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TITLE: Computation of Value-at-Risk and
Related Optimization

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Abstract

The paper is based mainly on introducing VaR and exploring its optimization portfolio by enhancing consumption level subject to required VaR and budgets. The concrete process of deriving the expressions of VaR for specific cases we set in the background will be made, which gives a basic notion and a better grasp of the definition of VaR. Furthermore, this paper will also involve the introduction to CVaR and some basic knowledge about bonds and stocks in the aspect of mathematics. We also analyze the results running from MATLAB to make further illustrations.

1 Introduction

VaR as a risk measurement method is widely used in various financial institutions, which is a percentile of a potential loss distribution. In other words, VaR itself is the expected maximum possible portfolio loss for a given confidence level and a certain holding period under normal market volatility. VaR, however, does not control scenarios exceeding Var [1]. There are two essential parameters need to know before introducing the concept of VaR: the holding period of the portfolio and the confidence level. The holding period is relative to the liquidity of assets and has a profound influence on VaR. Precisely, the stronger the asset liquidity, the shorter the holding period. Conversely, the weaker the asset liquidity, the longer the holding period. Furthermore, the choice of confidence level reflects the maximum loss acceptable to investors. The higher the confidence level, the greater the probability that the loss of the portfolio is less than VaR. That is, the less likelihood that VaR model will fail to predict the occurrence of extreme events.

CVaR also known as expected shortfall (ES) is a risk measurement technique superior to VaR proposed by Rockafellar and Uryasev [10], which accounts for losses exceeding VaR and hence depicts the risk in extreme tails. Hence, VaR and CVaR refer to different parts of the distribution [7]. From the statistical perspective, CVaR is actually the conditional expectation of losses over VaR.

The portfolio in the context involves two kinds of investment assets including bonds and stocks. A bond is a debt security (or a loan) that an investor makes either to a government agency or to

a corporation. The borrower promises to pay the lender interest at regular intervals and to repay the principal at the end of the loan. A stock is a type of ownership security representing a share in a cooperation. Importantly, geometric Brownian motion (GBM), which is a stochastic process in continuous time where the logarithm of a random variable follows a Brownian motion, now is used in financial mathematics and modelling stock prices.

This paper includes four sections in total. The first part introduces the formulas of bonds and stocks as well as their corresponding expressions of rate of return. Based upon the expressions of rate of return, we will construct two portfolios to be used to estimate VaR and CVaR in the next section. The first portfolio merely considered the return on investments in bonds and stocks. The other one still has regard to expenditure on consumption in addition to this. In the third section, we will introduce the process of deriving the expressions of VaR and CVaR on the basis of the two portfolios we constructed before. Finally, we will discuss how to distribute assets to enhance traders' consumption level with constraints on required VaR and budgets as far as possible.

2 Basic Notions of VaR and CVaR

2.1 Constructing a portfolio with a bond and a stock

2.1.1 Expressions of bond prices

In order to introduce a formula for the price of a bond at the end of t ($t > 0$) years $P(t)$ and explain its associated parameters, we consider a bond with an effective compound interest rate r , and $P(t)$ can be expressed as

$$P(t) = P(0)\left(1 + \frac{r}{n}\right)^{nt} \quad (1)$$

where $P(0)$ is the present value; n is the number of periods in a year; nt is the number of periods in the term of the bond.[13]

Remark:

If $n = 1$, this is a bond with an annually interest rate;

If $n = 2$, this is a bond with a semi-annually interest rate;

If $n = 4$, this is a bond with a quarterly interest rate.

Analysis for rate of bond with single period

For simplifying, we consider the single period under the following assumptions.

- The holding period is one year ($t=1$)
- Consider an annual interest of r for the bond, compounded annually (so, $nt = 1$ since $n = 1$) under the above assumptions $P(t) = P(0)(1+r)$, and the return in percentage is equal to

$$R_{bond} = \frac{p(t)-p(0)}{p(0)} = \frac{p(t)}{p(0)} - 1 = r \quad (2)$$

- Consider an annual interest of r for the bond, compounded semiannually (so, $nt = 2$ since $n = 2$) under the above assumptions $P(t) = P(0)(1 + \frac{r}{2})^2$, and the return in percentage is equal to

$$R_{bond} = \frac{p(t)-p(0)}{p(0)} = \frac{p(t)}{p(0)} - 1 = (1 + \frac{r}{2})^2 - 1$$

- Consider an annual interest of r for the bond, compounded quartely (so, $nt = 4$ since $n = 4$) under the above assumptions $P(t) = P(0)(1 + \frac{r}{4})^4$, and the return in percentage is equal to

$$R_{bond} = \frac{p(t)-p(0)}{p(0)} = \frac{p(t)}{p(0)} - 1 = (1 + \frac{r}{4})^4 - 1$$

2.1.2 Expressions of stock prices

According to Dunbar [12] and Dmouj [14], now, we suppose that the formula of the daily stock closing price follows a geometric Brownian motion (GBM) defined by

$$S(t) = S_0 e^{Y(t)} = S_0 e^{\sigma B(t) + \mu t} \quad (3)$$

where t is the unit interval for time (one day here); $S_0 = S(t = 0)$ is the initial value invested to the stock; S_t is the stock closing price of the day; $Y(t) = \sigma B(t) + \mu t$, for $t \geq 0$ is a Brownian motion (BM) with drift coefficient μ (instantaneous rate of return on a riskless asset) and standard deviation σ (volatility) and $\{B(t), t \geq 0\}$ is a standard BM. Hence, let $Z(t)$ represent the standard normal distribution,

$$Z(t) = \frac{B(t)}{\sqrt{t}} \sim N(0, 1) \text{ since } B(t) \sim N(0, t) \text{ with } \sigma = 1$$

Define the daily stock return in percentage is $R_{stock} = \frac{S(t)}{S_0} - 1 = e^{Y(t)} - 1$.

Note: GBM is a non-negative variation of BM.

Definition: Mathematically, a BM (A.K.A. Wiener process) is a continuous time random walk, $\{B(t), t \geq 0\}$ is a BM (process) satisfying the following properties:

1. $B(t=0)=0$ with probability 1;

-
2. $\{B(t), t \geq 0\}$ has stationary and independent increments;
 - Stationarity: For all $0 < s < t$, the increment $B(t) - B(s)$ corresponds to a normal distribution with mean 0 and variance $t - s$;
 - Independent increments: For non-overlapping intervals $[s_i, t_i]$, the random variables $B(t_i) - B(s_i)$ are independent of each other.
 3. $B(t) \sim N(0, \sigma^2 t)$, for every time $t > 0$.
- If $\sigma = 1$, the process is called a standard BM. Hence, $\frac{B(t)}{\sigma}$ is a standard BM.

Further remarks: For a large n , let S_n denote the stock closing price at day n . Now, let $X_n = \frac{S_n}{S_{n-1}}$ be the stock price changes in percentage

$$\Rightarrow S_n = X_n S_{n-1} = X_n X_{n-1} S_{n-2} = \dots = X_n X_{n-1} \dots X_1 S_0$$

Note: $X_1 \dots X_n$ are independent and identically random variables by the property of Markov process. Next, we take the natural logarithm of both sides of S_n equation and obtain the formula of $\log S_n$

$\Rightarrow \log S_n = \sum_{i=0}^n \log X_i + \log S_0$, where $\sum_{i=0}^n \log X_i \approx Y(t) \sim N(\mu t, \sigma^2 t)$. Now, raising a base e to the power of both sides, we get the initial formula of GBM again

$$S_n = S_0 e^{Y(t)} = S(t)$$

But it is noteworthy the $S(t)$ here represents the price of some stock at year t if we set $n = 252$ (because the trading days for one year are usually 252 days). Therefore, we can conclude that the annually stock closing price also follows a GBM.

2.2 An Introduction to Value at Risk

This section gives definition of VaR and explores the prerequisites for calculating VaR, as well as derives its expression under the condition of meeting prerequisites.

According to Saita [9], we know that Value-at-Risk (VaR) is the maximum loss in amount (dollar) or percentage for a desired confidence level $1 - \alpha$ within a certain holding period (t). Mathematically, the VaR with confidence level $(1 - \alpha) \in (0, 1)$ denoted by $VaR_{1-\alpha}$ is a positive number:

$$P(V_0 - V > VaR_{1-\alpha}) = \alpha \tag{4}$$

or alternatively,

$$P(V_0 - V \leq VaR_{1-\alpha}) = 1 - \alpha \quad (5)$$

As a more intuitive and specific description of the definition of VaR, Kisiala [10] stated several equivalent interpretations:

- VaR_α is the *minimum* loss that will not be exceeded with probability α .
- VaR_α is the α -quantile of the distribution of loss.
- VaR_α is the *smallest* loss in the $(1 - \alpha) \times 100\%$ worst cases.
- VaR_α is the *highest* loss in the $\alpha \times 100\%$ best cases.

