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HONOURS PROJECT

TITLE: Analysis of Point Schemes of Artin-Schelter Regular Algebras

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Chapter 1

Introduction

1.1 Introduction

We work over a field k, with $\text{char}k = 0$ and $k = \bar{k}$.

James J. Zhang and Jun Zhang's paper, *Double Extension Regular Algebras of Type (14641)*, constructs a large class of new associative Artin-Schelter regular algebras of global dimension four that are double Ore extensions. It labelled these 26 algebra families A, B, ..., Z.

Each algebra family has four generators: x_1, x_2, y_1, and y_2, and is defined by six relations; four of which mix the x_i's and y_i's and are referred to as the mixing relations, and two non-mixing relations.

Mixing Relations:

$$MR11 : y_1x_1 = a_{1111}x_1y_1 + a_{1112}x_2y_1 + a_{1211}x_1y_2 + a_{1212}x_2y_2$$

$$MR12 : y_1x_2 = a_{1121}x_1y_1 + a_{1122}x_2y_1 + a_{1221}x_1y_2 + a_{1222}x_2y_2$$

$$MR21 : y_2x_1 = a_{2111}x_1y_1 + a_{2112}x_2y_1 + a_{2211}x_1y_2 + a_{2212}x_2y_2$$

$$MR22 : y_2x_2 = a_{2121}x_1y_1 + a_{2122}x_2y_1 + a_{2221}x_1y_2 + a_{2222}x_2y_2$$

Non-Mixing Relations:

$$NRx : x_2x_1 = q_{12}x_1x_2 + q_{11}x_2^2 \qquad NRy : y_2y_1 = p_{12}y_1y_2 + p_{11}y_2^2$$

Each algebra family is then described through the form $\{\sum, P, Q\}$, with the 4x4 matrix $\sum = (a_{ijkl})$ providing the values for the variables in the four mixing relations, $Q = \{q_{12}, q_{11}\}$ providing those for NRx, and $P = \{p_{12}, p_{11}\}$ providing those for NRy.

From this, I computed an invariant for each algebra family. First, I created a polynomial ring S in 4x4 variables. Then, for convenience in computation, I redefined the variables as $w = x_1$, $x = x_2$, $y = y_1$ and $z = y_2$. I then defined an ideal T from each algebra family's six relations, created an invariant V of the alegbra family by combining S and T, and then I analyzed this invariant V by calculating the degree and dimension of it and the associated primes of its primary decomposition, if the invariant is a radical ideal, the degree and dimension of the associated primes of its radical if the invariant is not radical, the number of associated primes, and the singular locus of the invariant. Note that V is called the point scheme of the given algebra and is a subscheme of the projective variety $P^3 \times P^3$.

Definitions:

This definition is from page 6 of [6]. Let A be a k-algebra, and let $\sigma : A \to M_2(A)$ be an algebra homomorphism from A to the 2x2 matrix $M_2(A)$ over A such that $\sigma(rs) = \sigma(r) \times \sigma(s)$ for all r, $s \in A$. Then a k-linear map $\delta : A \to A^{\oplus 2}$ is a σ-derivation if:

$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all r, $s \in A$.

This definition is taken from [6]. Given an algebra A, an algebra automorphism σ of A and a σ-derivation δ of A, an Ore extension of A associated to (σ, δ) is obtained by adding a generator y to A such that, for all $r \in A$, $yr = \sigma(r)y + \delta(r)$.

A double Ore extension is a class of Artin-Schelter Regular Algebras of dimension 4 created by Zhang and Zhang, and is defined in [6, 7]. A double extension of an algebra A is denoted by $A_P[y_1, y_2; \sigma, \delta, \tau]$, and the double extension is a natural generalization of the Ore extension, as stated in [6].

As defined in [6], $P := \{p_{12}, p_{11}\}$ is the parameter of the double extension, and $\tau := \{\tau_1, \tau_2, \tau_0\}$ is the tail of the double extension, with $p_{12}, p_{11} \in k$ and $\tau_1, \tau_2, \tau_0 \in A$.

The map σ is an algebra homomorphism $A \to M_2(A)$ where $M_2(A)$ is the 2 x 2 matrix algebra over A, and δ is a σ-derivation $A \to A^{\oplus 2}$ where $A^{\oplus 2}$ is the 2 x 1 matrix over A satisfying:

$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all r, $s \in A$.

