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Abstract

Much work has been done in the field of quantum computing in recent years, however, with the increase of the complexity of quantum algorithms comes the need for a larger body of quantum logic gates. Here we summarize the Weyl representation of the Heisenberg group with the associated phase space formalism and introduce the Clifford Hierarchy, a recursively defined hierarchy of operators that can be applied fault tolerantly in quantum computations. Additionally, a simple example of the process of quantum gate teleportation is shown for the implementation of certain higher level gates with fewer necessary resources.

1 Introduction

At the advent of quantum mechanics in the early 20^{th} century, there were two equally appealing yet fundamentally different interpretations. One coming from Schrödinger's wave mechanics picture of quantum mechanics in which the system describing a quantum mechanical object, such as an electron, would vary in time and space, where as through the Heisenberg matrix picture, it was instead the measurements and operators acting on the system that carried this time dependence with the state remaining fixed. While these two interpretations lead to very different views of how the physics at such small scales are carried out, they were later shown to be equivalent through the Stone-von Neumann theorem, which asserts that any representation of a quantum system satisfying the Weyl commutation relations are unitarily equivalent.

In what follows, we summarize the Weyl representation and use it to present the aforementioned Stone-von Neumann theorem, which will establish the equivalence of any conjugate pair of unitary operators (Definition 7) on finite abelian groups. After demonstrating how these operators can be viewed as quantum logic gates, we will naturally be led to the Clifford hierarchy which provides a largely expanded set of computational resources for operations on quantum systems. Additionally, a simplified implementation of operators of the third level of the Clifford hierarchy through quantum gate teleportation, proposed by de Silva [1], is presented, which can be expanded to higher levels of the hierarchy.

We assume that the reader is familiar with concepts in group theory up to MATH3601, as well as some basic linear algebra. An elementary understanding of quantum mechanics is helpful but not necessary as all related topics will be explained when needed.

2 Background

We will begin our discussion of the Weyl representation and the associated phase space formalism by first introducing the basic postulates of quantum mechanics. This will act as the foundation for all that is to follow and will be necessary to understand the many applications to quantum systems.

Postulate 1. *Any isolated quantum system is represented by a Hilbert space H. The state of the system is represented by a unit vector $\psi \in H$, called the state vector.*

Here the normalization gives that $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$ for any state vector, where $\langle\cdot|\cdot\rangle : H \times H \to \mathbb{C}$ is the sesquilinear product on H.

Of particular importance in quantum computing is the state space describing, for example, the spin of an electron. By this we do not mean the angular moment analogous to a ball or spinning top rotating about a given axis. Rather, to each electron there is an intrinsic angular momentum that can take one of two values; $\frac{1}{2}$, or $-\frac{1}{2}$ (when setting $\hbar = 1$) with respect to the given axis of measurement. It is more common to refer to the spin $\frac{1}{2}$ state as the 'spin-up' state and the spin $-\frac{1}{2}$ state as the 'spin-down' state, each of which are denoted by $|\uparrow\rangle$, and $|\downarrow\rangle$ respectively.

For an arbitrary spin system of an electron, the spin state need not be only spin-up or spin-down. In reality, an electron can be in a mixture of the spin-up and spin-down state. For this reason, we can view the $|\uparrow\rangle$ and $|\downarrow\rangle$ as orthogonal basis vectors in the two-dimensional Hilbert space representing the electron's spin. So the state of an electron can then be described by a unitary vector $|\psi\rangle$ such that $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

Here we are using what is know as the Bra-Ket notation in which we denote an arbitrary vector x in a Hilbert space H by $|x\rangle$. This $|x\rangle$ is known as the ket of x, where as the corresponding bra is denoted by $\langle x| = (|x\rangle)^* = \overline{(|\mathrm{x}\rangle)}^T$, hence the conjugate transpose of $|x\rangle$. This notation is particularly useful, with respect to a given basis which will be clear from context, since we can represent the inner product of two arbitrary vectors $|\psi\rangle$ and $|\phi\rangle$ in H as $\langle\phi|\psi\rangle = (\langle\phi|)(|\psi\rangle) = \langle\psi|\phi\rangle^*$.

Returning to the spin state of an electron, with this notation and the requirement that the state vector be unitary, we find that $1 = \langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2$. These values of $|\alpha|^2$ and $|\beta|^2$ actually have a physical meaning to the spin system beyond simply describing the state vector. While the spin state of an electron may be represented as a superposition of the spin-up and spin-down state, when a measurement of the electron's spin is performed, the result of the measurement can only be either spin-up or spin-down, not both. In this regard, the quantity $|\alpha|^2$ represents the probability that a measurement of spin will return a spin-up value and the quantity $|\beta|^2$ represents an analogous quantity for the probability of measuring a spin-down value.

When it comes to using these quantum systems to perform calculations, there is a natural connection that can be made between the spin states above and the bits of a classical computer. In a classical logic gate, the constituent bits can only take on one of two states; 0 or 1. So identifying the spin-up state with 0 and the spin-down state as 1, we can then envision our electron as a quantum bit, or *qubit*, with state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. The important distinction here being that in general a qubit is a linear combination of the 0 and 1 state, where as the classical bit can only take on one of the two values.

It should be noted here that while all of the previous discussion has been related to the spin state of a given electron, we needn't restrict ourselves to this. In fact any quantum mechanical system with a two-dimensional Hilbert space (the polarization of a photon for another example) can be represented in this way. To bring this further, there exists quantum systems such as the spin state of a spin-$\frac{3}{2}$ baryon in which the particle can be observed to have spin values of $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, or $\frac{3}{2}$. In this case, the spin state system of the spin-$\frac{3}{2}$ baryon would be described by a four-dimensional Hilbert space, in which the state of the system would be given by a linear combination of four basis states corresponding to each possible value of the particle spin (this kind of system is not practical for computation due to the incredibly short half-life of these baryons which are on the order of 10^{-20} seconds, but they do demonstrate the potential to move past the 2 state system). A more practical example of a quantum system, available for computation, with greater than two dimensions comes from [7] in which the orbital angular momentum of a photon was used to generate a 7 dimensional system. For this reason, we will instead focus on the abstract *qudit* (analogue of a qubit with an arbitrary finite number of basis states) object rather than the physical states of any individual quantum system.

Postulate 2. *The evolution of a quantum system is given by* unitary transformations *acting on the Hilbert space of the system. Hence, a state $|\psi\rangle$ at a time t_1, is related to the state at a time t_2, $|\psi'\rangle$ through a unitary transformation $U : H \to H$ satisfying:*

$$|\psi'\rangle = U|\psi\rangle$$

Note here that the requirement for the transformation to be unitary ensures that U maps state vectors to state vectors, since:

$$\langle\psi'|\psi'\rangle = \langle U\psi|U\psi\rangle = \langle\psi|U^*U|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Of particular importance for quantum information is the analogue of the classical logic gate, a *quantum gate.* These gates, similar to the classical counterpart take in a certain quantum system and perform an operation on that system resulting in a new state. Since we can represent the state space of a quantum system as a Hilbert space, then from the above postulate we can see that any quantum logic gate can be represented as a unitary operator on the Hilbert space describing the system. A quantum circuit is then constructed by a series of unitary transformations acting on an input state. However, this picture does not provide us with the full richness that quantum computing makes available to us. Other important ingredients comes as consequences of the next two postulates.

Postulate 3. *A measurement on a quantum system is described by a set of projection operators* $\{M_i\}_{i=1}^m \subseteq L(H)$*, the set of linear operators from* H *to itself, with the property that* $\sum_{i=1}^m M_i^* M_i = I_H$ *and* $M_i M_j = M_j M_i$ *for all* i *and* j*. Here* i *referees to the possible outcomes of the measurement, and for a given initial state* $|\psi\rangle$*, the probability that an observation will result in* i *is:*

$$Prob(i, \psi) = \langle\psi|M_i^* M_i|\psi\rangle = \|M_i \psi\|^2$$

If a measurement is performed, and outcome i *is observed then the state of the system after the measurement is given by:*

$$|\psi'\rangle = \frac{M_i|\psi\rangle}{\|M_i \psi\|}$$

This last point is of particular importance as it demonstrates that the simple act of observing a quantum state results in the altering of the state. Additionally, the results of a measurement of a quantum system is purely statistical in nature. Hence, two identical systems may produce different results after an observation, leading to different subsequent states.

As a simple example, take for instance a spin state of an electron given by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. If we wanted to measure the spin of the electron, then our measurement system would be given by $\{M_0, M_1\}$ in which $M_i|j\rangle = \delta_{i,j}|j\rangle$ for $i = 0, 1$. So performing a measurement of the spin of the electron would return 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$. If 0 was observed, then the spin state after observation is then $\frac{M_0|\psi\rangle}{\|M_0|\psi\rangle\|} = \frac{\alpha}{|\alpha|}|0\rangle$. Similarly, if 1 was observed, then the resulting spin state would be $\frac{\beta}{|\beta|}|1\rangle$. What is important to note, is that in both cases, the spin state that results from the measurement is no longer a mixture of the $|0\rangle$ and the $|1\rangle$ state. Hence, any subsequent measurement of the spin would reveal the same value as the previous measurement with complete certainty.

Additionally, since the probability of a measurement giving an outcome i is $\|M_i \psi\|^2$, then multiplying a state by a complex phase, a complex number of unit length, does not have any effect on the probability of observing a particular outcome of a measurement.

Postulate 4. *For* n *quantum systems described by the Hilbert spaces* $H_1, H_2, \ldots, H_n$ *respectively, the Hilbert space describing the composite system is given by the tensor products of the* H_i*. In other words* $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$

The two previous postulates give rise to an extensively used phenomenon in quantum computing known as entanglement. This is the idea that the outcomes of the observation on one part of a composite system can influence the outcomes of a measurements on the other parts of the system. To demonstrate this, consider the composite quantum system given by the spin states of two electrons described by $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$. Now if one were to perform an observation of the spin of the first electron in the system, there would be an equal chance that the result would be a 0 or a 1. Say for example that the value 0 is measured, then by Postulate 3, the composite spin system would collapse to $|\psi'\rangle = |0\rangle \otimes |0\rangle$, meaning that any subsequent measurement of the second electron's spin would return a value of 0 with complete certainty. On the other hand, if one were to start by observing the spin of the second electron, then by similar logic, the result of a subsequent measurement of the first qubit would be completely determined. Because of this, one can influence the probabilistic nature of the either one of the spin states by first observing the other electron's spin.

It should be noted that not all composite quantum systems show this entanglement property. In fact, for a Hilbert space $H = H_1 \otimes H_2$, then any state described by $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where $|\psi_1\rangle \in H_1$ and $|\psi_2\rangle \in H_2$, any measurement performed on either parts with have no influence on the measurements of the other. In this case, we would say that the two constituent parts of the composite system are *unentangled*, or that the original state $|\psi\rangle$ is *separable*.

By strategically using entanglement, one opens up many addition possibilities for computation that are not possible in a classical system, such as the ability to teleport the state of a system onto an entangled system even when separated by an arbitrary distance [8]. This particular application will be discussed further when a modification of the standard teleportation procedure is introduced in a later section.

3 The Heisenberg Group

Now that we have the framework set out, we will now begin to discuss the Heisenberg group and the associated Weyl representation. For the remainder of this section, we will let G represent an arbitrary finite abelian group. In fact many of the following statements can be generalized for G being any locally compact abelian group as shown by Weil [2], however this approach is not necessary when discussing our particular applications of interest.

Definition 1. *For a finite abelian group G, The Pontryagin dual of G is defined as*

$$\widehat{G} = \{\gamma : G \to \mathbb{T} \mid \gamma \text{ is a homomorphisim}\}$$

Where $\mathbb{T}$ denotes the group of complex numbers with modulus one.

Remark 1. *$\widehat{G}$ with the operation $\circ : \widehat{G} \times \widehat{G} \to \widehat{G}$ defined by $(\gamma \circ \gamma')(x) = \gamma(x)\gamma'(x)$ for all $x \in G$, forms a group.*

Definition 2. *The Heisenberg group on G is defined as $H(G) := G \times \widehat{G} \times \mathbb{T}$ with group operation given by the law:*

$$(x, \gamma, \lambda) \circ (x', \gamma', \lambda') = (xx', \gamma \circ \gamma', \sigma(x, \gamma, x', \gamma')\lambda\lambda')$$

Where the mapping $\sigma : G \times \widehat{G} \times G \times \widehat{G} \to \mathbb{T}$ is given by:

$$\sigma(x, \gamma, x', \gamma') = \gamma'(x)$$

This notation for the Heisenberg group may seem bulky at first, however this provides a clean way of denoting the elements when we consider automorphisims of this group. Our representation of the elements in the Heisenberg group can simplified quite nicely as a consequence of the following results.

