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ACYLINDRICALLY HYPERBOLIC GROUPS

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1. Introduction

In this paper we motivate a modern generalization of hyperbolic groups, *acylindrically hyperbolic groups* . We will define a hyperbolic group and discuss a few of it's interesting properties, in particular the containment of F_2 subgroups and SQ-universality in the non-virtually cyclic case. We will then explain that these characteristics are not unique to hyperbolic groups, although still the arguments we make for the non-hyperbolic cases will be motivated similarly. We then define a Weak Properly Discontinuous (WPD) element, and demonstrate a few simple proofs that every one of the aforementioned groups do, in fact have loxodromic WPD elements. We finally finish off the paper by exploring the work of Genevois, and how he uses acylindrical hyperbolicity and WPD elements to explore CAT(0) cube complexes, one of the richest modern topics of research in group theory.

In this paper the reader will be exposed to many spaces common in geometric group theory, such as the Cayley graph, Bass-Serre trees and CAT(0) cube complexes. The reader will also be exposed to more classical combinatorial constructions like free groups, right-angled Artin groups and amalgamated products. The reader will also be exposed to many different standard proof methods. The ping pong lemma will be introduced early on and used throughout the paper to demonstrate the existence of F_2 subgroups. We will also spend some time exploring properties of groups acting on trees so that we may better understand what WPD elements look like in the case where G is an amalgamated product of two groups. The reader will finally, at the end of the paper, be exposed to the heavily geometric machinery used in Genevois' classification of elements who satisfy weak proper discontinuity in the acylindrical action of groups on CAT(0) cube complexes.

Overall, we hope that the reader comes away from this paper better understanding some common proof methods in group theory and will have enlightening examples to justify their intuition.

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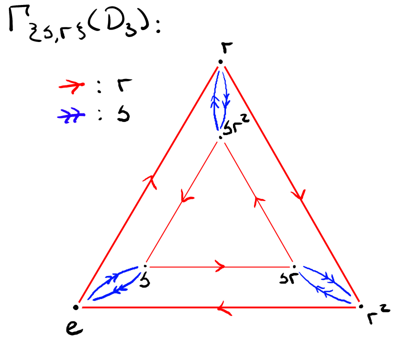
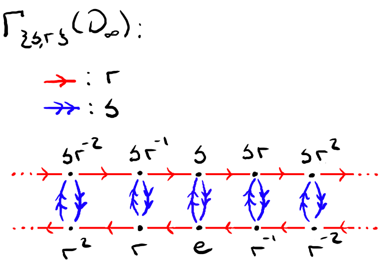
2. Preliminaries

Let G be a group generated by the set X. Throughout this paper we will only consider groups with finite generating sets. We can construct the oriented and labeled Cayley graph $\Gamma = \Gamma_X(G)$ of G by considering each element of G a vertex of Γ and declare two elements g, h of G adjacent if $g^{-1}h = x \in X^{\pm 1}$. We can then label the edge from g to h by x. As an example, $\Gamma_{\{s,r\}}(D_3)$ and $\Gamma_{\{s,r\}}(D_\infty)$ are illustrated below.





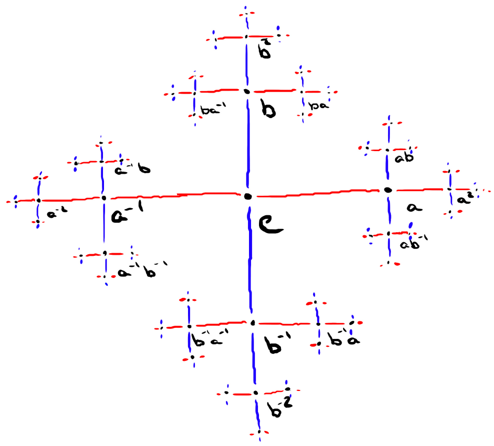
It is important to note that the Cayley graph of G encodes all the information of G. Every element of G is just a vertex of $\Gamma(G)$, and the multiplication of g and h in G is defined by looking at the image of h under the label preserving map which sends e to g. The group G then acts on it's Cayley graph by left-multiplication, i.e. $g.x = gx$. The action is, hence, faithful.

Given a set X we can define the set of all reduced words on X, $F(X)$, to be the set of all finite strings $x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} ... x_n^{\varepsilon_n}$ where $x_i \in X$, $\varepsilon_i \in \{+1, -1\}$ and $x_i = x_{i+1} \implies \varepsilon_i = \varepsilon_{i+1}$. This set forms a group under reduced concatenation $\cdot$, where given $x = x_1^{\varepsilon_1} x_2^{\varepsilon_2} ... x_n^{\varepsilon_n}$ and $y =$

$y_1^{\delta_1}y_2^{\delta_2}...y_m^{\delta_m}$, $x \cdot y = x_1^{\varepsilon_1}x_2^{\varepsilon_2}...x_k^{\varepsilon_k}y_{n-k}^{\delta_{n-k}}y_{n-k+1}^{\delta_{n-k+1}}...y_m^{\delta_m}$, we have that $x_k = y_{n-k} \implies \varepsilon_k = \delta_{n-k}$, and $x_i = y_{n-i}$ and $\varepsilon_i = -\delta_{n-i}$ for all $k+1 \leq i \leq n$. There will be a unique natural homomorphism $\varphi : F(X) \to G$ from this group F_2 to any group G generated by X such that for all $x \in X$, $\varphi_F(x) = x$ [1, Proposition 6.8.1]. It should be noted that this homomorphism is onto.

Given a group G generated by X, a *relator* R is a word in $F(X)$ such that $\varphi_F(R) = e$ (i.e, $R \in \ker\varphi$). By the First Homomorphism Theorem, $G \cong F(X)/\ker(\varphi_F)$. Therefore, a group G can be determined by a generating set paired with a normal generating set for $\ker\varphi$. This pair is called the *presentation* of a group and is written $< X|\mathcal{R} >$ where $\mathcal{R}$ generates $\ker\varphi$ as a normal subgroup. To learn more about generators and relators, the reader is encouraged to read [1, Chapter 6].

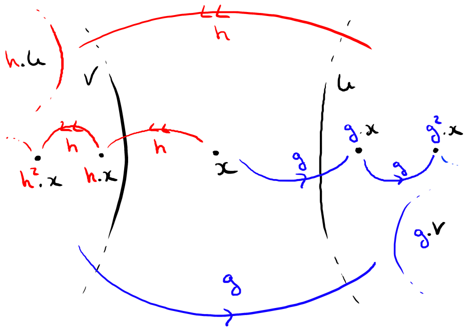
$F(X)$ then has no relators and so we will write it's presentation $< X|- >$. The Cayley graph of the free group with respect to a basis is then a tree. As an example, the Cayley graph of $F_2 = F(a, b)$ is illustrated below.



One of the most important tools when proving a given group is free is the ping-pong lemma, which is stated below [9, Proposition 1.1].

Lemma 2.1 (Ping-Pong Lemma ; 9 Proposition 1.1). *Let G be a group acting on a set S. If there are infinite order elements $g, h \,\epsilon\, G$ and disjoint, non-empty subsets U, V of S such that $g.U \subseteq V$ and $h.V \subseteq U$, then $< g, h > \cong F_2$.*

Proof. We can simplify the general proof while still capturing the main idea of the lemma by supposing that there was an element x such that $g.x \,\epsilon\, U$, $h.x \,\epsilon\, V$, and also that $h.U \subseteq U$ and $g.V \subseteq V$. While this does not give a proof in the full generality of the lemma as stated, it does allow us to draw the picture below.



We then have that if w is a reduced word in g and h then $w.x \neq x$ unless w is the empty word. Since $w.x$ is in either U or V, w cannot be trivial if w is not the empty word. Therefore, the natural map $\varphi_F : F_2 \to < g, h >$ must have a trivial kernel.

$\square$

Given two groups $G_1 =< X_1|\mathcal{R}_1 >$ and $G_2 =< X_2|\mathcal{R}_2 >$, two isomorphic subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, and an isomorphism $\varphi : H_1 \to H_2$, we can define the *amalgamated product* of G_1 and G_2 over φ to be $G_1 \underset{H_1 \cong_\varphi H_2}{\star} G_2 =< X_1 \sqcup X_2 | \mathcal{R}_1 \sqcup \mathcal{R}_2 \sqcup \{h^{-1}\varphi(h) | h \,\epsilon\, H_1\} >$. Sometimes, we will demand that the embedded subgroup H, the inclusion of H_1 (equivalently H_2) into G, be *malnormal* in G_1 and G_2. By this, we mean that given $g \,\epsilon\, G_i - H_i$, $H_i \cap gH_ig^{-1} = e$.