The two generators y_1 and y_2 satisfy the relation:

$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0$ where $p_{12}, p_{11} \in k$ and $\tau_2, \tau_1, \tau_0 \in A$.

There are other conditions relating y_1 and y_2 to $\{P, \sigma, \delta, \tau\}$. The example provided in [6] states that we need $\binom{y_1 r}{y_2 r} = \sigma(r)\binom{y_1}{y_2} + \delta(r)$ for all $r \in A$.

As stated in [7], a double Ore extension of type (14641) can be defined as $A_P[y_1, y_2; \sigma]$, with $\delta = 0$ and $\tau = (0, 0, 0)$.

As stated on page 4 of [7], the global dimension of an Artin-Schelter Regular Algebra is equivalent to the dimension of the algebra, so a regular algebra of dimension 4 is a regular algebra with global dimension $d = 4$.

As defined in [7], an algebra A is Gorenstein if there exists an integer ℓ such that: $\mathrm{Ext}^i_A({}_Ak, A) = k(\ell)$ if $i = d$ and 0 if $i \neq d$, where k is the trivial A-module, and the same condition holds for the right trivial A-module ${}_kA$.

As defined in [7], an algebra A has finite Gelfand-Kirillov dimension if there exists a positive number c such that $\dim A_n < c * n^c$ for all $n \in \mathbf{N}$.

This definition is taken from [7]. An algebra A is connected graded if $A = k \oplus A_1 \oplus A_2 \oplus ...$, where $1 \in k = A_0$ and $A_i A_j \subset A_{i+j}$ for all i, j, and throughout k is an algebraically closed commutative field.

This definition is taken from [7]. An Artin-Schelter Regular Algebra is a connected graded algebra A that satisfies the following three conditions:

a) A has finite global dimension d

b) A is Gorenstein.

c) A has finite Gelfand-Kirillov dimension.

This definition is from page 145 of [4]. Let $R = \oplus_{k \geq 0} R_k$ be a graded commutative ring with unity, and let $\mathrm{Proj}(R)$ be the set:

$\{p \subset R : p \text{ is a homogeneous ideal of } R, \oplus_{k>0} R_k \not\subset p\}$

Such that the closed subsets are the $V(a)$ for a homogeneous ideal of R, where $V(a) := \{p \in \mathrm{Proj}(R) : p \supset a\}$. Then $P^n_k = \mathrm{Proj}(k[x_1, ..., x_n])$ is the Projective Space of dimension n over k.

This definition is from [2]. Consider an Artin-Schelter Regular Algebra of dimension 4 as defined above; for example, consider the algebra family A, which is defined by the following six relations:

$wx - xw,\ yy + yz - zy,\ wy - yw,\ xy + wz - yx,\ zw - wz,\ zx + 2xy + wz - xz$

The multilinear relations corresponding to this are the following; note that 1 indicates the variable on the left, and 2 indicates the variable on the right:

$w_1x_2 - x_1w_2,\ y_1y_2 + y_1z_2 - z_1y_2,\ w_1y_2 - y_1w_2,\ x_1y_2 + w_1z_2 - y_1x_2,\ z_1w_2 - w_1z_2,\ z_1x_2 + 2x_1y_2 + w_1z_2 - x_1z_2$

where the coordinates in $P^3 \times P^3$ are labelled as $(w_1, x_1, y_1, z_1; w_2, x_2, y_2, z_2)$. The Point Scheme $V \subset P^3 \times P^3$ of an Artin Schelter Regular algebra A denotes the locus of common zeroes of these equations.

One reason that these point schemes are of interest can be found in Corollary 3.13 on page 47 of [2].

As defined in [4], for every n, let P^n_k be the projective space over k of dimension n. Let m, $n \in \mathbf{N}$. The Segre Embedding on $P^n \times P^m$ is the map:

$s_{n,m} : P^n \times P^m \to P^{(n+1)(m+1)-1},\ ([x_0, ..., x_n], [y_0, ..., y_m]) \mapsto [..., x_iy_j, ...]$

i.e. the map sending $([x_0, ..., x_n], [y_0, ..., y_m])$ to the point of $P^{(n+1)(m+1)-1}$ whose coordinates are the product $x_i y_j$ for all $i = 0, 1, ..., n$, $j = 0, 1, ... m$.

