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Dynamical Systems: Bowen's Theorem

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1 Introduction

In this paper, we provide a rigorous proof of Bowen's theorem, a result in the study of a particular class of dynamical systems called Smale spaces. Dynamical systems model time evolution through the iteration of a self-map on a set, particularly a metric space. For example, one can form a metric space out of all bi-infinite sequences $(x_n)_{n\in\mathbb{Z}}$ where each x_n belongs to some fixed set $\mathcal{A}$, then consider the "evolution" function σ that shifts the sequence forwards: $\sigma(x)_i = x_{i+1}$. This shift space is a simple example of a dynamical system but others can be more complicated. Given a dynamical system (X, f), partitioning X into finitely many pieces and following point's trajectories through the partition relates (X, f) to a subshift (related to shift spaces) on the partition. Then one can study the subshift and sometimes deduce properties of (X, f) from it. One of the strongest types of partitions is called a Markov partition. Given a Markov partition, one always has a finite-to-one semiconjugacy from the subshift to (X, f), which is furthermore one-to-one almost everywhere. Semiconjugacy is a map between dynamical systems that preserves a lot of topological information. A conjugacy on the other hand is an equivalence of dynamical systems. Smale spaces are dynamical systems that have local hyperbolic coordinates. Bowen's theorem says that all Smale spaces have Markov partitions.

2 Preliminaries

We begin with a review of some definitions regarding metric spaces and dynamical systems.

Definition 2.1. Metric space
A metric space (X, d) is a set X together with a function $d : X \times X \to [0, \infty)$ called the metric, satisfying the following properties for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

Definition 2.2. Open balls
Let (X, d) be a metric space. For x in X, and $\epsilon > 0$, let $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$. These sets are called open balls.

Definition 2.3. Compactness
A metric space (X, d) is compact if for every collection of open sets $\{U_i\}_{i\in I}$ of X such that $\bigcup_{i\in I} U_i = X$, then X can be covered by finitely many of them: there exists $i_1, \dots i_n \in I$ such that $\bigcup_{j=1}^{n} U_{i_j} = X$.

Definition 2.4. γ-dense set
Let (X, d) be a compact metric space. For all $\gamma > 0$ there exists a finite set $\{x_1, \dots x_n\}$ such that $\bigcup_{j=1}^{n} B(x_j, \gamma) = X$. Such a set is called a γ-dense set.

Definition 2.5. Dynamical system
Let (X, d) be a metric space. Let $f : X \to X$ be a continuous function. Then the pair (X, f) is called a topological dynamical system. d will always be used to denote the metric, so it is omitted, and we usually drop the "topological" as well.

In most of our cases, f will be a homeomorphism, and X will be compact.

Definition 2.6. Orbits
Let (X, f) be a dynamical system, and let x be a point in X. We define the following sets:

1. $\mathcal{O}^+(x) = \{f^n(x) \mid n \geq 0\}$ (Forward orbit)
2. $\mathcal{O}^-(x) = \{f^n(x) \mid n \leq 0\}$ (backward orbit)
3. $\mathcal{O}(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$ (Full orbit)

Definition 2.7. Expansive
Let (X, f) be a dynamical system, with f a homeomorphism. If there exists some $\epsilon > 0$ such that for any two distinct x, y in X, there is an $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \epsilon$, we say that X is ϵ-expansive. Here, ϵ is called an expansiveness constant for X. (not *the*, since it is not unique: anything between 0 and ϵ will work just as easily.)

Definition 2.8. Semiconjugacy
Let (X, f) and (Y, g) be dynamical systems. A topological semiconjugacy from g to f is a surjective continuous map $h : Y \to X$ such that $f \circ h = h \circ g$. If h is a homeomorphism, it is called a topological conjugacy, and f and g are said to be topologically conjugate.
Topologically conjugate dynamical systems have identical topological properties.

We have everything we need to now go ahead and define what a Smale space is.

3 Smale Spaces

First, we begin with a heuristic definition of a Smale space. The technical example is a bit opaque, while the heuristic definition is clearer. It says that X has "local hyperbolic coordinates."

Definition 3.1. Heuristic definition of a Smale space.
Let X be a compact metric space, and $f : X \to X$ be a homeomorphism. For each $x \in X$ we'll have two sets E_x "local stable set" and F_x "local unstable set" corresponding to it. These sets will satisfy the following properties (approximately):
P1: $E_x \cap F_x = \{x\}$
P2: $E_x \times F_x \simeq V$ for some neighborhood V of x.
P3: $f(E_x) \cap V = E_{f(x)} \cap V$ and $f(F_x) \cap V = F_{f(x)} \cap V$ (approximately)
And there exists some constant λ with $0 < \lambda < 1$ such that:
P4: For all $y, z \in E_x$, $d(f(y), f(z)) \leq \lambda d(y, z)$ "contracting, stable"
P5: For all $y, z \in F_x$, $d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z)$ "stretching, unstable"

Let us give an example of a space satisfying these properties.

Example 3.1. Hyperbolic toral automorphism (Our prototypical example)
Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$, given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

A is an integer matrix with $\det(A) = 1$, thus A^{-1} is also an integer matrix. Thus $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$ and $A^{-1}\mathbb{Z}^2 \subseteq \mathbb{Z}^2$. Therefore, they both induce well-defined maps on the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. These two induced maps are inverses of each other, and thus are homeomorphisms of the quotient space. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map. We define the metric d on $X = \mathbb{T}^2$ by $d(\pi(x), \pi(y)) = \min\{|a - b| : \pi(x) = \pi(a), \pi(y) = \pi(b)\}$

Note that the matrix in question has two real eigenvalues, the first being $\lambda = \frac{3-\sqrt{5}}{2} \in (0, 1)$ the other being $\lambda^{-1} = \frac{3+\sqrt{5}}{2}$. $v_1 = ((1 - \sqrt{5})/2, 1)^T$ and $v_2 = (1, (1 + \sqrt{5})/2)^T$ are eigenvectors corresponding to these eigenvalues.

Now, where are these properties in the Hyperbolic toral automorphism example? Let $\epsilon > 0$ such that $\epsilon \leq \lambda/2$, so that $d(x,y) < \epsilon \implies d(f(x), f(y)) < 1/2$.
Let $E_x = \{x + \frac{t}{||v_1||}v_1 : |t| < \epsilon\}$ and let $F_x = \{x + \frac{t}{||v_2||}v_2 : |t| < \epsilon\}$. Then we have:
P1: $E_x \cap F_x = \{x\}$, since $x + cv_1 = x + dv_2 \implies c = d = 0$, as v_1, v_2 are linearly independent
P2: $E_x \times F_x \simeq (x-\epsilon, x+\epsilon) \times (x-\epsilon, x+\epsilon)$. E_x,F_x are like intervals, their product homeomorphic to an open square neighborhood of x.
P3: $f(E_x) = \{f(x) + \frac{\lambda t}{||v_1||}v_1 \mid |t| < \epsilon\} \subseteq E_{f(x)}$ and $f(F_x) = \{f(x) + \frac{\lambda^{-1} t}{||v_1||}v_1 \mid |t| < \epsilon\} \supseteq F_{f(x)}$.
For the last two parts, we already have $\lambda = \frac{3-\sqrt{5}}{2} \in (0,1)$. These are the axioms that talk about the specifics of the "contracting, stableness" and "stretching, unstableness"
P4: For all $y, z \in E_x$, indeed, $d(f(y), f(z)) \leq \lambda d(y,z)$
P5: For all $y, z \in F_x$, indeed, $d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y,z)$.

Now we come to the precise formulation of a Smale space.
Let (X,d) be a compact metric space. Let $f : X \to X$ be a homeomorphism. Let $\epsilon_X > 0$, and let $\Delta_{\epsilon_X} = \{(x,y) \in X^2 \mid d(x,y) \leq \epsilon_X\}$ be the fat diagonal, the set of all pairs of points whose distance is at most ϵ_X. Suppose we have a (jointly) continuous map $[,] : \Delta_{\epsilon_X} \to X$ satisfying the following properties:
B1: $[x,x] = x$,
B2: $[x,[y,z]] = [x,z]$ whenever both sides are defined,
B3: $[[x,y],z] = [x,z]$ whenever both sides are defined,
B4: $[f(x), f(y)] = f([x,y])$ whenever both sides are defined.
Furthermore suppose there is a constant $\lambda \in (0,1)$ such that for all $x \in X$ the following two conditions hold:
C1: For all y, z such that $d(x,y), d(x,z) \leq \epsilon_X$ and $[y,x] = x = [z,x]$, we have

$$d(f(y), f(z)) \leq \lambda d(y,z),$$

C2: For all y, z such that $d(x,y), d(x,z) \leq \epsilon_x$ and $[x,y] = x = [x,z]$, we have

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y,z)$$

Definition 3.2. Precise definition of a Smale space
If a quadruple $(X, d, f, [,])$ satisfies all the above axioms B1, B2, B3, B4, C1, and C2, then it is called a Smale space.

Axioms B1, B2 and B3 effectively ensure that the bracket will function like a coordinate system, while axiom B4 is ensuring the bracket's compatibility with the map f.

Theorem 3.1. *Let $(X, d, f, [,])$ be a Smale space. Define $[,]^{-1} : \Delta_{\epsilon_X} \to X$ by $[x,y]^{-1} = [y,x]$ for all x, y in X such that $d(x,y) \leq \epsilon_X$. Then $(X, d, f^{-1}, [,]^{-1})$ is a Smale space.*

Proof. Since $[,]^{-1}$ is defined from $[,]$ and has the same domain; we get its continuity and satisfying axioms B1 to B3 for free. Similarly, C1 and C2 come for free; they're just swapped. For B4, suppose x and y are points in X such that $[y,x]$, $[f^{-1}(x), f^{-1}(y)]^{-1}$ and $f^{-1}([x,y]^{-1})$ are all defined. By definition of $[,]^{-1}$ these are equal to $[f^{-1}(y), f^{-1}(x)]$ and $f^{-1}([y,x])$ respectively. Then $f(f^{-1}([y,x])) = [y,x]$ and $f([f^{-1}(y), f^{-1}(x)]) = [f(f^{-1}(y), f(f^{-1}(x))] = [y,x]$. Since f is a homeomorphism, it is injective, and thus $[f^{-1}(y), f^{-1}(x)] = f^{-1}([y,x])$. Therefore $[f^{-1}(x), f^{-1}(y)]^{-1} = f^{-1}([x,y]^{-1})$. $(X, d, f^{-1}, [,]^{-1})$ satisfies all the axioms, so it is a Smale space. ☐

Definition 3.3. Local stable and unstable sets
For each $x \in X$ and $0 < \epsilon \leq \epsilon_X$, we define

$$W^s(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon, [y,x] = x\}$$

$$W^u(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon, [x,y] = x\}$$

The former will be referred to as local stable sets, and the latter as local unstable sets.