The free product of two groups $G_1 =< X_1|\mathcal{R}_1 >$ and $G_2 =< X_2|\mathcal{R}_2 >$, written as $G_1 \star G_2$, is the group with presentation $< X_1 \sqcup X_2 | \mathcal{R}_1 \sqcup \mathcal{R}_2 >$. The free product can also be realized as the amalgamated product of G_1 and G_2 over the isomorphism of their trivial subgroups.

2.1. **Basic Geometry.**

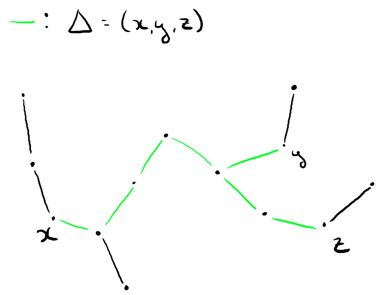
Definition 2.2 (Geodesics)**.** A *geodesic* in S is a path $\gamma : [0, l] \to S$ such that $d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1$ for all $t_1 < t_2$ in $[0, l]$. We say that γ as above is a geodesic between x and y if $\gamma(0) = x$ and $\gamma(l) = y$, and will often simply write $[x, y]$ to denote a geodesic between x and y. Finally, a *geodesic ray* is an infinite path $\gamma : [0, \infty) \to S$ with the property that $d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1$ for all $t_1 < t_2$. If $\gamma(0) = x$, we say γ is a geodesic ray emanating from x.

Definition 2.3 (Geodesic Triangles)**.** A geodesic triangle is a triplet of points (x, y, z) paired with a triplet of geodesics $([x, y], [y, z], [z, x])$.

Let $\delta \geq 0$. A geodesic triangle $\Delta = (x, y, z)$ is said to be δ-*slim* if every point on any one of the sides is within distance δ of one of the other two sides. A metric space S is then said to be δ-hyperbolic if every geodesic triangle in S is δ-slim.

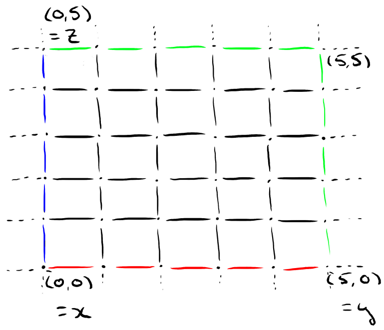
Definition 2.4 (Hyperbolic Space)**.** We say a geodesic metric space is *hyperbolic* if it is δ hyperbolic for some δ.

For some examples, notice that trees are 0-hyperbolic. This is because every geodesic triangle in a tree is a tripod.



Also, notice that any finite graph Γ will be hyperbolic, as you can choose $\delta = |\Gamma|$

The $\mathbf{Z}^2$-lattice is not hyperbolic. Look at the points $(0, 0), (n + 1, 0)$ and $(0, n + 1)$. The triangle drawn below is clearly geodesic, however the point $(n + 1, n + 1)$ is distance $n + 1$ from the other two sides. Hence, there are triangles in the $\mathbf{Z}^2$-lattice which are not n-slim for any n.



Now, let S be a hyperbolic metric space with metric d. Fix some $x \in S$. If S is proper (i.e. every closed ball is compact) then we may define the *boundary* of S to be the set of all geodesic rays emanating from x under the equivalence $\gamma_1 \cong \gamma_2$ if there is some natural K for which for all $t \geq 0$ $d(\gamma_1(t), \gamma_2(t)) \leq K$ (as is done in [2, Definition III.H.3.12]).

This definition of the boundary of a space works well in our initial context, where we are only dealing with Cayley graphs of finitely generated groups. Later in the paper we will however need to work with hyperbolic spaces which are not, in general, proper. It is for this reason that we will work with the following definition.

Definition 2.5 (The (Gromov) Boundary ; 2, Definition III.H.3.12)**.** Let S be a hyperbolic space. A sequence $(x_n)_{x \in \mathbf{N}}$ in S *diverges to infinity* if $d(x_i, z) + d(x_j, z) - d(x_i, x_j) \to \infty$ for some (and so any) fixed z. The boundary of S, denoted ∂S, is the set of all divergent sequences in S under the relation that $(x_n)_{n \in \mathbf{N}} \cong (y_n)_{n \in \mathbf{N}}$ if $d(x_i, z) + d(y_j, z) - d(x_i, y_j) \to \infty$. The set $S \cup \partial S$ is given the topology induced by the pseudo-metric given in [2, Proposition III.H.3.21].

We will not go through the construction of this pseudo metric, however we need to know that we have a topology to use certain properties of neighborhoods of boundary points later.

Also, if we define $g.(x_i)_{n \in \mathbf{N}} = (g.x_i)_{n \in \mathbf{N}}$ we have $d(g.x_i, z) + d(g.x_j, z) - d(g.x_i, g.x_j) = d(x_i, g^{-1}z) + d(x_j, g^{-1}z) - (x_i, x_j) \to \infty$, so any element $g \in G$ sends divergent sequences to divergent sequences. We can, therefore, extend the action of G on S to an action of G on ∂S.

It should be noted that in the case that S is proper both the boundaries defined above coincide. Why it is necessary to redefine the boundary of a space then might not be so clear. If S is not proper it may be that not every point in S will necessarily be connected to a point on ∂S by some geodesic ray, however every limit of a geodesic can be seen as a limit of points along that geodesic. This means that the Gromov boundary will always be larger than the boundary of geodesic rays. It is for this reason that we chose to use the latter definition, despite the first definition being perhaps more intuitive.

Given a group G acting on a hyperbolic space S we can classify the elements of G by how they behave on ∂S. If $g \in G$ fixes no point on ∂S, then we say G is *elliptic* . If g fixes one points on ∂S it is called parabolic and if it fixes two points on ∂S then it is said to be loxodromic. Note that g cannot fix any more than two points on ∂S unless g is identity, since points on ∂S that g fixes correspond to cluster points of $(g^n.x)_{n \in \mathbf{Z}}$, i.e cluster points of $(g^n.x)_{n \in \mathbf{N}}$ or $(g^{-n})_{n \in \mathbf{N}}$ [3, Theorem 2.3]. If g is loxodromic, we denote $g^+ = (g^n)_{n \in \mathbf{N}}$ and $g^- = (g^{-n})_{n \in \mathbf{N}}$ (note that these two sequences are divergent, as shown in [3, Theorem 2.3]).

Finally, we are ready to define hyperbolic groups.

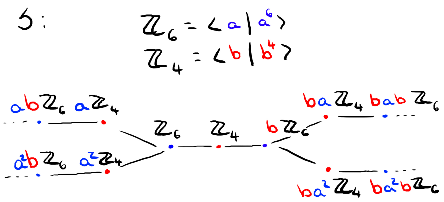
Definition 2.6. A group G is said to be *hyperbolic* if its Cayley graph $\Gamma_X(G)$ is hyperbolic with respect to some (and hence any) finite generating set X.

Infinite hyperbolic groups, aside from when G is virtually cyclic ($\partial G = 2$), have some interesting properties. Firstly, there is a subgroup H of G which is isomorphic to F_2. From [2, Lemma III.H.3.6] we can see that if g and h are two *independent loxodromic elements* (elements for whom $\{g^+, g^-\}$ and $\{h^+, h^-\}$ are disjoint) then there are disjoint open neighborhoods U and V around g^+ and h^+ respectively and some integer M so that $g^n.V \subseteq U$ and $h^n.U \subseteq V$ for all $n \geq M$. The group G does, in general, have two independent loxodromic elements as demonstrated in [3, Theorem 2.7]. The proof Hamann gives is not too difficult to follow, however we will skip over the details. First, G is not virtually cyclic and so there are loxodromic elements g, h for which $g^+ \neq h^+$. The set $B = \{h^n g^+ | n \in \mathbf{N}\}$ is however an infinite set, since $(h^n g^+)_{n \in \mathbf{N}} \to h^+$ and $h^n g^+ \neq h^+$ for any n. Therefore, we have $|\partial S| = \infty$. We then see that if $f_1^+ \neq f_2^-$ for $f_1^+, f_2^- \in B$, then $(f_1 f_2)^n.f_2^+ \to f_1^+$ and $(f_1 f_2)^{-n}.f_2^+ \to f_2^-$ and so we have constructed infinitely many loxodromic elements that fix distinct points on ∂S. Finally, applying the ping pong lemma to the two sets U and V we can see that the subgroup H generated by g^M and h^M is isomorphic to F_2.