Lemma 3.1. *For any cyclic group $G = \mathbb{Z}_d$ for some $d \in \mathbb{N}$, G is isomorphic to its Pontryagin dual, $G \cong \widehat{G}$.*

Proof. First we note that for any $\gamma \in \widehat{G}$,

$$1 = \gamma(0) = \gamma(d) = (\gamma(0))^d$$

since γ is a homomorphism. So $\gamma(0) = z$ for some $z \in \mathbb{T}$ and

$$z^d = 1 \iff z \in \{e^{\frac{2\pi ik}{d}} \mid k \in \mathbb{Z}_d\}$$

Letting $z = e^{\frac{2\pi ik}{d}}$ for some $k \in \mathbb{Z}_d$, it follows that for any $x \in \mathbb{Z}_d$,

$$\gamma(x) = (\gamma(0))^x = z^x = e^{\frac{2\pi ikx}{d}}$$

Let $\phi : G \to \widehat{G}$ be the mapping that such that $\phi(k) = \gamma_k$ where $\gamma_k(x) = e^{\frac{2\pi ikx}{d}}$. Then clearly ϕ is a bijection and

$$[\phi(n+m)](x) = e^{\frac{2\pi i(n+m)x}{d}} = e^{\frac{2\pi inx}{d}} e^{\frac{2\pi imx}{d}} = [\phi(n)\phi(m)](x)$$

Therefore ϕ is an isomorphism from G onto $\widehat{G}$. □

Theorem 3.2. *For any finite abelian group G, $G \cong \widehat{G}$.*

Proof. By the fundamental theorem of finite abelian groups, G is isomorphic to a direct product of cyclic subgroups of prime power order. In other words $G \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}$ for some $n \in \mathbb{N}$ and $d_i = p_i^{n_i}$ where p_i is a prime and $n_i \in \mathbb{N}$ for all $i \in \{1, \dots, n\}$. So fixing an isomorphism $\Phi : G \to \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}$, for any $g \in G$ we can represent g as $\Phi(g) = (a_1, \dots, a_n)$ where $a_i \in \mathbb{Z}_{d_i}$ for all i. Now define $G_i = \{g \in G \mid \Phi(g) = (0, \dots, 0, a_i, 0, \dots, 0)\}$ and $\phi_i : G \to \mathbb{Z}_{d_i}$ by $\phi_i(g) = a_i$ when $\Phi(g) = (a_1, \dots, a_n)$. Then $G_i \cong \mathbb{Z}_{d_i}$ and ϕ_i is the projection of the image of Φ onto $\mathbb{Z}_{d_i}$. Let $\gamma \in \widehat{G}$, then $\gamma|_{G_i}$ is a homomorphism from a cyclic group G_i to $\mathbb{T}$, so by the previous lemma, $\gamma|_{G_i} = \gamma^i_{k_i} \circ \phi_i$ for $k_i \in \mathbb{Z}_{d_i}$

where $\gamma^i_{k_i}(x) = e^{\frac{2\pi i k_i x}{d_i}}$ for all $x \in \mathbb{Z}_{d_i}$. Writing $g = g_1 \cdots g_n$ where $g_i \in G_i$ for all i, we then have that

$$\gamma(g) = \prod_{i=1}^{n} \gamma(g_i) = \prod_{i=1}^{n} \gamma|_{G_i}(g_i) = \prod_{i=1}^{n} (\gamma^i_{k_i} \circ \phi_i)(g_i)$$

Hence $\widehat{G} \cong \widehat{\mathbb{Z}}_{d_1} \oplus \cdots \oplus \widehat{\mathbb{Z}}_{d_n} \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n} \cong G$ □

Now with Theorem 3.2, for any finite abelian group G, by fixing an isomorphism from $\Phi : G \to \widehat{G}$, we can index the elements of $\widehat{G}$ by the elements of G such that $\gamma_g = \Phi(g)$. So letting $Z = G \times G$, the elements of $H(G)$ are denoted as (z, λ) for some $z = (z_1, z_2) \in Z$ (here $z_1, z_2 \in G$) and $\lambda \in \mathbb{T}$ with composition law given by

$$(z, \lambda) \circ (z', \lambda') = (zz', \sigma(z, z')\lambda\lambda')$$

where now σ is a mapping from $G \times G \times G \times G$ to $\mathbb{T}$ given by $\sigma(z, z') = \sigma((z_1, z_2), (z'_1, z'_2)) = \gamma_{z'_2}(z_1)$. For consistency, we will continue to let z_1 and z_2 represent the first and second component of the element $z \in Z$.

As a result of this new notation for elements in the Heisenberg group, we can think of each element as a point in the 'phase space' Z together with an additional complex phase λ. With this we now have the tools necessary to introduce the symplectic form acting on the phase space Z. This will be of great importance in the discussions to come, especially with the introduction of the Weyl representation.

Definition 3. *We denote by $\Delta : Z \times Z \to \mathbb{T}$ the symplectic form given by:*

$$\Delta(z, z') = \sigma(z, z')\sigma(z', z)^{-1}$$

For all $z, z' \in Z = G \times G$.

Remark 2. *By the definition of σ and the group operation on G and $\widehat{G}$, we can see that $\sigma(zz', z'') = \sigma(z, z'')\sigma(z', z'')$ and $\sigma(z, z'z'') = \sigma(z, z')\sigma(z, z'')$. From this it follows that $\Delta(zz', z'') = \Delta(z, z'')\Delta(z', z'')$ and $\Delta(z, z'z'') = \Delta(z, z')\Delta(z, z'')$, meaning that Δ is a bicharacter. Additionally, $\Delta(z^{-1}, z') = \Delta(z, z')^{-1} = \Delta(z, z'^{-1})$*

Remark 3. *Suppose that for some $z \in Z$, that $\Delta(z, w) = 1$ for all $w \in Z$. Then,*

$$\begin{aligned} & 1 = \Delta(z, w) \\ \Rightarrow \quad & 1 = \sigma(z, w)\sigma(w, z)^{-1} \\ \Rightarrow \quad & \sigma(z, w) = \sigma(w, z) \\ \Rightarrow \quad & \gamma_{w_2}(z_1) = \gamma_{z_2}(w_1) \end{aligned}$$

Since this must be true for all $w \in Z$, then we can fix w_2 and vary w_1 through every value of G and the equality must still hold. In this case the left side of the *equation is constant, which means the right side of the equation must also be constant. This only happens if z_2 is the identity element in G which would force both sides to be equal to 1. Then by varying w_2, the left hand side of the equation must remain equal to 1, which can only happen if z_1 is also the identity in G. Hence, this relation can hold only if z is the identity element in Z, meaning that Δ is non-degenerate.*

Before we introduce the Weyl representation, it is beneficial to discuss some additional properties of the Heisenberg group, as well as develop the relationship between the automorphisms of $H(G)$ and the symplectic transformations of the phase space Z. We first begin with some basic observations about $H(G)$

Lemma 3.3. *The center of the Heisenberg group, $Z(H(G))$, is given by:*

$$Z(H(G)) = \{(1, \lambda) \in H(G) \mid \lambda \in \mathbb{T}\}$$

Where $1 \in Z$ denotes the identity element in Z.

Proof. Suppose $(z, \lambda) \in Z(H(G))$, then for all $(z', \lambda') \in H(G)$ we have,

$$\begin{aligned} & (z, \lambda) \circ (z', \lambda') = (z', \lambda') \circ (z, \lambda) \\ \iff \quad & (zz', \sigma(z, z')\lambda\lambda') = (z'z, \sigma(z', z)\lambda'\lambda) \\ \iff \quad & \sigma(z, z') = \sigma(z', z) \\ \iff \quad & \Delta(z, z') = 1 \end{aligned}$$

Since this must be true for all $z' \in Z$, then for (z, λ) to be in $Z(H(G))$, we must have $z = 1$. Noting that $\mathbb{T}$ is abelian under multiplication and $\sigma(1, z) = 1 = \sigma(z, 1)$ for all $z \in Z$, it follows that $(1, \lambda) \in Z(H(G))$ for any choice of $\lambda \in \mathbb{T}$. Hence, $Z(H(G)) = \{(1, \lambda) \in H(G) \mid \lambda \in \mathbb{T}\}$ □

Remark 4. *For any $(z, \lambda) \in H(G)$, the inverse element is given by:*

$$(z, \lambda)^{-1} = (z^{-1}, \gamma_{z_2}(z_1)\lambda^{-1})$$

Lemma 3.4. *For any $(z, \lambda), (z', \lambda') \in H(G)$, we have the following relation:*

$$(z, \lambda) \circ (z', \lambda') = (z', \lambda') \circ (z, \lambda) \circ (1, \Delta(z, z'))$$

Proof. This result follows by simply expanding out both sides of the equation. □

We will now introduce the concept of a *symplectic transformation* and begin working towards the relationship between these transformations and automorphisms of the Heisenberg group.

Definition 4. *A transformation $S : Z \to Z$ is* symplectic *if it preserves the symplectic form Δ. That is, for all $z, z' \in Z$,*

$$\Delta(Sz, Sz') = \Delta(z, z')$$

We denote by $Sp(G)$ the set of all symplectic transformations on $Z = G \times G$.

Proposition 1. *$Sp(G)$ is a subgroup of $Aut(Z)$.*

Proof. First, suppose $S \in Sp(G)$ and that for $z, z' \in Z$, $Sz = Sz'$. then for all $w \in Z$ we have,

$$\begin{aligned} & \Delta(Sz, Sw) = \Delta(Sz', Sw) \\ \Rightarrow \quad & \Delta(z, w) = \Delta(z', w) \\ \Rightarrow \quad & \Delta(z, w)\Delta(z, w)^{-1} = \Delta(zz'^{-1}, w) = 1 \\ \Rightarrow \quad & zz'^{-1} = 1 \\ \Rightarrow \quad & z = z' \end{aligned}$$

by Remark 3. Thus, S is a bijective mapping from Z to Z and is therefore invertible. Hence, for any $w \in Z$, there exists $w' \in Z$ such that $w = Sw'$. Now consider $z, z' \in Z$, then for any $w \in Z$,

$$\begin{aligned} \Delta(S(zz'), w) &= \Delta(S(zz'), Sw') \\ &= \Delta(zz', w') \\ &= \Delta(z, w')\Delta(z', w') \\ &= \Delta(Sz, Sw')\Delta(Sz', Sw') \\ &= \Delta((Sz)(Sz'), w) \end{aligned}$$

So by Remark 3, it follows that $(Sz)(Sz') = S(zz')$. Therefore, each $S \in Sp(G)$ is a bijective homomorphisim of Z onto $Z \Rightarrow Sp(G) \subseteq Aut(Z)$.

Now, let $S, S' \in Sp(G)$ be any symplectic transformation on Z. Then,

$$\begin{aligned} & \Delta(SS'z, SS'z') = \Delta(S'z, S'z') = \Delta(z, z') \\ \textit{and}, \quad & \Delta(z, z') = \Delta(SS^{-1}z, SS^{-1}z') = \Delta(S^{-1}z, S^{-1}z') \end{aligned}$$

for all $z, z' \in Z$. Hence, $Sp(G)$ is closed under composition and $S^{-1} \in Sp(G)$ for each $S \in Sp(G) \Rightarrow Sp(G)$ forms a group under composition. Putting everything together, we get that $Sp(G)$ is a subgroup of $Aut(Z)$. □

With this, we will now begin to examine the automorphisms of the Heisenberg group. In our case we are not interested in the full set of automorphisms from $H(G)$ onto itself, but rather only the subset of those automorphisms that act trivially on the center of the group.

Definition 5. *We denote by $Aut_0(H(G))$ the subgroup of $Aut(H(G))$ that act trivially on the center. In other words,*

$$Aut_0(H(G)) = \{\rho \in Aut(H(G)) \mid \rho(I, \lambda) = (I, \lambda) \textit{ for all } \lambda \in \mathbb{T}\}$$

Some time will now be spent analysing how these automorphisms act on the Heisenberg group.