An example to calculate VaR

Next, we consider a relatively simple model to derive the expression of VaR based upon its definition.

Suppose Y is normally distributed with mean μ and variance σ^2 :

$$V = V_0(1 + Y), \text{ for } V_0 \text{ known and } \mu \geq 0, \sigma^2 < \infty$$

Substitute $V = V_0(1 + Y)$ into equation (1), we get

$$\begin{aligned} P(-V_0Y > VaR_{1-\alpha}) &= \alpha \quad \text{since } V_0 - V = V_0 - V_0(1 + Y) = -V_0Y > 0 \\ \Rightarrow p(-Y > \frac{VaR_{1-\alpha}}{V_0}) &= \alpha \\ \Rightarrow p(Y < -\frac{VaR_{1-\alpha}}{V_0}) &= \alpha \quad \text{for } Y \sim N(\mu, \sigma^2) \end{aligned}$$

Note: $-\frac{VaR_{1-\alpha}}{V_0} < 0$ since both of $VaR_{1-\alpha}$ and V_0 are positive here. Therefore, $y \in (-\infty, 0)$ for each $y \in Y$.

and we can create a new random variable Z such that $Z = \frac{Y-\mu}{\sigma} \sim N(0, 1)$ is the standard normal distribution.

$$\Rightarrow P \left(Z < \frac{VaR_{1-\alpha} - \mu V_0}{\sigma V_0} \right) = \alpha$$

$$\Rightarrow P \left(Z < \frac{-VaR_{1-\alpha} - \mu V_0}{\sigma V_0} \right) = \alpha$$

$$\Rightarrow \frac{-(VaR_{1-\alpha} + \mu V_0)}{\sigma V_0} = \Phi^{-1}(\alpha)$$

where $\Phi^{-1}(\bullet)$ is the inverse cumulative standard normal distribution.

$$\Rightarrow \frac{(VaR_{1-\alpha} + \mu V_0)}{\sigma V_0} = -\Phi^{-1}(\alpha)$$

$$\Rightarrow VaR_{1-\alpha} = -\Phi^{-1}(\alpha)V_0\sigma - \mu V_0 = -(\Phi^{-1}(\alpha)V_0\sigma + \mu V_0)$$

where the value of V_0 and σ are always positive; the percentage drift μ could be either positive or negative.

Note: $VaR_{1-\alpha}$ should be positive number, so the value of $\Phi^{-1}(\alpha)V_0\sigma + \mu V_0$ is negative. $\Phi^{-1}(\alpha) \leq 0$ if $\alpha \leq 0.5$ There are three values of α are commonly considered: 0.10, 0.05, and 0.01. Hence, the values of $\Phi^{-1}(\alpha)$ we need are usually negative.

Example: Now consider a \$100 portfolio and suppose that the trader wants to estimate the maximum potential loss at 95% confidence level. He estimates that $\mu = 0.05$ and $\sigma = 0.2$ on a annually basis. Then $VaR_{0.95} = -(-1.645 \times 100 \times 0.2 + 0.05 \times 100) = \27.9 .

The potential loss, obviously, depends upon μ and σ for a given confidence level $(1 - \alpha)$ within a certain trading period. By Rockafellar and Uryasev [9], the random variable X representing loss follows a Normal distribution with mean μ and variance $V_0^2\sigma^2$.

$$-X \sim N(V_0\mu, V_0^2\sigma^2)$$

It can be easily understand that, if $X \sim N(\mu, \sigma^2)$

$$VaR_{1-\alpha}(X) = \mu + \Phi^{-1}(\alpha)\sigma \quad (6)$$

So, the standard deviation σ representing the volatility is proportional to VaR.

This is an approach to calculate VaR of a portfolio with single asset for which asset return (i.e., loss = negative return) is a normal distribution. Similarly, VaR for a portfolio with multi-assets can be also expressed as $VaR_{1-\alpha}(Y) = \mu + \Phi^{-1}(\alpha)\sigma$, where the asset return Y is jointly normally distributed, both of the percentage drift μ and the percentage volatility σ are constant and derived from the historical data. We have an introduce of the computations of drift μ and volatility σ of stocks in the part of Appendix.

2.2.1 Computation of VaR for 2 cases

In this section, we will construct two model to estimate VaR: the first one is based on the portfolio that merely considers return on investments in bonds and stocks; the second one still has regard to expenditure on consumption in addition to this.

Before deriving the expressions of VaR, one point worth mentioning here is the background in the context. In this paper, we suppose that the holding period is 1 year and also, there are 3 given confidence levels: 90%, 95% and 99% used to test models and make further illustrations. Moreover, the rest of variables still need to be fixed to estimate the models we need include the initial value of a portfolio $V_0 = \$1000$; the annually interest rate $r = 0.05$; the drift and volatility of stocks on an annual basis are 0.15 and 0.2 respectively.

Case 1: Now, consider a trader with initial asset V_0 decides to split his assets into two portions with weights β and $1 - \beta$ ($0 \leq \beta \leq 1$), and returns R at the end of the holding period (one year) in mathematical expression is

$$\begin{aligned}
R &= (1 - \beta)V_0R_{bond} + \beta V_0R_{stock} \\
&= (1 - \beta)V_0r + \beta V_0(e^{Y(t)} - 1) \\
&= V_0r - \beta V_0(r + 1 - e^{Y(t)}) \\
&= V - V_0
\end{aligned}$$

Then, suppose return is non-positive. That is,

$$\begin{aligned}
R = V_0[r - \beta(r + 1 - e^{Y(t)})] &\leq 0 \\
\Rightarrow r - \beta(r + 1 - e^{Y(t)}) &\leq 0 \text{ since } V_0 > 0 \\
\Rightarrow \frac{r}{r+1-e^{Y(t)}} &\leq \beta \leq 1
\end{aligned}$$

This is the **prerequisite** satisfying $R = V - V_0 \leq 0$. So, the loss equation we set before $V_0 - V = -V_0Y \geq 0$, for $Y \sim N(\mu, \sigma^2)$.

For $\frac{r}{r+1-e^{Y(t)}} \leq \beta \leq 1$, the return function $R = V_0r - \beta V_0(r + 1 - e^{Y(t)}) = V - V_0 \leq 0$. Then, $V_0 - V = \beta V_0(r + 1 - e^{Y(t)}) - V_0r = V_0[\beta(r + 1) - \beta e^{Y(t)} - r] \geq 0$. Now, we substitute this equation $V_0 - V$ into equation (1) and get,

$$\begin{aligned}
P(V_0 - V > VaR_{1-\alpha}) &= \alpha \\
\Rightarrow P(V_0[\beta(r + 1) - \beta e^{Y(t)} - r] > VaR_{1-\alpha}) &= \alpha \\
\Rightarrow P(\beta(r + 1) - \beta e^{Y(t)} - r > \frac{VaR_{1-\alpha}}{V_0}) &= \alpha \\
\Rightarrow P(-\beta e^{Y(t)} > \frac{VaR_{1-\alpha}}{V_0} + r - \beta(r + 1)) &= \alpha
\end{aligned}$$

$$\Rightarrow P(e^{Y(t)} < (r+1) - \frac{r}{\beta} - \frac{VaR_{1-\alpha}}{\beta V_0}) = \alpha$$

$$\Rightarrow P(Y(t) < \ln[(r+1) - \frac{r}{\beta} - \frac{VaR_{1-\alpha}}{\beta V_0}]) = \alpha \quad \text{where } Y(t) = \sigma B(t) + \mu t$$

$$\Rightarrow P(B(t) < \frac{\ln[(r+1) - \frac{r}{\beta} - \frac{VaR_{1-\alpha}}{\beta V_0}] - \mu t}{\sigma}) = \alpha \quad \text{where } B(t) \sim N(0, t), \text{ for } t \geq 0$$

$$\Rightarrow P(Z(t) < \frac{\ln[(r+1) - \frac{r}{\beta} - \frac{VaR_{1-\alpha}}{\beta V_0}] - \mu t}{\sigma \sqrt{t}}) = \alpha$$

for $\frac{\ln[(r+1) - \frac{r}{\beta} - \frac{VaR_{1-\alpha}}{\beta V_0}] - \mu t}{\sigma \sqrt{t}} = \Phi^{-1}(\alpha)$, after some computations, we get the expression of $VaR_{1-\alpha}$ with a constraint $\frac{r}{r+1-e^{Y(t)}} \leq \beta \leq 1$