Another interesting property of hyperbolic groups is that they are *SQ-universal* , meaning that every countable group can be embedded into a quotient of G. The proof is given in [12].

Hyperbolic groups tend to have some very nice properties. In general, however, the restriction of being hyperbolic seems too strong. As we will demonstrate, these properties are shared by a larger class of groups which may not in general be hyperbolic. We will begin by discussing a particular type of amalgamated product.

The amalgamated product of two groups $G_1 \underset{H_1 \cong_\varphi H_2}{\star} G_2$ acts on a hyperbolic space called the *Bass-Serre* tree corresponding to $G_1 \underset{H_1 \cong_\varphi H_2}{\star} G_2$ [11, Theorem 4.9]. The Bass-Serre tree is constructed by starting with a tree of groups $\mathcal{T}_0$, with the integer-valued metric on its vertices. The Bass-Serre tree is constructed by starting with an edge labeled by H, with two vertices labeled G_1 and G_2. We then construct the tree $\mathcal{T}_n$ by creating copy of $\mathcal{T}_{n-1}$ for each coset of G_i at each vertex of valence 1 labeled G_j, $i \neq j$ and label appropriately. The Bass-Serre tree of $\mathbf{Z}_6 \underset{a^3 \mapsto b^2}{\star} \mathbf{Z}_4$ over $\mathbf{Z}_2$ and is illustrated below.



As a consequence of this construction, $g \epsilon G = G_1 \underset{H}{\star} G_2$ acts on this tree by sending x to the unique point labeled hx where $g^{-1}h \epsilon G_i$, where $x = kG_i$ for $K \epsilon G$. Now that we understand this action, we can demonstrate the following.

Proposition 2.7. *The amalgamated product of two groups over a malnormal embedded subgroup has an F_2 subgroup.*

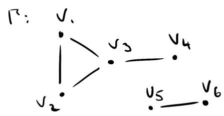
Proof. Let S the connected subtree U whose points are labelled by a word beginning in an element from G_1 and egdes are inherited from the tree, and the connected subtree V who is defined similarly U, except with labels from G_2. These two subsets are disjoint and it's clear that acting on U by any word beginning with a letter from $G_2 \setminus H$ will send U into V, and acting on V by any word beginning with a letter from $G_1 \setminus H$ will send V into U. We can then apply the ping-pong lemma, and see that as long as $G_1 \underset{H}{\star} G_2$ has two loxodromic elements beginning in letters from G_1 and G_2 then $G_1 \underset{H}{\star} G_2$ will contain a subgroup isomorphic to F_2.

Finally, $G_1 \underset{H_1 \cong_\varphi H_2}{\star} G_2$ does indeed contain two such loxodromic elements. As H is a proper subgroup of G_1 and G_2, we can find elements $g_1 \in G_1$ and $g_2 \in G_2$ which are not in H. So, $g_1 g_2$ and $g_2 g_1$ are not in conjugates of H or G_i and so are not elliptic. We also have that since S is a tree it has no parabolic automorphism [14, Lemma 2.2]. Hence, these two elements are loxodromic. □

2.2. **Right Angled Artin Groups.** Let Γ be a finite graph. The right angled Artin group (RAAG) $A(\Gamma)$ corresponding to Γ is the group with presentation $< V(\Gamma) | [v_1, v_2]$ if $(v_1, v_2) \in E(\Gamma) >$.

We can then view the free group on n generators as the RAAG associated with the graph with n vertices and no edges. Similarly, the free abelian group of order n would be the RAAG associated with the complete (i.e. every two points are joined by an edge) graph with n vertices.

Another example is illustrated below.



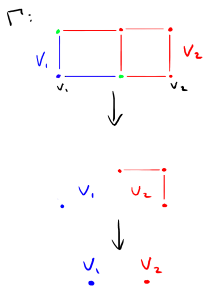
$$G: \langle v_1, \ldots, v_2 \mid [v_1, v_2], [v_2, v_3], [v_3, v_1], [v_3, v_4], [v_5, v_6] \rangle$$
$$\cong \langle a_1, b_1, c_1 \mid [a_1, b_1][b_1, c_1], [c_1, a_1] \rangle \underset{c_1 \mapsto a_2}{*} \langle a_2, b_2 \mid [a_2, b_2] \rangle * \langle a_3, b_3 \mid [a_3, b_3] \rangle$$
$$\cong \mathbb{Z}^3 \underset{c_1 \mapsto a_2}{*} \mathbb{Z}^2 * \mathbb{Z}^2$$

RAAGs have some interesting properties, one of which we will now demonstrate.

Proposition 2.8. *The Right-Angled Artin group of a complete graph with at least 2 vertices is SQ-universal.*

Proof. Suppose Γ is not complete and has at least 2 vertices. Let v_1 and v_2 be vertices of Γ which do not commute in $A(\Gamma)$. Then, as done in [8], we let V_1 be the set of vertices adjacent to v_1 along with v_1 and let V_2 be all vertices other than v_1. Note then that V_1 and V_2 are non-empty. We may then quotient $A(\Gamma)$ by the normal subgroup $<< v \epsilon V_1 \cap V_2, vw^{-1}$ where $v, w \epsilon V_i$ for $i = 1, 2 >>$. This quotient is analagous to deleting $V_1 \cap V_2$ from Γ then identifying the remainder and V_2. This group will then be isomorphic to $A(\Gamma')$, where Γ' the graph with two vertices and no edges. However, $A(\Gamma')$ is just the free group on V_1, V_2 which is SQ-universal [15].

Now let G be a countable group. Since F_2 is SQ-universal G embeds into a quotient of F_2. However since F_2 is a quotient of $A(\Gamma)$, G embeds into a quotient of $A(\Gamma)$. Hence, $A(\Gamma)$ is SQ-universal. □



Notice, however that the groups we have just discussed are not in general hyperbolic. The group $BS_{a,b}(1,2) =< a, b | bab^{-1}a^{-2} >$ is the (1-2) Baumslag-Solitar group, and we may let $G = BS_{a,b}(1,2) \underset{b \mapsto d}{\star} BS_{c,d}(1,2)$ be the amalgamated product of groups where $\mathbf{Z}$ embeds malnormally into both copies of $BS(1,2)$. However, hyperbolic groups cannot have Baumslag-Solitar subgroups [17, Proposition 2.4 (6)]. Hence, $BS_{a,b}(1,2) \underset{b \mapsto d}{\star} BS_{c,d}(1,2)$ cannot be hyperbolic.

Note that RAAGs are not in general hyperbolic either. Let Γ be a graph. If Γ is not totally disconnected then there will be two vertices v_1 and v_2 which commute in $G = A(\Gamma)$. Then $A(\Gamma)$ has a subgroup isomorphic to $\mathbf{Z}^2$. However hyperbolic groups cannot have $\mathbf{Z}^2$ subgroups [17, Proposition 2.4 (6)], and so not all RAAGs are hyperbolic.

3. Weak Proper Discontinuity

The above discussion seems to indicate that the above groups all belong to a common class of groups which satisfies SQ-universality and contains an F_2 subgroup. In light of this, we introduce the folowing definition.

Definition 3.1 (Weak Proper Discontinuity (16, Definition 2.5))**.** Let G be a group acting on a metric space, S. Let $h \in G$. Then h is said to satisfy *Weak Proper Discontinuity* (or be a WPD element) if for every $x \in S$ and $\varepsilon \geq 0$ there is a natural number M so that $\{g \in G | d(x, g.x) \leq \varepsilon \text{ and } d(h^M.x, g(h^M.x)) \leq \varepsilon\}$ is a finite set.

Let G_1, G_2 be two groups with an isometry φ between subgroups $H \cong H_1 \subset G_1$ and $H \cong H_2 \subset G_2$. By the construction outlined above, the group $G = G_1 \underset{H_1 =_\varphi H_2}{\star} G_2$ acts on its Bass-Serre tree (which is a tree, and so is hyperbolic). Every element of G is either elliptic or loxodromic [3, Lemma 2.2]. For our purposes, we will only consider the situation where the subgroup H is malnormal in G_1 and G_2. This leads us to the following lemma.

Lemma 3.2. *Let $G_1 \underset{H}{\star} G_2$ be the amalgamated product of G_1 and G_2 with a malnormal embedding of H acting on its Bass Serre tree. Two loxodromic elements commute if and only if they both belong to some cyclic subgroup.*

Proof. By replicating the classical proof that commuting elements in the free group belong to a common cyclic subgroup (as given in [4, Lemma 1.2]) with the normal form theorem for amalgamated products (as given in [5, Theorem 4.2.6]), we will see that g_1 and g_2 must both then belong to some cyclic subgroup.