Observation 3.3.1. From Theorem 3.1 we see that $W^s_f = W^u_{f^{-1}}$ and $W^u_f = W^u_{f^{-1}}$.

It turns out that these sets will correspond to the E_x and F_x sets from the heuristic definition. Some properties:

Lemma 3.2. *Suppose $x, y \in X$ are such that $d(x, y) \leq \epsilon_X$. Then*

$$1.\ [x, y] = x \iff [y, x] = y$$
$$2.\ [x, y] = y \iff [y, x] = x$$

Proof. 1. $[x, y] = x \implies [y, x] = [y, [x, y]] = [y, y] = y$.
$[y, x] = y \implies [x, y] = [x, [y, x]] = [x, x] = x$.
2. $[x, y] = y \implies [y, x] = [[x, y], x] = [x, x] = x$.
$[y, x] = x \implies [x, y] = [[y, x], y] = [y, y] = y$. Done

□

Lemma 3.3. *Suppose $x, y \in X$ are such that $d(x, y), d(x, [x, y]), d(y, [x, y]) < \epsilon \leq \epsilon_X$. Then*

$$[x, y] \in W^s(x, \epsilon) \cap W^u(y, \epsilon)$$

Proof. We check: $[[x, y], x] = [x, x] = x$. Thus $[x, y] \in W^s(x, \epsilon)$.
On the other hand, $[y, [x, y]] = [y, y] = y$. Thus $[x, y] \in W^u(y, \epsilon)$. We are done.

□

The following theorem corresponds to heuristic axiom P2, that products of the local stable and unstable sets of points form neighborhoods around the points.

Theorem 3.4. *There is a constant $0 < \epsilon'_X \leq \epsilon_X/2$ such that, for every $0 < \epsilon \leq \epsilon'_X$ and for every $x \in X$ the map*

$$[,] : W^u(x, \epsilon) \times W^s(x, \epsilon) \to X$$

is a homeomorphism onto its image, which is an open set containing x.

Proof. First note that the function is well-defined, as for any $y \in W^u(x, \epsilon)$ and any $z \in W^s(x, \epsilon)$, we'll have $d(y, z) \leq d(x, y) + d(x, z) < \epsilon + \epsilon \leq \epsilon_X$. Moreover, since $[,]$ is jointly continuous and $[x, x] = x$, there exists $0 < \delta \leq \epsilon_X$ such that for all y with $d(x, y) \leq \delta$, we have $d(x, [x, y]) \leq \epsilon_X/2$ and $d([y, x], x) \leq \epsilon_X/2$. We choose $0 < \epsilon'_X \leq \epsilon_X/2$ so that for all y, z with $d(x, y), d(x, z) \leq \epsilon'_X$, we have $d(x, [y, z]) \leq \delta$. Then we can define a map h on a neighborhood of x by $h(y) = ([y, x], [x, y])$. By the choice of ϵ'_X, this map is defined on the range of $[,]$. The map is continuous since $[,]$ itself is continuous. Next, we verify that h is the inverse of the map in the definition. For y with $d(x, y) < \epsilon'_X$, we have

$$\begin{aligned}
[,] \circ h(y) &= [[y, x], [x, y]] \\
&= [y, [x, y]], \text{ since } y \in W^u(x, \epsilon), \text{ thus } [y, x] = y \text{ by Lemma 3.2} \\
&= [y, y] \\
&= y,
\end{aligned}$$

where we have used Lemma 3.2 and axioms B2 and B1.
Moreover, if we begin with $y \in W^u(x, \epsilon)$ and $z \in W^s(x, \epsilon)$, then we have

$$\begin{aligned}
h([y, z]) &= ([[y, z], x], [x, [y, z]]) \\
&= ([y, x], [x, z]) && \text{by Axioms B2 and B3} \\
&= (y, z) && \text{by Lemma 3.2}
\end{aligned}$$

Finally, we must check that the image of our map, $W^u(x, \epsilon), W^s(x, \epsilon)]$, is open. Let $y \in W^u(x, \epsilon)$ and let $z \in W^s(x, \epsilon)$. Since the bracket is jointly continuous, we can choose $\delta_1 > 0$ such that $d(w, [y, z]) < \delta_1$

implies $d(x,[w,x]) < \epsilon - d(x,y)$. Similarly, we can choose $\delta_2 > 0$ such that $d(w,[x,z]) < \delta_2$ implies $d(x,[x,w]) < \epsilon - d(x,z)$. Pick $\delta' = \min(\delta_1, \delta_2)$. Then

$$h(B([y,z],\delta')) \subseteq B(x, \epsilon - d(x,y)) \times B(x, \epsilon - d(x,z))$$

From this, it follows that $h(B([y,z],\delta'))$ actually lies in the domain of $[,]$ and so $B([y,z],\delta')$ lies in the range of $[,]$. Thus, for point $[y,z]$ in $[W^u(x,\epsilon), W^s(x,\epsilon)]$ we have an open ball $B([y,z],\delta')$ around it contained in $[W^u(x,\epsilon), W^s(x,\epsilon)]$, so this set is open. This completes the proof. □

Corollary 3.4.1. *For every $\epsilon > 0$ there exists some $\gamma > 0$ such that $[W^u(x,\gamma), W^s(x,\gamma)]$ is contained in $B(x,\epsilon)$.*

Proof. In the proof of theorem 3.4, we can choose $0 < \delta < \epsilon$ and $\epsilon'_X > 0$ small enough so that for all y, z with $d(x,y), d(x,z) \leq \epsilon'_X$, we have $d(x,[y,z]) \leq \delta < \epsilon$. Choose $\gamma = \epsilon'_X$. Then for all $y \in W^u(x,\gamma)$ and $z \in W^s(x,\gamma)$, we'll have $d(x,[y,z]) < \epsilon$. Thus $[W^u(x,\gamma), W^s(x,\gamma)] \subseteq B(x,\epsilon)$. □

Lemma 3.5. *Suppose that $(X, d, f, [,])$ is a Smale space. Then there exists a constant $0 < \epsilon_1 \leq \epsilon_X$ such that for all $0 < \epsilon \leq \epsilon_1$, we have:*
1. For $x, y \in X$, $d(f^n(x), f^n(y)) \leq \epsilon$ for all $n \geq 0$ if and only if $y \in W^s(x,\epsilon)$
2. For $x, z \in X$, $d(f^{-n}(x), f^{-n}(z)) \leq \epsilon$ for all $n \geq 0$ if and only if $z \in W^u(x,\epsilon)$

Proof. First, choose $0 < \epsilon_1 \leq \epsilon_X$ such that for all $x, y \in X$, $d(x,y) < \epsilon_1$ implies $d([y,x],x) < \epsilon_X$. Now first of all, $y \in W^s(x,\epsilon) \implies d(y,x) < \epsilon$ and thus for all $n \geq 0$ we have $d(f^n(y), f^n(x)) \leq \lambda^n d(y,x) \leq \epsilon$. Similarly for part 2, $z \in W^u(x,\epsilon) \implies d(z,x) < \epsilon$ and thus for all $n \geq 0$ we have $d(f^{-n}(z), f^{-n}(x)) \leq \lambda^n d(z,x) \leq \epsilon$. Conversely, suppose x, y are in X such that $d(f^n(x), f^n(y)) \leq \epsilon \leq \epsilon_1$ for all $n \geq 0$. It follows that $[f^n(y), f^n(x)]$ is defined for all $n \geq 0$. By axioms B1 and B2 of Smale spaces, we see that $[f^n(x), [f^n(y), f^n(x)]] = f^n(x)$ and thus $[f^n(y), f^n(x)] \in W^u(f^n(x), \epsilon_X)$ for all $n \geq 0$. Now we apply axiom B4 (perhaps, together with the fact that $(X, d, f^{-1}, [,]^{-1})$ forms a Smale space, with $[x,y]^{-1} = [y,x]$) to note that for all $n > 0$ we have $f^{-1}[f^n(y), f^n(x)] = [f^{n-1}(y), f^{n-1}(x)]$. We apply axiom C2 to assert that

$$\begin{aligned} d(f^{n-1}(x), [f^{n-1}(y), f^{n-1}(x)]) &= d(f^{-1}(f^n(x)), f^{-1}[f^n(y), f^n(x)]) \\ &\leq \lambda d(f^n(x), [f^n(y), f^n(x)]) \end{aligned}$$

Thus $d(x,[y,x]) \leq \lambda^n d(f^n(x), [f^n(y), f^n(x)]) < \lambda^n \epsilon_X$ for all $n \geq 0$. Since $0 < \lambda < 1$, we get that $x = [y,x]$, thus $y \in W^s(x,\epsilon)$. This completes the proof of part 1.
For part 2, suppose x, z are in X such that $d(f^{-n}(x), f^{-n}(z)) \leq \epsilon \leq \epsilon_1$ for all $n \geq 0$. It follows through a similar argument as above that $[f^{-n}(x), f^{-n}(z)]$ is defined and $[f^{-n}(x), f^{-n}(z)] \in W^s(f^n(x), \epsilon_X)$ for all $n \geq 0$. Using C1 this time, we get that for all $n > 0$ we have

$$\begin{aligned} d(f^{1-n}(x), [f^{1-n}(x), f^{1-n}(z)]) &= d(f(f^{-n}(x)), f[f^{-n}(x), f^{-n}(z)]) \\ &\leq \lambda d(f^{-n}(x), [f^{-n}(x), f^{-n}(z)]) \end{aligned}$$

Thus $d([x,z],x) \leq \lambda^n d([f^{-n}(x), f^{-n}(z)], f^{-n}(x)) < \lambda^n \epsilon_X$ for all $n \geq 0$ so we get $x = [x,z]$, thus $z \in W^u(x,\epsilon)$. This completes the proof of part 2. □

The following theorem strengthens the previous lemma, and also confirms the heuristic property P1 of Smale spaces: that $E_x \cap F_x = \{x\}$.