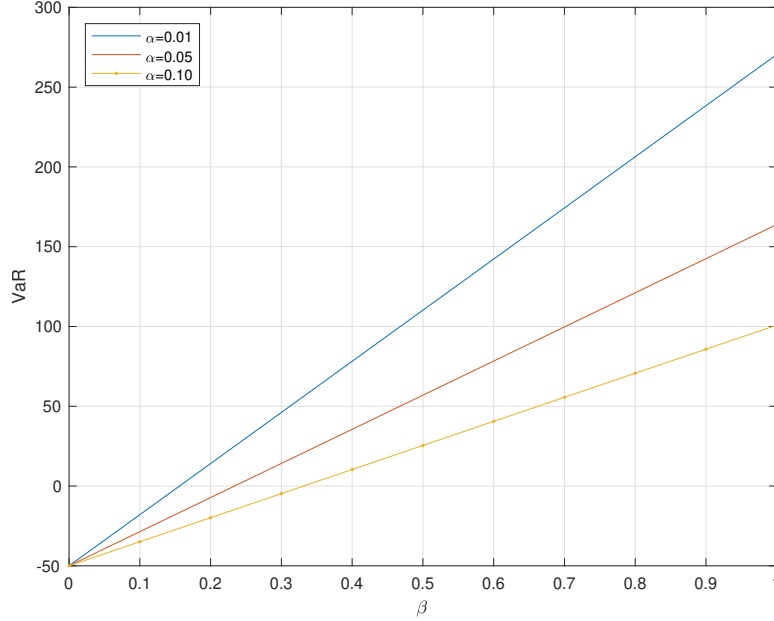
$$VaR_{1-\alpha} = V_0[\beta(r+1) - r - \beta e^{\Phi^{-1}(\alpha)\sigma\sqrt{t} + \mu t}] \quad (7)$$

where the initial value V_0 and the effective compound interest rate r are known. Now, suppose α is given and we can find $VaR_{1-\alpha}$ for the different cases of different given μ and σ . Both of values have been set and fixed in this paper.

Scenario for case 1: Consider a portfolio with \$1000 and suppose that the trader wants to estimate the maximum potential loss at 90%, 95% and 99% confidence level, where $\mu = 0.15$ and $\sigma = 0.2$ on an annual basis, the annually interest rate of bond, compounded annually ($n=1$) is 5% and here $t = 1$.

Prerequisite for estimating VaR: On the basis of definition of VaR mentioned above, the constraint β should be $\frac{r}{r+1-e^{Y(t)}} \leq \beta \leq 1$. From MATLAB code, we get the desired value of β when VaR is equal to zero:

α	0.01	0.05	0.10
betaDesired	0.1561	0.2338	0.3314



Importantly, the constraints of β of different preferred confidence levels for different traders are:

- $\beta \geq 0.1561$ at a confidence level of 99%
- $\beta \geq 0.2338$ at a confidence level of 95%
- $\beta \geq 0.3314$ at a confidence level of 90%

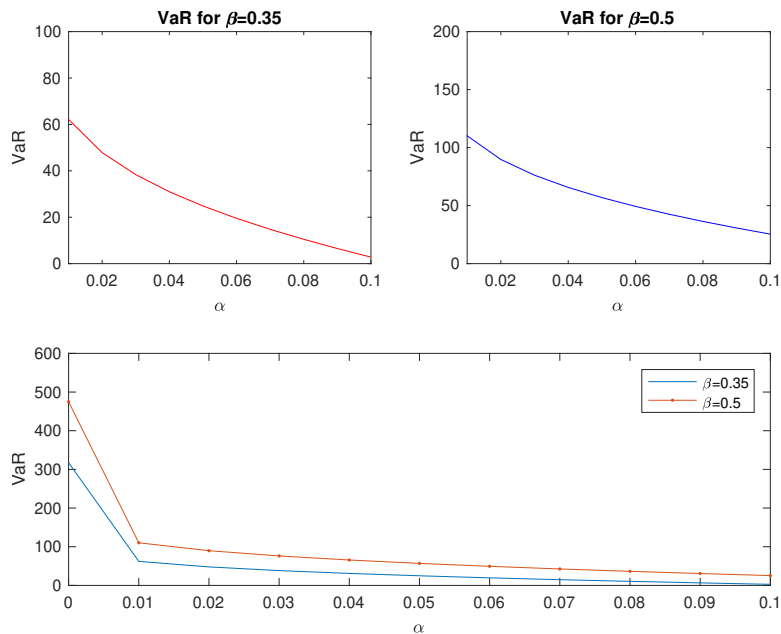
Based upon the definition of VaR and to make our research be meaningful, the value of β needs to satisfy $\beta \geq 0.3314$. This is due to the fact that VaR brings down with the increase in confidence level. Consequently, for $\beta \geq 0.3314$, there is always a negative return ($VaR \geq 0$) at any confidence level from 90% to 99%. Therefore, we set $\beta = 0.35$ and $\beta = 0.50$ in the next analysis, as well as the corresponding VaR shows as following:

These data running from MATLAB are a strong proof of deduction of β constraints. For $\beta = 0.35$ (approximate to 0.3314), VaR at 90% confidence level is 2.80, which is also very close to zero. Furthermore, both of VaR and the constraint of β varies with the investors' confidence level. Precisely, if investors are willing to take on a relatively great risk, they will assign more asset to stock. Conversely, they will assign more asset to bond. This conclusion will be also supported by

Value at Risk		
α	$\beta=0.35$	$\beta=0.50$
0.01	62.14	110.20
0.05	24.85	56.93
0.10	2.80	25.43

Table 1: VaR for different values of α and β

the relationship between investors' expected return and the value of β , which will show at the next subsection.



From the above figure, it is obviously to know that

- There is a negative relationship between VaR and the value of α , which implies that VaR is proportional to confidence level $(1 - \alpha)$. As the figure and the output from MATLAB code shown, if this investor puts 35% assets of this portfolio into stock ($\beta = 0.35$), there are 95% and 99% chances he won't lose more than \$56.93 and \$25.43 respectively at the end of one year.
- There is a positive relationship between VaR and the value of β and hence under the condition of the same level of confidence level α , VaR for $\beta = 0.50$ is definitely greater than the case of $\beta = 0.35$.

Case 2: Continue to another cases that the consumption is considered and accounts for c ($0 \leq$

$c \leq 1$). Let the proportion of investment in stock stay at level β and that of bond, we use $(1-c-\beta)$ instead of $(1-\beta)$. Then, as the same processes calculating VaR shown before, we firstly write out the return equation

$$\begin{aligned} R &= (1 - c - \beta)V_0R_{bond} + \beta V_0R_{stock} - cV_0 \\ &= (1 - c - \beta)V_0r + \beta V_0(e^{Y(t)} - 1) - cV_0 \\ &= V_0[r - c(r + 1) - \beta(r + 1 - e^{Y(t)})] \\ &= V - V_0 \end{aligned}$$

Prerequisite for estimating VaR: Now we need to find a relationship between β and c satisfying

$$V_0[r - c(r + 1) - \beta(r + 1 - e^{Y(t)})] \leq 0 \Rightarrow \frac{r-(r+1)c}{r+1} \leq \frac{r-(r+1)c}{r+1-e^{Y(t)}} < \beta \leq 1 \Rightarrow \frac{r}{1+r} \leq \beta + c \leq 1$$

Note that the lower bound of $\beta+c$ is larger than $\frac{r}{1+r}$, however. Hence, the actual restriction of $\beta+c$, in fact, should be approximately $\frac{2r}{1+r} \leq \beta + c \leq 1$.

We will have a negative return as well as $V_0 - V = V_0[c(r + 1) + \beta(r + 1 - e^{Y(t)}) - r] = V_0[(\beta + c)(r + 1) - \beta e^{Y(t)} - r] \geq 0$. Substituting $V_0 - V$ into equation (4), we get

$$P(V_0 - V > VaR_{1-\alpha}) = \alpha$$

$$\Rightarrow P(V_0[(\beta + c)(r + 1) - \beta e^{Y(t)} - r] > VaR_{1-\alpha}) = \alpha$$

$$\Rightarrow P(-\beta e^{Y(t)} > \frac{VaR_{1-\alpha}}{V_0} + r - (\beta + c)(r + 1)) = \alpha$$

$$\Rightarrow P(e^{Y(t)} < \frac{1}{\beta}[(\beta + c)(r + 1) - r - \frac{VaR_{1-\alpha}}{V_0}]) = \alpha$$

$$\Rightarrow P(Y(t) < \ln\{\frac{1}{\beta}[(\beta + c)(r + 1) - r - \frac{VaR_{1-\alpha}}{V_0}]\}) = \alpha \quad Y(t) = \sigma B(t) + \mu t$$

$$\Rightarrow P(B(t) < \frac{\ln\{\frac{1}{\beta}[(\beta + c)(r + 1) - r - \frac{VaR_{1-\alpha}}{V_0}]\} - \mu t}{\sigma}) = \alpha \quad \text{where } B(t) \sim N(0, t), \text{ for } t \geq 0$$

$$\Rightarrow P(Z(t) < \frac{\ln\{\frac{1}{\beta}[(\beta + c)(r + 1) - r - \frac{VaR_{1-\alpha}}{V_0}]\} - \mu t}{\sigma\sqrt{t}}) = \alpha$$