First by the normal form theorem, $g_1 = s_{1,1}t_{1,1}s_{1,2}...t_{1,n}h_1$ and $g_2 = s_{2,1}t_{2,1}s_{2,2}...t_{2,m}h_2$ for where the $s_{i,j}$ are fixed representatives of left cosets of H in G_1, the $t_{i,j}$ are fixed representatives of left cosets of H in G_2, $h_i \in H$ and we allow only $s_{i,1}$, $t_{i,n}$ or h_i to be trivial. In this case however, we have that $s_{i,1}t_{i,1}s_{i,2}...t_{i,k}$ contains at least two terms for $k = n, m$, as otherwise $s_{i,1}t_{i,1}s_{i,2}...t_{i,k}h_i$ would be in one of G_1 or G_2, and so would fix that point in the Bass Serre tree S. Since both g_1 and g_2 are loxodromic, this is impossible.

We will proceed by induction on $n + m$. In the case where $s_{i,1}t_{i,1}s_{i,2}...t_{i,k}$ has only two terms, without loss of generality $g_1 = s_{1,1}t_{1,1}h_1$. $g_2 = s_{2,1}t_{2,1}s_{2,2}h_2$ with one of $s_{2,1}$ or $s_{2,2}$ trivial. However, $g_1g_2 = g_2g_1$, and so $s_{1,1}t_{1,1}h_1s_{2,1}t_{2,1}s_{2,2}h_2 = s_{1,1}t_{1,1}s'_{2,1}t'_{2,1}s'_{2,2}h'_1h_2 = s_{2,1}t_{2,1}s_{2,2}s'_{1,1}t'_{1,1}h'_2h_1 = s_{2,1}t_{2,1}s_{2,2}h_2s_{1,1}t_{1,1}h_1$. By the normal form theorem we then have $s_{1,1} = s_{2,1}$ and $t_{1,1} = t_{2,1}$, and so $g_2 = s_{1,1}t_{1,1}h_2$. Finally, we have that $h_2^{-1}h_1s_{1,1}t_{1,1} =$

$s_{1,1}t_{1,1}h_1h_2^{-1}$. However, both $h_2^{-1}h_1$ and $h_1h_2^{-1}$ stabilize the edge H in S, and so it must too stabilize the edge $s_{1,1}t_{1,1}.H$. Since $s_{1,1}t_{1,1}$ is loxodromic, $h_2^{-1}h_1$ must then stabilize 2 distinct edges of S (In particular the edge stabilized by H and the edge stabilized by $(s_{1,1}t_{1,1})^{-1}H(s_{1,1}t_{1,1})$). But H is malnormal, and so this product $h_2^{-1}h_1$ is therefore trivial. Finally, $h_2 = h_1$, giving that g_1 and g_2 belong to a common cyclic subgroup.

Now, without loss of generality suppose $n \leq m$. We again have that $s_{1,1}t_{1,1}...t_{1,n-r_1} \cdot \cdot s'_{2,r_1+1}t'_{2,r_2+1}...t'_{2,m}h'_1h_2 = s_{2,1}t_{2,1}s_{2,2}...t_{2,m-r_2}s'_{1,r_2+1}t'_{1,r_2+1}...t'_{1,n}h'_2h_1$ is $g_1g_2 = g_2g_1$ written in reduced form. By the normal form theorem, $r_1 = r_2 = r$. We then have three cases.

i) $r = 0$. Then $s_{1,j} = s_{2,j}$ and $t_{1,j} = t_{2,j}$ for all $1 \leq j \leq n$. Then $g_2 = g_1f$ for some where then length of f, l in its normal form is $m - n < m$ and so $n + l \leq n + m$. Finally, $g_1f = g_2 = g_1^{-1}g_2g_1$ since g_1 and g_2 commute. Continuing, $g_1^{-1}g_1fg_1 = fg_1$ and so by the induction hypothesis f and g_1 belong to a common cyclic subgroup, and so g_2 does also.

ii) $r = n$. Since g_1 and g_2 are given in their normal forms, $s_{2,j} = s^{-1}_{1,n-j+1}$ are the identical coset representatives and $t_{2,j} = t^{-1}_{1,n-j+1}$ are also the identical coset representatives so $g_2 = g_1^{-1}p$. By above, g_1^{-1} and p belong to a common cyclic subgroup, and so g_2 does also.

iii) $0 < r < n$. Here $s_{1,1}t_{1,1} = s_{2,1}t_{2,1}$, $s_{1,n}t_{1,n} = s_{2,m}t_{2,m}$, $s_{1,n}t_{1,n} = (s_{2,1}t_{2,1})^{-1}$ and $s_{2,m}t_{2,m} = (s_{1,1}t_{1,1})^{-1}$ as above. We can then write $g_1 = s_{1,1}t_{1,1}g'_1(s_{1,1}t_{1,1})^{-1}$ and $g_2 = s_{1,1}t_{1,1}g'_2(s_{1,1}t_{1,1})^{-1}$.

By the induction hypothesis, g'_1 and g'_2 belong to a common cyclic subgroup. g_1 and g_2 then both belong to the conjugate of this cyclic subgroup by $s_{1,1}t_{1,1}$.

□

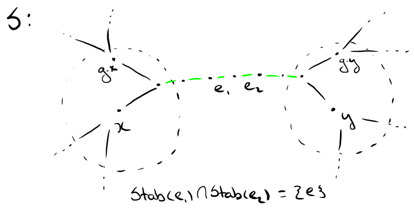
Now that we have this lemma, we are able to proceed.

Proposition 3.3. *Let $\varepsilon \geq 0$ If x and y are point in the Bass-Serre tree corresponding to an amalgamated product over a malnormal embedded subgroup such that $d(x,y) \geq 2\varepsilon + 2$, then there are only finitely many elliptic elements g for which $d(x,g.x) \leq \varepsilon$ and $d(y,g.y) \leq$ varepsilon.*

Proof. Let x and y be two distinct points in the Bass-Serre tree S and fix some $\varepsilon \geq 0$. Without loss of generality x is stabilized by either G_1 or G_2. If g is elliptic and $g.x \,\epsilon\, B_x(\varepsilon)$ then g must fix a point in $B_x(\varepsilon)$, as both x and $g.x$ must be distance $d(x,g.x)/2 \leq \varepsilon/2$

from $Fix(g)$. The same holds if $g.y \,\epsilon\, B_y(\varepsilon)$. If any elliptic element g fixes a point in $B_x(\varepsilon)$ and $B_y(\varepsilon)$ it necessarily fixes a path between these two points. This is because any path between two points is mapped to a path between their images, and S is a tree.

If $d(x, y) \geq 2\varepsilon + 2$, then g must fix two distinct edges since it fixes a path between points at least distance 2 away. Hence, g must belong to two distinct conjugates of H. Since H is malnormal in G, g must be trivial. Therefore at most finitely many elliptic elements can shift both x an y by less than ε provided $d(x, y) \geq 2\varepsilon + 2$. □

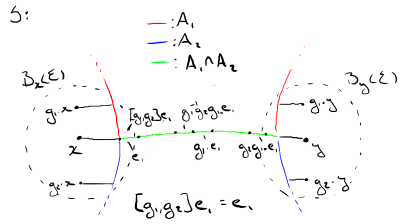


We now have a similar result for loxodromic elements.