Theorem 3.6. *Let $(X, d, f, [,])$ be a Smale space, and let ϵ_1 be as in Lemma 3.5. If $x, y \in X$ such that $d(x,y) \leq \epsilon_X$ and $d(x,[x,y]), d(y,[x,y]) < \epsilon_1$, then we have*

$$\begin{aligned} \{[x,y]\} &= W^s(x,\epsilon_1) \cap W^u(x,\epsilon_1) \\ &= \bigcap_{n \geq 0} \{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z) < \epsilon_1\} \end{aligned}$$

Proof. By the previous lemma, we have:

$$\begin{aligned}
&\bigcap_{n\geq 0}\{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\} \\
&= \{z \mid d(f^n(x), f^n(z)) < \epsilon_1 \text{ and } d(f^{-n}(y), f^{-n}(z)) < \epsilon_1, \text{ for all } n \geq 0\} \\
&= \{z \mid d(f^n(x), f^n(z)) < \epsilon_1, \text{ for all } n \geq 0\} \cap \{z \mid d(f^{-n}(y), f^{-n}(z)) < \epsilon_1, \text{ for all } n \geq 0\} \\
&= W^s(x, \epsilon_1) \cap W^u(y, \epsilon_1)
\end{aligned}$$

We already know that $[x, y]$ is in this intersection by Lemma 3.3. If z is any point in it, then $[z, x] = x$ and $[y, z] = y$. By Lemma 3.2 these imply $[x, z] = z$ and $[z, y] = z$. Thus $z = [x, z] = [x, [z, y]] = [x, y]$ by axiom B2. □

Theorem 3.7. *Let (X, f) be a Smale space, and let ϵ_1 be as in Lemma 3.5. Then the map f is ϵ_1-expansive.*

Proof. Suppose x, y are in X such that $d(f^n(x), f^n(y)) < \epsilon_1$ for all $n \in \mathbb{Z}$. Then by Theorem 3.6, x, y are in

$$\bigcap_{n\geq 0}\{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\}$$

and hence $x = [x, y] = y$. □

4 Examples of Smale Spaces

Example 4.1. Hyperbolic Toral Automorphism
We saw earlier in Example 3.1 that this particular space satisfies the heuristic axioms of a Smale space. Here we see that it indeed satisfies the precise formulation as well, and this does in fact form a Smale space. First, choose $\epsilon_X = \lambda/2$. For $0 < \epsilon \leq \epsilon_X$, define $E_x(\epsilon) = \{x + \frac{t}{||v_1||}v_1 : |t| < \epsilon\}$ and $F_x(\epsilon) = \{x + \frac{t}{||v_2||}v_2 : |t| < \epsilon\}$. For x, y such that $d(x, y) \leq \epsilon_X$, define the bracket to be $[x, y] = z$, where z is the unique element in $E_x \cap F_y$. Uniqueness follows from the eigenvectors v_1, v_2 being linearly independent. Then we see:

B1: $E_x \cap F_x = \{x\}$, as $x \in E_x$ and $x \in F_x$, and v_1, v_2 are linearly independent. Thus $[x, x] = x$.
B2: Suppose $E_x \cap (E_y \cap F_z)$ and $E_x \cap F_z$ are both nonempty. Then, since $E_x \cap E_y \cap F_z \subseteq E_x \cap F_z$ and both are singletons, they must be equal. Thus $[x, [y, z]] = [x, z]$ whenever both sides are defined.
B3: Suppose $(E_x \cap F_y) \cap F_z$ and $E_x \cap F_z$ are both nonempty. Then, since $E_x \cap F_y \cap F_z \subseteq E_x \cap F_z$ and both are singletons, they must be equal. Thus $[[x, y], z] = [x, z]$ whenever both sides are defined.
B4: Suppose $E_{f(x)} \cap F_{f(y)}$ and $E_x \cap F_y$ are both nonempty. If z is the unique element in $E_x \cap F_y$, then $z = x + \frac{t_1}{||v_1||}v_1 = y + \frac{t_2}{||v_2||}v_2$ for some unique $t_1, t_2 \in (-\epsilon, \epsilon)$. Then $f(z) = f(x) + \lambda\frac{t_1}{||t_1||}v_1 = f(y) + \lambda^{-1}\frac{t_2}{||v_2||}v_2 \in E_{f(x)} \cap F_{f(y)}$. Thus $[f(x), f(y)] = f([x, y])$ provided both sides are defined.
Now, $[y, x] = x$ implies $y = x + \frac{t}{||v_2||}v_2$ for some $t \in (-\epsilon, \epsilon)$. Thus $W^s(x, \epsilon) = E_x(\epsilon)$. Similarly, $[x, y] = x$ implies $y = x + \frac{t}{||v_1||}v_1$ for some $t \in (-\epsilon, \epsilon)$. Thus $W^u(x, \epsilon) = F_x(\epsilon)$.
Therefore C1 is identical to P4 and C2 is identical to P5. Thus, HTA forms a Smale space.

Example 4.2. Subshifts of Finite Type (SFT)
Here will we introduce an important class of examples of Smale spaces, called the shifts of finite type. We will need these for the proof of Bowen's theorem.
Let $\mathcal{A}$ denote a finite non-empty set, sometimes called the alphabet. We consider the space

$$\mathcal{A}^{\mathbb{Z}} = \{(a_n)_{n\in\mathbb{Z}} \mid a_n \in \mathcal{A}, \text{ for all } n \in \mathbb{Z}\}$$

and give it the metric

$$d_{\mathcal{A}}(a, b) = \inf\{1, 2^{-n} \mid n \geq 1, a_i = b_i, \text{ for } |i| < n\}$$

where $a = (a_n), b = (b_n)$ are in $\mathcal{A}^{\mathbb{Z}}$

That is, $d_{\mathcal{A}}(a,b) = 0$ precisely when $a_i = b_i$ for all i, $d_{\mathcal{A}}(a,b) = 1$ when $a_0 \neq b_0$. Otherwise, $d_{\mathcal{A}}(a,b) = 2^{-n}$ when $a_i = b_i$ for all $-n < i < n$, but $a_i \neq b_i$ at either $i = n$ or $i = -n$. It turns out that this is more than a metric, it is an *ultrametric*: it satisfies the stronger condition

$$d_{\mathcal{A}}(a,b) \leq \max\{d_{\mathcal{A}}(a,c), d_{\mathcal{A}}(c,b)\}$$

Fact: $(\mathcal{A}^{\mathbb{Z}}, d_{\mathcal{A}})$ is a compact metric space.

Definition 4.1. We define a shift map $\sigma_{\mathcal{A}} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$(\sigma_{\mathcal{A}}(a))_n = a_{n+1}$$

for any a in $\mathcal{A}^{\mathbb{Z}}$ and n in $\mathbb{Z}$. Sometimes for emphasis, we call $\sigma_{\mathcal{A}}$ the (italic) left shift.
Fact: $\sigma_{\mathcal{A}}$ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$. Thus $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is a dynamical system.

Definition 4.2. If $w = (w_1, \ldots, w_n)$ is a finite sequence of elements of $\mathcal{A}$, we say that w is a word in $\mathcal{A}$ and that n is the length of the word w. Given an element a in $\mathcal{A}^{\mathbb{Z}}$, we say that w appears in a if, for some $k \in \mathbb{Z}$,

$$(a_{k+1}, \ldots, a_{k+n}) = (w_1, \ldots, w_n).$$

If $\mathcal{F}$ is a finite (possibly empty) collection of words in $\mathcal{A}$, we define

$$X_{\mathcal{F}} = \{a \in \mathcal{A}^{\mathbb{Z}} \mid \text{ no word in } \mathcal{F} \text{ appears in } a\}$$

Fact: $X_{\mathcal{F}}$ is a closed subset of $\mathcal{A}^{\mathbb{Z}}$. Moreover, it is invariant under $\sigma : a$ is in $X_{\mathcal{F}}$ if and only if $\sigma(a)$ is.

Definition 4.3. Let $\mathcal{F}$ be a finite collection of words in $\mathcal{A}$. The restriction of σ to $X_{\mathcal{F}}$ is denoted by $\sigma_{\mathcal{F}}$. Any non-empty system obtained as $(X_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is called a shift of finite type.

Now we get to the Smale space part.

Definition 4.4. Let $\mathcal{F}$ be a finite collection of words in $\mathcal{A}$. Let N be the maximum length of the words in $\mathcal{F}$ (or $N = 1$ if $\mathcal{F}$ is empty). For a, b in $X_{\mathcal{F}}$ with $d_{\mathcal{A}}(a,b) \leq 2^{-N}$, define $[a,b]$ in $\mathcal{A}^{\mathbb{Z}}$ by

$$([a,b]_{\mathcal{F}})_n = \begin{cases} b_n & n \leq 0 \\ a_n & n \geq 1 \end{cases}$$

Theorem 4.1. *If $\mathcal{F}$ is a finite collection of words in $\mathcal{A}$, then $(X_{\mathcal{F}}, \sigma_{\mathcal{F}}, d_{\mathcal{A}}, [,]_{\mathcal{F}})$ is a Smale space.*

Proof. We go through the axioms above. For all x, y, z in $X_{\mathcal{F}}$ such that $d(x,y), d(y,z), d(x,z) \leq \epsilon_{X_{\mathcal{F}}} = 2^{-N}$:
B1:

$$([x,x]_{\mathcal{F}})_n = \begin{cases} x_n & n \leq 0 \\ x_n & n \geq 1 \end{cases} = x_n$$

Thus $[x,x]_{\mathcal{F}} = x$.
B2:

$$([x,[y,z]]_{\mathcal{F}})_n = \begin{cases} ([y,z]_{\mathcal{F}})_n & n \leq 0 \\ x_n & n \geq 1 \end{cases} = \begin{cases} z_n & n \leq 0 \\ x_n & n \geq 1 \end{cases} = ([x,z]_{\mathcal{F}})_n$$

Thus $[x,[y,z]] = [x,z]$.
B3:

$$([[x,y],z]_{\mathcal{F}})_n = \begin{cases} z_n & n \leq 0 \\ ([x,y]_{\mathcal{F}})_n & n \geq 1 \end{cases} = \begin{cases} z_n & n \leq 0 \\ x_n & n \geq 1 \end{cases} = ([x,z]_{\mathcal{F}})_n$$

Thus $[[x,y],z] = [x,z]$.
B4:

$$([\sigma_{\mathcal{F}}(x), \sigma_{\mathcal{F}}(y)]_{\mathcal{F}})_n = \begin{cases} \sigma_{\mathcal{F}}(y)_n & n \leq 0 \\ \sigma_{\mathcal{F}}(x)_n & n \geq 1 \end{cases} = \begin{cases} y_{n+1} & n \leq 0 \\ x_{n+1} & n \geq 1 \end{cases} = \sigma_{\mathcal{F}}([x,y]_{\mathcal{F}})$$

Now, let $\lambda = 1/2$. We have:
C1: If $[y,x] = x = [z,x]$, then $y_n = x_n = z_n$ for $n \geq 0$. Thus $d(y,z) = 2^{-i}$ where i is the least positive integer such that $y_{-i} \neq z_{-i}$. This implies that $d(\sigma(y), \sigma(z)) = 2^{-(i+1)} = \lambda 2^{-i}$ as $i+1$ will be the least positive integer such that $y_{-(i+1)} \neq z_{-(i+1)}$. Therefore we have $d(\sigma(y), \sigma(z) \leq \lambda d(y,z)$ as desired.
C2: An identical argument. If $[x,y] = x = [x,z]$, then $y_n = x_n = z_n$ for $n \leq 0$. Thus $d(y,z) = 2^{-i}$ where i is the least positive integer such that $y_i \neq z_i$. This implies that $d(\sigma^{-1}(y), \sigma^{-1}(z)) = 2^{i+1} = \lambda 2^{-i}$ as $i+1$ will be the least positive integer such that $y_{i+1} \neq z_{i+1}$. There fore we have $d(\sigma^{-1}(y), \sigma^{-1}(z)) \leq \lambda d(y,z)$ as desired.