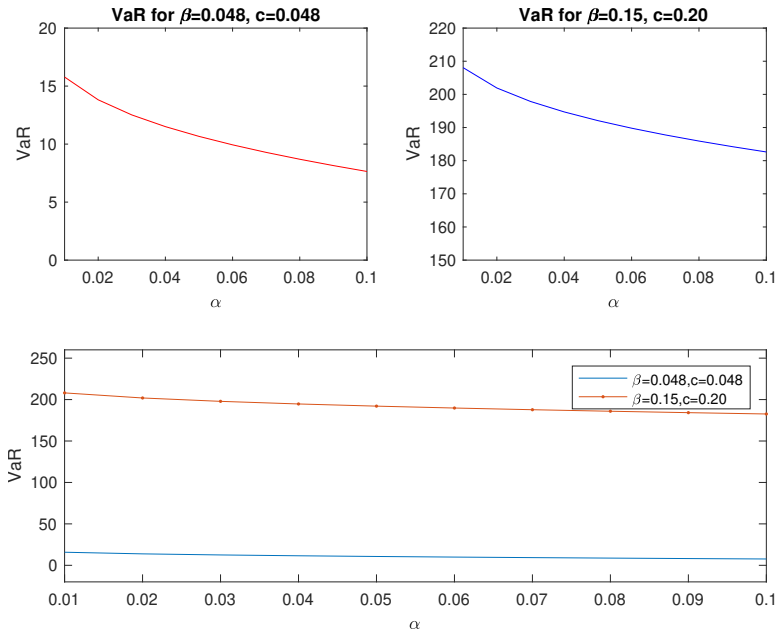
where $\frac{\ln\{\frac{1}{\beta}[(\beta + c)(r + 1) - r - \frac{VaR_{1-\alpha}}{V_0}]\} - \mu t}{\sigma\sqrt{t}} = \Phi^{-1}(\alpha)$ after some computations, we have

$$VaR_{1-\alpha} = V_0[(\beta + c)(r + 1) - r - \beta e^{\Phi^{-1}(\alpha)\sigma\sqrt{t} + \mu t}] \quad (8)$$

Scenario for case 2: Substituting $V_0 = \$1000$, $r = 0.05$, $t = 1$, $\mu = 0.15$, and $\sigma = 0.2$ into equation (8), all of data and plots show as follows. It is noteworthy that the proportion of asset assigned to stock β , in this case, need to satisfy $0.095 \approx \frac{2 \times 0.05}{1 + 0.05} = \frac{2r}{1+r} \leq \beta + c \leq 1$. We analyze the changes in VaR by setting $c = 0.048$ $\beta=0.048$ and $c = 0.20$, $\beta=0.15$

Value at Risk		
α	$\beta=0.048; c=0.048$	$\beta=0.15; c=0.20$
0.01	15.78	208.06
0.05	10.67	192.08
0.10	7.64	182.63

Table 2: VaR for different values of α, β and c



From Table 2 with $\beta = 0.15$ and $c = 0.20$, emphasized on the weights spent on consumption (c) is 20%, that is \$200. Then there is a 99% chance that the trader will lose equal to or less than \$208.06, which implies that the trader has a 99% confidence that his loss will not exceed \$8.06 for the part of investment in stocks. Similarly, there will be a minimum gain of \$7.92 and \$17.37 at $\alpha = 0.05$ and 0.10 levels.

2.2.2 Computation of Expected Return

In order to have a clear view of the relationship between the proportion of investment in stocks β and the expected return, we obtain their linear relationship and plot it by MATLAB.

As Saita[12] and Dmouj[14] demonstrated, a Brownian motion with drift denoted by $Y(t) = \sigma B(t) + \mu t$. Its distribution in any length t satisfies the normal distribution with mean μt and variance $\sigma^2 t$. Considering the stochastic differential equation (SDE)

$$dY(t) = \sigma dB(t) + \mu dt \quad (9)$$

where $dB(t)$ represents the change of Brownian motion over an infinitesimal time interval. $Y(t)$ can be negative so that it can be used to describe the rate of return. While the stock price $S(t) = S_0 e^{Y(t)}$ is always positive. Then $dS(t)$ is the amount of change in the stock price in infinitesimal intervals, $\frac{dS(t)}{S(t)}$ is the rate of return during this interval. So

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \sigma dB(t) + \mu dt = dY(t) \\ \Rightarrow dS(t) &= \sigma S(t) dB(t) + \mu S(t) dt \end{aligned}$$

This is called $It\bar{o}$ drift-diffusion process.

Set $f(t) = \ln(S(t))$, and $\frac{\partial f(t)}{\partial S(t)} = \frac{1}{S(t)}$, $\frac{\partial^2 f(t)}{\partial S(t)^2} = -\frac{1}{S^2(t)}$, $\frac{\partial f(t)}{\partial t} = 0$

$$\begin{aligned} df(t) &= d\ln(S(t)) \\ &= \left(\frac{\partial f(t)}{\partial S(t)}\sigma S(t)\right)dB(t) + \left(\frac{\partial f(t)}{\partial t} + \frac{\partial f(t)}{\partial S(t)}\mu S(t) + \frac{1}{2}\frac{\partial^2 f(t)}{\partial S(t)^2}\sigma^2 S^2(t)\right)dt \\ &= \left(\frac{1}{S(t)}\sigma S(t)\right)dB(t) + \left(\frac{1}{S(t)}\mu S(t) + \frac{1}{2}\left(-\frac{1}{S^2(t)}\right)\sigma^2 S^2(t)\right)dt \\ &= \sigma dB(t) + \left(\mu - \frac{1}{2}\sigma^2\right)dt \end{aligned} \quad (10)$$

Then, we can rewrite the geometric Brownian motion:

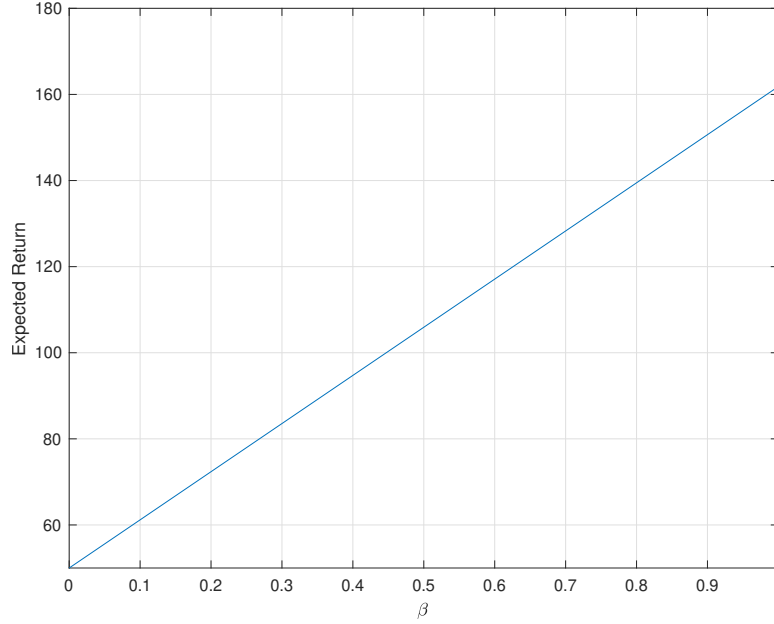
$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)} \quad (11)$$

$$\Rightarrow E[S(t)] = E[S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}] = S_0 e^{(\mu - \frac{\sigma^2}{2})t} E[e^{\sigma B(t)}] = S_0 e^{(\mu - \frac{\sigma^2}{2})t} e^{\frac{\sigma^2}{2}t} = S_0 e^{\mu t}$$

$$\begin{aligned} \Rightarrow E[R] &= E[V_0(r - \beta(r + 1) + \beta e^{Y(t)})] = V_0(r - \beta(r + 1) + \beta V_0 E[e^{Y(t)}]) \\ &= V_0(r - \beta(r + 1) + \beta V_0 e^{\mu t}) = V_0[r - \beta(r + 1 - e^{\mu t})] \end{aligned} \quad (12)$$

β	Expected return
0.1	61.18
0.2	72.37
0.5	105.92

Table 3: The relationship between β and ER



Above plot gives us a clear view that the expected return is proportional to β . This implies the more investors put on stock, the higher return they expected. At the same time, traders have to bear a relatively great risk, however.

2.2.3 Analysis of different trading periods

Now, we try to analyze the changes in VaR and CVaR for different holding period of this portfolio. The tables and plots illustrate those variance in detail.