As defined in [4], let $z_{i,j}$ for $i = 0, 1, ..., n$ and $j = 0, 1, ..., m$ be the coordinates on $P^{(n+1)(m+1)-1}$. The image of the Segre Embedding $s_{n,m}$ is the zero locus of the polynomials of degree 2: $z_{i,j}z_{k,\ell}$ - $z_{i,\ell}z_{k,j}$, with i, $k \in 0, 1, ..., n$ and j, $\ell \in 0, 1, ..., m$. This image is a smooth variety of dimension $m + n$ and degree $\binom{m+n}{n}$, called the Segre Variety.

This definition is from page 116 of [1]. Let $A = \bigoplus_{n=0}^{\infty}$ be a graded algebra. the Hilbert Series of A, $\mathrm{H}_A(t)$, is the series $\sum_{i \in \mathbf{Z}} \dim_k(A_i)t^i$.

This definition is from the theorem on pages 81 and 82 of [5] and from Theorem 11.1 and corollary 11.2 on page 117 of [1]. Consider the algebra A above as a projective variety and consider A_d as above for $d >> 0$. We have that $\dim_k(A_d)$ is equal to a polynomial $\mathrm{h}_A(d) = e_0t^d + e_1t^{d-1} + ... + e_d$ called the Hilbert Polynomial, with the degree$(h_A(d)) = \dim A$, and the leading coefficient $e_0 = \frac{\text{degree}(A)}{d!}$.

This definition is from page 41 of [4]. Let X be an algebraic variety of dimension n in P^m. The degree of X is the number of intersection points between X and a generic $(m - n)$-dimensional linear subspace. As stated on the same page of [4], this is equivalent to the degree stated above with X as the A above.

This definition is from page 51 of [1]. Let R be a ring and I be an ideal of R. A primary decomposition is an expression of I as a finite intersection of primary ideals $I = \bigcap_{i=0}^{n} J_i$. If all J_i are distinct and $J_i \not\supseteq \bigcap_{j \neq i} J_j \ \forall \ i = 1, ..., n$, then the primary decomposition is minimal. Any primary decomposition can be reduced to a minimal primary decomposition by removing superfluous J_i's. A primary decomposition need not necessarily exist, and an ideal I is called decomposable if $\exists$ a primary decomposition of I.

As defined in page 74 of [1], A ring R is Noetherian if it fulfills the ascending chain condition ($a.c.c$): for any ascending chain of subrings $R_1 \subseteq R_2 \subseteq ... \subseteq R$, $\exists$ $m \geq 1$ such that $N_i = N_m \ \forall \ m \geq i$.

As stated in a theorem on page 81 of [4], every ideal I of a Noetherian Ring is decomposable, and as proven in [7], the 26 algebra families that were constructed in it are Noetherian. Thus, they are all decomposable.

Definition from page 57 of [1]. Let R be a ring and let $I \subset R$ be an ideal of R. Then $\sqrt{I} = \{a \in R : a^n \in I$ for some $n \in \mathbf{N}\}$ is the Radical of I in R. If $I = \sqrt{I}$, then I is a Radical Ideal in R.

This definition is from pages 85 to 86 of [5]. Consider a line ℓ in V that passes through the point $p = (p_1, ..., p_n)$ and has the direction $(q_1, ..., q_n)$. Then the line ℓ can be written as $\{(tq_1 + p_1, ..., tq_n + p_n) : t \in \mathbf{C}\}$.

Let $F_1, ..., F_r$ be the generators of the radical ideal $I(V)$ defining V. Then $V \cap \ell$ is found by solving for t in the system of linear equations $\{F_i(tq_1, ..., tq_n) = 0 : i = 1, ..., r\}$. Since the q_is are fixed, each $F_i(tq_1, ..., tq_n)$ is a polynomial with variable

t. The intersection $V \cap \ell$ corresponds to common roots, and the multiplicity of $V \cap \ell$ is the exponent of the highest power of t that divides all the polynomials.

We have that ℓ is tangent to V at p if the multiplicity of $V \cap \ell$ at p exceeds 1.

The definition is from page 86 of [5]. The tangent space T_pV of V at p is the union of all points lying on lines tangent to V at p.

The definition is from page 92 of [5]. A point p on a projective variety V is smooth if the dimension of the tangent space at p is equal to the dimension of V at p, i.e. $\dim T_pV = \dim_p V$. Otherwise, the point p is singular.