Lastly, we need to check that $[,]$ is (jointly) continuous! That is, continuous in each argument when the other is fixed. Let $x \in X_{\mathcal{F}}$ be given. Need to prove that for all y such that $d(x,y) \leq \epsilon_X$, and for all $0 < \epsilon \leq \epsilon_X$ there exists $\delta > 0$ such that $d(y,z) < \delta \implies d([x,y],[x,z]) < \epsilon$. This would prove that the map $[x,-]$ is continuous at y for every y. The proof for $[-,y]$ would be similar. It suffices to choose $\delta = \epsilon$. Indeed, let N be the least positive integer such that $2^{-N} < \epsilon = \delta$. then $d(y,z) < \delta$ implies $d(y,z) \leq 2^{-N}$, which implies $y_i = z_i$ for all $-N < i < N$, and possibly (but not necessarily) disagree at $i = \pm N$. Since $\delta \leq \epsilon_X$, the bracket is defined on all pairs of x, y, z. By the definition of the bracket, we have $([x,y]_{\mathcal{F}})_n = x_n = ([x,z]_{\mathcal{F}})_n$ for all $n \geq 1$. On the other side, we have $([x,y]_{\mathcal{F}})_n = y_n = z_n = ([x,z]_{\mathcal{F}})_n$ for all $-N < n \leq 0$. Thus $d([x,y],[x,z]) \leq 2^{-N} < \epsilon$ as desired. The proof for the continuity of $[-,y]$ is nearly identical and is omitted.
$\square$

Example 4.3. Edge shift
Let $E = (E^1, E^0, r, s)$ be a finite directed graph, where E^1 denotes the edges, E^0 denotes the vertices, and $r, s : E^1 \to E^0$ denote the range and source maps, respectively. We assume here that $r^{-1}(v)$ and $s^{-1}(v)$ are both nonempty for all $v \in E^0$. The corresponding (italic) edge shift is

$$(3.1)\ \Sigma_E := \{(x_i)_{i \in \mathbb{Z}} : x_i \in E^1 \text{ and } r(x_i) = s(x_{i+1}) \text{ for all } i \in \mathbb{Z}\}$$

Then the product topology on Σ_E is generated by the metric $d(x,y) = 2^{-N_{x,y}}$ where

$$N_{x,y} = \begin{cases} 1 + \max\{k : x_j = y_j \text{ for all } |j| \leq k\} & x_0 = y_0 \\ 0 & x_0 \neq y_0 \end{cases}$$

So sequences that don't agree at 0 are distance 1 apart, sequences that agree at 0 but not at ± 1 are distance $1/2$ apart, and so on. The shift map $\sigma : \Sigma_E \to \Sigma_E$ defined by

$$\sigma(x)_i = x_{i+1}$$

is a homeomorphism.
Take $\epsilon_{\Sigma_E} = \frac{1}{2}$ and for $d(x,y) \leq \frac{1}{2}$ define $[x,y] \in \Sigma_E$ by

$$[x,y]_i = \begin{cases} x_i & i \geq 0 \\ y_i & i \leq 0 \end{cases}$$

Then (Σ_E, σ) is a Smale space with this bracket map.
Fact: Every shift of finite type is conjugate to an edge shift. [BS02, Corollary 3.2.2]

5 Shadowing

In this section, we prove that every Smale space satisfies the *shadowing property*, which roughly says that every pseudo-orbit, a sequence that's almost an orbit within some margin of error, is close to an actual orbit.

Definition 5.1. Pseudo orbit
Let (x_n) be a sequence in a Smale space (X, d, f). We say (x_n) is an ϵ-pseudo orbit over an interval $I \subseteq \mathbb{Z}$ if for every n in I, $d(f(x_n), x_{n+1}) \leq \epsilon$. Observation: a 0-pseudo orbit over $\mathbb{Z}$ is precisely a full orbit.

Definition 5.2. Shadowing
Let $\epsilon > 0$ and $\delta > 0$. If (x_n) and (y_n) are ϵ-pseudo-orbits over the same interval I, then we say that one δ-shadows the other if $d(x_n, y_n) \leq \delta$ for all n in I. If x is in X, we also say that x_n is δ-shadowed by (the orbit of) x, if $d(x_n, f^n(x)) \leq \delta$ for all n in I.

Lemma 5.1. *Suppose that $0 < \delta_1 \leq \epsilon_X$. Then there is an $\epsilon > 0$ such that, if $d(f(x), x') < \epsilon$, then for all z in $W^s(x, \delta_1)$, $d(x', f(z)) \leq \epsilon_X$ and $[x', f(z)]$ is in $W^s(x', \delta_1)$.*

Proof. Consider the set
$$A = \{(x, y, z) \mid d(x, y), d(y, z) \leq \epsilon_X/2, [y, z] = z\}$$
in $X \times X \times X$. Consider a sequence (x_n, y_n, z_n) in A that converges to some (x, y, z) in $X \times X \times X$. Then $d(x_n, y_n)$ and $d(y_n, z_n)$ are convergent sequences in $[0, \epsilon_X/2]$, and so $d(x, y), d(y, z) \leq \epsilon_X/2$. Since the bracket $[,]$ is continuous, we have that $\lim_{n\to\infty}[y_n, z_n] = [y, z] = z$, and thus (x, y, z) is in A. Thus A is a closed subset of a compact space, and therefore A is compact.
Consider the function $h : A \to \mathbb{R}$, defined by
$$h(x, y, z) = d(x, [x, z]) - d(y, z)$$
which is continuous on A and hence uniformly continuous. On the set $B = \{(x, y, z) \in A \mid x = y\}$ which is compact by a similar argument, we have
$$\begin{aligned} h(x, y, z) &= h(y, y, z) \\ &= d(y, [y, z]) - d(y, z) \\ &= d(y, z) - d(y, z) \\ &= 0 \end{aligned}$$
Therefore, there is some $\epsilon > 0$ such that, if $d(x, y) < \epsilon$ and $(x, y, z) \in A$, then
$$|h(x, y, z)| < \delta_1(1 - \lambda).$$
Choose ϵ such that $\epsilon < \delta_1(1 - \lambda)$. Now consider x, x', z as in the statement. First we have $d(x, z) < \epsilon$ and hence
$$\begin{aligned} d(x', f(z)) &\leq d(x', f(x)) + d(f(x), f(z)) \\ &< \epsilon + \lambda d(x, z) \\ &< \delta_1(1 - \lambda) + \delta_1 \\ &\leq \delta_1 \\ &\leq \epsilon_X \end{aligned}$$
Also, we have that $(x', f(x), f(z))$ is in A and so we can conclude that
$$\begin{aligned} d(x', [f(x), f(z)]) &\leq h(x' f(x), f(z)) + d(f(x), f(z)) \\ &< \delta_1(1 - \lambda) + (x, z) \\ &< \delta_1(1 - \lambda) + \lambda\delta_1 \\ &= \delta_1 \end{aligned}$$

This completes the proof.

□

We say that a dynamical system (X, f) satisfies the *shadowing property* if every ϵ-pseudo orbit is δ-shadowed by an orbit. Now we can prove that every Smale space has the shadowing property.

Lemma 5.2. *Let (X, d, f) be a Smale space. For any $\delta > 0$, there is an $\epsilon > 0$ such that every ϵ-pseudo-orbit in X is δ-shadowed by an orbit of X.*

Proof. First choose $0 < \delta_1 \leq \epsilon_X/2$ such that

$$[W^u(x, \delta_1), W^s(x, \delta_1)] \subset X(x, \delta),$$

for all x in X. Next, we choose an $\epsilon > 0$ as in Lemma 5.1 which holds for both (X, d, f) and (X, d, f^{-1}).
We will first show the conclusion holds for finite intervals $I = (a, b) \cap \mathbb{Z}$, and a a negative integer, b a positive integer. For each $a < i < b - 1$, we define a map $g_i : W^s(x_i, \delta_1) \to W^s(x_{i+1}, \delta_1)$ by $g_i(z) = [x_{i+1}, f(z)]$. Theorem 3.1 proves that each g_i is well defined, by substituting $x = x_i$ and $x' = x_{i+1}$. In an analogous fashion, for $a + 1 < i < b$ we may define maps $h_i : W^u(x_i, \delta_1) \to W^u(x_{i-1}$ given by $h_i(z) = [f^{-1}(z), x_{i-1}]$. Applying Theorem 3.1 to the Smale space (X, d, f^{-1}) will show that each of these h_i are also well defined. For integers i, j such that $a < i < j < b$, let $g_{i,j} = g_{j-1} \circ g_{j-2} \circ \cdots \circ g_i$ denote the $j - i + 1$ fold composition of successive g maps, and similarly let $h_{j,i} = h_{i+1} \circ h_{i+2} \circ \cdots h_j$ be the $j - i + 1$ fold composition of successive h maps. Then we see that

$$g_{a+1,i} : W^s(x_{a+1}, \delta_1) \to W^s(x_i, \delta_1)$$
$$h_{b-1,i} : W^u(x_{b-1}, \delta_1) \to W^u(x_i, \delta_1)$$