Let $\alpha \in Aut_0(H(G))$ be arbitrary. Then for any $(z, \lambda) \in H(G)$ we have that,

$$\begin{aligned}\alpha(z,\lambda) &= \alpha((z,1)\circ(1,\lambda))\\ &= \alpha(z,1)\circ\alpha(1,\lambda)\\ &= (S_\alpha z, f_\alpha(z))\circ(1,\lambda)\\ &= (S_\alpha z, f_\alpha(z)\lambda)\end{aligned}$$

Where $S_\alpha : Z \to Z$ is some transformation acting on Z and $f_\alpha : Z \to \mathbb{T}$ is some mapping from the phase space into $\mathbb{T}$. We can therefore represent elements in $Aut_0(H(G))$ as pairs (S, f) for some mappings S and f. Since $Aut_0(H(G))$ is a subgroup of $Aut(H(G))$ with the operation of composition, it follows that for all $(z, \lambda) \in H(G)$

$$\begin{aligned}[(S,f)\cdot(S',f')](z,\lambda) &= (S,f)[(S',f')(z,\lambda)]\\ &= (S,f)(S'z, f'(z)\lambda)\\ &= (SS'z, f(S'z)f'(z)\lambda)\end{aligned}$$

Therefore, $(S, f) \cdot (S', f') = (SS', f'')$ where $f''(z) = f(S'z)f'(z)$ for all $z \in Z$.

We can, however, be more specific about the identity of these mappings. First, by applying α to both sides of the relation given in Lemma 3.4, we find that

$$\begin{aligned}&\alpha((z,\lambda)\circ(z',\lambda')) = \alpha((z',\lambda')\circ(z,\lambda)\circ(1,\Delta(z,z')))\\ \Rightarrow\quad &\alpha(z,\lambda)\circ\alpha(z',\lambda') = \alpha(z',\lambda')\circ\alpha(z,\lambda)\circ\alpha(1,\Delta(z,z'))\\ \Rightarrow\quad &(Sz,f(z)\lambda)\circ(Sz',f(z')\lambda') = (Sz',f(z')\lambda')\circ(Sz,f(z)\lambda)\circ(1,\Delta(z,z'))\end{aligned}$$

But then applying Lemma 3.4 again to the left side of the equation we get

$$(Sz,f(z)\lambda)\circ(Sz',f(z')\lambda') = (Sz',f(z')\lambda')\circ(Sz,f(z)\lambda)\circ(1,\Delta(Sz,Sz'))$$

Setting the right side of both expressions equal to each other, we find that

$$\Delta(Sz, Sz') = \Delta(z, z')$$

Hence, $S \in Sp(G)$. We also require that α perseveres the operation in $H(G)$, and hence

$$\begin{aligned}\alpha((z,\lambda)\circ(z',\lambda')) &= \alpha(z,\lambda)\circ\alpha(z',\lambda')\\ &= (Sz,f(z)\lambda)\circ(Sz',f(z')\lambda')\\ &= (S(zz'),\sigma(Sz,Sz')f(z)f(z')\lambda\lambda')\\ \text{and,}\quad \alpha((z,\lambda)\circ(z',\lambda')) &= \alpha(zz',\sigma(z,z')\lambda\lambda')\\ &= (S(zz'),f(zz')\sigma(z,z')\lambda\lambda')\end{aligned}$$

From this we find that

$$\begin{aligned}&\sigma(Sz,Sz')f(z)f(z') = f(zz')\sigma(z,z')\\ \iff\quad &f(zz')f(z)^{-1}f(z')^{-1} = \sigma(Sz,Sz')\sigma(z,z')^{-1} \qquad (1)\end{aligned}$$

Thus, the functions $f : Z \to \mathbb{T}$ for a given automorphism α is dependent on the symplectic transformation associated α. To get a better idea of how these mappings act on Z, consider the set of inner automorphisms of $H(G)$. For $(z, \lambda), (y, \phi) \in H(G)$, the inner automorphism induced by (z, λ) is given by:

$$\begin{aligned}
(z^{-1}, \gamma_{z_2}(z_1)\lambda^{-1}) \circ (y, \phi) \circ (z, \lambda) &= (z^{-1}, \gamma_{z_2}(z_1)\lambda^{-1}) \circ (yz, \gamma_{z_2}(y_1)\phi\lambda) \\
&= (z^{-1}yz, \gamma_{y_2 z_2}(z_1^{-1})\gamma_{z_2}(y_1)\gamma_{z_2}(z_1)\lambda^{-1}\phi\lambda) \\
&= (y, \gamma_{y_2}(z_1^{-1})\gamma_{z_2}(z_1^{-1})\gamma_{z_2}(y_1)\gamma_{z_2}(z_1)\phi) \\
&= (y, \gamma_{y_2}(z_1)^{-1}\gamma_{z_2}(y_1)\phi) \\
&= (y, \Delta(y, z)\phi)
\end{aligned}$$

Hence, $Inn(H(G))$ are of the form (I, f) where $f(z) = \Delta(z, w)$ for some $w \in Z$. So for $w \in Z$, defining $\Delta_w : Z \to \mathbb{T}$ as the homomorphism given by $\Delta_w(z) = \Delta(z, w)$ for all $z \in Z$, we can denote the elements of $Inn(H(G))$ as (I, Δ_w) for some w. From this we see that for any $(S, f) \in Aut_0(H(G))$, we have $(S, f) \cdot (I, \Delta_w) = (S, f') \in Aut_0(H(G))$ where $f'(z) = f(z)\Delta_w(z) = f(z)\Delta(z, w)$ for each $w \in Z$.

Now consider two mapping f and f' such that $(S, f), (S, f') \in Aut_0(H_\sigma(G))$. Then by taking the quotient of relation (1) for both f and f', we get that:

$$\begin{aligned}
&\frac{f(zz')f(z)^{-1}f(z')^{-1}}{f'(zz')f'(z)^{-1}f'(z')^{-1}} = \frac{\sigma(Sz, Sz')\sigma(z, z')^{-1}}{\sigma(Sz, Sz')\sigma(z, z')^{-1}} = 1 \\
\iff \quad &\frac{f(zz')}{f'(zz')} = \frac{f(z)f(z')}{f'(z)f'(z')}
\end{aligned}$$

So the mapping f/f' is a homomorphism from Z to $\mathbb{T} \Rightarrow f/f' \in \widehat{Z}$. Since Z is a finite abelian group, then by Theorem 3.2, $Z \cong \widehat{Z}$, and it follows from Remarks 2 and 3 that $\Delta_w \in \widehat{Z}$ is a unique homomorphism for each choice of $w \in Z$. Therefore, for some $w \in Z$, $f/f' = \Delta_w$ or in other words $f(z) = f'(z)\Delta_w(z)$. So for each $S \in Sp(G)$ we need only to find one single function, which we will from here on denote as f_S, such that f_S satisfies relation (1) then all mappings f with $(S, f) \in Aut_0(H(G))$ can be obtained by multiplying f_S with Δ_w for each $w \in Z$.

Remark 5. *The mapping from $Aut_0(H(G))$ to $Sp(G)$ given by $(S, f) \mapsto S$ is a surjective homomorphism with kernel given by $\{(I, \Delta_w) | w \in Z\} = Inn(H(G)) \cong Z$. So by the first isomorphisim theorem*

$$Sp(G) \cong Aut_0(H(G))/Inn(H(G)) \cong Aut_0(H(G))/Z$$

Under certain additional conditions on the group G we can get more specific about the identity of these particular solutions, f_S, to equation (1). In particular, when we can define the square root of a group element, we can find a relatively simple f_S for any simplectic transformation $S \in Sp(G)$. This, however, only happens when the group G is 2-regular.

Definition 6. *A group G is 2-regular if the mapping given by $g \mapsto g^2$ is an automorphism of G onto G. When a group is 2-regular, we denote by $\sqrt{\ } : G \to G$ the bijective mapping such that $\sqrt{g}\sqrt{g} = g$ for all $g \in G$. In this case, $\sqrt{\ }$ is the inverse mapping of $g \mapsto g^2$.*

Remark 6. *When a group G is 2-regular, then the direct product $Z = G \times G$ is also 2-regular.*

One of the simplest examples of a 2-regular group is the finite cyclic group $\mathbb{Z}_d$, when d is odd. In this case, our group operation would be addition, hence the analogue of the mapping $z \mapsto z^2$ with multiplication, would be the mapping $\phi : \mathbb{Z}_d \mapsto \mathbb{Z}_d$ defined as $\phi(z) = 2z$. In this context, $\frac{d+1}{2}$ is the multiplicative inverse of 2, so the mapping $z \mapsto \frac{d+1}{2}z$ would be the inverse of ϕ, from which it follows that ϕ is an automorphism and therefore $\mathbb{Z}_d$ is 2-regular.

For the remainder of this section, we will restrict ourselves to only those finite abelian groups G which are 2-regular. In this case we have the following lemma:

Lemma 3.5. *When Z is 2-regular, for any $S \in Sp(G)$, a particular solution to equation (1) is given by $f_S(z) = \sigma(Sz, \sqrt{Sz})\sigma(z, \sqrt{z})^{-1}$*

Proof. This follows by simply substituting f_S for f in equation (1) and expanding, while noting the that equation (1) is symmetric in z and z'. □

Remark 7. *Using the particular solution equation (1) given in lemma 3.5, we note that;*

$$\begin{aligned} f_S(S'z)f_{S'}(z) &= \left[\sigma(SS'z, \sqrt{SS'z})\sigma(S'z, \sqrt{S'z})^{-1}\right]\left[\sigma(S'z, \sqrt{S'z})\sigma(z, \sqrt{z})^{-1}\right] \\ &= \sigma(SS'z, \sqrt{SS'z})\sigma(z, \sqrt{z})^{-1} = f_{SS'}(z) \end{aligned}$$

So for any $S, S' \in Sp(G)$ we have that,

$$\left[(S, f_S) \cdot (S', f_{S'})\right](z, \lambda) = (SS', f_S(S'z)f_{S'}(z)\lambda) = (SS', f_{SS'})(z, \lambda)$$

As a consequence of Remark 5, for a general finite abelian group (even those that need not be 2-regular), it is clear that we can construct the short exact sequence

$$\{e\} \longrightarrow Z \longrightarrow Aut_0(H(G)) \longrightarrow Sp(G) \longrightarrow \{e\}$$

However, when we impose the restriction that Z be 2-regular, we can then construct the homomorphism $\rho : Sp(G) \mapsto Aut_0(H(G))$ defined as $\rho(S) = (S, f_S)$ for all $S \in Sp(G)$. This homomorphism composed with the projection of $Aut(H(G))$ onto $Sp(G)$ is just the identity mapping on $Sp(G)$, meaning that this is a split exact sequence, from which we can conclude that $Aut_0(H(G)) \cong Sp(G) \ltimes Z$.

4 The Weyl Representation

A particularly important representation of the Heisenberg group is the representation as unitary operators acting on the Hilbert space $L^2(G)$ of functions $\psi : G \mapsto \mathbb{C}$ with inner product $\langle \psi, \phi \rangle = \sum_{g \in G} \psi(g)\overline{\phi(g)}$. Note that for any function $\psi \in L^2(G)$ we can represent ψ as $\psi(h) = \sum_{g \in G} \psi(g)\delta_g(h)$ for all $h \in G$, where $\delta_g(h) = 1$ when $h = g$ and zero otherwise. In this way, $\langle \delta_g, \delta_h \rangle = \delta_g(h)$, so $\{\delta_g \mid g \in G\}$ forms an orthonormal basis of $L^2(G)$.

It should also be noted, however, that in the case when G is finite, all of the conditions for an arbitrary mapping from G to $\mathbb{C}$ to be in $L^2(G)$ are trivially satisfied since any integral of a function defined in the group becomes a sum of the values of the function on each of the group elements. Since the group is finite then the integral of the modulus squared of any function is also finite. This is not true in the more general case where the group G can be any locally compact abelian group.

Due to the simplification mentioned above, we can instead fix an ordering of the elements in G and view the functions in $L^2(G)$ as vectors in $\mathbb{C}^{|G|}$, and with this, any linear operator on the set $L^2(G)$ can be represented as a matrix in $M_{|G|}(\mathbb{C})$ after fixing a basis. This picture can be very helpful in developing some of the properties of these operators, and as such we will use both views interchangeably. In what is to follow, we will always assume the standard basis of $M_{|G|}(\mathbb{C})$ unless otherwise stated. To be clear about notation we will continue to use ψ to denote a mapping from G to $\mathbb{C}$ with the inner product defined above, and we will adopt the bra-ket notation when discussing the interpretation of these functions as a vector in $\mathbb{C}^{|G|}$. Hence, $\psi \in L^2(G)$ and $\langle \psi, \phi \rangle$ is an inner product of elements in $L^2(G)$, where as $|\psi\rangle \in \mathbb{C}^{|G|}$ and $\langle \psi | \phi \rangle$ is the standard complex inner product in $\mathbb{C}^{|G|}$. The identity of an operator acting on ψ as an element of $B(L^2(G))$ or as a matrix in $M_{|G|}(\mathbb{C})$ will be clear from context.