Proposition 3.4. *Let $\varepsilon \geq 0$ If x and y are point in the Bass-Serre tree corresponding to an amalgamated product over a malnormal embedded subgroup such that $d(x, y) \geq 5\varepsilon$, then there are only finitely many loxodromic elements g for which $d(x, g.x) \leq \varepsilon$ and $d(y, g.y) \leq$ varepsilon.*

Proof. Suppose g_1, g_2 are two loxodromic elements for which $g.x \,\epsilon\, B_x(\varepsilon)$ and $g.y \,\epsilon\, B_y(\varepsilon)$. Then, since if g_i is loxodromic, $d(x, g_i.x) = 2d(x, A_i) + L_i$ where $L_i = min_{x \,\epsilon\, \S}$ is the *translation length* of g_i and A_i is the connected subset of all points in S shifted by L_i, called the *axis* of g_i. In this case we have that both x and y lie within $\varepsilon/2$ of the axes of g_1 and g_2, and that the translation length of g is at most ε. If $\varepsilon < 1$ we can see that $g.x = x$, which is impossible since g is loxodromic. Now suppose that x and y are at least 5ε apart. Then we see that the axes of g_1 and g_2 must intersect along a path P of length at least 4ε between x and y, since we both have that g_1 and g_2 connect two points distance $\varepsilon/2$ from x and y and we know the path $[x, y]$ between x and y is unique. There will then be two edges of P sent to edges of P distance $L_1 - 1$ from themselves by g_1, which will in turn be sent to edges of P distance $L_2 - 1$ from themselves by g_2. Finally, we have that both g_1^{-1} and g_2^{-1} will send all of these edges back L_1 and L_2 respectively along P giving that $g_1 g_2 g_1^{-1} g_2^{-1}$ must stabilize two distinct edges of Γ. Since H is malnormal, g_1 and g_2 must commute. By the above Lemma 3.2, we see that g_1 and g_2 must both then belong to some common cyclic subgroup. At most 2ε loxodromic elements from a given cyclic subgroup can have translation length less than ε, and so in our case only finitely loxodromic many elements of G can shift both x and y by less than ε given that x and y are at least distance 5ε away.

□



Finally, we can combine these two results together.

Proposition 3.5. *Any loxodromic element in the action of an amalgamated product of groups over a malnormally embedded subgroup is WPD.*

Interestingly, any RAAG $A(\Gamma)$ which is not cyclic and not directly decomposable (i.e, there are no two distinct subgraphs Γ_1, Γ_2 with $V(\Gamma_1) \cup V(\Gamma_2) = V(\Gamma)$ for which every vertex of Γ_1 is connected to every vertex of Γ_2) will act on a hyperbolic space, and have a WPD loxodromic element under that action [16 Example, 8.(d)]. This is difficult to demonstrate, and so a general proof will be omitted.

We can however observe the case where Γ is a graph with two connected components and v_1, v_2 are in distinct components. The group $A(\Gamma)$ will act on the quotient of its Cayley graph that identifies all $\mathbf{Z}^k$-lattice subsets, and this action will render the element $v_1 v_2$ loxodromic and WPD. This is because this action analogous to $A(\Gamma)$ acting on the Cayley Graph one of it's free quotient groups. In this quotient v_1 and v_2 will be mapped to generators of the free group, and so the element $v_1 v_2$ will be loxodromic and WPD.

It is however very easy to show that hyperbolic groups have WPD elements. Let g be an infinite order element in a hyperbolic group G, and consider G acting on its Cayley graph ,Γ.

Consider the two boundary points $g^+ = \{g^n\}_{n \,\epsilon\, \mathbf{N}}$ and $g^- = \{g^{-m}\}_{m \,\epsilon\, \mathbf{N}}$. We can see that $g.g^+ = \{g^{n+1}\}_{n \,\epsilon\, \mathbf{N}} \cong g^+$ and $g.g^- = \{g^{-m+1}\}_{m \,\epsilon\, \mathbf{N}} \cong g^-$. Note that if $g^+ = g^-$, then we would have that for a subsequence $(g^{n_i})_{i \,\epsilon\, \mathbf{N}}$ $d(g^{n_i}.x, x) + d(g^{-n_j}.x, x) - d(g^{n_i}.x, g^{-n_j}.x) \to \infty$. However, then $d(g^{n_i}.x, x) + d(g^{n_j}, x) - d(g^{n_i+n_j}.x, x) \to \infty$. This, however, cannot possibly be true, as $d(g^{n_i}.x, x) + d(g^{n_j}, x) - d(g^{n_i+n_j}.x, x) = n_i L_g + 2d(x, A_g) + n_j L_g + 2d(x, A_g) - (n_i + n_j) L_g - 2d(x, A_g) = 2d(x, A_g)$ where L_g is the translation length of g and A_g axis of G. Hence, g^+ and g^- are distinct and g is loxodromic.

Now, recall that the action of a group on its Cayley graph is faithful. Also, hyperbolic groups are finitely generated and so $|B_x(\varepsilon)| < \infty$, meaning then $\{h \,\epsilon\, G | d(x, hx) \leq \varepsilon\} \leq \infty$. Then, trivially, g is a WPD element.

Because of the work done above, we can generalize the notion of a hyperbolic group to a acylindrically hyperbolic group, which is given by the following definition.

Definition 3.6. Let G be an infinite non-virtually cyclic group. Then G is said to be *acylindrically hyperbolic* if there is some hyperbolic space S such that there is a WPD loxodromic element g in the action of G on S.

Now that we have highlighted the significance of these WPD elements, we will follow the work of Genevois, and see how he classifies WPD elements in the action of Groups on CAT(0) cube complexes.

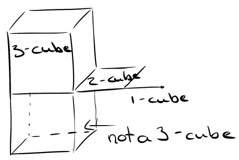
4. CAT(0) Cube Complexes

We will first need a few definitions.

Definition 4.1 (n-Cube)**.** A 0-cell is a vertex.
An *n-cell* for $n > 0$ in a metric space S is the continuous image of $[-1, 1]^n$ for which the image of the boundary of $[-1, 1]^n$ is a set of $(n-1)$-cells.
An *n-cube* is an n-cell for which $\partial[-1, 1]^n$ is a set of $2n$ $(n-1)$-cells (note that every 0-cell is a 0-cube). We will often refer to 1-cells as edges and 2-cells as faces

Definition 4.2 (Cube Complex)**.** A metric space X is said to be a *cube complex* if it is isometric to a "combinatorial gluing of n-cubes".

This definition can be made precise, however it is more enlightening to first look at a picture.

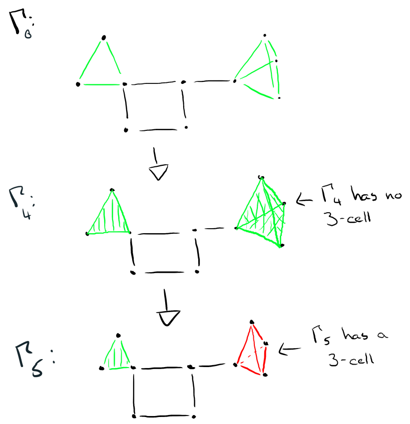


In the picture above X can be realized as several 1, 2, and 3-cubes glued together, where the gluing only happens between faces, edges, or vertices. In general a cube complex X could be constructed from n-cubes for any n, and the gluing could only occur along any m-cube. Also, notice there is the graph metric d on the subset X^1 of X that consists of all the edges and vertices of X. This metric can be extended to a metric on X, but for our purposes we will require that geodesics between vertices in X are just geodesics in X^1.

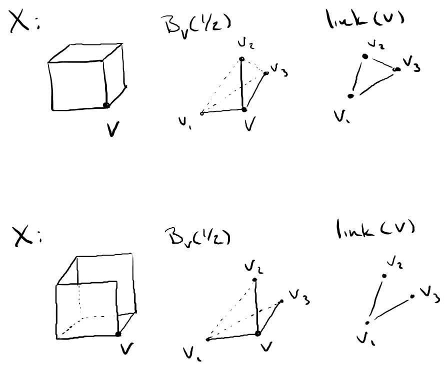
Now that we have established what a cube complex is, we will be looking at cube complexes which satisfy the CAT(0) criterion. This criterion is primarily a curvature condition although is also a minor homotopy restriction. The usefulness of curvature conditions for group actions is highlighted in the situation when a group G is a hyperbolic group - G acts naturally on it's Cayley graph, which is hyperbolic and thus has negative curvature.

Definition 4.3 (Simplicial complex)**.** Let Γ_0 be a graph. For every n pairwise adjacent n-cells in Γ_i, let N be an $(n+1)$-cell with a boundary consisting of the n given n-cells. Then $\Gamma_{i+1} = \Gamma_i \cup N$. Any space in this sequence Γ_i is called a simplicial complex.

Definition 4.4 (Simplex)**.** An n-simplex is a simplicial complex consisting of $n+1$ adjacent vertices which boarder a n-cell.



Let X be a cube complex and let v be a vertex of X. The *link* of v, $link(v)$ is the simplicial complex whose n-cells are the $n+1$ cells in X which contain v, where the embedding relation is preserved. The link of v can be visualized as the $1/2$-sphere around v, and it describes the local behavior of X around v.



Now that we have a way of describing local behavior at vertices, we can start describing the curvature of X.