For every $a < i < b$, We define sets

$$S_i = [h_{b-1,i}(W^u(x_{b-1}, \delta_1), g_{a+1,i}(W^s(x_{a+1}, \delta_1)]$$

Since δ_1 was chosen so that $[W^u(x, \delta_1), w^s(x, \delta_1)] \subset X(x, \delta)$ for all x in X, we have that $S_i \subset X(x_i, \delta)$ for all i. If we can show that $f(S_i) = S_{i+1}$, then any orbit of an element in S_0 will δ-shadow our ϵ-pseudo-orbit on this interval I.
Choose y in $W^s(x_{a+1}, \delta_1)$ and z in $W^u(x_{b-1}, \delta_1)$. Let $y' = g_{a+1,i}(y)$ and $z' = h_{b-1,i+1}(z)$ so that $[h_{i+1}(z'), y'] = [h_{b-1,i}(z), g_{a+1,i}(y)] \in S_i$ and $[z', g_i(y')] = [h_{b-1,i+1}(z), g_{a+1,i+1}(y)] \in S_{i+1}$. This covers every element of S_i and S_{i+1}. We will show that $f([h(z'), y']) = [z', g(y')]$, proving that $f(S_i) = S_{i+1}$.

$$\begin{aligned}
f([h(z'), y']) &= f([[f^{-1}(z'), x_i], y']) \\
&= [f([f^{-1}(z'), x_i]), f(y')], \text{ since } d(h(z'), y') < d(h(z'), x_i) + d(x_i, y') < \delta_1 + \delta_1 \leq \epsilon_X \\
&= [[z', f(x_i)], f(y')], \text{ since both } d(f^{-1}(z'), x_i), d(z', f(x_i)) \leq \epsilon_X \text{ by our choice of } \epsilon \\
&= [z', f(y')] \text{ by axiom B3 of Smale spaces} \\
&= [z', [x_{i+1}, f(y')]], \text{ as } f(y') \in W^s(x_{i+1}, \delta_1) \\
&= [z', g(y')]
\end{aligned}$$

as desired. Thus $f(S_i) = S_{i+1}$ for all relevant i and so any orbit of some $s \in S_0$ will δ-shadow our ϵ-pseudo-orbit (x_n) on I.
Now, suppose $I = \mathbb{Z}$. We begin by considering $I_n = (-n, n) \cap \mathbb{Z}$ for any positive integer n. Notice that the choice of δ_1 is independent of n. This also means that every point of

$$S^{(n)} = [h_{n-1,0}(W^u(x_{n-1}, \delta_1)), g_{-n+1,0}(W^s(x_{-n+1}, \delta_1))]$$

will δ-shadow the pseudo-orbit over the interval I_n. The same is true of the closure of $S^{(n)} = \overline{S^{(n)}}$. It follows directly from the definitions that $\overline{S^{(n)}} \supseteq \overline{S^{(n+1)}}$ for all n. By Cantor's intersection theorem, $\bigcap_{n \geq 1} \overline{S^{(n)}}$ is non-empty. Any point in this intersection will δ-shadow the pseudo-orbit over all of $\mathbb{Z}$. A similar argument will work for half open intervals $U = [a, \infty) \cap \mathbb{Z}$ and $L = (-\infty, b] \cap \mathbb{Z}$ for $a, b \in \mathbb{Z}$. If $U_n = [a, a + n]$ and $L_n = [b - n, b]$ for positive integer n, then if we define

$$S_U^{(n)} = [h_{a+n,a}(W^u(x_{a+n}, \delta_1)), W^s(x_a, \delta_1)]$$

$$S_L^{(n)} = [W^u(x_b, \delta_1), g_{b-n,b}(W^s(x_{b-n}, \delta_1))]$$

and work with the closures of these sets, it follows by definition of these sets that $\overline{S_U^{(n)}} \supseteq \overline{S_U^{(n+1)}}$ and $\overline{S_L^{(n)}} \supseteq \overline{S_L^{(n)}}$ for all n. By cantor's intersection theorem, both $\bigcap_{n\geq 1} \overline{S_U^{(n)}}$ and $\bigcap_{n\geq 1} \overline{S_L^{(n)}}$ are non-empty. Any point in these intersections will δ-shadow the pseudo-orbit over U or L respectively.

□

Corollary 5.2.1. *If $I = \mathbb{Z}$, and $\delta < \epsilon_1/2$, where ϵ_1 is the expansiveness constant of X, then there is an $\epsilon > 0$ such that every ϵ-pseudo-orbit in X is δ-shadowed by a unique orbit of X.*

Proof. By Lemma 5.2, every point in the intersection $S = \bigcap_{n\geq 1} \overline{S^{(n)}}$ will δ-shadow the pseudo-orbit over all of $\mathbb{Z}$. Suppose x, y are two distinct points in S. Then there exists some $m \in \mathbb{Z}$ such that $d(f^m(x), f^m(y)) \geq \epsilon_1 > 2\delta$. But $d(f^m(x), f^m(y)) \leq d(f^m(x), x_m) + d(x_m, f^m(y)) \leq 2\delta$. This is a contradiction. Thus S consists of a single point, making the shadowing orbit unique.

□

Before leaving the section, I would remark that the Shadowing Theorem can also help prove other powerful results about Smale spaces: for instance, one can show that in a non-wandering Smale space, the periodic points are dense. [Pu15, Theorem 4.4.1]

6 Rectangles and Markov Partitions

Bowen's theorem says that every Smale space has a Markov partition. Having a Markov partition gives you a nice semiconjugacy from a shift of finite type into the your Smale space. In this section, we prove results about rectangles and Markov partitions that we'll need for the proof of the main theorem.

Definition 6.1. Rectangle
A subset of R is called a rectangle if $\mathrm{diam}(R) \leq \epsilon_X$ and $[R, R] = R$.

Definition 6.2. Proper rectangle and more
R is called proper if R is closed and $R = \overline{\mathrm{int}(R)}$.
For $x \in R$, let

$$W^s(x, R) = \{y \in R \mid [y, x] = x\}$$

$$W^u(x, R) = \{y \in R \mid [x, y] = x\}$$

Lemma 6.1. *Let R be a closed rectangle. As a subset of X, R has boundary*

$$\partial R = \partial^s R \cup \partial^u R$$

where

$$\partial^s R = \{x \in R \mid W^u(x, \delta) \nsubseteq R \text{ for all } \delta > 0\}$$

$$\partial^u R = \{x \in R : W^s(x, \delta) \nsubseteq R \text{ for all } \delta > 0\}$$

Proof. Suppose $x \in R$.
If $x \notin \partial R$, then $x \in \mathrm{int}(R)$. By Corollary 3.4.1 there's some $\delta > 0$ such that $[W^u(x, \delta), W^s(x, \delta)] \subseteq R$. Recall that by Lemma 3.2, $y \in W^u(x, \delta)$ implies $[y, x] = y$, and so $W^u(x, \delta) = [W^u(x, \delta), x] \subseteq R$. Similarly, $y \in W^s(x, \delta)$ implies $[x, y] = y$, and so $W^s(x, \delta) = [x, W^s(x, \delta)] \subseteq R$. Thus $x \notin \partial^s R \cup \partial^u R$. Conversely if $x \notin \partial^s R \cup \partial^u R$, then you can find $\delta > 0$ such that both $W^u(x, \delta)$ and $W^s(x, \delta)$ are subsets of R, and since R is a rectangle, $[W^u(x, \delta), W^s(x, \delta)]$ is a subset of R as well. Hence x is in the interior of R. □

Lemma 6.2. *Suppose $x, y \in X$ with $d(x, y) < \epsilon_X$ and that $[x, y] = x$. then there exists $\alpha, \delta > 0$ such that the map $z \mapsto [y, z]$ restricts to a homeomorphism of $W^s(x, \alpha)$ onto a relatively open subset of $W^s(y, \delta)$ containing y.*

Proof. Let $\delta = \frac{1}{2}(\epsilon_X - d(x,y))$. Then using the triangle inequality, one gets that $[y,z]$ and $[x,w]$ are defined for all $z \in W^s(x,\delta)$ and all $w \in W^s(y,\delta)$. The map

$$f : W^s(x,\delta) \to X$$

$$f(z) = [y,z]$$

is continuous, and since $f(x) = y$ we can find $\alpha > 0$ such that $z \in W^s(x,\alpha)$ implies $f(z) = [y,z] \in B(y,\delta)$. Since $[[y,z],y] = [y,y]$ for all such z, the range of this map is contained in $W^s(y,\delta)$. Note also that f is injective, for if $[y,z_1] = [y,z_2]$ we have

$$z_1 = [x,z_1] = [x,[y,z_1]] = [x,[y,z_2]] = [x,z_2] = z_2.$$

Likewise, the map

$$g : W^s(y,\delta) \to X$$

$$g(w) = [x,w]$$

is continuous and injective. Note that $g \circ f$ is defined on $W^s(x,\alpha)$ and $g \circ f(z) = [x,[y,z]] = [x,z] = z$ for all $z \in W^s(x,\alpha)$. Since $W^s(x,\alpha)$ is an open set in the range of g, $g^{-1}(W^s(x,\alpha))$ is open in $W^s(y,\delta)$, and since f is injective this is $f(W^s(x,\alpha))$. Since g is continous and the inverse map of f on $W^s(x,\alpha)$, f is a homeomorphism there. ☐

Lemma 6.3. *Let R be a closed rectangle with* $\operatorname{diam} R < \epsilon_X$*, and let $x,y \in R$. Then*

1. *If $[x,y] = x$, then $x \in \partial^u R \iff y \in \partial^u R$*
2. *If $[x,y] = y$, then $x \in \partial^s R \iff y \in \partial^s R$*

Proof. We prove the first statement; the argument to prove the second is similar. Let $x,y \in R$. Since $\operatorname{diam} R < \epsilon_X$, we can apply Lemma 6.2 to find $\alpha,\delta > 0$ such that $z \mapsto [y,z]$ restricts to a homeomorphism of $W^s(x,\alpha)$ onto a relatively open subset of $W^s(y,\delta)$ containing y.
If we suppose that $x \notin \partial^u R$, then WLOG we can assume that $W^s(x,\alpha) \subseteq R$. Then $[y, W^s(x,\alpha)]$ contains a relatively open subset of $W^s(y,\delta)$, and since R is a rectangle, $[y, W^s(x,\alpha)] \subseteq R$. Every relatively open subset of $W^s(y,\delta)$ containing y contains a set of the form $W^s(y,\beta)$ for some $\beta > 0$, and so we have $W^s(y,\beta) \subseteq R$. Thus, $y \notin \partial^u R$.
This argument is symmetric, so $x \notin \partial^u R \iff y \notin \partial^u R$. This is equivalent to first statement. ☐

Lemma 6.4. *Let $\mathfrak{T} = \{T_1, \dots T_j\}$ be a finite covering of X by closed rectangles with diameters less than ϵ_X. Then for pairs (j,k) such that $T_j \cap T_k \neq \emptyset$, the sets*

$$C^u(j,k) = \{x \in T_j : W^u(x,\epsilon) \cap \partial^u T_k \neq \emptyset\}$$

$$C^s(j,k) = \{x \in T_j : W^u(x,\epsilon) \cap \partial^s T_k \neq \emptyset\}$$

are closed and have empty interior.