Now for any finite abelian group G, we denote by $W : H(G) \to B(L^2(G))$ the representation of the Heisenberg group such that:

$$W(z, \lambda)\psi(g) = \lambda M_{z_2} T_{z_1} \psi(g) = \lambda \gamma_{z_2}(g)\psi(gz_1)$$

for $\psi \in L^2(G)$, where T_g is the translation operator by the group element $g \in G$ and M_h is the multiplication operator by the character γ_h.

Remark 8. *For any $g, h \in G$ and $\psi \in L^2(G)$,*

$$\begin{aligned} T_g M_h \psi(k) &= \gamma_h(kg)\psi(kg) \\ &= \gamma_h(kg)\gamma_h(k)^{-1}\gamma_h(k)\psi(kg) \\ &= \gamma_h(kg)\gamma_h(k^{-1})\left(M_h T_g \psi(k)\right) \\ &= \gamma_h(kgk^{-1})\left(M_h T_g \psi(k)\right) \\ &= \gamma_h(g) M_h T_g \psi(k) \end{aligned}$$

Therefore $T_g M_h = \gamma_h(g) M_h T_g$ or $M_h T_g = \gamma_h(g)^{-1} T_g M_h$. Using these commu-

tation relations, we see that for any $(z,\lambda),(z',\lambda')\in H(G)$,

$$\begin{aligned}
W(z,\lambda)W(z',\lambda') &= (\lambda M_{z_2}T_{z_1})\left(\lambda' M_{z_2'}T_{z_1'}\right)\\
&= \lambda\lambda' M_{z_2}T_{z_1}M_{z_2'}T_{z_1'}\\
&= \gamma_{z_2'}(z_1)\lambda\lambda' M_{z_2 z_2'}T_{z_1 z_1'}\\
&= \sigma(z,z')\lambda\lambda' M_{z_2 z_2'}T_{z_1 z_1'}\\
&= W(zz',\sigma(z.z')\lambda\lambda')\\
&= W((z,\lambda)\circ(z',\lambda'))
\end{aligned}$$

Hence, W *is a homomorphism as required for being a valid representation of* $H(G)$.

From the definition of W, we see that for any $\psi,\phi\in L^2(G)$ and $(z,\lambda)\in H(G)$,

$$\begin{aligned}
\langle W(z,\lambda)\phi, W(z,\lambda)\psi\rangle &= \sum_{g\in G} W(z,\lambda)\phi\overline{W(z,\lambda)\psi}\\
&= \sum_{g\in G} \lambda\gamma_{z_2}(g)\phi(gz_1)\overline{\lambda\gamma_{z_2}(g)\psi(gz_1)}\\
&= \sum_{g\in G} |\lambda|^2|\gamma_{z_2}(g)|^2\phi(gz_1)\overline{\psi(gz_1)}\\
&= \sum_{g\in G} \phi(g)\overline{\psi(g)}\\
&= \langle\phi,\psi\rangle
\end{aligned}$$

Hence $W(z,\lambda)$ is a unitary transformation of $L^2(G)$ for all $(z,\lambda)\in H(G)$.

Our main point of interest, however, does not lie with the representation W but rather with its restriction to Z, otherwise known as the Weyl representation.

Definition 7. *For a finite abelian group* G*, the Weyl representation of* $Z = G\times G$ *is given by* $W: Z\to B(L^2(G))$*, where* $W(z) = M_{z_2}T_{z_1}$ *for all* $z\in Z$.

Both the unitary representation of the Heisenberg group and the Weyl representation are denoted by W. To be clear about notation, for all that follows, if a complex phase factor is included in the argument, the Heisenberg representation is assumed, while if a phase factor is absent, then the Weyl representation is used.

From Remark 8, we can see that the Weyl representation is not a homomorphism from Z onto $B(L^2(G))$, rather, it is a homomorphism up to a complex phase, in other words a projective representation on Z. More precisely,

$$\begin{aligned}
W(z)W(z') &= M_{z_2}T_{z_1}M_{z_2'}T_{z_1'}\\
&= \gamma_{z_2'}(z_1)M_{z_2}M_{z_2'}T_{z_1}T_{z_1}\\
&= \sigma(z,z')M_{z_2 z_2'}T_{z_1 z_1'}\\
&= \sigma(z,z')W(zz')
\end{aligned}$$

From the lines above we also see how the members of the Weyl representation commute since,

$$W(z)W(z') = \sigma(z,z')W(zz') = \sigma(z,z')\left[\sigma(z',z)^{-1}W(z')W(z)\right] = \Delta(z,z')W(z')W(z)$$

Another important property of the Weyl representation comes as a result of the following lemma.

Lemma 4.1. *for any $z \in Z$ with $z \neq 1$, $Tr(W(z)) = 0$.*

Proof. Suppose $z \in Z$ with $z \neq 1$, then

$$\begin{aligned} Tr(W(z)) &= \sum_{g \in G} \langle W(z)\delta_g, \delta_g \rangle \\ &= \sum_{g \in G} \langle \gamma_{z_2}(g)\delta_{gz_1^{-1}}, \delta_g \rangle \\ &= \sum_{g \in G} \gamma_{z_2}(g)\langle \delta_{gz_1^{-1}}, \delta_g \rangle \end{aligned}$$

When $z_1 \neq 1$ then $gz_1^{-1} \neq g \Rightarrow \langle \delta_{gz_1^{-1}}, \delta_g \rangle = 0$ for all $g \in G$, and from this we find that $Tr(W(z)) = 0$. When $z_1 = 1$ (and so we must have $z_2 \neq 1$), then we have $Tr(W(z)) = \sum_{g \in G} \gamma_{z_2}(g)$. Now since $z_2 \neq 1$ then there exists some $h \in G$ such that $\gamma_{z_2}(h) \neq 1$. With this we get the following

$$\begin{aligned} \sum_{g \in G} \gamma_{z_2}(g) &= \sum_{g \in G} \gamma_{z_2}(hg) \\ &= \sum_{g \in G} \gamma_{z_2}(h)\gamma_{z_2}(g) \\ &= \gamma_{z_2}(h)\left(\sum_{g \in G} \gamma_{z_2}(g)\right) \end{aligned}$$

But the only way this can hold is if $\sum_{g \in G} \gamma_{z_2}(g) = 0 \Rightarrow Tr(W(z)) = 0$. Thus, for any choice of $z \in Z$ with $z \neq 1$, we have $Tr(W(z)) = 0$.

□

Theorem 4.2. *$\{W(z) \mid z \in Z\}$ is an orthogonal basis of operators in $B(L^2(G))$ with respect to the Hilbert-Schmidt inner product $\langle A|B \rangle_{HS} = Tr(A^*B)$.*

Proof. First note that since $W(z)$ is unitary for all $z \in Z \Rightarrow W(z)^*W(z) = I \Rightarrow \langle W(z)|W(z) \rangle_{HS} = Tr(W(z)^*W(z)) \neq 0$. Now suppose $z, z' \in Z$ with $z \neq z'$. Then,

$$\begin{aligned} \langle W(z)|W(z') \rangle_{HS} &= Tr(W(z)^*W(z')) \\ &= Tr(W(z^{-1})W(z')) \\ &= \sigma(z^{-1}, z')Tr(W(z^{-1}z')) = 0 \end{aligned}$$

Hence, $\{W(z)|z \in Z\}$ forms an orthogonal set of operators in $B(L^2(G))$. Since there are a total of $|G|^2$ operators in the $\{W(z)|z \in Z\}$ and $dim(M_{|G|}(\mathbb{C})) = |G|^2 = dim(B(L^2(G)))$, then this set also forms a basis of $B(L^2(G)) \cong M_{|G|}(\mathbb{C})$. □

After noting the following lemma, we will see how Theorem 4.2 leads us to some important properties of the Weyl representation that will be used extensively in what is to come.

Lemma 4.3. $Z(M_d(\mathbb{C})) = \{\alpha I_d \mid \alpha \in \mathbb{C}\}$

Proof. Suppose $|\psi\rangle \in \mathbb{C}^d$ with $\langle\psi|\psi\rangle = 1$ and $A \in Z(M_d(\mathbb{C}))$. Let $\lambda = \langle\psi|A|\psi\rangle$, then

$$A|\psi\rangle = A|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle\langle\psi|A|\psi\rangle = \lambda|\psi\rangle$$

Now, suppose $|\phi\rangle \in \mathbb{C}^d$ is any other unit vector. Then,

$$\langle\phi|A|\phi\rangle = \langle\phi|A|\phi\rangle\langle\psi|\psi\rangle = \langle\phi|\phi\rangle\langle\psi|A|\psi\rangle = \lambda$$

Thus, for any $|\xi\rangle \in \mathbb{C}^d$ with $\langle\xi|\xi\rangle = 1$, we have that $A|\xi\rangle = \lambda|\xi\rangle$ for some $\lambda \in \mathbb{C}$. Since the set of unit vectors in $\mathbb{C}^d$ generate all of $\mathbb{C}^d$, then this can be extended to any vector in $\mathbb{C}^d \Rightarrow A = \lambda I_d$. Finally, noting that any αI_d commutes with all of $M_d(\mathbb{C})$, then we get that $Z(M_d(\mathbb{C})) = \{\alpha I_d \mid \alpha \in \mathbb{C}\}$. □

Corollary 4.4. *The representation W of the Heisenberg group is faithful and irreducible*

Proof. First note that for any $(z,\lambda) \in H(G)$ we have that $W(z,\lambda) = \lambda W(z)$ where the left side is the representation of the Heisenberg group and the right is a multiple of an element in the Weyl representation. Since the Weyl representation of Z forms a basis of $B(L^2(G))$, then for any $A \in (W(H(G)))'$, A must commute with all elements in $B(L^2(G))$. This is only the case when $A = \alpha I$ where α is some complex scalar, by Lemma 4.3. Hence, $(W(H(G)))' = \mathbb{C}I \Rightarrow$ the representation is irreducible as a result of Schur's Lemma.

Additionally, suppose $(z,\lambda),(z',\lambda') \in H(G)$ with $W(z,\lambda) = W(z',\lambda')$. Then for all $\psi \in L^2(G)$ we have that for all $g \in G$

$$\begin{aligned} & W(z,\lambda)\psi(g) = W(z',\lambda')\psi(g) \\ \iff \quad & \lambda\gamma_{z_2}(g)\psi(gz_1) = \lambda'\gamma_{z_2'}(g)\psi(gz_1') \end{aligned}$$

Since this holds for every $\psi \in L^2(G)$, then in particular it holds for every δ_h. In this case we get that for some $k \in G$, $kz_1 = h = kz_1' \Rightarrow z_1 = z_1'$. Hence $gz_1 = gz_1'$ for all $g \in G$, which allows us to reduce the above expression to

$$\begin{aligned} & \lambda\gamma_{z_2}(g) = \lambda'\gamma_{z_2'}(g) \\ \iff \quad & \gamma_{z_2}(g)\gamma_{z_2'}(g)^{-1} = \frac{\lambda'}{\lambda} \\ \iff \quad & \Delta((g,z_2'),(g,z_2)) = \frac{\lambda'}{\lambda} \end{aligned}$$

Where the right side is a constant and the left side depends on g. Since this holds for all $g \in G$, then we must have $\frac{\lambda'}{\lambda} = 1 \Rightarrow \lambda = \lambda'$, from which we find $\gamma_{z_2}(g) = \gamma_{z_2'}(g) \Rightarrow z_2 = z_2'$. Therefore, $(z, \lambda) = (z', \lambda') \Rightarrow$ the mapping W is injective, hence the representation is faithful. □

Corollary 4.5. *For any $(S, f) \in Aut_0(H(G))$, the set $\{f(z)W(Sz)|z \in Z\}$ forms an orthogonal basis of operators in $B(L^2(G))$ with respect to the Hilbert-Schmidt inner product.*

Proof. This follows immediately from Theorem 4.2 by noting that for $z, z' \in Z$,

$$\langle f(z)W(Sz)|f(z')W(Sz')\rangle_{HS} = f(z)^* f(z')Tr(W(Sz)^* W(Sz'))$$

And since each $S \in Sp(G)$ is a bijection, then the inner product above is non-zero if and only if $z = z'$. Hence, $\{f(z)W(Sz)|z \in Z\}$ forms an orthogonal set of operators. By a similar counting argument from the previous theorem, we find that $\{f(z)W(Sz)|z \in Z\}$ also forms a basis of $B(L^2(G))$. □

From the last few results, it becomes clear that by relabelling the points in the phase space according to a symplectic transformation $S \in Sp(G)$, we can maintain the properties of the Weyl representation as long as we also multiply the elements by a phase factor given by some f associated to the transformation S. This can be seen more concretely by noting the following;

$$\begin{aligned}[f(z)W(Sz)]\,[f(z')W(Sz')] &= f(z)f(z')\sigma(Sz, Sz')W((Sz)(Sz')) \\ &= \sigma(z, z')\,[f(zz')W(S(zz'))]\end{aligned}$$

by equation (1), and;

$$\begin{aligned}[f(z)W(Sz)]\,[f(z')W(Sz')] &= f(z)f(z')\,[\Delta(z, z')W(Sz')W(Sz)] \\ &= \Delta(z, z')\,[f(z')W(Sz')]\,[f(z)W(Sz)]\end{aligned}$$

On the other hand, in our operator picture, a relabelling of the phase points according to S just corresponds to a change of basis transformation such that $W(z) \mapsto f(z)W(Sz)$. Developing this gives a way to translate between equivalent representations given by automorphisms of the Heisenberg group. However, before this is done, we require the following fundamental result.