The definition is from page 92 of [5]. The Singular Locus of a projective variety is the set of singular points of the variety.

Chapter 2

Results

2.1 Results

Note: Deg = Degree, Dim = Dimension, PD = Primary Decomposition of V, Degree PD/Dimension PD is the degree/dimension of the associated primes in the primary decomposition of V, # PD is the number of associated primes in the primary decomposition of V, Intersections = the sums of the associated primes ($p_0 + p_1$, etc.). (add a bit more detail; i.e. explain what's in table A as an example for all of these), and SingularLocus = set of all the singular points. IsRadical = boolean value of whether or not the ideal is equal to its radical. p, g and f are parameters, and w, x, y, and z are algebra generators. The . in the tables refer to any data that was unable to be obtained, either due to a loading error when running the code or for another problem when running.

For example, for the invariant of Algebra A below, there are no special parameters p, q, or f, the degree is 10, the dimension is 2, the degree of each associated prime is 2, 2, 6, 3, and 8 respectively, the dimension of each associated prime is 2, 2, 2, 1, and 1 respectively, the invariant is not a radical ideal, the degree and dimension of the associated primes of the primary decomposition of the radical are 2, 2, and 2 respectively, and the minimum number of associated primes is 3. Let p_0, p_1 and p_2 be these respective associated primes, then the degree of $p_0 + p_1$ is 1, the degree of $p_0 + p_2$ is 1, and the degree of $p_1 + p_2$ is 1. The dimension of $p_0 + p_1$ is 1, the degree of $p_0 + p_2$ is 1, and the degree of $p_1 + p_2$ is 0. Finally, pd-SingularLocus is the dimension of the singular locus of the associated primes.

Note that degree = 1 and dimension = 1 means that $p_0 + p_1$ and $p_0 + p_2$ are lines, while degree = 1 and dimension = 0 means that $p_1 + p_2$ is a point. For another example, if degree = 2 and dimension = 1, then the intersection of the associated primes is a conic.

From here, I would first continue by computing the remaining data that was not obtained for the invariant of each of the 26 algebra families, including the data marked by . in the tables and the data for the invariant of each of the algebra families H, S, U, V, W, X, and Y. I would then compute the invariant of algebras that are not one of the 26 for comparison. An example of algebras from other papers that would provide a useful comparison would be those described in Andrew Davies' *Cocycle Twists of Algebras* thesis [3].

Relations	$wx - xw$ $yy + yz - zy$ $wy - yw$ $xy + wz - yx$ $zw - wz$ $zx + 2xy + wz - xz$
Degree Dimension	10 2
Degree PD Dim. PD	2, 2, 6, 3, 8 2, 2, 2, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 2 2, 2, 2 3
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1 1, 1, 0 0, 0, 0

Table 2.1: Algebra A

Relations	$p^2 = -1$ $xw - pwx$ $zy - pyz$ $yw - xz$ $yx - wz$ $zw + xy$ $zx - wy$
Degree Dimension	4 2
Degree PD Dim. PD	2, 2, 4, 4, 4, 4 2, 2, 1, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2 2, 2 2
Deg Intersections Dim Intersections pd-SingularLocus	1 0 0, 0

Table 2.2: Algebra B

Relations	$p^2 + p + 1 = 0$ $zy - pyz$ $xw - pwx$ $yw + wy - p^2xy - wz + pxz$ $yx + pwy - xy - wz + pxz$ $zw + pwy + 2p^2xy - pwz + pxz$ $zx + pwy - p^2xy - wz + xz$
Degree Dimension	10 2
Degree PD Dim. PD	2, 2, 2, 2, 2, 2, 4 2, 2, 2, 2, 2, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 2, 2, 2 2, 2, 2, 2, 2 5
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1, 1, 1, 1, 1, 1, 1, 1 0, 1, 1, 1, 1, 1, 1, 0, 0, 0 0, 0, 0, 0, 0

Table 2.3: Algebra C

Relations	p is a general parameter which could be ± 1 $xw + wx$ $zy - pyz$ $yw + pwy$ $yx + p^2xy - wz$ $zw - pwz$ $zx - wy - xz$
Degree Dimension	8 2
Degree PD Dim. PD	2, 2, 2, 2, 4, 4, 4 2, 2, 2, 2, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 2, 2 2, 2, 2, 2 4
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1, 1, 1, 1 1, 1, 0, 1, 1, 1 0, 0, 0, 0

Table 2.4: Algebra D

Relations	$p^2 = -1$ $xw + wx$ $zy - pyz$ $yw - wz - xz$ $yx - wz + xz$ $zw + wy - xy$ $zx - wy - xy$
Degree Dimension	12 2
Degree PD Dim. PD	2, 2, 2, 2, 2, 2 2, 2, 2, 2, 2, 2
Is Radical	true
# PD	6
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1 .