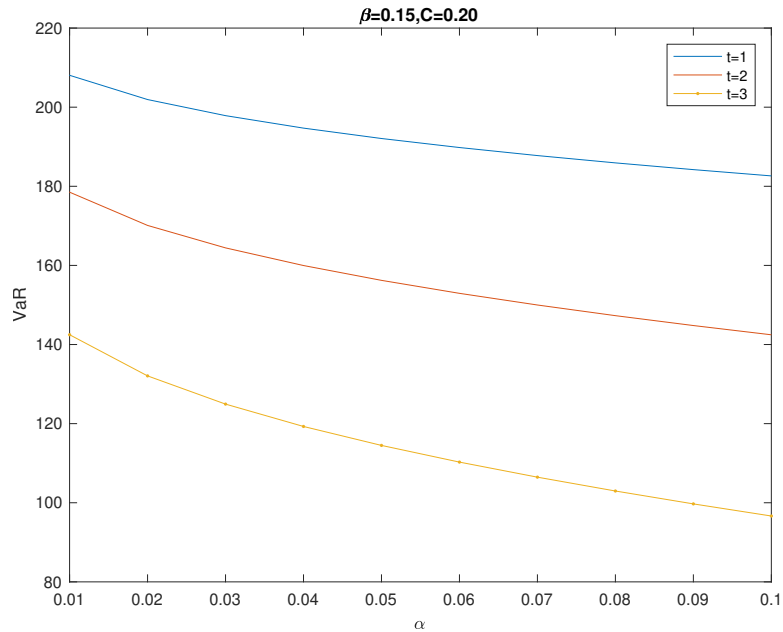
Note: It is important to note that rate of bonds will change with the increases in holding period t . From equation (1), we have for $t = 1$, $R_{bond} = r = 0.05$

$$\text{for } t = 2, R_{bond} = (1 + \frac{r}{1})^{1 \times 2} - 1 = (1 + \frac{0.05}{1})^2 - 1 = 0.1025$$

$$\text{for } t = 3, R_{bond} = (1 + \frac{r}{1})^{1 \times 3} - 1 = (1 + \frac{0.05}{1})^3 - 1 \approx 0.1576$$

Value-at-Risk			
α	t=1	t=2	t=3
0.01	208.06	178.51	142.47
0.05	192.08	156.22	114.49
0.10	162.63	142.26	96.65

Table 4: VaR for $\beta = 0.15$, $c=0.20$



The above results depicts that VaR has a positive trend associate with trading periods. VaR drops off but with the growth of holding period of portfolio. In addition, if confidence level was fixed, an increase in holding period of a portfolio bumps VaR up as well. It is easily and intuitively to see from the plot shown above . Hence, for traders who are preferring small risks, they can choose a relatively long holding period.

2.3 An Introduction to Conditional Value at Risk

Conditional Value at Risk (CVaR) also known as Expected Shortfall (ES), Average Value-at-Risk, or Tail Conditional Expectation is a measure that can be used to calculate the average size of losses in the worst α region of the given distribution. The term of conditional value-at-risk was stated by Rockfellar and Uryasev [10]. Consider the case that ES is measured with the same confidence level

$(1 - \alpha)$ as the VaR that we mentioned at the beginning. Then, its mathematical expression is

$$CVaR_{1-\alpha} := E[V_0 - V | V_0 - V > VaR_{1-\alpha}] \quad (13)$$

which is in case that $P(V_0 - V > VaR_{1-\alpha}) = \alpha = 1 - F_{(V_0 - V)}(VaR_{1-\alpha})$. Note that the loss $V_0 - V$ is positive and so the return function we set before is $R = V - V_0 = -(V_0 - V) < 0$. Hence, $CVaR_{1-\alpha}$ can also be expressed as

$$CVaR_{1-\alpha} := E[R | R < -VaR_{1-\alpha} < 0] \quad (14)$$

Based on the Acerbi's Integral Formula [11], we can have another expression of CVaR by solving the following integral formul

$$\begin{aligned} CVaR_{1-\alpha} := E[V_0 - V | V_0 - V > VaR_{1-\alpha}] &= \frac{1}{\alpha} \int_{1-\alpha}^1 VaR_{1-\beta} d\beta \\ &= \frac{1}{\alpha} \int_0^\alpha VaR_{1-\beta} d\beta \end{aligned} \quad (15)$$

Computation of CVaR for Example:

On the basis of the conclusion from example, we have if X is the loss distribution and follows a normal distribution with mean μ and variance σ^2 , and $VaR_{1-\alpha}(X) = \Phi^{-1}(\alpha)\sigma$. As well as the example at the beginning gives us $-X \sim N(V_0\mu, V_0^2\sigma^2)$ since $VaR_{1-\alpha}(X) = -(\Phi^{-1}(\alpha)V_0\sigma + \mu V_0)$. So we are going to compute its equation of $CVaR_{1-\alpha}(X)$. By definition and Uryasev et al.[3], we have,

$$\begin{aligned} CVaR_{1-\alpha}(X) &= E[X | X > VaR_{1-\alpha}(X)] \\ &= \int_{-\infty}^{\infty} (V_0 - V) dF_X^{1-\alpha}(V_0 - V) \end{aligned}$$

where $F_X^{1-\alpha}(V_0 - V) = P(V_0 - V > VaR_{1-\alpha}(X))$

$$\begin{aligned} &= P(V_0 - V > -V_0\mu - V_0\sigma Z) \\ &= P(Z > \frac{(V_0 - V) + V_0\mu}{-V_0\sigma}) \\ &= 1 - \Phi\left(\frac{(V_0 - V) + V_0\mu}{-V_0\sigma}\right) = \alpha \end{aligned}$$

$$\Rightarrow \Phi\left(\frac{(V_0 - V) + V_0\mu}{-V_0\sigma}\right) = 1 - \alpha$$

where $\Phi(\cdot)$ is a cumulative standard normal distribution. We can then deduce that

$$VaR_{1-\alpha}(X) = V_0 - V = -\Phi^{-1}(1 - \alpha)V_0\sigma - V_0\mu$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative standard normal distribution. Then, by Acerbi's Integral

Formula, $CVaR_{1-\alpha}(X) = \frac{1}{\alpha} \int_{1-\alpha}^1 VaR_{1-\beta}(X) d\beta$. Now, we can rewrite above equation by using the previous result.

$$\begin{aligned} CVaR_{1-\alpha}(X) &= \frac{1}{\alpha} \int_{1-\alpha}^1 -(\Phi^{-1}(1-\beta)V_0\sigma - V_0\mu) d\beta \\ &= \frac{1}{\alpha} \int_{1-\alpha}^1 -\Phi^{-1}(1-\beta)V_0\sigma d\beta - \frac{1}{\alpha} \int_{1-\alpha}^1 V_0\mu d\beta \\ &= \frac{1}{\alpha} \int_{1-\alpha}^1 -\Phi^{-1}(1-\beta)V_0\sigma d\beta - V_0\mu \end{aligned}$$

To solve the integral part of $\int_{1-\alpha}^1 -\Phi^{-1}(1-\beta)V_0\sigma d\beta$, we use the change of variable introducing $y = \Phi^{-1}(1-\beta)$, $1-\beta = \Phi(y) \Rightarrow \beta = 1-\Phi(y)$, $d\beta = -\phi(y)dy$, where $\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ is the standard normal density function.

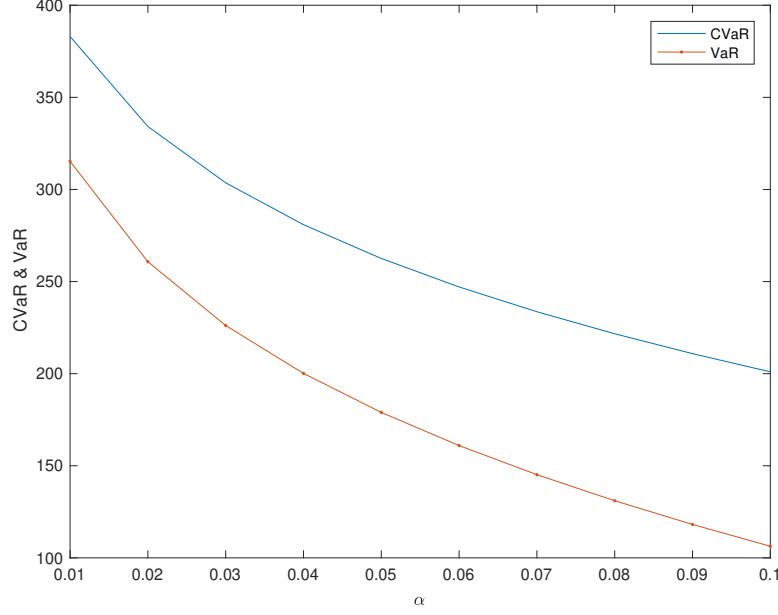
$$\begin{aligned} \Rightarrow \int_{1-\alpha}^1 -\Phi^{-1}(1-\beta)V_0\sigma d\beta &= \int_{\Phi^{-1}(1-\alpha)}^{\Phi^{-1}(1)} -yV_0\sigma(-\phi(y))dy \\ &= \int_{\Phi^{-1}(1-\alpha)}^{\infty} yV_0\sigma\phi(y)dy \\ &= \int_{\Phi^{-1}(1-\alpha)}^{\infty} yV_0\sigma \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} dy \\ &= \frac{V_0\sigma}{\sqrt{2\pi}} \int_{\Phi^{-1}(1-\alpha)}^{\infty} ye^{-\frac{y^2}{2}} dy \\ &= \frac{V_0\sigma}{\sqrt{2\pi}} \left\{ -e^{-\frac{y^2}{2}} \Big|_{\Phi^{-1}(1-\alpha)}^{\infty} \right. \\ &= V_0\sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{(\Phi^{-1}(1-\alpha))^2}{2}} \\ &= V_0\sigma\phi(\Phi^{-1}(1-\alpha)) \end{aligned}$$