Lemma 4.6. *For any $n \in \mathbb{N}$, the algebra given by $M_n(\mathbb{C})$ with addition and multiplication, is a central simple algebra. In other words, $M_n(\mathbb{C})$ has no non-trivial proper two sided ideals, and $Z(M_n(\mathbb{C})) \cong \mathbb{C}$.*

Proof. Suppose I is a non-trivial two sided ideal of $M_n(\mathbb{C})$. Then there exists a non-zero element $A \in I$ which can be written as $A = \sum_{i,j=1}^{n} a_{i,j}|e_i\rangle\langle e_j|$ for some $a_{i,j} \in \mathbb{C}$. Since A is non-zero, then there is some p, q such that $a_{p,q} \neq 0$. Now since I is a two sided ideal, then for any $C, D \in M_n(\mathbb{C})$, $CAD \in I$. In particular, for $C = |e_k\rangle\langle e_p|$ and $D = \frac{1}{a_{p,q}}|e_q\rangle\langle e_l|$ for any $k, l \in \{1, \ldots, n\}$, then

$$CAD = |e_k\rangle\langle e_p| \left(\sum_{i,j=1}^{n} a_{i,j}|e_i\rangle\langle e_j|\right)\left(\frac{1}{a_{p,q}}|e_q\rangle\langle e_l|\right) = |e_k\rangle\langle e_l| \in I$$

Hence, any rank-1 projection is in I, and by extension any multiple of a rank-1 projection, from which we can generate all of $M_n(\mathbb{C}) \Rightarrow I = M_n(\mathbb{C}) \Rightarrow M_n(\mathbb{C})$ is simple. Additionally, $Z(M_n(\mathbb{C})) = \{\alpha I \mid \alpha \in \mathbb{C}\} \cong \mathbb{C}$, so $M_n(\mathbb{C})$ is central. $\square$

With this, we now have the tools to prove the following theorem.

Theorem 4.7. *For any automorphism $(S, f) \in Aut_0(H(G))$, there exists a unitary transformation $U_{(S,f)}$ such that*

$$U^*_{(S,f)} W(z) U_{(S,f)} = f(z) W(Sz)$$

for all $z \in Z$.

Proof. Let (S, f) be any automorphism of the Heisenberg group. From Theorem 4.2 and Corollary 4.5, we have that $\{W(z)|z \in Z\}$ and $\{f(z)W(Sz)|z \in Z\}$ are two orthogonal bases of $M_{|G|}(\mathbb{C})$. Hence, we can define the mapping $\rho : M_{|G|}(\mathbb{C}) \to M_{|G|}(\mathbb{C})$, by $\rho(W(z)) = f(z)W(Sz)$ for all $z \in Z$. This mapping then has a unique linear extension to all of $M_{|G|}(\mathbb{C})$. Noting that,

$$\begin{aligned}
\rho(W(z)W(z')) &= \rho(\sigma(z, z')W(zz')) \\
&= \sigma(z, z')\rho(W(zz')) \\
&= \sigma(z, z')f(zz')W(S(zz')) \\
&= \sigma(z, z')[\sigma(Sz', Sz)\sigma(z', z)^{-1} f(z)f(z')]W((Sz)(Sz')) \\
&= \sigma(Sz', Sz)\Delta(z, z')f(z)f(z')\sigma(Sz, Sz')^{-1}W(Sz)W(Sz') \\
&= \Delta(Sz', Sz)\Delta(z, z')[f(z)W(Sz)][f(z')W(Sz')] \\
&= \Delta(z', z)\Delta(z, z')\rho(W(z))\rho(W(z')) \\
&= \rho(W(z))\rho(W(z'))
\end{aligned}$$

we can view ρ as a ring automorphism of $M_{|G|}(\mathbb{C})$ onto itself. Now since $M_{|G|}(\mathbb{C})$ is a central simple algebra over $\mathbb{C}$, then as a result of the Skolem-Noether theorem, ρ is an inner automorphism. In other words, there exists an invertible $U_{(S,f)} \in M_{|G|}(\mathbb{C})$ such that

$$U^{-1}_{(S,f)} W(z) U_{(S,f)} = \rho(W(z)) = f(z)W(Sz)$$

Note also that we have the following,

$$\begin{aligned}
\rho(W(z))^* &= (f(z)W(Sz))^* \\
&= (f(z)W(Sz))^{-1} \\
&= \rho(W(z))^{-1} \\
&= \rho(W(z)^{-1}) \\
&= \rho(W(z)^*)
\end{aligned}$$

Hence, ρ is a *-automorphism, from which we see that,

$$\begin{aligned}
& \rho(W(z)) = \rho(W(z)^*)^* \\
\iff \quad & U_{(S,f)}^{-1} W(z) U_{(S,f)} = [U_{(S,f)}^{-1} W(z)^* U_{(S,f)}]^* \\
\iff \quad & U_{(S,f)}^{-1} W(z) U_{(S,f)} = U_{(S,f)}^* W(z) (U_{(S,f)}^*)^{-1} \\
\iff \quad & W(z) U_{(S,f)} U_{(S,f)}^* = U_{(S,f)} U_{(S,f)}^* W(z)
\end{aligned}$$

And so $U_{(S,f)}U_{(S,f)}^*$ commutes with $W(z)$ for all $z \in Z$. But since $\{W(z) \mid z \in Z\}$ forms a basis of $M_{|G|}(\mathbb{C})$, then this means that $U_{(S,f)}U_{(S,f)}^* = \alpha I$ for some scalar α by Lemma 4.3. Noting that,

$$\begin{aligned}
det\left(U_{(S,f)}U_{(S,f)}^*\right) &= det\left(U_{(S,f)}\right) det\left(U_{(S,f)}^*\right) \\
&= det\left(U_{(S,f)}\right) \overline{det\left(U_{(S,f)}\right)} \\
&= \left|det\left(U_{(S,f)}\right)\right|^2 \in \mathbb{R}
\end{aligned}$$

we see that we can scale $U_{(S,f)}$ by some real scalar such that $U_{(S,f)}U_{(S,f)}^* = I \Rightarrow U_{(S,f)}$ is unitary. With this newly scaled U, our first relation becomes,

$$U_{(S,f)}^* W(z) U_{(S,f)} = \rho(W(z)) = f(z)W(Sz)$$

□

5 The Stone-Von Neumann Theorem

The representation W and the associated Weyl representation give us a clean way to represent the structure of the Heisenberg group in the context of operators on a Hilbert space, however, this representation is not unique. In fact, from the last few results we can see that for any $S \in Sp(G)$ with associated f then $W' : Z \to B(L^2(G))$ given by $W'(z) = f(z)W(Sz)$ is a equivalent to the standard Weyl representation. But we can bring this further still.

If we let $M : G \to B(L^2(G))$ and $T : G \to B(L^2(G))$ be the mappings defined by $M(g) = M_g$ and $T(g) = T_g$ for all $g \in G$, then we can write the Weyl representation as $W(z) = M(z_2)T(z_1)$. From this, we get that T and M satisfy the commutation relation $T(g)M(h) = \gamma_h(g)M(h)T(g)$ as shown in the previous section. However, any two different homomorphisms M' and T' mapping G to unitary operators on $L^2(G)$ that satisfy the same commutation relations should have the same underlying structure (To keep the notation consistent with the previous section, for any mapping $U : G \to B(L^2(G))$ we will let $U_g = U(g)$). The Stone-von Neumann Theorem makes this precise.

Definition 8. *Let $A, B : G \to B(L^2(G))$ be homomorphisms. We call the pair (A, B) a conjugate pair if $B_g A_h = \gamma_h(g) A_h B_g$ and A_h, B_g are unitary for all $g, h \in G$.*

We begin by generalizing some results from the previous section. Lemma 4.1 showed that for any $z \in Z$ with $z \neq 1$, then $Tr(W(z)) = 0$ by using the specific identity of the operators M and T. This property, however, can be seen as a more general result that comes from the commutation relation of the given operators. The following lemma shows this for any homomorphisms that commute in the same way as those defining the Weyl representation.

Lemma 5.1. *Let $W' : Z \to B(L^2(G))$ be defined by $W'(z) = M'_{z_2}T'_{z_1}$ where (M', T') is a conjugate pair. Then $Tr(W'(z)) = 0$ for all $z \neq 1$.*

Proof. First suppose that $z_1 \neq 1$. Then there exist some $w \in G$ such that $\gamma_w(z_1) \neq 1$. So,

$$\begin{aligned} Tr(W'(z)) &= Tr(M'_{z_2}T'_{z_1}) \\ &= Tr(M'_w M'_{w^{-1}} M'_{z_2} T'_{z_1}) \\ &= Tr(M'_{w^{-1}} M'_{z_2} T'_{z_1} M'_w) \quad \textit{(cyclic property of the trace)} \\ &= \gamma_w(z_1) Tr(M'_{w^{-1}} M'_{z_2} M'_w T'_{z_1}) \\ &= \gamma_w(z_1) Tr(M'_{z_2} T'_{z_1}) \\ &= \gamma_w(z_1) Tr(W'(z)) \end{aligned}$$

Since we choose w such that $\gamma_w(z_1) \neq 1$, the only way this can hold is if $Tr(W'(z)) = 0$. Next, we consider the case when $z_2 \neq 1$. Then there exists some $w' \in G$ with $\gamma_{z_2}(w') \neq 1$, so by following the same reasoning as above, we find;

$$Tr(W'(z)) = \gamma_{z_2}(w') Tr(W'(z))$$

From which we conclude again that $Tr(W'(z)) = 0$. □

Corollary 5.2. *For any $W' : Z \to B(L^2(G))$ defined by $W'(z) = M'_{z_2}T'_{z_1}$ where (M', T') is a conjugate pair, the set $\{W'(z) \mid z \in Z\}$ forms an orthogonal basis of $B(L^2(G))$.*

Proof. Follows from the exact same reasoning as the proof of Theorem 4.2. □

Theorem 5.3. *(Discrete Stone-von Neumann Theorem). Let (A, B) and (A', B') be two conjugate pairs and define the mappings $V, V' : Z \to B(L^2(G))$ by $V(z) = A_{z_2}B_{z_1}$ and $V'(z) = A'_{z_2}B'_{z_1}$. Then there exists a unitary mapping U such that $V'(z) = U^*V(z)U$ for all $z \in Z$. In particular, $A'_h = U^*A_hU$ and $B'_h = U^*B_hU$ for all $h \in G$.*

Proof. First, we note that $\{V(z) \mid z \in Z\}$ and $\{V'(z) \mid z \in Z\}$ are two orthogonal bases of $B(L^2(G)) \cong M_{|G|}(\mathbb{C})$. Hence we can define the mapping $\rho : M_{|G|}(\mathbb{C}) \to M_{|G|}(\mathbb{C})$ by $\rho(V(z)) = V'(z)$ for all $z \in Z$, which extends to a unique linear bijection of $M_{|G|}(\mathbb{C})$. Following the same steps as in the proof of Theorem 4.6, we find that the ρ is an inner automorphism induced by some unitary $U \in M_{|G|}(\mathbb{C})$ which is the matrix representation of some unitary operator $U \in B(L^2(G))$. Hence,

$$V'(z) = U^*V(z)U$$

We get the last statement of the theorem by noting that when $z = (z_1, 1)$, the above relation simplifies to $B'_{z_1} = U^* B_{z_1} U$ and when $z = (1, z_2)$, the relation becomes $A'_{z_2} = U^* A_{z_2} U$. □

The Stone-von Neumann theorem is incredibly powerful as it shows that any representation given by a conjugate pair is unitarily equivalent to the Weyl representation. At the moment, however, we do not have much information about the mapping U that would relate two equivalent conjugate pairs. To be able to translate between two equivalent representations, one needs to determine the identity of the unitary U and for that we require the following result.