Proof. We first show closed. We claim that

$$C^u(j,k) = [T_j, \partial^u T_k \cap T_j]. \tag{1}$$

First note that the above set is well defined because we are taking the bracket of two sets which are contained in the rectangle T_j. First suppose $x \in C^u(j,k)$. Let $y \in T_j \cap T_k$ and $z \in W^u(x,\epsilon) \cap \partial^u T_k$. Then $[y,z] = [y,[z,x]] = [y,x]$ since $[x,z] = x \implies [z,x] = x$. Since $y,z \in T_k$, we must have $[y,z] = [y,x] \in T_k$. Let $w = [y,z] = [y,x]$ which must be in $T_j \cap T_k$. Since $[z,w] = [z,[y,z]] = [z,z] = z$ and $z \in \partial^u T_k$, Lemma 6.3 implies $w \in \partial^u T_k$. Then $w \in \partial^u T_k \cap T_j$, and so $x = [x,w] \in [T_j, \partial^u T_k \cap T_j]$. Thus $C^u(j,k) \subseteq [T_j, \partial^u T_k \cap T_j]$.

On the other hand, suppose $x = [a,b]$ with $a \in T_j$ and $b \in \partial^u T_k \cap T_j$. Then, since $a,b \in T_j$, so is $x = [a,b]$, as T_j is a rectangle. Then $[x,b] = [[a,b],b] = [a,b] = x$, so $b \in W^u(x,\epsilon) \cap \partial^u T_k$. Thus $x \in C^u(j,k)$. Equality is established.

Since the bracket and equation (1) shows that $C^u(j,k)$ is the image of a compact set under $[,]$, we have that $C^u(j,k)$ is closed. We also note that equation (1) implies that $C^u(j,k)$ is a subrectangle of T_j and that $\partial^u T_k \cap T_j \subseteq C^u(j,k)$.

We now show that $C^u(j,k)$ has empty interior. Suppose that $x \in \mathrm{int}(C^u(j,k))$, so that Lemma 6.1 implies there exists $\delta > 0$ such that $W^s(x,\delta) \subseteq C^u(j,k)$. Use (1) to write $x = [x,w]$ for $w \in \partial^u T_k \cap T_j$. Lemma 6.2 implies that $[w, W^s(x,\delta)]$ contains a relatively open subset of $W^s(x,\delta)$ containing w. We also have that $[w, W^s(x,\delta)] \subseteq C^u(j,k)$ by (1).
Find $0 < \alpha < \epsilon$ such that $W^s(x,\alpha) \subseteq C^u(j,k)$. For each $z \in W^s(x,\alpha)$ we can find $w_z \in \partial^u T_k \cap T_j$ such that $z = [z, w_z]$ and so $[w_z, z] = w_z$. Since T_k is a rectangle, we have

$$T_k \ni [w, w_z] = [w, [w_z, z]] = [w, z] = z \tag{2}$$

Hence $W^s(x,\alpha) \subseteq T_k$ which contradicts $w \in \partial^u T_k$. This shows that $C^u(j,k)$ has empty interior.
An almost identical argument shows that $C^s(j,k)$ is closed and has empty interior as well. □

Lemma 6.5. *Let (X,f) be a Smale space, let $\mathfrak{T} = \{T_1, \dots T_j\}$ be a covering of X by closed rectangles with diameters less than ϵ_X. For $x \in X$ write $\mathfrak{T}^*(x) = \{T_k \in \mathfrak{T} : T_j \cap T_k \neq \emptyset$. Then the set*

$$Z^* = \{x \in X : W^u(x,\epsilon) \cap \partial^u T_k = \emptyset \textit{ and } W^s(x,\epsilon) \cap \partial^s T_k = \emptyset \textit{ for all } T_k \in \mathfrak{T}^*(x)\}$$

is open and dense in X.

Proof. Referring to Lemma 6.4, one sees that

$$Z^* = X \setminus \Big(\bigcup_{T_j \cap T_k \neq \emptyset} C^u(j,k) \cup C^s(j,k) \Big)$$

so Z^* is the complement of a finite union of closed sets with empty interior. The Baire category theorem then implies that Z^* is open and dense. □

Definition 6.3. Markov partition
A Markov partition of X is a finite covering $\mathcal{R} = \{R_1, \dots, R_m\}$ by proper rectangles such that

1. $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$,
2. $f(W^u(x, R_i)) \supseteq W^u(f(x), R_j)$ and
   $f(W^s(x, R_i)) \subseteq W^s(f(x), R_j)$ when $x \in \mathrm{int}(R_i), f(x) \in \mathrm{int}(R_j)$.

This second condition effectively means that f maps points in the interiors of rectangles to the interiors of rectangles.

Examples of Markov partitions

Example 6.1. Hyperbolic toral automorphism
For the hyperbolic toral automorphism, we have A as given above in section 3. We have a Markov partition consisting of five rectangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$. Below is a nice diagram drawn to aid with visualization. In it, points are labelled, such that p_0 is the origin, $f(p_0) = p_0$. With respect to the origin, p_1, p_4 are in the direction of the contracting eigenvector v_1, and p_2, p_3 are in the direction of the expanding eigenvector v_2. $f(p_1) = p_4 = \lambda p_1$ and $f(p_2) = p_3 = \lambda^{-1} p_2$. The rectangles are such that $\Delta_1, \Delta_3, \Delta_4$ have the same width a and Δ_2, Δ_5 have the same smaller width b. It's a bit messy to explicitly parametrize these rectangles, but it can be done:
If $\alpha = \arctan(\sqrt{\lambda})$, the slope of v_2, then we can set up a system of equations:

$$\begin{aligned} -\sin\alpha &= -a - 2b \\ \cos\alpha - \sin\alpha &= a + b \\ \cos\alpha - 2\sin\alpha &= -b \\ 2\cos\alpha - 3\sin\alpha &= a \end{aligned}$$

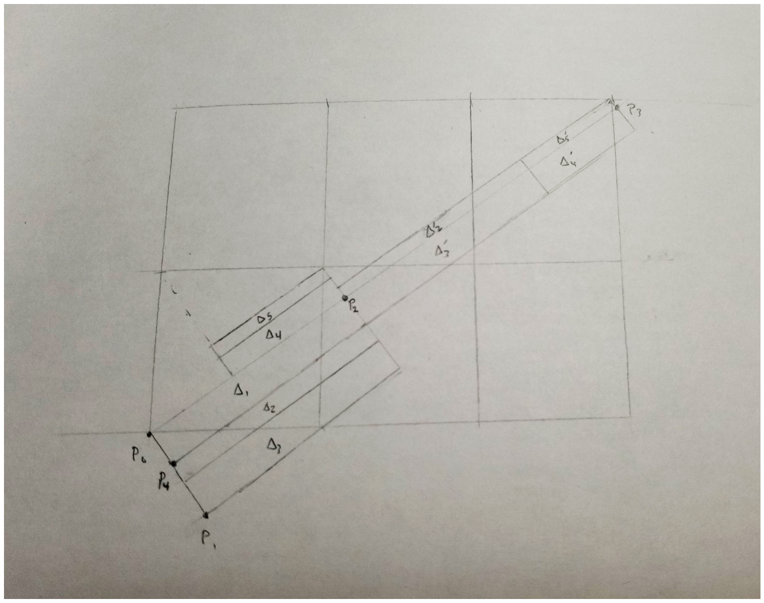


Figure 1: A hand-drawn sketch of the Markov partition for the hyperbolic toral automorphism

Where $\alpha = \arctan(\sqrt{\lambda})$, $a =$ width of Δ_2, Δ_5, and $b =$ width of $\Delta_1, \Delta_3, \Delta_4$.
With this we can solve for a and b, and it turns out that we get:

$$a = \frac{2 - 3\sqrt{\lambda}}{\sqrt{\lambda + 1}} \text{ and } b = \frac{1 - 2\sqrt{\lambda}}{\sqrt{\lambda + 1}}$$

It's quite relieving to see that the dimensions of these literal rectangles can be expressed as elementary functions of the eigenvalue λ. So yeah, the parametrization is doable, but yucky.
One can form an adjacency matrix for the intersections of these rectangles:

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Where $B_{ij} = 1$ if $f(\operatorname{int}(\Delta_i)) \cap \operatorname{int}(\Delta_j) \neq \emptyset$ and $B_{ij} = 0$ otherwise.
Let's verify this adjacency matrix.
Note that p_1 is a vector in the direction of the eigenvector v_1, and that $f(p_1) = p_4$. Also, p_2 is a vector in the direction of the eigenvector v_2, and that $f(p_2) = p_3$. Thus for all $x \in \Delta_1 \cup \Delta_2 \cup \Delta_3$, $x = c_1 p_1 + c_2 p_2$ and $f(x) = c_1 p_4 + c_2 p_3$ for $c_1, c_2 \in [0, 1]$. Thus $f(\Delta_1 \cup \Delta_2 \Delta_3) = \Delta_1 \cup \Delta_3' \cup \Delta_4'$. Since f is a homeomorphism, this also shows that $f((\Delta_1 \cup \Delta_2 \cup \Delta_3)^c) \cap (\Delta_1 \cup \Delta_3' \cup \Delta_4') = \emptyset$. It is easy to see that each of $f(\operatorname{int} \Delta_1), f(\operatorname{int} \Delta_2), f(\operatorname{int} \Delta_3)$ will intersect each of $\operatorname{int} \Delta_1, \operatorname{int} \Delta_3', \operatorname{int} \Delta_4'$. Since $\operatorname{int}(\Delta_4 \cup \Delta_5) \subseteq (\Delta_1 \cup \Delta_2 \cup \Delta_3)^c$, we get that the corresponding individual intersections are empty as well. This verifies the first three rows of the adjacency matrix. Similarly, taking complements will also help to verify the last two rows, using that f is a bijection.
The proof that $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}$ satisfies the second Markov partition properties is omitted.