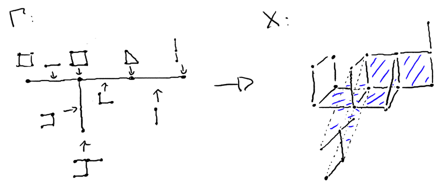
A *flag complex* is a simplicial complex such that any $n+1$ vertices span an n simplex if and only if they are pairwise adjacent. Thus, a flag complex is completely determined by its 1-skeleton. Essentially, the condition that $link(v)$ is a flag ensures that not too many n-cubes can be embedded into the cube complex without having an $n+1$ cube containing them all underlying graph. The addition of n-cubes into our cube complex can be interpreted as increasing the area in a given region, for example in the case where X is a hollow cube. The space X will have 6 "maximal" n-cells, whereas the cube (which satisfies the flag condition) only has one, and will hence have a smaller "area". From this perspective this condition can be interpreted as a linear isoperimetric condition on our space. Isoperimetric conditions are a common in classifications of hyperbolic (negatively curved) spaces, and so perhaps we will be able to classify non-positively curved spaces this way. Indeed, this is how we proceed.

Definition 4.5. A cube complex X is said to be *non-positively curved* if the link of any vertex v of X is a flag complex. We say X is a *CAT(0) cube complex* if it is non-positively curved and simply connected.

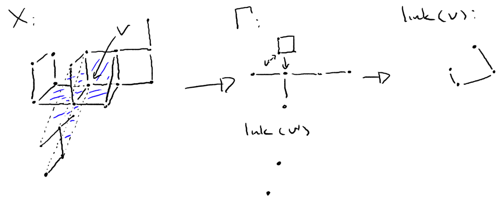
Note that when we declare a group G to act on a CAT(0) cube complex it must also respect the combinatorial structure of the complex.

From here on, we will use $\widetilde{X}$ to denote a CAT(0) cube complex. The motivation for this notation is that given a non-positively curved cube complex X, its universal cover $\widetilde{X}$ will be a CAT(0) space [10]. We will now look at some examples.

Let Γ be a graph. To each vertex v of Γ assign a graph Γ_v, and to each edge e of Γ assign a graph Γ_e and two combinatorial maps $\varphi_- : \Gamma_e \to \Gamma_{e_-}$, and $\varphi_+ : \Gamma_e \to \Gamma_{e_+}$. We can then define a space X by associating to each edge e of Γ the space $\Gamma_e \times [-1,1]$, and glue the subspaces $\Gamma_e \times \{-1\}$ to Γ_{e_-} and $\Gamma_e \times \{1\}$ to Γ_{e_+} under the inclusions given by the maps φ_- and φ_+. We call this space a graph of graphs.

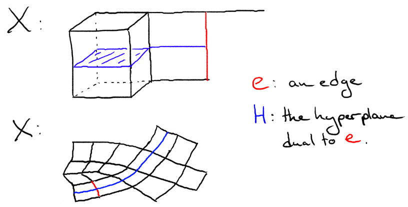


If the maps $\varphi_\pm$ are locally injective (which we will call *immersions*) for each edge e in Γ, then we have that X is CAT(0). This is because if w is a vertex in X (and so then a vertex in some Γ_v), then $link_X(w)$ has a copy of $link_\Gamma(v)$, plus additional edges between vertices not both in $link_\Gamma(v)$. This graph is certainly a flag, as no new triangle can possibly be created. The space X will hence be a non-positively curved cube complex. We can, using [10], conclude that the universal cover $\widetilde{X}$ of X is a CAT(0) cube complex.



Now that we have introduced a few useful geometric constructions, we have some examples to look at while moving into material more focused on the developments in the paper by Genevois [7].

A *midcube* of an n-cube $[-1,1]^n$ is the subspace obtained by restricting one of its coordinates to zero. A *hyperplane* is then a connected subspace of a CAT(0) space $\widetilde{X}$ whose intersection with any n-cube is either a midcube or is empty. If H is a hyperplane which intersects some edge, we say that H is *dual* to the edge.

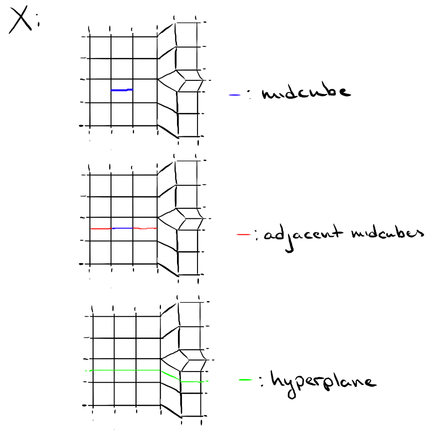


Hyperplanes have some basic properties, which will be listed here unproven [6, Theorem 2.13].

Proposition 4.6. *Let $\widetilde{X}$ be a CAT(0) cube complex.*

i) *Every midcube of $\widetilde{X}$ is contained in some hyperplane of $\widetilde{X}$*
ii) *The neighborhood $N(\mathcal{H}) = \mathcal{H} \times [-1,1]$ of a hyperplane $\mathcal{H}$ is a convex subcomplex of $\widetilde{X}$ called the* carrier *of H*
iii) *$\widetilde{X} \setminus \mathcal{H}$ has two components*

In i), the intuition is that a hyperplane is just the union of many adjacent midcubes. So, in constructing a hyperplane containing a midcube M we take the union of M with all midcubes adjacent to M and call it $\mathcal{H}_1$. Next, we union $\mathcal{H}_1$ with all the midcubes adjacent to it, and call it $\mathcal{H}_2$. We will then have that the union of all the $\mathcal{H}_n$ will be a hyperplane $\mathcal{H}$ which contains M.



Properties *ii*) and *iii*) are more difficult to demonstrate, however the above picture should grant reader with some intuition behind the result.

We now have the tools to investigate hyperplanes in some CAT(0) space, $\widetilde{X}$. We will use a process called "bigon removal". In this process, we use the fact that any hexagon (which in a cube complex will be 3 pairwise adjacent cubes) in $\widetilde{X}$ is a corner of a cube. What this gives is that when looking at the 2-cubes of $\widetilde{X}$, we can invert any hexagons we see to eliminate any bigons in any diagram of a CAT(0) cube complex. This process is illustrated in the diagram below.

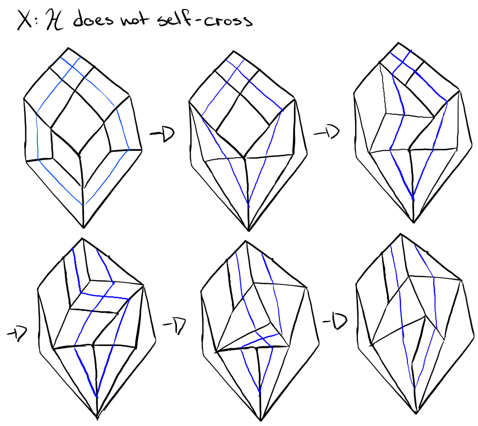
Corollary 4.7 (6, Corollary 3.3)**.** *Let $\mathcal{H}$ be a hyperplane in $\widetilde{X}$. Then*

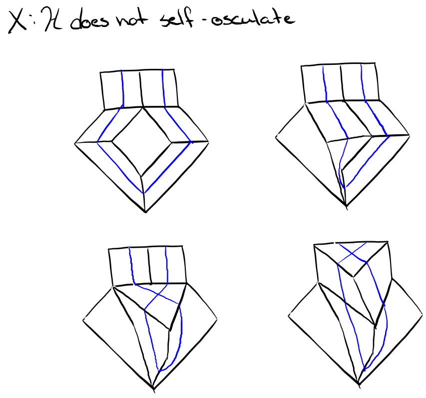
i) *$\mathcal{H}$ does not self-cross*

ii) *$\mathcal{H}$ does not self-osculate*

iii) *$\mathcal{H}$ is simply connected*

A hyperplane $\mathcal{H}$ is said to self-cross if there is a n-cube M for which $\mathcal{H} \cap M$ is more than one midcube. Similarly, $\mathcal{H}$ is said to self-osculate if $\mathcal{H}$ does not self-cross but does pass through adjacent n-cubes.





Additionally, simple connectedness of H follows immediately from i) and ii).

We will now introduce some definitions.

Definition 4.8 (Separated)**.** Two subsets U and V of a cube complex $\widetilde{X}$ are said to be *separated* by a hyperplane $\mathcal{H}$ if U and V lie in different components of $\widetilde{X} \setminus \mathcal{H}$.

Definition 4.9 (Facing Triples)**.** [13, Definition 3.1] A *facing triple* in a CAT(0) cube complex is a collection of three hyperplanes such that no two of them are separated by the third hyperplane.