Table 2.5: Algebra E

Relations	$p^2 = -1$ $xw + wx$ $zy - pyz$ $yw - wy + pxy - wz + xz$ $yx + pwy - xy - wz - xz$ $zw + pwy - pxy - pwz - xz$ $zx + pwy + pxy - wz + pxz$
Degree Dimension	6 2
Degree PD Dim. PD	2, 2, 2, 2, 2, 2, 2, 4 2, 2, 2, 1, 1, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 2, 2 2, 2, 2, 2 3
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1 1, 1, 0 0, 0, 0

Table 2.6: Algebra F

Relations	p is general, $f \neq 0$ $xw - wx$ $zy - pyz$ $yw - pwy$ $yx - pwy - p^2xy - wz$ $zw - pwz$ $zx - fwy + wz - xz$
Degree Dimension	8 2
Degree PD Dim. PD	2, 2, 2, 2, 4, 4, 4 2, 2, 2, 2, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 2, 2 2, 2, 2, 2 4
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1, 1, 1, 1 1, 1, 0, 1, 1, 1 0, 0, 0, 0

Table 2.7: Algebra G

Relations	$q^2 = -1$ $xw - qwx$ $zy + yz$ $yw + qwy + qxy - wz + qxz$ $yx - wy - xy - wz + qxz$ $zw - wy - qxy - qwz + qxz$ $zx + wy + qxy - wz + xz$
Degree Dimension	4 2
Degree PD Dim. PD	2, 2, 2, 2, 4, 4, 2, 2 2, 2, 1, 1, 1, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2 2, 2 2
Deg Intersections Dim Intersections pd-SingularLocus	1 0 0, 0

Table 2.8: Algebra I

Relations	$q^2 = -1$ $xw - qwx$ $zy + yz$ $yw - xy - xz$ $yx + wy - xy$ $zw - xy + xz$ $zx - wy - xy$
Degree Dimension	8 2
Degree PD Dim. PD	1, 2, 1, 2, 2, 6, 2, 2 2, 2, 2, 2, 2, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 1, 2, 1, 2 2, 2, 2, 2, 2 5
Deg Intersections Dim Intersections pd-SingularLocus	. . 0, -1, 0, -1, 0

Table 2.9: Algebra J

Relations	$q = \pm 1,\ f \neq 0$ $xw - qwx$ $zy + yz$ $yw - wy$ $yx - xz$ $zw - wz$ $zx - fxy$
Degree Dimension	12 2
Degree PD Dim. PD	4, 2, 2, 2, 2 2, 2, 2, 2, 2
Is Radical	true
# PD	5
Deg Intersections Dim Intersections pd-SingularLocus	. . .

Table 2.10: Algebra K

Relations	$q = \pm 1,\ f \neq 0$ $xw - qwx$ $zy + yz$ $yw - fwz$ $yx - xz$ $zw - fwy$ $zx - xy$
Degree Dimension	12 2
Degree PD Dim. PD	4, 2, 4, 2 2, 2, 2, 2
Is Radical	true
# PD	4
Deg Intersections Dim Intersections pd-SingularLocus	. . 1, 0, 1, 0

Table 2.11: Algebra L

Relations	$f \neq 1$ $xw + wx$ $zy + yz$ $yw - xy - wz$ $yx - fwy + xz$ $zw - wy + xz$ $zx + xy + fwz$
Degree Dimension	12 2
Degree PD Dim. PD	2, 2, 8 2, 2, 2
Is Radical	true
# PD	3
Deg Intersections Dim Intersections pd-SingularLocus	1, 2, . 0, 1, . .