Proposition 2. *For a fixed ordering of the elements of G, let $|g\rangle$ for $g \in G$, be the vector in $M_{|G|}(\mathbb{C})$ with a 1 in the g component and 0 everywhere else. Then the unitary mapping U that brings the conjugate pair (A, B) to the standard pair (M, T) is given by*

$$U|g\rangle = B_{g^{-1}}|u_1\rangle$$

where $|u_1\rangle = U|1\rangle$ is the simultaneous eigenvector of A_g for all $g \in G$ with eigenvalue 1 and norm 1.

Proof. From the relations in Theorem 5.3, we get that

$$\begin{aligned} |1\rangle &= M_g|1\rangle \\ &= U^* A_g U|1\rangle \\ \iff \quad U|1\rangle &= A_g U|1\rangle \\ \iff \quad A_g|u_1\rangle &= |u_1\rangle \end{aligned}$$

Hence, $|u_1\rangle = U|1\rangle$ is a simultaneous eigenvector of A_g with eigenvalue 1 for all $g \in G$, and by definition of $|1\rangle$ and since U is unitary, $\langle u_1|u_1\rangle = 1$. Additionally, we note that

$$\begin{aligned} |h\rangle &= T_{h^{-1}}|1\rangle \\ &= U^* B_{h^{-1}} U|1\rangle \\ \iff \quad U|h\rangle &= B_{h^{-1}} U|1\rangle = B_{h^{-1}}|u_1\rangle \end{aligned}$$

□

It is important to note that since A_g and M_g are related by unitary conjugation, then they both have the same eigenvalues, only the eigenvectors of A_g are shuffled around by U. By a similar process to above, we can see that $U|h\rangle$ is an eigenvector of A_g with eigenvalue $\gamma_g(h)$ for all $g, h \in G$. From this observation, it is clear that multiples of $|u_1\rangle = U|1\rangle$ are the only vectors with simultaneous eigenvalues of 1 for all A_g. Therefore, in practice, one needs only to find one of these vectors, then re-scale such that the norm is 1 to obtain $|u_1\rangle$. Once this is found, the relations above can be used to determine how U acts on the whole space.

6 Quantum Systems and the Clifford Hierarchy

As mentioned in the preliminaries, any quantum system can be described by a state vector in a given Hilbert space and through the fourth postulate of quantum mechanics, composite systems, those formed by combining multiple systems, can be described by state vectors in the Hilbert space given by the tensor product of the constituent systems. So to construct the analogue to a classical computer performing operations on a string of bits, we can look at how we can operate on a composite system of spin state of a particle, for example.

As mentioned previously, the Hilbert space representing the spin of a arbitrary particle with n spin states is composed of the n basis states $|1\rangle, \ldots, |n\rangle$. The spin state of any particle, $|\psi\rangle$, is then simply a linear combination of the basis spin states such that $\langle\psi|\psi\rangle = 1$. In a form more reminiscent of the previous sections, a particle's spin can be described by a function $\psi \in L^2(\mathbb{Z}_n)$, with $\psi(x) = \sum_{i=1}^{n} c_i \delta_i(x)$ for $x \in \mathbb{Z}_n$ and the condition $\langle \psi, \psi \rangle = 1$. So the Hilbert space that describes the the spin state of a composite system containing m such particles is given by the m-fold tensor product of $L^2(\mathbb{Z}_n)$ which is isomorphic to $L^2(\mathbb{Z}_n^m)$. In this way, the canonical basis states of our qudit system are given by the functions $\delta_{k_1} \otimes \cdots \otimes \delta_{k_m}$ where each $k_i \in \mathbb{Z}_n$, and arbitrary spin states are normalized combinations of those basis states.

In practice, it is much more convenient to work with the bra-ket notation of the spin vectors. So for any $\hat{a} \in \mathbb{Z}_n^m$, we let $\hat{a} = (a_1, \ldots, a_m)$ and make the identification $\delta_{a_1} \otimes \cdots \otimes \delta_{a_m} \mapsto |\hat{a}\rangle$, such that $\langle \hat{a}|\hat{b}\rangle = \langle a_1|b_1\rangle \ldots \langle a_m|b_m\rangle$ is equal to one whenever $\hat{a} = \hat{b}$ and zero otherwise.

It is instructive to see how the matrices in the standard Weyl representation act when we restrict ourselves to $G = \mathbb{Z}_n$. First we note that by Lemma 3.1, since G is cyclic, then for all $g \in G = \mathbb{Z}_n$ we have that $\gamma_g(h) = e^{\frac{2\pi i g h}{n}}$ for $h \in G$. To simplify notation, we will denote $\omega_n = e^{\frac{2\pi i}{n}}$ such that $\gamma_g(h) = \omega_n^{gh} = \gamma_h(g)$. So in the single qudit case, for a given basis state $|g\rangle$, the operator $M(1) = M_1$ acts as $M_1|g\rangle = \gamma_1(g)|g\rangle = \omega_n^g|g\rangle$. Since M is a homomorphisim and G is cyclic, then we note that for all $k \in G$, $M(k)$ is just the product of $M(1)$ k times. Hence, $M(k) = M_k = M_1^k$, meaning that for a basis state $|g\rangle$ we have $M_k|g\rangle = M_1^k|g\rangle = \omega_n^{kg}|g\rangle = \gamma_k(g)|g\rangle$. Similarly, we can represent $T(k) = T_k = T_1^k$ with $T_k|g\rangle = T_1^k|g\rangle = |g-k\rangle$. With this realization, we see that any element of the Weyl representation can be generated by the operators M_1 and T_1. For clarity, we will suppress the 1 in these operators such that $M = M_1$ and $T = T_1$ and write $W(z) = M^{z_2}T^{z_1}$ which can be interpreted as the repeated application of T and M.

When we consider a composite system of multiple qudits, we can make similar conclusions but with a bit more work. Suppose we have a system of m qudits, each with n basis states, then as mentioned above, the spin states of this system can be represented by a function $\psi \in L^2(\mathbb{Z}_n^m)$ or as a linear combination of vectors $|\hat{a}\rangle = |a_1 \ldots a_m\rangle$. So by setting our finite abelian group $G = \mathbb{Z}_n^m$, then the Weyl operators on the Hilbert space take the form $W(\hat{z}) = M_{\hat{z}_2}T_{\hat{z}_1}$ where

$\hat{z} = (\hat{z_1}, \hat{z_2}) \in \mathbb{Z}_n^m \times \mathbb{Z}_n^m$. Now for a basis state $|\hat{g}\rangle$ and $\hat{h} \in \mathbb{Z}_n^m$, we have

$$\begin{aligned} T_{\hat{h}}|\hat{g}\rangle &= |\hat{g} - \hat{h}\rangle \\ &= |(g_1 - h_1) \dots (g_m - h_m)\rangle \\ &= (T^{h_1} \otimes \dots \otimes T^{h_m})|\hat{g}\rangle \end{aligned}$$

Similarly, we can use Theorem 3.2 to write $\gamma_{\hat{h}}(\hat{g}) = \gamma_{h_1}(g_1) \dots \gamma_{h_m}(g_m)$ to get that

$$\begin{aligned} M_{\hat{h}}|\hat{g}\rangle &= \gamma_{\hat{h}}(\hat{g})|\hat{g}\rangle \\ &= \gamma_{h_1}(g_1) \dots \gamma_{h_m}(g_m)|\hat{g}\rangle \\ &= \omega_n^{h_1 g_1} \dots \omega_n^{h_m g_m}|\hat{g}\rangle \\ &= (M^{h_1} \otimes \dots \otimes M^{h_m})|\hat{g}\rangle \end{aligned}$$

Note that we can also write $\omega_n^{h_1 g_1} \dots \omega_n^{h_m g_m} = \omega_n^{h_1 g_1 + \dots + h_m g_m} = \omega_n^{\langle \hat{h}, \hat{g} \rangle}$ where $\langle \cdot, \cdot \rangle : \mathbb{Z}_n^m \times \mathbb{Z}_n^m \to \mathbb{Z}_n$ is the standard inner product on $\mathbb{Z}_n^m \times \mathbb{Z}_n^m$. Then we get $M_{\hat{h}}|\hat{g}\rangle = \omega_n^{\langle \hat{h}, \hat{g} \rangle}|\hat{g}\rangle$.

From above, it follows that $W(\hat{z}) = M^{(z_2)_1}T^{(z_1)_1} \otimes \dots \otimes M^{(z_2)_m}T^{(z_1)_m} = W((z_1)_1, (z_2)_1) \otimes \dots \otimes W((z_1)_m, (z_2)_m)$. Therefore, for a composite system of spin states, the Weyl operators can be represented as the tensor product of the single qudit Weyl operators. This notation can get quite bulky, so we again simplify these expressions using the following.

Notation 1. *For $G = \mathbb{Z}_n^m$, we let $\widehat{M_i} = I \otimes \dots \otimes I \otimes M \otimes I \otimes \dots \otimes I$ be the tensor product of m operators such that the i^{th} operator is M and all others are the identity operator I. We define $\widehat{T_i} = I \otimes \dots \otimes I \otimes T \otimes I \otimes \dots \otimes I$ in a similar way. Additionally, for $\hat{p} \in \mathbb{Z}_n^m$ we define $M^{\hat{p}} = \prod_{i=1}^m \widehat{M}_i^{p_1}$ and $T^{\hat{p}} = \prod_{i=1}^m \widehat{T}_i^{p_1}$ such that $M^{\hat{p}} = \otimes_{i=1}^m M^{p_i}$ and $T^{\hat{p}} = \otimes_{i=1}^m T^{p_i}$*

Note that with this notation, for all $i \neq j$ we have $\widehat{T}_j\widehat{M}_i = \widehat{M}_i\widehat{T}_j$ and when $i = j$ we have $\widehat{T}_i\widehat{M}_i = \omega_n\widehat{M}_i\widehat{T}_i$. Hence we can write $W(\hat{z}) = M^{\hat{z_2}}T^{\hat{z_1}}$ and all the commutation relations for the Weyl representation follow naturally.

We now have the foundation to begin discussing the Pauli group as well as the Clifford Hierarchy. We note again that the operators in the Weyl representation do not form a group, rather the composition of Weyl operators give a Weyl operator up to some phase determined by the commutation relations. It is clear from the discussion above that in the case of $G = \mathbb{Z}_n^m$, the phases introduced by the commutation relations are always of the form ω_n^h for some $h \in \mathbb{Z}_n^m$. So by carefully choosing our phase factors, we should be able to construct a group using the Weyl operators as a set of generators. In particular we have the following remark.

Remark 9. *For any $z \in \mathbb{Z}_n \times \mathbb{Z}_n$ and any $h \in \mathbb{Z}_n$ we can write the operator $\omega_n^h W(z)$ as the product of Weyl operators for $G = \mathbb{Z}_n$. This can be shown in the following way:*

$$\omega_n^h W(z) = (W(1, n-1)W(n-1, 1))^h W(z)$$

Now since Weyl representation for $G = \mathbb{Z}_n^m$ is formed by the tensor product of the Weyl operator of a single qudit, then this previous remark can be extended to show that for any $\hat{z} \in \mathbb{Z}_n^m \times \mathbb{Z}_n^m$ and any $h \in \mathbb{Z}_n$, we can write $\omega_n^h W(\hat{z})$ as a product of Weyl operators for $G = \mathbb{Z}_n^m$. It is exactly the operators of this form that define the group generated by the Weyl representation, which we will refer to as the Pauli group.