Definition 4.10 (Well-Separated)**.** [13, Definition 3,8] If $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint hyperplanes in a CAT(0) cube complex and $L \geq 0$, we say $\mathcal{H}_1$ and $\mathcal{H}_2$ are L-*well separated* if any family of hyperplanes that cross both $\mathcal{H}_1$ and $\mathcal{H}_2$ and does not contain a facing triple has cardinality at most L. The two hyperplanes are said to be *well-separated* if they are L-well-separated for some L.

Definition 4.11 (The Join of Hyperplanes)**.** [13, Definition 3.2] The *join of hyperplanes* $(\mathfrak{H}_1, \mathfrak{H}_2)$ is a pair of families of hyperplanes without facing triples and the property that any hyperplane of $\mathfrak{H}_1$ has discrete intersection with any hyperplane of $\mathfrak{H}_2$. We say the join is C*-thin* if one of $\mathfrak{H}_1$ or $\mathfrak{H}_2$ has cardinality at less than C.

Definition 4.12 (Skewer Hyperplanes ; 7 Definition 3.12)**.** Some isometry g *skewers* hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ if for some half-spaces H^0 and H^1 (one of the components of $X \setminus \mathcal{H}_1$ or $X \setminus \mathcal{H}_2$ respectively) $g^n H^0 \subset H^1$ for some natural n.

We will now work through the proof from Genevois classifying WPD loxodromic elements on a group acting on a CAT(0) cube complex. First, some lemmas need to be established.

Lemma 4.13 (7, Theorem 17)**.** *Let G be a group acting on some metric space (S, d). Then g is WPD if and only if there exists some $x \in S$ such that for all $\varepsilon \geq 0$ there is some natural m such that $\{h \in G | d(x, hx) \leq \varepsilon$ and $d(hx, hg^m x) \leq \varepsilon\}$ is finite.*

Proof. The following proof is due to [7].

The forward implication is clear. If this condition holds for all x then it holds for some x. Conversely, fix $\varepsilon \geq 0$ and $y \in S$. Then we can use our assumption to find $x \in S$ with which the set $\{h \in G | d(x, g^m x) \leq 2d(x, y)\varepsilon$ and $d(hx, hg^m x) \leq 2d(x, y)\varepsilon\}$ is finite. The rest of the proof follows from the triangle inequality. First observe that $d(x, hx) \leq d(x, y) + d(y, hy) + d(hy, hx)$. However h is an isometry, and so $d(x, hx) \leq 2d(x, y) + d(y, hy)$. Similarly, $d(g^m x, hg^m x) \leq d(g^m x, g^m y) + d(g^m y, hg^m y) + d(g^m hy, hg^m x) = 2d(x, y) + d(g^m y, hg^m y)$ as g^m and hg^m are isometries. However we can then see that $\{h \in G | d(y, g^m y) \leq \varepsilon$ and $d(hy, hg^m y) \leq \varepsilon\}$ is a subset of $\{h \in G | d(x, g^m x) \leq 2d(x, y)\varepsilon$ and $d(hx, hg^m x) \leq 2d(x, y)\varepsilon\}$ and is thus finite, completing the proof.

□

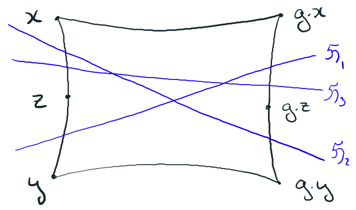
This lemma simplifies the WPD condition, which will come in handy moving forward.

Lemma 4.14 (7, Lemma 20)**.** *Let X be a CAT(0) cube complex and γ a geodesic between two vertices x, y such that the join of hyperplanes that intersect γ is C-thin. If an isometry g of X satisfies $d(x, gx)$ and $d(y, gy) \leq \varepsilon$, then for every $z \in \gamma$, $d(z, gz) \leq C + 6\varepsilon$*

7. First, fix two geodesics $[x, gx]$ and $[y, gy]$. We will define some families of hyperplanes:
- Let $\mathfrak{H}_1$ be the set of hyperplanes separating $\{gx, z\}$ and $\{gz, y\}$
- Let $\mathfrak{H}_2$ be the set of hyperplanes separating $\{gz, x\}$ and $\{gy, z\}$

- Let $\mathfrak{H}_3$ be the set of hyperplanes separating $\{gx, gx\}$ and $\{gz, z\}$

Now, let $\mathcal{H}$ be a hyperplane separating x and z as pictured below.



We can see, from the picture above, that $\mathcal{H}$ either belongs to $\mathfrak{H}_2 \cup \mathfrak{H}_3$, or it separates one of $\{x, gx\}$ or $\{y, gy\}$. So, we see that $d(x, z)$ is less that the number of hyperplanes that separate x and z, since the geodesic from x to z has at least one hyperplane passing through each edge. Each of these hyperplanes will separate x and z. By above, $d(x, z)$ is less than $\#\mathfrak{H}_2 + \#\mathfrak{H}_3 + d(x, gx) + d(y, gy)$. However, both $d(x, gx)$ and $d(y, gy) \leq \varepsilon$, and hence $|d(x, z) - \#\mathfrak{H}_2 - \#\mathfrak{H}_3 \leq 2\varepsilon$.

We can similarly conclude that $|d(gx, gz) - \#\mathfrak{H}_1 - \#\mathfrak{H}_3| \leq 2\varepsilon$. These two inequalities can be used to get a bound on the relative sizes of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, through the following computation.

$$|\#\mathfrak{H}_1 - \#\mathfrak{H}_2| = |\#\mathfrak{H}_1 + \#\mathfrak{H}_3 - d(x, z) + d(x, z) - \#\mathfrak{H}_2 - \#\mathfrak{H}_3| =$$

$$= |\#\mathfrak{H}_1 + \#\mathfrak{H}_3 - d(gx, gz) + d(x, z) - \#\mathfrak{H}_2 - \#\mathfrak{H}_3|, \text{ as } g \text{ is an isometry. Continuing,}$$

$$|\#\mathfrak{H}_1 + \#\mathfrak{H}_3 - d(gx, gz) + d(x, z) - \#\mathfrak{H}_2 - \#\mathfrak{H}_3| \leq$$

$$\leq |\#\mathfrak{H}_1 + \#\mathfrak{H}_3 - d(gx, gz)| + |d(x, z) - \#\mathfrak{H}_2 - \#\mathfrak{H}_3| \leq 2\varepsilon + 2\varepsilon = 4\varepsilon$$

Referring again to the picture above, a hyperplane separating z and gz belongs to $\mathfrak{H}_1 \cup \mathfrak{H}_2$ or separates one of $\{x, gx\}$ or $\{y, gy\}$ and so following in the same steps as before, we conclude that $d(z, gz) \leq \#\mathfrak{H}_1 + \#\mathfrak{H}_2 + 2\varepsilon \leq min(\#\mathfrak{H}_1, \#\mathfrak{H}_2) + 4\varepsilon + 2\varepsilon$.

Our final observation is that $(\mathfrak{H}_1, \mathfrak{H}_2)$ is a join of hyperplanes that intersect γ, which by assumption is C-thin. Therefore $min(\#\mathfrak{H}_1, \#\mathfrak{H}_2) \leq C$. Also, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ do not contain any facing triples as every hyperplane in $\mathfrak{H}_i$ passes through a common geodesic and so cannot cross, since we cannot have three hyperplanes in $\mathfrak{H}_1 \cup \mathfrak{H}_2$ which all pairwise intersect without two of them belonging to one $\mathfrak{H}_\mathfrak{i}$.So, we have our result. □

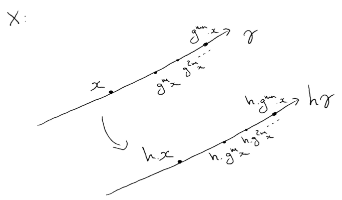
Now, another proposition.

Definition 4.15 (Quasi-convex)**.** An element g of a group G acting on a metric space S is said to be *quasi-convex* if there is some $k \epsilon \mathbf{N}$ for which for every $x \epsilon S$, every geodesic between two points g^i and g^j with $i, j \epsilon \mathbf{Z}$ will lie within $\bigcup_{n \epsilon \mathbf{Z}} B_{g^n}(k)$. In this case the union of geodesics $[g^n.x, g^{n+1}.x]$ is said to be the axis of g.