Table 2.12: Algebra M

Relations	$f^2 \neq g^2$ $xw + wx$ $zy + yz$ $yw + gxy - fxz$ $yx - gwy - fwz$ $zw - fxy + gxz$ $zx - fwy - gwz$
Degree Dimension	4 2
Degree PD Dim. PD	2, 2, 4, 4, 4, 4 2, 2, 1, 1, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2 2, 2 2
Deg Intersections Dim Intersections pd-SingularLocus	1 0 0, 0

Table 2.13: Algebra N

	$f = 0$	$f \neq 1$
Relations	$xw + wx$ $zy + yz$ $yw - wy - fxz$ $yx + xy - wz$ $zw - fxy + wz$ $zx - wy - xz$	
Degree Dimension	4 3	12 2
Degree PD Dim. PD	2, 4, 8 2, 3, 2	2, 2, 8 2, 2, 2
Is Radical	false	true
DegPD Radical DimPD Radical # PD	4, 2 3, 2 2	. . 3
Deg Intersections Dim Intersections pd-SingularLocus	. . .	. . .

Table 2.14: Algebra O

	$f = 0$	$f \neq 1$
Relations	$xw + wx$ $zy + yz$ $yw - wz - fxz$ $yx - wz - xz$ $zw - wy + fxy$ $zx + wy - xy$	
Degree Dimension	8 2	4 2
Degree PD Dim. PD	2, 4, 2, 8, 2, 2 2, 2, 2, 1, 1, 1	2, 2, 4, 4, 4, 4 2, 2, 1, 1, 1, 1
Is Radical	false	false
DegPD Radical DimPD Radical # PD	4, 2, 2 2, 2, 2 3	2, 2 2, 2 2
Deg Intersections Dim Intersections pd-SingularLocus	1, 2, . 1, 1, . 1, 0, 0	1 0 0, 0

Table 2.15: Algebra P

Relations	$xw + wx$ $zy + yz$ $yw - wz$ $yx - wy - xy - wz$ $zw + wy$ $zx - wy + wz - xz$
Degree Dimension	12 2
Degree PD Dim. PD	2, 2, 2, 2, 2, 2 2, 2, 2, 2, 2, 2
Is Radical	true
# PD	6
Deg Intersections Dim Intersections pd-SingularLocus	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0 0, 0, 0, 0, 0, 0

Table 2.16: Algebra Q

Relations	$xw + wx$ $zy + yz$ $yw - wy - xy - wz$ $yx - wz$ $zw - xy$ $zx + xy + wz - xz$
Degree Dimension	8 3
Degree PD Dim. PD	8 3
Is Radical	true
# PD	1
Deg Intersections Dim Intersections pd-SingularLocus	. . .

Table 2.17: Algebra R

Relations	$xw + wx$ $zy + yz$ $yw + wy - xy - wz - xz$ $yx - wy + xy - wz - xz$ $zw - wy - xy - wz + xz$ $zx - wy - xy + wz - xz$
Degree Dimension	14 2
Degree PD Dim. PD	8, 4, 2 2, 2, 2
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 8 2, 2, 2 3
Deg Intersections Dim Intersections pd-SingularLocus	. . .

Table 2.18: Algebra T

Relations	f is a square, $f(f+1) \neq 0$ $xw + wx$ $zy - yz$ $yw - wz - xz$ $yx - xy - wz$ $zw - fxy + wz$ $zx - fwz + xz$
Degree Dimension	6 2
Degree PD Dim. PD	2, 2, 1, 1, 6, 6 2, 2, 2, 2, 1, 1
Is Radical	false
DegPD Radical DimPD Radical # PD	2, 2, 1, 1 2, 2, 2, 2 4
Deg Intersections Dim Intersections pd-SingularLocus	. . .

Table 2.19: Algebra Z

Chapter 3

Appendix

3.1 Appendix

The following code was written in the Macaulay2 programming language.