Definition 9. *The m-qudit Pauli group, P_m, is the group generated by the Weyl representation of $\mathbb{Z}_n^m \times \mathbb{Z}_n^m$,*

$$P_m = \{\omega_n^h W(\hat{z}) \mid h \in \mathbb{Z}_n, \hat{z} \in \mathbb{Z}_n^m \times \mathbb{Z}_n^m\}$$

From this we also define the Clifford group.

Definition 10. *the m-qudit Clifford group, C_m, is the group of unitary operators that preserve P_m under conjugation. In other words,*

$$C_m = \{U \in U(L^2(\mathbb{Z}_n^m)) \mid UP_mU^* = P_m\}$$

The identity of these Clifford operators are nothing new, we have seen many examples of elements of the Clifford group in previous sections. In particular, in Theorem 4.7 we saw that any rearrangement of the phase space Z by a symplectic $S \in Sp(G)$ would give rise to an equivalent Weyl representation, that we could relate back to the standard Weyl representation through conjugation by a unitary. Each of these unitaries that shuffle up the standard Weyl representation are Clifford operators. In fact, we will now show all Clifford operators are equivalent to the unitaries in Theorem 4.7 up to phase.

Proposition 3. *For all $C \in C_m$ there exists some phase $\lambda \in \mathbb{T}$ and some automorphism, $(S, f) \in Aut_0(H(\mathbb{Z}_n^m))$, of the Heisenberg group such that $C = \lambda U_{(S,f)}$.*

Proof. Suppose $C \in C_m$. Note that the product group $\mathbb{T}P_m \cong H(\mathbb{Z}_n^m)$ and so the mapping defined by $A \mapsto C^*AC$ for all $A \in \mathbb{T}P_m$ is related to an automorphism of the Heisenberg group. Additionally, since $Z(H(\mathbb{Z}_n^m)) \cong \mathbb{C}I$ then we can see that conjugation by C acts trivially on the center. Hence, for some $(S, f) \in Aut_0(H(\mathbb{Z}_n^m))$, we have

$$\begin{aligned} C^*AC &= U_{(S,f)}^* A U_{(S,f)} \\ \iff \quad ACU_{(S,f)}^* &= CU_{(S,f)}^* A \end{aligned}$$

Hence $CU_{(S,f)}^*$ commutes with every $A \in \mathbb{T}P_m$. But since the $span\{\mathbb{T}P_m\} = B(L^2(\mathbb{Z}_n^m))$, then this means that $CU_{(S,f)}^* = \alpha I$ for some $\alpha \in \mathbb{C}$ by Lemma 4.3. Since both C and $U_{(S,f)}^*$ are unitary, then $det(CU_{(S,f)}^*) \in \mathbb{T} \Rightarrow \alpha \in \mathbb{T}$. Thus, $CU_{(S,f)}^* = \alpha I \Rightarrow C = \alpha U_{(S,f)}$ where α is some phase. □

With both the Pauli group and the Clifford group set out, we can now continue to define the rest of the sets in the Clifford Hierarchy. Starting with the Pauli group as the first level, and the Clifford's as the second level of the Hierarchy, we define all the subsequent levels recursively through the following definition.

Definition 11. *For $k > 1$ we define the k^{th} level of the m-qudit Clifford Hierarchy, C_m^k, as the following:*

$$C_m^k = \{U \in U(L^2(G)) \mid U^* C_m^1 U \subset C_m^{k-1}\}$$

Where $C_m^1 = P_m$ is the Pauli group.

Remark 10. *The sets in the Clifford Hierarchy form a nested sequence of operators such that $C_m^k \subset C_m^{k+1}$. This statement is clear when $k = 1$ since $C_m^1 = P_m$ being a group means that for any $A, B \in C_m^1$, we have that $A^*BA \in P_m = C_m^1$. Hence, $C_m^1 \subset C_m^2$. Assuming that $C_m^d \subset C_m^{d+1}$ for all $d \leq k$, then for any $C \in C_m^{k+1}$ and $A \in C_m^1$, we have that $C^*AC \in C_m^k$ by definition of C_m^{k+1}. But then by assumption, $C^*AC \in C_m^{k+1}$, hence $C \in C_m^{k+2} \Rightarrow C_m^{k+1} \subset C_m^{k+2}$. Therefore, by induction the claim holds for all $k \in \mathbb{N}$.*

It should be noted here, that while the first two levels of the Clifford Hierarchy form groups, in general products of Clifford operators of higher levels do not produce Clifford operators of the same level, and therefore do not form groups themselves.

A major point of interest in studying the Clifford Hierarchy comes from the application of the Clifford operators as logical gates in quantum circuits. Both the first and the second level Clifford operators can be implemented fault tolerantly [3], and through the work of Gottesman and Chuang [4], it was shown that any operators in the higher levels of the Clifford Hierarchy can be applied fault tolerantly through a process known as *quantum gate teleportation* (for $n = 2$). Any operators in the $k > 3$ level of the Hierarchy would have to be implemented recursively, hence requiring larger and larger amounts of resources for computation. This was later improved by Zeng, Chen, and Chuang [5] such that those higher level Clifford operators could be applied through quantum gate teleportation with much fewer resources if those operators are semi-Clifford.

Definition 12. *For a given m, a k^{th} level Clifford operator $U \in C_m^k$ is* semi-Clifford *if there exists some $C_1, C_2 \in C_m^2$ such that $U = C_1 D C_2$ with $D \in D_m^k$ where*

$$D_m^k = \{D \in C_m^k \mid D \text{ is diagonal }\}$$

In other words, k^{th} level semi-Clifford operators are those operators in the k^{th} level of the Clifford Hierarchy that can be diagonalized by 2^{nd} level Clifford operators. We denote by SC_m^k the set of all k^{th} level semi-Clifford operators.

This process of fault tolerant quantum gate teleportation was then expanded by de Silva [1] to the qudit case when m is prime. As a proof of concept, we demonstrate the protocol proposed by de Silva for the case of a single qudit in the following section.

7 Quantum Gate Teleportation For a Single Qudit

We will begin the discussion of single qudit gate teleportation by first showing that for all prime n, $SC_1^3 = C_1^3$. This will allow us to make use of the simplified teleportation protocol introduced by de Silva [1] for single qudit systems, which can be generalized to larger composite systems. Before we can do this, however, we will need to introduce a few results. For the remainder of this section we will assume that n is an odd prime.

Lemma 7.1. *For any pair $(p,q) \in \mathbb{Z}_n \times \mathbb{Z}_n$ and for any $k \in \mathbb{Z}_n$, there is some $C \in C_1^2$ such that $W(p,q) = C^* \omega_n^k W(0,1) C$.*

Proof. We start by first finding an $S \in Sp(\mathbb{Z}_n)$ such that $S(0,1) = (p,q)$. Since S is a homomorphism we can define how S acts by specifying where it sends $(1,0)$ and $(0,1)$. Suppose that $S(1,0) = (s_1, s_2)$ and $S(0,1) = (s_3, s_4)$, then for $S(0,1)$ to give (p,q) we need $s_3 = p$ and $s_4 = q$. Now since S is simplectic, then we must have

$$\begin{aligned}
\Delta(Sz, Sw) &= \Delta((s_1 z_1 + p z_2, s_2 z_1 + q z_2), (s_1 w_1 + p w_2, s_2 w_1 + q w_2)) \\
&= \omega_n^{(s_2 w_1 + q w_2)(s_1 z_1 + p z_2) - (s_2 z_1 + q z_2)(s_1 w_1 + p w_2)} \\
&= \omega_n^{(s_1 q - s_2 p)(z_1 w_2 - z_2 w_1)} \\
&= \left(\omega_n^{z_1 w_2 - z_2 w_1}\right)^{s_1 q - s_2 p} \\
&= (\Delta(z,w))^{s_1 q - s_2 p} \\
&= \Delta(z,w)
\end{aligned}$$

This only happens when $s_1 q - s_2 p \equiv 1 (\text{mod n})$. Since we assumed that n is prime then there exist some $c \in \mathbb{Z}_n$ such that $cq \equiv 1(\text{mod n})$, so we can choose $s_1 = c$ and $s_2 = 0$ then the resulting transformation, S, defined by $S(1,0) = (c,0)$ and $S(0,1) = (p,q)$ is a symplectic transformation of $\mathbb{Z}_n \times \mathbb{Z}_n$.

With this, there exists some mapping $f : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$ such that $(S,f) \in Aut_0(H(\mathbb{Z}_n))$. Since n is an odd prime, then by our construction of the mappings f corresponding to S in section 3, it follows that, for all $z \in \mathbb{Z}_n \times \mathbb{Z}_n$, $f(z) = \omega_n^b$ for some $b \in \mathbb{Z}_n$. In particular, $f(0,1) = \omega_n^d$ for some $d \in \mathbb{Z}_n$. Since $f \cdot \Delta_z$ is a valid mapping corresponding to S for each $z \in \mathbb{Z}_n \times \mathbb{Z}_n$, then $f \cdot \Delta_{(k+d,0)}$ is a valid mapping for S and $(f \cdot \Delta_{(k+d,0)})(0,1) = \omega_n^{-k}$. So letting $f' = f \cdot \Delta_{(k+d,0)}$, then $(S,f') \in Aut_0(H(\mathbb{Z}_n))$. Then, from Theorem 4.7 we get that there is a unitary, $U_{(S,f')}$, which by Proposition 2 is in C_1^2, such that

$$\begin{aligned}
U_{(S,f')}^* W(0,1) U_{(S,f')} &= f'(0,1) W(S(0,1)) \\
&= \omega_n^{-k} W(p,q) \\
\Rightarrow \qquad W(p,q) &= U_{(S,f')}^* \omega_n^k W(0,1) U_{(S,f')}
\end{aligned}$$

Hence, $U_{(S,f')}$ is our desired operator. ∎

Lemma 7.2. *For any operator $D \in M_n(\mathbb{C})$, we have that D is diagonal if and only if it commutes with M.*

Proof. First, if D is diagonal, then since each M is also diagonal, D commutes with M. Now suppose that D commutes with M. Note that for all $z \in \mathbb{Z}_n$ we can write the operator $|z\rangle\langle z|$ as $|z\rangle\langle z| = \frac{1}{n}\sum_{p=1}^{n} \omega_n^{-pz} M^p$, since for $g \neq z$,

$$\frac{1}{n}\sum_{p=1}^{n} \omega_n^{-pz} M^p |g\rangle = \frac{1}{n}\sum_{p=1}^{n} \omega_n^{pg-pz}|g\rangle = \frac{1}{n}\omega_n^{g-z}\sum_{p=1}^{n}\omega_n^p = 0$$

and when $g = z$ we get,

$$\frac{1}{n}\sum_{p=1}^{n} \omega_n^{-pz} M^p |z\rangle = \frac{1}{n}\sum_{p=1}^{n} \omega_n^{pz-pz}|z\rangle = \frac{1}{n}\sum_{p=1}^{n}|z\rangle = |z\rangle$$

So since D commutes with M, then by above, D must also commute with each projection operator of the form $|z\rangle\langle z|$ for all $z \in \mathbb{Z}_n$, which can only happen if D is diagonal. □

Lemma 7.3. *For any $G \in C_1^3$, let $U = G^*MG$ and $V = G^*TG$. Then G is semi-Clifford if there exist some $p, q \in \mathbb{Z}_n$ such that $U^pV^q \in C_1^1$.*

Proof. Suppose that there is some $p, q \in \mathbb{Z}_n$ such that $U^pZ^q \in C_1^1$. Then for some $a, b, k \in \mathbb{Z}_n$, $\omega_n^k W(a,b) = U^pV^q = G^*W(p,q)G$. Now by Lemma 7.1, there exists some $C_1, C_2 \in C_1^2$ such that $W(p,q) = C_1^*W(0,1)C_1$ and $W(a,b) = C_2\omega_n^{-k}W(0,1)C_2^*$. With this we can write,

$$\begin{aligned} & C_2W(0,1)C_2^* = G^*C_1^*W(0,1)C_1G \\ \Rightarrow \quad & M = C_2^*G^*C_1^*MC_1GC_2 \\ \Rightarrow \quad & C_1GC_2M = MC_1GC_2 \end{aligned}$$

Therefore, C_1GC_2 commutes with M so by Lemma 7.2 is diagonal. Let $D = C_1GC_2$, then $G = C_1^*DC_2^*$ from which we can conclude that G is semi-Clifford as long as $D \in C_1^3$. But since $C_1 \in C_1^2$, then $C_1^*TC_1 \in C_1^1 \Rightarrow G^*C_1^*TC_1G \in C_1^2$. Additionally, since $C_2 \in C_1^2$ and since C_1^2 is a group, then $C_2^*G^*C_1^*TC_1GC_2 \in C_1^2$. Hence, $D^*MD = M \in C_1^2$, $D^*TD \in C_1^2$ and D is clearly unitary by construction, so $D \in C_1^3 \Rightarrow G \in SC_1^3 \Rightarrow SC_1^3 = C_1^3$. □

Lemma 7.4. *For any prime $n \in \mathbb{N}$, we have that $|Sp(\mathbb{Z}_n)| = n(n^2-1)$.*

Proof. This result comes from a simple counting exercise after noting that any transformation S with $S(1,0) = (s_1, s_2)$ and $S(0,1) = (s_3, s_4)$ is symplectic if $s_1s_4 - s_2s_3 \equiv 1 (\text{mod n})$ as in the proof of Lemma 7.1. □

We now have all of the necessary results to show that every third level Clifford operator is semi-Clifford.