Proposition 4.16 (7, Proposition 18)**.** *If a group G acts on a CAT(0) cube complex X with a quasi-convex WPD element $g \epsilon G$, then for every natural n, g^n is WPD as well.*

Proof. [7] Let γ be the axis of g in the action of G on X^1. In this proof we will consider this axis geodesic to simplify some details. From [7, Proposition 14] we have that the join of any two families of hyperplanes crossing γ is C-thin for some C. If we now fix some point x on γ then for any $\varepsilon \geq 0$ since g is WPD we can find some natural m so that $\{h \epsilon \mathrm{G} | d(x, hx)$ and $d(g^m x, hg^m x) \leq C + 6\varepsilon\}$ is finite.
Suppose h satisfies $d(x, hx)$, $d(g^{mn}x, hg^{mn}x) \leq \varepsilon$. Since $g^m x$ lies on γ between x and $g^{mn}x$, we apply Lemma 4.10 and see that $d(g^m x, hg^m x) \leq C + 6\varepsilon$. Hence, $\{h \epsilon G | d(x, g^{mn}x)$ and $d(hx, hg^{mn}x) \leq \varepsilon\}$ is finite. By Lemma 4.9 g^n is also WPD.

□

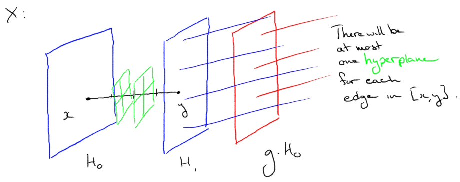


Definition 4.17 (Contracting Isometries)**.** [7, Theorem 16] An isometry of a CAT(0) cube complex X is said to be *contracting* if it skewers a pair of well-separated hyperplanes.

We are now ready to prove our main theorem.

Theorem 4.18 (7, Theorem 17)**.** *Let G be a group acting on a CAT(0) cube complex. Then $g \epsilon G$ is a WPD contracting isometry if and only if it skewers a pair of well-separated hyperplanes for which the intersection of the stabilizers of the two hyperplanes is finite.*

Proof. [7] First, suppose g skewers a pair of well-separated hyperplanes, $(\mathcal{H}_1, \mathcal{H}_2)$ such that the intersection of their stabilizers is finite. By definition g is a contracting isometry.
We may denote by H_1 and H_2 the half-spaces determined by $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfying that for some natural n, $g^n H_1 \subset H_2$ (as g skewers the two hyperplanes). Also, there exist finitely many hyperplanes separating H_1 and $g^n H_1$, demonstrated by the illustration below.



If we let $S = stab(\mathcal{H}_1, g^n\mathcal{H}_1)$ then we can see that if we denote $S_1 = \cap stab(\mathcal{H}_i)$ the subgroup of G that stabilizes all of these separating hyperplanes, S_1, is necessarily a finite index subgroup of S. We do, however, in addition have that $\mathcal{H}_2$ separates $\mathcal{H}_1$ and $g^n\mathcal{H}_1$, and so $S_1 \subseteq stab(\mathcal{H}_1) \cap stab\mathcal{H}_2)$, which is finite by assumption. Thus, S is necessarily finite, as it has a finite finite index subgroup.

Let γ be the axis of g in X^1. By what we have just shown, there is a hyperplane $\mathcal{H}$ which intersects this axis and $n \,\epsilon\, \mathbf{N}$ such that $\mathcal{H}$ and $g^n\mathcal{H}$ are disjoint, and the intersection of their stabilizers is finite (in particular, $\mathcal{H}_1$). Let $x \,\epsilon\, \gamma \cup N(\mathcal{H})$, (where $N(\mathcal{H})$ is the carrier from Proposition 4.5), $\varepsilon \geq 0$, and define $F = \{h \,\epsilon\, G | d(x, hx)$ and $d(g^{n+2\varepsilon}x, hg^{n+2\varepsilon}x) \leq \varepsilon\}$. In light of Lemma 4.12, we need only prove F is finite to show that g is WPD.

Let $W = \{g^k\mathcal{H} | 1 \leq k \leq n + 2\varepsilon\}$ where n is from above. We will show that for all but at most 2ε elements of F, some element of W is sent to some hyperplane which separates x and $g^{n+2\varepsilon}x$. For this section, we will chose to let $\mathfrak{H}_{x,g^{n+2\varepsilon}x}$ denote the set of hyperplanes which separates x and $g^{n+2\varepsilon}x$.

Let $f \,\epsilon\, F$, and $J \,\epsilon\, W$. Since J separates x and $g^{n+2\varepsilon}x$, fJ must separate fx and $fg^{n+2\varepsilon}x$. If $f\mathfrak{H}$ does not separate x and $g^{n+2\varepsilon}x$, it must of course separate one of $\{x, fx\}$ or $\{g^{n+2\varepsilon}x, fg^{n+2\varepsilon}x\}$. We have, however, that $d(x, hx)$ and $d(g^{n+2\varepsilon}x, hg^{n+2\varepsilon}x) \leq \varepsilon$. This gives that there are at most 2ε possible f that do not separate x and $g^{n+2\varepsilon}x$.

Now, let L denote the set of functions from some subset T of W with cardinality at least $|W| - 2\varepsilon$ into $\mathfrak{H}_{x,g^{n+2\varepsilon}x}$. Then any element of F determines some hyperplane in $\mathfrak{H}_{x,g^{n+2\varepsilon}x}$, and so defines a function in L. In light of this, let us suppose then that F is infinite. Notice

L is finite as it has cardinality at most $|\mathfrak{H}_{x,g^{n+2\varepsilon}x}|^{|W|}$, and so we must necessarily have an infinite collection of elements $f_0, f_1, f_2, ... \in F$ which all induce the same function in L. In particular, $f_0^{-1}f_1, f_0^{-1}f_2, ...$ stabilize all the hyperplanes in T. We must then have that there exists some natural k, $0 \leq k \leq 2\varepsilon$ such that infinitely many elements simultaneously stabilize $g^k\mathcal{H}$ and $g^{k+n}\mathcal{H}$. Notice, however, that $stab(g^k\mathcal{H}) \cap stab(g^{k+1}\mathcal{H}) = g^{-k}(stab(\mathcal{H}) \cap g^n\mathcal{H})g^k$, which we know to be finite. This is a contradiction, and thus F is finite.

Conversely, suppose that g is WPD and contracting. Since g is contracting, by Definition 4.15 it skewers a pair of well-separated hyperplanes, and so again there exists some hyperplane $\mathcal{H}$ and some natural n for which $\mathcal{H}$ and $g^n\mathcal{H}$ are L-well separated. In [7, Theorem 17] Genevois provides a proof that g is contracting. This result unfortunately uses a construction not otherwise needed in this paper. and so we will proceed as if g is contracting.

Fix some $z \in P$. By Proposition 5.7 since g is both WPD and contracting g^n is also WPD. We can find some natural m so that $\{h \in G | d(z, hz)$ and $d(g^{nm}z, hg^{nm}z\} \leq L\}$ is finite. We would like to prove that $S = \bigcap_{i=0}^{m+1} stab(g^{ni}\mathcal{H})$ is finite. Because S stabilizes $\mathcal{H}$ and $g^n\mathcal{H}$, we see that S stabilizes P whose diameter we know is at most L. S also however stabilizes $g^{nm}\mathcal{H}$ and $g^{n(m+1)}\mathcal{H}$, and so must similarly stabilize $g^{nm}P$ which too has diameter at most L. We have chosen $z \in P$ and shown that $g^{nm}z \in g^{nm}P$. We have however also concluded that $d(z, hz) \leq L$ and $d(g^{mn}x, hg^{mn}x) \leq L$ for every $h \in S$. Since g^n is WPD, S must then be finite.

Note that also there are only finitely many hyperplanes separating $\mathcal{H}$ and $g^{n(m+1)}\mathcal{H}$ and so S is, similarly to an argument from the reverse direction, a finite index subgroup of $stab(\mathcal{H}) \cap stab(g^{n(m+1)})\mathcal{H}$ which is thus finite. We have thus successfully classified WPD contractions on a CAT(0) cube complex.

□

5. Conclusions

In this paper we have seen how we can generalize the outstandingly nice properties of a hyperbolic group to the somewhat more realistic situation of an acylindrically hyperbolic group. We have demonstrated that right-angled Artin groups and amalgamated products with a malnormal embedded subgroup fit into this class of groups, and how we can classify WPD loxodromic elements to more easily identify if a group acting on a CAT(0) cube complex is acylindrically hyperbolic. Finally, the reader should understand how we can use both the classical and modern methods we have shown off to solve group theoretic problems.

We hope that the reader will have taken interest in what we have discussed, and if so we suggest that they reference the bibliography below for further reading on the topic.

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