Example 6.2. Edge shift

Let $E = (E^1, E^0, r, s)$ be a finite directed graph, and (Σ_E, σ) the corresponding edge shift.
For $e \in E^1$, let

$$C(e) := \{x \in \Sigma_E : x_0 = e\}.$$

Theorem 6.6. *Each $C(e)$ is a clopen (and hence regular) rectangle, and the set $\{C(e) : e \in E^1\}$ is a Markov partition for (Σ_E, σ).*

Proof. The proof that each $C(e)$ is clopen is standard. Since all elements of $C(e)$ agree at the origin, the diameter is at most $1/2$. It is clear that $C(e)$ is closed under the bracket, as $x_0 = [x, y]_0 = y_0 = e$. Thus each $C(e)$ is a clopen, regular rectangle. Each $C(e)$ is therefore equal to its interior, and if $e \neq f$ we have $C(e) \cap C(f) = \emptyset$. For the second part of the definition of a Markov partition, note that if $x \in C(e)$, we have

$$W^u(x, C(e)) = \{y \in C(e) : y_i = x_i \text{ for all } i \leq 0\}$$

$$W^s(x, C(e)) = \{y \in C(e) : y_i = x_i \text{ for all } i \geq 0\}$$

Now suppose that $x \in C(e)$ and $\sigma(x) \in C(f)$, so that $x_0 = e$ and $x_1 = f$. We need to show that

$$\sigma(W^u(x, C(e))) \supseteq W^u(\sigma(x), C(f)) \text{ and } \sigma(W^s(x, C(e))) \subseteq W^s(\sigma(x), C(f))$$

First suppose that $y \in W^u(\sigma(x), C(f))$. Then $y_i = \sigma(x)_i = x_{i+1}$ for all $i \leq 0$. Then $\sigma^{-1}(y_i) = y_{i-1} = x$ for all $i \leq 1$, which implies $y \in \sigma(W^u(x, C(e)))$.
The other inclusion is similar. ☐

7 Bowen's Theorem

The main result all of this has been working towards.

Theorem 7.1. *Bowen's Theorem*
Let (X, d, f) be a Smale space. Then X admits Markov partitions of arbitrarily small diameter. That is, for all $\epsilon > 0$, X admits a Markov partition $\mathcal{R}$ with diam$(R) \leq \epsilon$ for all $R \in \mathcal{R}$.

Proof. Let $\delta > 0$ be small enough so that Corollary 5.2.1 applies, and choose an $\epsilon > 0$ so that every ϵ-pseudo-orbit in X is δ-shadowed by a unique orbit of X. Choose γ in $(0, \epsilon/2)$ such that $d(f(x), f(y)) < \epsilon/2$ whenever $d(x, y) < \gamma$. Such a selection is possible by the continuity of f. Let $P = \{p_1, \ldots, p_r\}$ be a γ-dense subset of X. That is, $\bigcup_{i=1}^r X(p_i, \gamma) = X$.
Our plan is to first make a not-quite-Markov partition based on P, with diameters bounded by ϵ, then later refine it to form an actual Markov partition.

Let

$$\Sigma(P) = \{\underline{q} \in P^{\mathbb{Z}} \mid d(f(q_i), q_{i+1}) < \epsilon \text{ for all } i \in \mathbb{Z}\}$$

That is, $\Sigma(P)$ consists of all ϵ-pseudo-orbits of points in our γ-cover P. For each $\underline{q} \in \Sigma(P)$ Let $\theta(\underline{q}) \in X$ be the unique element whose orbit δ-shadows $\underline{q}$, whose existence and uniqueness is ensured by Corollary 5.2.1. This gives us a well-defined map $\theta : \Sigma(P) \to X$. Observe that $\Sigma(P)$ is a shift of finite type: $\mathcal{F} = \{(p_i, p_j) \mid d(p_i, p_j) \geq \epsilon\}$ is the set of banned words, which there are finitely many of by virtue of P being finite. The maximum length of words in $\mathcal{F}$ is 2. By Theorem 4.1, $(\Sigma(P), d_P, \sigma_{\mathcal{F}}, [,]_{\mathcal{F}})$ is a Smale space. We will prove desirable properties of the θ map and use them to form our initial not-quite-Markov partition.
If $\underline{q}, \underline{q}' \in \Sigma(P)$ such that $q_0 = q_0'$, let $\underline{q}^* = [\underline{q}, \underline{q}']$. Then $d(f^n(\theta(\underline{q}^*)), f^n(\theta(\underline{q}))) \leq 2\delta$ for all $n \geq 0$ and $d(f^n(\theta(\underline{q}^*)), f^n(\theta(\underline{q}'))) \leq 2\delta$ for all $n \leq 0$, by Lemma 3.5 and so $\theta(\underline{q}^*) \in W^s(\theta(\underline{q}), 2\delta) \cap W^u(\theta(\underline{q}'), 2\delta)$, i.e.,

$$\theta([\underline{q}, \underline{q}']) = [\theta(\underline{q}, \theta(\underline{q}')]$$

Claim: θ is continuous.
Suppose not. Then there is a $\gamma > 0$ such that for every $N \geq 0$ one can find $\underline{q}^N, \underline{p}^N \in \Sigma(P)$ (it's just more indexing) with $q_j^N = p_j^N$ for all $j \in [-N, N]$ but $d(\theta(\underline{q}^N), \theta(\underline{p}^N)) \geq \gamma$. Form sequences $(x_N), (y_N)$ in $\Sigma(P)$ by letting $x_N = \theta(\underline{q}^N)$ and $y_N = \theta(\underline{p}^N)$. Taking subsequences, we may assume (Compactness $\iff$ sequential compactness) $x_N \to x$ and $y_N \to y$. Then $d(f^j(x), f^j(y)) \leq 2\delta$ for all j , but $d(x, y) \geq \gamma > 0$. In particular, $x \neq y$. This contradicts expansiveness. Hence θ is continuous.

Claim: θ is surjective.
For each $x \in X$, choose q_j such that $f^j(x)$ is in $X(q_j, \gamma)$. Because of our choice of γ, for all $j \in \mathbb{Z}$ we have

that $d(f(q_j), q_{j+1}) \leq d(f(q_j), f(f^j(x))) + d(f^{j+1}(x), q_{j+1}) < 2\gamma < \epsilon$. Thus the sequence $\underline{q} = (q_n)$ is in $\Sigma(P)$, and $\theta(\underline{q}) = x$. Hence θ is surjective.

For all s such that $1 \leq s \leq r$, let $T_s = \{\theta(\underline{q}) \mid \underline{q} \in \Sigma(P), q_0 = p_s\}$. For $x, y \in T_s$, we write $x = \underline{q}, y = \underline{q}'$, with $q_0 = p_s = q_0'$. Then

$$[x, y] = \theta([\underline{q}, \underline{q}']) \in T_s$$

Each T_s is closed under the bracket, so they are rectangles. Furthermore, as the sets $\theta^{-1}(T_s)$ consist of pseudo orbits in P with the 0th index fixed; these sets must be closed. But closed implies compact, the image of a compact set under a continuous map is compact, and compact sets are closed. Thus $\theta(\theta^{-1}(T_s)) = T_s$ is closed.

Now suppose $x = \theta(\underline{q})$ with $q_0 = p_s$ and $q_1 = p_t$. Consider $y \in W^s(x, T_s), y = \theta(\underline{q}'), q_0' = p_s$. Then

$$y = [x, y] = \theta([\underline{q}, \underline{q}']) \text{ and}$$

$$f(y) = \theta(\sigma([\underline{q}, \underline{q}'])) \in T_t$$

as $\sigma([\underline{q}, \underline{q}'])$ has $q_1 = p_t$ in its zeroth position. It's worth mentioning at this point that θ is a continuous surjection satisfying $f \circ \theta = \theta \circ \sigma$. Thus θ is actually a semiconjugacy from Σ_P to X.
Since $f(y) \in W^s(f(x), \epsilon)$ and $f(y) \in T_t$, we have that $f(y) \in W^s(f(x), T_t)$. We have proved:
(i) $f(W^s(x, T_s)) \subset W^s(f(x), T_t)$.
Using a similar argument but replacing f with f^{-1} will show that $f^{-1}(W^u(f(x), T_t)) \subset W^u(x, T_s)$, i.e.
(ii) $f(W^u(x, T_s)) \supset W^u(f(x), T_t)$.
So $\mathcal{T} = \{T_1, \ldots, T_r\}$ is a covering by closed rectangles, and (i) and (ii) are like Markov partition property (b), but these rectangles are likely not proper, and likely intersect. A refinement is needed. For each $x \in X$, let

$$\mathcal{T}(x) = \{T_j \in \mathcal{T} \mid x \in T_j\} \text{ and } \mathcal{T}^*(x) = \{T_k \in \mathcal{T} \mid T_k \cap T_j \neq \emptyset \text{ for some } T_j \in \mathcal{T}(x)\}$$

Where $\mathcal{T}(x)$ is the set of rectangles in $\mathcal{T}$ containing the point x, and $\mathcal{T}^*(x)$ is the set of rectangles in $\mathcal{T}$ that don't necessarily contain x, but intersect a rectangle containing x. Let

$$Z^* = \{x \in X \mid W^s(x, \epsilon) \cap \partial^s T_k = \emptyset \text{ and } W^u(x, \epsilon) \cap \partial^u T_k = \emptyset \text{ for all } T_k \in \mathcal{T}^*(x)\}.$$

By Lemma 6.5, Z^* here is open and dense.
For $T_j \cap T_k \neq \emptyset$, let

$$\begin{aligned}
T_{j,k}^1 &= \{x \in T_j \mid W^u(x, T_j) \cap T_k \neq \emptyset, W^s(x, T_j) \cap T_k \neq \emptyset\} = T_j \cap T_k \\
T_{j,k}^2 &= \{x \in T_j \mid W^u(x, T_j) \cap T_k \neq \emptyset, W^s(x, T_j) \cap T_k = \emptyset\} \\
T_{j,k}^3 &= \{x \in T_j \mid W^u(x, T_j) \cap T_k = \emptyset, W^s(x, T_j) \cap T_k \neq \emptyset\} \\
T_{j,k}^4 &= \{x \in T_j \mid W^u(x, T_j) \cap T_k = \emptyset, W^s(x, T_j) \cap T_k = \emptyset\}
\end{aligned}$$

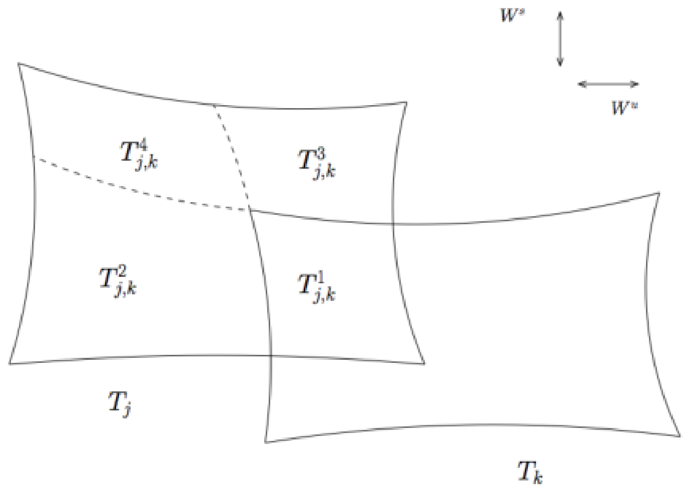
This is a partition of each T_j into 4 "quadrants" so-to-speak, of its intersections with T_k.