Code Used to Obtain the Results:

```
-- Ring and Segre Ideal function:

k = QQ

n = 3

rm = m->k[x_{0,0}..x_{m,m}]

r = rm n

s = minors(2, genericMatrix(r,n+1,n+1))

ww = x_{0,0}

wx = x_{0,1}

wy = x_{0,2}

wz = x_{0,3}

xw = x_{1,0}

xx = x_{1,1}

xy = x_{1,2}

xz = x_{1,3}

yw = x_{2,0}
```

```
yx = x_{2,1}

yy = x_{2,2}

yz = x_{2,3}

zw = x_{3,0}

zx = x_{3,1}

zy = x_{3,2}

zz = x_{3,3}

-- General Relations (Q = (q12, q11), P = (p12, p11)):

NRx:  x2x1 = q12x1x2 + q11x1x1

NRy:  y2y1 = p12y1y2 + p11y1y1

t = ideal(the six relations of the algebra family)

v = s + t

-- Analysis of v:

v = saturate(v)

degree(v)

dim(v)

pdv = primaryDecomposition(v)

-- only if v == radical(v) returns false

apply(pdv, degree)

apply(pdv, dim)

v == radical(v)

v = radical(v)

pd = primaryDecomposition(v)

apply(pd, degree)

apply(pd, dim)

#primaryDecomposition(v)

-- dim singularLocus(v)
```

```
apply(pd, x->(dim singularLocus(x)))

pds = primaryDecomposition(singularLocus(v))

degree(pds)

dim(pds)
```

The code for the algebras (these would be the t above):

– Algebra A

```
ideal(wx - xw, yy + yz - zy, wy - yw, xy + wz - yx, zw - wz, zx +
2*xy + wz - xz)
```

– Algebra B

```
ideal(xw - p*wx, zy - p*yz, yw - xz, yx - wz, zw + xy, zx - wy)
```

– Algebra C

```
ideal(zy-p*yz, xw-p*wx, yw+wy-(p^2)*xy-wz+p*xz, yx+p*wy-xy-wz+p*xz,
zw+p*wy+2*(p^2)*xy-p*wz+p*xz, zx+p*wy-(p^2)*xy-wz+xz)
```

– Algebra D

```
ideal(xw + wx, zy - p*yz, yw + p*wy, yx + (p^2)*xy - wz, zw - p*wz,
zx - wy - xz)
```

– Algebra E

```
ideal(xw + wx, zy - p*yz, yw - wz - xz, yx - wz + xz, zw + wy - xy,
zx - wy - xy)
```

– Algebra F

```
ideal(xw + wx, zy - p*yz, yw - wy + p*xy - wz + xz, yx + p*wy - xy
- wz - xz, zw + p*wy - p*xy - p*wz - xz, zx + p*wy + p*xy - wz + p*xz)
```

– Algebra G

```
ideal(xw - wx, zy - p*yz, yw - p*wy, yx - p*wy - (p^2)*xy - wz, zw
- p*wz, zx - f*wy + wz - xz)
```

– Algebra I

```
ideal(xw - q*wx, zy + yz, yw + q*wy + q*xy - wz + q*xz, yx - wy -
xy - wz + q*xz, zw - wy - q*xy - q*wz + q*xz, zx + wy + q*xy - wz + xz)
```

– Algebra J

```
ideal(xw - q*wx, zy + yz, yw - xy - xz, yx + wy - xy, zw - xy + xz,
```

zx - wy - xy)

– Algebra K

ideal(xw - q*wx, zy + yz, yw - wy, yx - xz, zw - wz, zx - f*xy)

– Algebra L

ideal(xw - q*wx, zy + yz, yw - f*wz, yx - xz, zw - f*wy, zx - xy)

– Algebra M

ideal(xw + wx, zy + yz, yw - xy - wz, yx - f*wy + xz, zw - wy + xz, zx + xy + f*wz)

– Algebra N

ideal(xw + wx, zy + yz, yw + g*xy - f*xz, yx - g*wy - f*wz, zw - f*xy + g*xz, zx - f*wy - g*wz)

– Algebra O

ideal(xw + wx, zy + yz, yw - wy - f*xz, yx + xy - wz, zw - f*xy + wz, zx - wy - xz)

– Algebra P

ideal(xw + wx, zy + yz, yw - wz - f*xz, yx - wz - xz, zw - wy + f*xy, zx + wy - xy)

– Algebra Q

ideal(xw + wx, zy + yz, yw - wz, yx - wy - xy - wz, zw + wy, zx - wy + wz - xz)

– Algebra R

ideal(xw + wx, zy + yz, yw - wy - xy - wz, yx - wz, zw - xy, zx + xy + wz - xz)

– Algebra T

ideal(xw+wx, zy+yz, yw+wy-xy-wz-xz, yx-wy+xy-wz-xz, zw-wy-xy-wz+xz, zx-wy-xy+wz-xz)

– Algebra Z

ideal(xw + wx, zy - yz, yw - wz - xz, yx - xy - wz, zw - f*xy + wz, zx - f*wz + xz)

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