Theorem 7.5. *Every operator of the third level of the Clifford Hierarchy for a single qudit system is semi-Clifford. In other words, $C_1^3 = SC_1^3$.*

Proof. Suppose $G \in C_1^3$ and let $U = G^*TG$ and $V = G^*MG$. Now since n is an odd prime, then $\mathbb{Z}_n$ is 2-regular, so $Aut_0(H(\mathbb{Z}_n)) \cong Sp(\mathbb{Z}_n) \ltimes \mathbb{Z}_n$. Additionally, from Proposition 3, it follows that $Aut_0(H(\mathbb{Z}_n)) \cong C_1^2/\mathbb{T}$, so by letting $[H] = \mathbb{T}H$ for all $H \in C_1^2$, we can define an isomorphism ϕ from $Aut_0(H(\mathbb{Z}_n))$ to $C_1^2/\mathbb{T}$. Let $(S, f), (R, g) \in Aut_0(H(\mathbb{Z}_n))$ be such that $\phi(S, f) = [U]$ and $\phi(R, g) = [V]$. From the definition of U and V, it follows that $UV = \omega_n VU$, hence $[U][V] = [V][U]$ which means that (S, f) and (R, g) must commute in $Aut_0(H(\mathbb{Z}_n))$ and in particular $SR = RS$. Because of this, we can define a homomorphism $\varphi : \mathbb{Z}_n \times \mathbb{Z}_n \mapsto Sp(\mathbb{Z}_n)$ such that $\varphi(p, q) = S^p R^q$. Now since $|Sp(\mathbb{Z}_n)| = n(n^2 - 1)$ with n prime, then $|\mathbb{Z}_n^2| \nmid |Sp(\mathbb{Z}_n)|$, so φ cannot be injective. This means that there must be some non-trivial $(p, q) \in \mathbb{Z}_n^2$ with $\varphi(p, q) = I \Rightarrow (S, f)^p \circ (R, g)^q = (I, \Delta_h)$ for some $h \in \mathbb{Z}_n$.

Now the unitary transformation $U_{(I,\Delta_h)}$ that brings $W(z) \mapsto \Delta_h(z)W(Iz) = \Delta_h(z)W(z)$ through conjugation satisfies the following,

$$\begin{aligned}
U^*_{(I,\Delta_h)} W(z) U_{(I,\Delta_h)} &= \Delta_h(z)W(z) \\
&= \Delta(z, h)W(h)^*W(h)W(z) \\
&= \Delta(z, h)\Delta(h, z)W^*(h)W(z)W(h) \\
&= W(h)^*W(z)W(h)
\end{aligned}$$

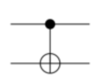
So by a similar line of reasoning from Proposition 3, it follows that $U_{(I,\Delta_h)} = \lambda W(h)$ for some phase $\lambda \in \mathbb{T} \Rightarrow [U_{(I,\Delta_h)}] = [W(h)]$. From this we get that,

$$\begin{aligned}
[U^p V^q] &= \phi(S, f)^p \phi(R, g)^q \\
&= \phi((S, f)^p (R, g)^q) \\
&= \phi(I, \Delta_h) \\
&= [U_{(I,\Delta_h)}] \\
&= [W(h)]
\end{aligned}$$

And so $U^p V^q$ is a Pauli operator for some non-trivial $(p, q) \in \mathbb{Z}_n^2$, therefore by Lemma 7.3, G is semi-Clifford. □

With the previous theorem, we can now make use of the simplified gate teleportation protocol proposed by de Silva to implement any third level Clifford operator of one qudit with less computational resources than originally proposed by Gottesman and Chuang [4]. We provide below the description of the additional quantum gates required.

Notation 2. *The symbol below denotes the CX gate which is the analogue to the qubit CNOT gate. The action of this gate on the tensor product of two basis states* $|g\rangle \otimes |h\rangle$ *is given by* $CX(|g\rangle \otimes |h\rangle) = |g\rangle \otimes |h + g\rangle$



Additionally, we will require The following symbol to denote an observation of the state of a bit in the standard basis.

Here, the two lines on the right denote the classical output of the observation performed on the state, which as we will see, will be fed into other series of gates.

Finally, we will denote by H the Hadamard gate which brings the basis state $|g\rangle \mapsto \frac{1}{\sqrt{n}} \sum_{h\in\mathbb{Z}_n} \omega_n^{gh}|h\rangle$. From this definition it follows that applying the Hadamard transformation twice has the effect, $H^2|g\rangle = |-g\rangle$.

Now, suppose that $G \in C_1^3$ is a third level Clifford operator and that $G = C_1 D C_2$ for some $C_1, C_2 \in C_1^2$ and $D \in D_1^3$. Then the circuit which implements the operator G on an arbitrary state $|\psi\rangle$ is the following:

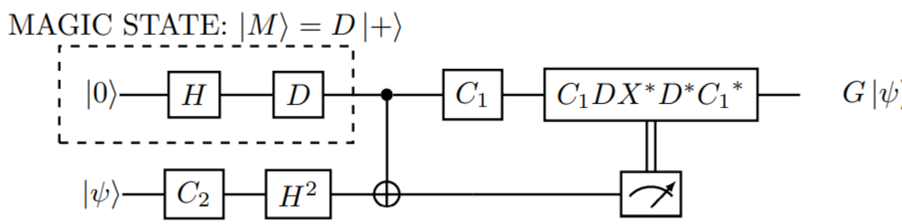


Figure 1: Quantum Gate Teleportation protocol for a single qudit, third level Clifford gate

Where X represents the operator T^{-K} where K is the result of the observation performed on the second qudit. To ensure the validity of the circuit presented, we will now walk through each step. We begin with the input state of $|0\rangle \otimes |\psi\rangle = \sum_{i=0}^{n-1} \psi_i |0\rangle \otimes |i\rangle$, to which the first set of gates, $(DH) \otimes (H^2 C_2)$, is applied. The result is:

$$\begin{aligned}
& [(DH) \otimes (H^2 C_2)](|0\rangle \otimes |\psi\rangle) \\
= & \sum_{i=0}^{n-1} \psi_i (DH|0\rangle) \otimes (H^2 C_2 |i\rangle) \\
= & \sum_{i=0}^{n-1} \psi_i \left(\sum_{j=0}^{n-1} \frac{1}{\sqrt{n}} D|j\rangle \right) \otimes \left(\sum_{k=0}^{n-1} [C_2]_{k,i} H^2 |k\rangle \right) \\
= & \sum_{i=0}^{n-1} \psi_i \left(\sum_{j=0}^{n-1} \frac{1}{\sqrt{n}} [D]_{j,j} |j\rangle \right) \otimes \left(\sum_{k=0}^{n-1} [C_2]_{k,i} |-k\rangle \right) \\
= & \sum_{i,j,k=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{k,i} [D]_{j,j} |j\rangle \otimes |-k\rangle
\end{aligned}$$

Next the CX gate is applied, giving:

$$\sum_{i,j,k=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{k,i} [D]_{j,j} |j\rangle \otimes |j-k\rangle$$

Then, moving further to the right we apply the operator $(C_1 \otimes I)$ to get:

$$\begin{aligned}
& \sum_{i,j,k=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{k,i} [D]_{j,j} \, (C_1 |j\rangle) \otimes |j-k\rangle \\
= & \sum_{i,j,k=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{k,i} [D]_{j,j} \left(\sum_{q=0}^{n-1} [C_1]_{q,j} |q\rangle \right) \otimes |j-k\rangle \\
= & \sum_{i,j,k,q=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{k,i} [D]_{j,j} [C_1]_{q,j} |q\rangle \otimes |j-k\rangle
\end{aligned}$$

At this point we then make an observation of the state of the second qubit. The outcome of this observation is probabilistic and is related to the modulus squared of the coefficient for a given pure state as mentioned in the preliminaries. Once an observation is made, say the result K is observed, then by postulate 3, the state of the system collapses into a linear combination of the composite state in which the second bit is in state K. So, if we observe a state K, we have that $j - k = K \Rightarrow k = j - K$ and the state becomes:

$$\sum_{i,j,q=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{j-K,i} [D]_{j,j} [C_1]_{q,j} |q\rangle \otimes |K\rangle$$

It should be noted that the specific value observed in the second qudit does not matter, however this value will tell us how many times to apply the T

operator in the last set of gates to properly perform the teleportation. Once this value is observed, the second qudit can be discarded, leaving us with just the state of the first qudit:

$$\sum_{i,j,q=0}^{n-1} \frac{1}{\sqrt{n}} \psi_i [C_2]_{j-K,i} [D]_{j,j} [C_1]_{q,j} |q\rangle$$

This state is just $C_1 D T^{-K} C_2 |\psi\rangle$, which can be verified by expanding out the operators. The last set of operators apply $(C_1 D T^K D^* C_1^*)$ to the state above giving a final state of:

$$(C_1 D T^K D^* C_1^*)(C_1 D T^{-K} C_2 |\psi\rangle) = C_1 D C_2 |\psi\rangle = G|\psi\rangle$$

And hence the semi-Clifford gate G has been applied to the state $|\psi\rangle$. One last point of note here is that all of the operators presented in the circuit are Pauli gates or second level Clifford gates, and can therefore be implemented fault tolerantly, except for the operator D. This does not cause much problems because $D \in C_1^3$ and so $DX^*D^* \in C_1^2$ since X^* is a Pauli gate. Therefore, the last set of gates applied, $C_1 D X^* D^* C_1^*$, is a second level Clifford operator due to C_1^2 being a group. So the only point of difficulty comes with the D gate in what is denoted as the magic state in Figure 1. This D gate cannot in general be implemented fault tolerantly, however $DH \in C_1^3$, and as shown by Zhou, Leung, and Chuang [6], through a series of second level Clifford gates and observations, the result of $A|0\rangle$ for any $A \in C_1^3$ can be prepared fault tolerantly. Hence, $DH|0\rangle$ can be prepared in advance of the required computation, and entangled with the desired input state when needed.

8 Conclusion

We have seen, starting from the very foundation of the Heisenberg group, how one can derive many of the important properties of the Weyl representation. In addition, the unitary equivalence of any conjugate pair of operators with the Weyl Representation was established through the Stone von-Neumann theorem, giving a clean way to translate between equivalent phase space pictures of operators which act on a given Hilbert space. It should be emphasised again that many of the results presented can be generalized to the case of locally compact abelian groups with some added technicalities [2]. This generalization is of historical importance and has many applications such as demonstrating the equivalence of the Heisenberg and Schrodinger picture of quantum mechanics.

The relation between the Weyl representation and unitary operations acting as quantum logic gates was also established and the Clifford hierarchy was presented. Through the work of [1, 4, 5, 6], this recursively defined Hierarchy of operators provides a large bank of operations one could utilize fault tolerantly in computations with quantum systems. Many of the higher level gates, however, do not have efficient implementations and instead are performed by recursive processes greatly increasing the resources needed for computation. In the case of semi-Clifford gates, this process has been simplified by [1], and the implementation of this simplified circuit for the simple case of third level, single qudit gate was presented in the previous section. This however was done only for prime n, as shown by de Silva, yet many of the results presented can be extended to odd n. Hence, in the future it would be natural to pursue a generalization of the gate teleportation protocol for 2-regular finite abelian groups. The improvements introduced by this simplified gate teleportation procedure can have drastic impacts in areas such as quantum error correcting code, giving much more control over operations on strings of qudits.

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