To form Z^*, we need to remove the boundaries of the T_j's, but also we need to remove the dotted lines of the edges of the $T_{j,k}^n$'s as well. This leaves just the interiors of the $T_{j,k}^n$'s. Recall by Theorem 3.6, x, y close enough implies $\{[x, y]\} = W^s(x, \epsilon) \cap W^u(y, \epsilon)$. Thus if $x, y \in T_j$, then $W^s([x, y], T_j) = W^s(x, T_j)$ and $W^u([x, y], T_j) = W^u(y, T_j)$ since the diameter of T_j is small enough compared to ϵ; This implies that $T_{j,k}^n$ is a rectangle whose interior is an open rectangle in X, and each $x \in T_j \cap Z^*$ lies in $\text{int}(T_{j,k}^n)$ for some n (because Z^* removes the boundaries). For $x \in Z^*$ define

$$R(x) = \bigcap \{\text{int}(T_{j,k}^n) \mid x \in T_j, T_j \cap T_k \neq \emptyset, \text{ and } x \in T_{j,k}^n\}$$

The intersection of a collection of rectangles is a rectangle, so each $R(x)$ is an open rectangle. Suppose $y \in R(x) \cap Z^*$. Since $R(x) \subset T_j(x)$ for all $T_j \in \mathcal{T}(x)$ and $R(x) \cap T_j = \emptyset$ for $T_j \notin \mathcal{T}(x)$, we have that y is in precisely the same rectangles that x is in; that is, $\mathcal{T}(y) = \mathcal{T}(x)$. Now for $T_j \in \mathcal{T}(x) = \mathcal{T}(y)$ and $T_j \cap T_k \neq \emptyset$,

Figure 2: This figure is taken from [Bow08] on page 56



y lies in the same $T^n_{j,k}$ as x does, as $T^n_{j,k} \supset R(x) \ni y$; hence $R(y) = R(x)$ (Since the R sets do not include boundary). As there are only finitely many $T^n_{j,k}$'s, there are only finitely many $R(x)$'s. At this point, the set R= $\{R(x) \mid x \in Z^*\}$ is a partition of Z^* into open rectangles. Let

$$\mathcal{R} = \{\overline{R(x)} : x \in Z^*\} = \{R_1, \dots, R_m\}$$

For $x' \in Z^*$, either $R(x') = R(x)$ or $R(x') \cap R(x) = \emptyset$; hence $(\overline{R(x)} \setminus R(x)) \cap Z^* = \emptyset$. Since Z^* is dense, $\overline{R(x)} \setminus R(x)$ has no interior.

Claim: $R(x) = \text{int}(\overline{R(x)})$

We always have the $\subseteq$ inclusion. So suppose $y \in \text{int}(\overline{R(x)})$, so that we can find an open set V with $y \in V \subseteq \overline{R(x)}$.

Going for a contradiction, suppose that $y \in \overline{R(x)} \setminus R(x)$. Then $y \notin Z^*$. From our observation in Lemma 6.5, we see that $y \in C^u(j,k)$ or $y \in C^s(j,k)$ for some j, k with $T_j \cap T_k \neq \emptyset$. We consider the case where $y \in C^u(j,k)$. By definition of $C^u(j,k)$, we must have that y is in either $T^1_{j,k}$ or $T^2_{j,k}$.

Find $\delta > 0$ such that $W^s(y, \delta) \subseteq V$. Then the same argument leading to 2 and subsequent contradiction show that $W^s(y, \delta)$ must contain points z such that $W^u(z, T_j) \cap T_k = \emptyset$. Hence $W^s(y, \delta)$ contains points in both $T^1_{j,k}$ and $T^3_{j,k}$ or it contains points in both $T^2_{j,k}$ and $T^4_{j,k}$.

In either case, we must have that V intersects $R(x')$ for some $R(x') \neq R(x)$. Since V is contained in $\overline{R(x)}$, we have $\overline{R(x)} \cap R(x') \neq \emptyset$. Since $R(x')$ is an open set which intersects $\overline{R(x)}$, it must intersect $R(x)$, which contradicts $R(x') \neq R(x)$. This contradiction implies $y \in R(x)$, giving us that $R(x) = \text{int}(\overline{R(x)})$.

Thus $\mathcal{R}$ is a covering by proper rectangles satisfying property (a) of Markov partitions. The rest of this proof is for verifying that it actually does satisfy property (b), and that $\mathcal{R}$ is our desired Markov partition.

Suppose $x, y \in Z^* \cap f^{-1}(Z^*)$, $R(x) = R(y)$ and $[y, x] = x$. We will show that $R(f(x)) = R(f(y))$. Suppose $f(x) \in T_j$ and $f(y) \notin T_j$. Write $f(x) = \theta(\sigma(\underline{q}))$ with $q_1 = p_j$ and $q_0 = p_s$. Then $x = \theta(\underline{q}) \in T_s$ and by inclusion (i) from before in this proof we have that

$$f(y) \in f(W^s(x, T_s)) \subset W^s(f(x), T_j) \subset T_j$$

contradicting our assumption. Thus $f(x) \in T_j \implies f(y) \in T_j$. If we assume instead that $f(y) \in T_j$ and $f(x) \notin T_j$, then we get the corresponding contradiction, proving $f(y) \in T_j \implies f(x) \in T_j$. Thus $\mathcal{T}(x) = \mathcal{T}(y)$. Now let $f(x), f(y) \in T_j$ and $T_j \cap T_k \neq \emptyset$. We want to show that $f(x), f(y)$ belong to the same $T^n_{j,k}$. First, note that $[y, x] = x \implies [f(y), f(x)] = f(x)$. Thus $W^s(f(x), T_j) = W^s(f(y), T_j)$. Thus $f(x)$ and $f(y)$ are both in $T^2_{j,k} \cup T^4_{j,k}$ or both in $T^1_{j,k} \cup T^3_{j,k}$. We show further that they are in the same $T^n_{j,k}$. Suppose that (for contradiction)

$$W^u(f(y), T_j) \cap T_k = \emptyset, \text{ and } f(z) \in W^u(f(x), T_j) \cap T_k.$$

Recall that $f(x) = \theta(\sigma(\underline{q}))$ and $q_1 = p_j$ and $q_0 = p_s$. Then by inclusion (ii),

$$f(z) \in W^u(f(x), T_j) \subset f(W^u(x, T_s)) \text{ or } z \in W^u(x, T_s)$$

Let $f(z) = \theta(\sigma(\underline{q}'))$; $q'_1 = p_k$ and $q_0 = p_t$. Then $z \in T_t$ and $f(W^s(z, T_t)) \subset W^s(f(z), T_k)$. Now $T_s \in \mathcal{T}(x) = \mathcal{T}(y)$ and $z \in T_t \cap T_s \neq \emptyset$.
Now $z \in W^u(x, T_s) \cap T_t$ and so there is some $z' \in W^u(y, T_s) \cap T_t$ as x, y are in the same $T^n_{s,t}$. Then

$$z'' = [z, y] = [z, z'] \in W^s(z, T_t) \cap W^u(y, T_s),$$

and $f(z'') = [f(z), f(y)] \in W^s(f(z), T_k) \cap W^u(f(y), T_j)$ (using $f(z), f(y) \in T_j$ a rectangle), a contradiction. So $R(f(x)) = R(f(y))$.
For small $\delta > 0$ the sets

$$Y_1 = \bigcup \{W^s(z, \delta) : z \in \bigcup_j \partial^s R_j\} \text{ and } Y_2 = \bigcup \{W^u(z, \delta) : z \in \bigcup_j \partial^u R_j\}$$

are closed and nowhere dense, thus $Z^* \supset X \setminus (Y_1 \cup Y_2)$ is open and dense by Baire category theorem. Furthermore if $x \notin (Y_1 \cup Y_2) \cup f^{-1}(Y_1 \cup Y_2)$ then $x \in Z^* \cap f^{-1}(Z^*)$ and the set $y \in W^s(x, R(x))$ with $y \in Z^* \cap f^{-1}(Z^*)$ is open and dense in $W^s(x, \overline{R(x)})$ (as a subset of $W^s(x, \epsilon)$). By the previous paragraph $R(f(y)) = R(f(x))$ for such y; by continuity

$$f(W^s(x, \overline{R(x)})) \subset \overline{R(f(x))}.$$

As $f(W^s(x, \overline{R(x)})) \subset W^s(f(x), \epsilon)$, $f(W^s(x, \overline{R(x)})) \subset W^s(f(x), \overline{R(f(x))})$.
If $\mathrm{int}(R_i) \cap f^{-1}(\mathrm{int}(R_j) \neq \emptyset$, then this open subset contains some x satisfying the above conditions, $R_i = \overline{R(x)}$ and $R_j = \overline{R(f(x))}$. For any $x' \in R_i \cap f^{-1}(R_j)$ one has $W^s(x', R_i) = \{[x', y] : y \in W^s(x, R_i)\}$ and

$$\begin{aligned} f(W^s(x', R_i)) &= \{[f(x'), f(y)] : y \in W^s(x, R_i)\} \\ &\subset \{[f(x'), z] : z \in W^s(f(x), R_j)\} \\ &\subset W^s(f(x'), R_j). \end{aligned}$$

This completes the proof of half of the Markov conditions (b). The other half is proved similarly by applying the above to f^{-1}, noting that $W^u_f = W^s_{f^{-1}}$.

$\square$

One can find a shift of finite type Σ and a semiconjugacy from Σ to X which is one-to-one on an open dense set and finite-to-one elsewhere. [Bow08, Theorem 3.18]

References

[Pu15] Ian Putnam. *Lecture Notes on Smale Spaces* University of Victoria, 2015.

[Bow08] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.

[BS02] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.