We can get the expression of $CVaR_{1-\alpha}(X)$ as follows

$$CVaR_{1-\alpha}(X) = \frac{1}{\alpha} \int_{1-\alpha}^1 -\Phi^{-1}(1-\beta)V_0\sigma d\beta - V_0\mu = \frac{1}{\alpha} [V_0\sigma\phi(\Phi^{-1}(1-\alpha))] = -V_0\mu + V_0\sigma \frac{\phi(\Phi^{-1}(1-\alpha))}{\alpha}$$

α	VaR	CVaR
0.01	315.27	383.04
0.05	178.97	262.54
0.10	106.31	201.00

Table 5: VaR & CVaR



From table 3, it can be concluded that, VaR and CVaR both decreases with the climb in confidence level. Moreover, at the same confidence level, CVaR is strictly greater than VaR, because VaR and CVaR .This is also a further illustration of the definition of VaR.

2.3.1 Computation of CVaR for 2 cases

Case 1: From equation(7), $VaR_{1-\alpha} = V_0 [\beta(r + 1) - r - \beta e^{\Phi^{-1}(\alpha)\sigma\sqrt{t}+\mu t}]$

where $\Phi^{-1}(\alpha)\sigma\sqrt{t} + \mu t = \sigma\sqrt{t}Z + \mu t = Y(t) \sim N(\mu t, \sigma^2 t)$, for $Z \sim N(0, 1)$

Now, to compute CVaR, we use the formula of VaR $P(V_0 - V > VaR_{1-\alpha}) = \alpha$
 $\Rightarrow P(V_0 - V > VaR_{1-\alpha}) = P(V_0 - V > V_0 [\beta(r + 1) - r - \beta e^{\Phi^{-1}(\alpha)\sigma\sqrt{t}+\mu t}])$

$$= P(V_0 - V > V_0 [\beta(r + 1) - r - \beta e^{\sigma\sqrt{t}Z+\mu t}])$$

$$= P(\beta V_0 e^{\sigma\sqrt{t}Z+\mu t} > V_0 [\beta(r + 1) - r] - (V_0 - V))$$

$$= P(e^{\sigma\sqrt{t}Z+\mu t} > \frac{1}{\beta}[\beta(r + 1) - r] - \frac{1}{\beta V_0}(V_0 - V))$$

$$= P(\sigma\sqrt{t}Z + \mu t > \ln \left\{ \frac{1}{\beta}[\beta(r + 1) - r] - \frac{1}{\beta V_0}(V_0 - V) \right\})$$

$$= P(Z > \frac{\ln \left\{ \frac{1}{\beta}[\beta(r+1)-r] - \frac{1}{\beta V_0}(V_0-V) \right\} - \mu t}{\sigma\sqrt{t}})$$

$$= 1 - \Phi\left(\frac{\ln\left\{\frac{1}{\beta}[\beta(r+1)-r]-\frac{1}{\beta V_0}(V_0-V)\right\}-\mu t}{\sigma\sqrt{t}}\right) = \alpha$$

where $\Phi(\cdot)$ is a cumulative standard normal distribution. We can then deduce that the $VaR_{1-\alpha}$ under the condition $V_0 - V > VaR_{1-\alpha}$ is

$$\begin{aligned} VaR_{1-\alpha} &= V_0 - V = -\beta V_0 \left\{ e^{\Phi^{-1}(1-\alpha)\sigma\sqrt{t}+\mu t} + \frac{r}{\beta} - (r+1) \right\} \\ &= -\beta V_0 \left\{ e^{\Phi^{-1}(1-\alpha)\sigma\sqrt{t}+\mu t} \right\} - V_0 r + \beta V_0 (r+1) \end{aligned}$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative standard normal distribution. By Acerbi's Integral Formula, we have

$$CVaR_{1-\alpha} = \frac{1}{\alpha} \int_{1-\alpha}^1 VaR_{1-u} du \quad (16)$$

Now, we can rewrite CVaR using the previous result,

$$\begin{aligned} CVaR_{1-\alpha} &= \frac{1}{\alpha} \int_{1-\alpha}^1 (-\beta V_0 \{ e^{\Phi^{-1}(1-u)\sigma\sqrt{t}+\mu t} \} - V_0 r + \beta V_0 (r+1)) du \\ &= \frac{1}{\alpha} (-\beta V_0 e^{\mu t}) \int_{1-\alpha}^1 \{ e^{\Phi^{-1}(1-u)\sigma\sqrt{t}} \} du + \frac{1}{\alpha} \int_{1-\alpha}^1 -V_0 r + \beta V_0 (r+1) du \\ &= \frac{1}{\alpha} (-\beta V_0 e^{\mu t}) \int_{1-\alpha}^1 \{ e^{\Phi^{-1}(1-u)\sigma\sqrt{t}} \} du - V_0 r + \beta V_0 (r+1) \end{aligned}$$

To solve the integral part of $\int_{1-\alpha}^1 e^{\Phi^{-1}(1-u)\sigma\sqrt{t}} du$, we need to use a change of variable introducing $y = \Phi^{-1}(1-u) \Rightarrow 1-u = \Phi(y) \Rightarrow du = -\phi(y)dy$, where $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ is the standard normal density function

$$\begin{aligned} \int_{1-\alpha}^1 e^{\Phi^{-1}(1-u)\sigma\sqrt{t}} du &= \int_{\Phi^{-1}(1-\alpha)}^{\Phi^{-1}(1)} e^{y\sigma\sqrt{t}} (-\phi(y)) dy = -\frac{1}{\sqrt{2\pi}} \int_{\Phi^{-1}(1-\alpha)}^{\infty} e^{-\frac{y^2}{2}} e^{y\sigma\sqrt{t}} dy \\ &= -\frac{1}{\sqrt{2\pi}} \left\{ \frac{\sqrt{\pi}}{\sqrt{2}} e^{\frac{\sigma^2 t}{2}} \operatorname{erf}\left(\frac{y-\sigma\sqrt{t}}{\sqrt{2}}\right) \Big|_{\Phi^{-1}(1-\alpha)}^{\infty} \right\} \end{aligned}$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = 2\Phi(z\sqrt{2}) - 1$

$$\begin{aligned} &= \frac{1}{2} e^{\frac{\sigma^2 t}{2}} (2\Phi(y - \sigma\sqrt{t}) - 1) \Big|_{-\infty}^{\Phi^{-1}(\alpha)} \\ &= e^{\frac{\sigma^2 t}{2}} \{ \Phi(\Phi^{-1}(\alpha) - \sigma\sqrt{t}) \} \end{aligned}$$

$$\begin{aligned} \Rightarrow CVaR_{1-\alpha} &= \frac{1}{\alpha} (-\beta V_0 e^{\mu t}) \int_{1-\alpha}^1 \{ e^{\Phi^{-1}(1-u)\sigma\sqrt{t}} \} du - V_0 r + \beta V_0 (r+1) \\ &= \frac{1}{\alpha} (-\beta V_0 e^{\mu t}) e^{\frac{\sigma^2 t}{2}} \{ \Phi(\Phi^{-1}(\alpha) - \sigma\sqrt{t}) \} - V_0 r + \beta V_0 (r+1) \end{aligned} \quad (17)$$

This result can be supported by the proposition of CVaR formula for the log-normal distribution [16]. For the log-normal distribution the CVaR @ α can be calculated as $CVaR = 1 - e^{\mu + \frac{\sigma^2}{2} \frac{\Phi(\Phi^{-1}(\alpha) - \sigma)}{\alpha}}$ with VaR @ α equal to $1 - e^{\mu + \Phi^{-1}(\alpha)\sigma}$. This is equivalent to that for the log-normal distribution, $CVaR_{1-\alpha} = e^{\mu + \frac{\sigma^2}{2} \frac{\Phi(\Phi^{-1}(\alpha) - \sigma)}{\alpha}}$ with $VaR_{1-\alpha} = e^{\mu + \Phi^{-1}(\alpha)\sigma}$

