( c_j = \frac{\mathbf{v} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \).

We also know that an orthogonal basis must be linearly independent.

We saw the Gram-Schmidt process that can be applied to a basis of \( V \) (a subspace of \( \mathbb{R}^n \)) and yields an orthogonal basis for \( V \). If \( \mathbf{x}_1, ..., \mathbf{x}_p \) is a basis of \( V \), we define

\[
\begin{align*}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
\mathbf{v}_4 &= \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - ... - \frac{\mathbf{x}_4 \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1} \\
\end{align*}
\]

The \( \mathbf{v}_1, ..., \mathbf{v}_p \) is an orthogonal basis of \( V \).

### Inner Product and Orthogonality Examples

**Example 1:**

Let \( \mathbf{u} = (1, 3, 2) \) and \( \mathbf{v} = (2, 2, 4) \).

(a) Compute ||\(\mathbf{u}||\), ||\(\mathbf{v}||\), and \( \mathbf{u} \cdot \mathbf{v} \).

(b) Are \( \mathbf{u} \) and \( \mathbf{v} \) orthogonal?

(c) Find a unit vector that is a scalar multiple of \( \mathbf{v} \).

(d) Since \( \mathbf{u} \) and \( \mathbf{v} \) are not scalar multiples of each other, \( V = \text{Span}\{\mathbf{u}, \mathbf{v}\} \) is a 2 dimensional subspace of \( \mathbb{R}^3 \). use the Gram-Schmidt process to find an orthogonal basis for \( V \).

**Example 2:**

Let \( \mathbf{u} = (2, 1, 6) \) and \( \mathbf{v} = (-3, 0, 1) \).

(a) Show that \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal.

(b) Find all vectors in \( \mathbb{R}^3 \) that are orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

(c) Let \( \mathbf{w} \) be one of the vectors you found in (b). Explain why \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) are an orthogonal basis of \( \mathbb{R}^3 \).

(d) Find the coefficient to use on \( \mathbf{v} \) when \( (1,1,1) \) is written as a linear combination of \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \).

### Equivalent Statements about Square Matrices

- \( A \) is invertible.
- The RREF of \( A \) is \( I_n \).
- \( A \) has \( n \) pivot positions.
- The equation \( A\mathbf{x} = \mathbf{0} \) has only the trivial solution.
• The columns of $A$ form a linearly independent set.
• The equation $Ax = b$ has at least one solution for each $x \in \mathbb{R}^n$.
• The columns of $A$ span $\mathbb{R}^n$.
• $A^T$ is an invertible matrix.
• $\det(A)$ is nonzero.

• Nul $A = \{ 0 \}$

**Complex Numbers**

We saw how to add, subtract, multiply and divide complex numbers. We also saw how to write them in $a + bi$ form where $a, b \in \mathbb{R}$.

• For addition and subtraction: we group the real numbers together and the imaginary numbers together.

• For multiplication we expand and use the fact that $i^2 = -1$.

• For division we multiply the number and denominator by the conjugate of the denominator.

The conjugate of a complex number $a + bi$ is $a - bi$. The sign of the imaginary part has switched. So when you multiply a complex number by its conjugate you get a real number.

**Complex Number Examples**

**Example 1:**
Let $z_1 = 1 + 2i$, and $z_2 = 5 - 2i$, $z_3 = 8 - 6i$
(a) Find $z_1 + z_2 - z_3$.
(b) Find $z_1^2 + z_3$
(c) Find $z_1(z_2 + z_3)$
(d) Find $\frac{z_2}{z_3 - z_1}$

**Example 2:**
Find $i^{1894}$