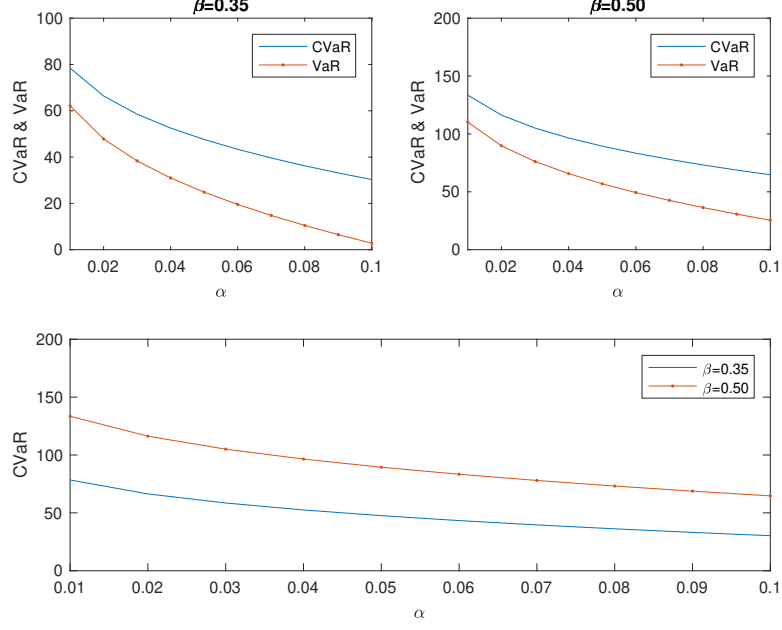
We test the model of CVaR using the same values of β as testing the model of VaR of case 1. Thus, set $beta=0.35$ and $\beta=0.50$, we then obtain results from MATLAB showing as follows:

$\beta=0.35$		
α	VaR	CVaR
0.01	62.14	78.43
0.05	24.85	47.60
0.10	2.80	30.29

Table 6: VaR vs CVaR

$\beta=0.50$		
α	VaR	CVaR
0.01	110.20	133.47
0.05	56.93	89.43
0.10	25.43	64.71

Table 7: VaR vs CVaR



By analyzing table 5 and the plots shown above, we can have results that is consistent with the conclusions of example. In addition, CVaR is the measure estimating risk that is more sensitive to the value of β (assets assigned to stocks in percentage) than VaR. All of tables and plots give us a clear view that the difference between the values of VaR and CVaR gradually expands with declines in confidence level. This is due to the fact that VaR and CVaR refer to different part of the given distribution. Furthermore, we have concluded that from above analysis that VaR has a positive relationship with confidence level, as well as by the definition of CVaR, which is equal to conditional expectation of losses exceeding VaR. Intuitively, the smaller VaR, the greater expectations losses over VaR.

Case 2: Repeating the same procedures as case 1, we obtain the equation of CVaR for case 2:

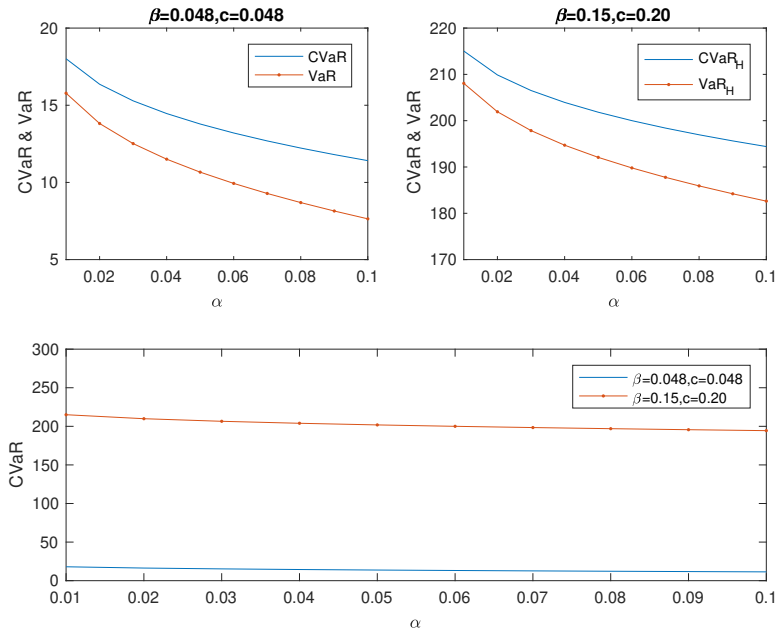
$$CVaR_{1-\alpha} = \frac{1}{\alpha}(-\beta V_0 e^{\mu t}) e^{\frac{\sigma^2 t}{2}} \{\Phi(\Phi^{-1}(\alpha) - \sigma\sqrt{t})\} - V_0 r + (\beta + c)V_0(r + 1) \quad (18)$$

$\beta=0.048; c=0.048$		
α	VaR	CVaR
0.01	15.78	18.01
0.05	10.67	13.78
0.10	7.64	11.41

Table 8: VaR vs CVaR

$\beta=0.15; c=0.20$		
α	VaR	CVaR
0.01	208.06	215.04
0.05	192.08	201.83
0.10	182.63	194.41

Table 9: VaR vs CVaR



The results shown above are consistent with the conclusions we made in case 1. One other point worth mentioning here is that VaR at level $\alpha=0.10$ is 7.64 with $\beta=c=0.048$ ($\beta + c \approx \frac{2}{21}$). So the deduction above with respect to the constraints of $\beta+c$ where $VaR \geq 0$ is feasible.

3 Portfolio Optimization

3.1 Background for optimal portfolio

In this section, we refer to Gaivoronski and Pflug [2], Cid, Soler and Blanco [4], and conclude that the optimal portfolio can be obtained by the two main processes:

1. Constructing objective function and explore or setting the constraints of required equations or variables for the questions we are exploring.
2. Minimizing or maximizing objective function subject to constraints on required formulas or variables.

The objective of portfolio optimization is to deal with the problem of asset allocation in this paper and it can generally be illustrated as

$$\begin{aligned} \text{min or max} \quad & f(x) \leftarrow \text{objective function} \\ \text{subject to} \quad & g(x) \geq 0 \leftarrow \text{constraints} \\ & x \in D \leftarrow \text{definition domain} \end{aligned}$$

3.2 Model Construction

This section focuses mainly on enhancing the consumption level as far as possible. We define a utility function regarding for consumption level.

$$\begin{aligned} \max_c u(c) &= c^\gamma, \text{ for } 0 < \gamma < 1 \\ \text{subject to} \quad & \frac{VaR(\beta, c)}{V_0} \leq w \quad \text{where } 0 < w < 1 \\ & VaR(\beta, c) > 0, \\ & 0 < \beta, c < 1 \end{aligned}$$

3.3 Scenarios for Portfolio Optimization

As a further illustration of our model, we consider an example of $w = 15\%$. Furthermore, we set $\gamma = 0.8$ and **utility function** is $u(c) = c^{0.8}$. Now we discuss the first constraint $\frac{VaR(\beta, c)}{V_0} \leq w$ by three confidence levels 90%, 95% and 99%. After some computations, we get **the first constraints**,

$$\begin{aligned} 1050c + 320.408\beta &\leq 200, \text{ for } \alpha = 0.01 \\ 1050c + 213.87\beta &\leq 200, \text{ for } \alpha = 0.05 \\ 1050c + 150.854\beta &\leq 200, \text{ for } \alpha = 0.10 \end{aligned}$$

This is equivalent to $Ac + B\beta \leq 200$, where $A = [1050 \quad 1050 \quad 1050]'$ and $B = [320.408 \quad 213.87 \quad 150.854]'$

The second constraint is

$$k \leq \beta + c \leq 1 \tag{19}$$

where k is a constant but varies with the changes in confidence level.

This example next will be extended to two cases:

1. The trader don't want to take a large risk. It can be explained that the loss 15% we assumed in the background is the maximum acceptable for them. Consequently, they hope that there is only 1% possibility of loss exceeding 15% of the total.
2. The trader is willing to take a larger risk. In other words, it is good for them if they have 90% or 95% confidence level that the maximum loss is 15%.

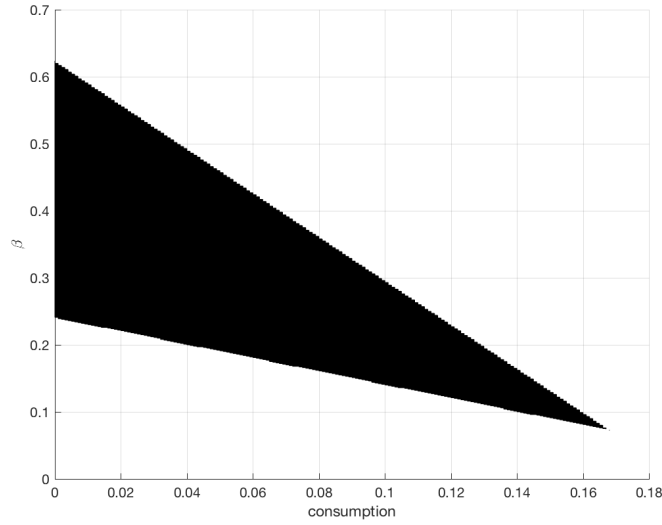
We next construct the optimization model for the first-case scenario

$$\begin{aligned} \max_c u(c) &= c^{0.8} \\ \text{subject to} \quad & 1050c + 320.408\beta \leq 200 \\ & 0.24 \leq \beta + c \leq 1, \end{aligned}$$

where $0.24 \approx \frac{0.1561+0.2338+0.3314}{3}$. Recall that the numbers 0.1561, 0.2338 and 0.3314 are the values of β when $\text{VaR} = 0$ for case 1.

c (the proportion of consumption)	β (weight assigned to stock)
0.1687	0.0713

Table 10: Optimization for $\text{VaR} \leq 15\%$ with the highest confidence level 99%



$c = 0.1687, \beta = 0.0713$		
α	VaR	CVaR
0.01	149.98	153.30
0.05	142.38	147.02
0.10	137.89	143.49

Table 11: Optimization Test for $VaR \leq 15\%$ with the highest confidence level 99%

For $c = 16.87\%$ and $\beta = 7.13\%$, it is obviously shown that VaR for $\alpha = 0.01$ is approximately \$150 because of the negligible investment in stock.

Comparing with the model of optimization for the two-case scenarios, this one (confidence level = 95%) is differ from the second constraints of the first one.

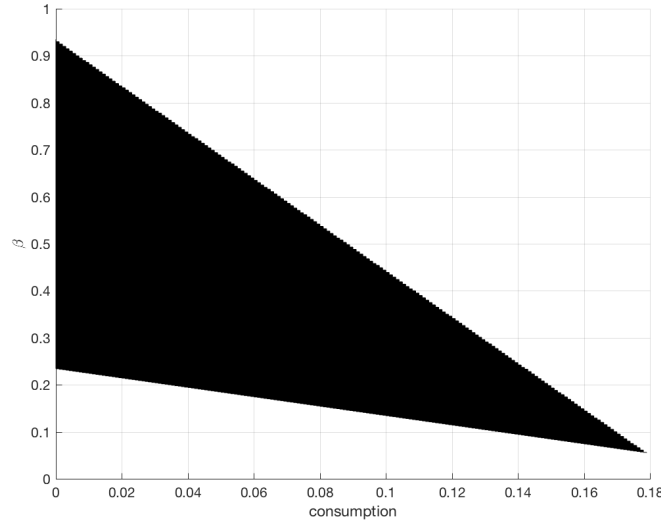
$$\max_c u(c) = c^{0.8}$$

$$\text{subject to } 1050c + 213.87\beta \leq 200$$

$$0.2338 \leq \beta + c \leq 1$$

c (the proportion of consumption)	β (weight assigned to stock)
0.1794	0.0544

Table 12: Optimization for $VaR \leq 15\%$ with the highest confidence level 95%



$c = 0.1794, \beta = 0.0544$		
α	VaR	CVaR
0.01	155.80	158.33
0.05	150.00	153.54
0.10	146.58	150.85

Table 13: Optimization Test for $VaR \leq 15\%$ with the highest confidence level 95%

Table 15 gives us a clear view that traders will be with 95% confidence level that their maximum loss is \$150 when consumption and assets invested in stock take up 17.94% and 5/44% respectively.

The model of optimization at 90% confidence level is

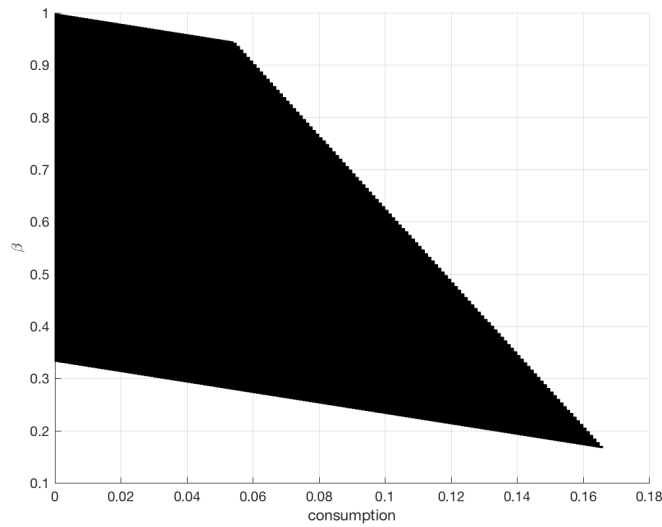
$$\max_c u(c) = c^{0.8}$$

$$\text{subject to } 1050c + 150.854\beta \leq 200$$

$$0.3314 \leq \beta + c \leq 1$$

c (the proportion of consumption)	β (weight assigned to stock)
0.1668	0.1646

Table 14: Optimization for $VaR \leq 15\%$ with the highest confidence level 90%



$c = 0.1668, \beta = 0.1646$		
α	VaR	CVaR
0.01	177.88	185.54
0.05	160.34	171.04
0.10	149.97	162.90

Table 15: Optimization Test for $VaR \leq 15\%$ with the highest confidence level 90%

To sum up, there is a 90% chance that the losses will not exceed \$149.97 when traders' consumption level accounts for 16.68% , the investment in stock and bond approximately achieve up to 16.46% and 66.86%. Because asset in stock, in this case achieves up to 16.46%, the likelihood of high returns on stock is desired to expect. Thus, this part of assets can be used for consumption.

Comparing with the CVaR for the 3 cases with different optimization model, it can be clearly seen that the difference of VaR and CVaR are quite small. More importantly, after some calculations by hand, we obtained that the mean of the difference of the three optimal portfolios are 4.52, 3.45 and 10.43 respectively. Precisely speaking, the values of β which represents risk experience a decrease and a relatively significant increase, so does the differences between VaR and CVaR. Thus, we can conclude that CVaR is more sensitive to the changes in β (the weights of stock). In addition, With these values at hand for comparison purposes, it is crucial to note that the consumption level (values of c) for the three cases are quite close, while the changes in assets invested in stock (values of β) are relatively great.

4 Conclusion

For the portfolio with 2 assets of bonds and stocks, we have obtained the distinct formulas of VaR and CVaR in the two cases that whether the expenses on consumption is considered. We can then get the corresponding values of different confidence level $(1 - \alpha)$ from MATLAB. These results are consistent with the concepts of VaR and CVaR: CVaR strictly lies above VaR at the same confidence level on the plot. Intuitively, VaR and CVaR are in inverse ration with confidence level of traders. Moreover, both of VaR and CVaR increase if there is a climb in the value of β (assets invested into stock in percentage). CVaR, however, is more sensitive to the estimation error than VaR. In the light of the different values in percentage of β and c after optimizing, we found that, with an alteration in α , there is no significant fluctuations in the value of c comparing with the relatively great changes in β .

5 Appendix

5.1 Stock price as a Markov process

As mentioned above, $S(t) = S_0 e^{Y(t)}$ for $t \geq 0$ is a GBM and let it represents the present state (price) and $\{S(u), 0 \leq u < t\}$ denote the process before time t, $S(t+h)$ is the future (next) state, where $h > 0$ is the length of time interval between two consecutive trading periods expressed in yearly. Then

$$\begin{aligned} S(t+h) &= S_0 e^{Y(t+h)} = S_0 e^{Y(t)+Y(t+h)-Y(t)} \\ &= S_0 e^{Y(t)} e^{Y(t+h)-Y(t)} \\ &= S(t) e^{Y(t+h)-Y(t)} \end{aligned}$$

It is clearly that the future state $\{Y(t+h) - Y(t)\}$ is independent of the past state $\{S(u), 0 \leq u < t\}$ and only depends on the present state $S(t)$. In other words, the current value of the process contains all the information needed to make predictions about the future. Hence, GBM is a Markov process. [12]

5.2 Computation of volatility and drift

There are two types of volatility, one is backward looking and the other is forward looking. The former one is calculated from historical data, and the latter is derived from the Black-Scholes (B-S) option pricing model based on the current option price. We focus mainly on exploring the algorithm of Backward looking. [14]

Backward looking volatility:

We define

$n+1$: number of observations;

S_t : stock closing price at the end of time t;

τ : the length of interval in year ($\tau = \frac{1}{n}$).

The daily log return on day i denoted by R_i is

$$R_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad i = 1, \dots, n$$

The mathematical expression of daily volatility is usually expressed as

$$\sigma_{daily} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (R_i - \bar{R})^2}$$

Or alternatively,

$$\sqrt{\frac{1}{n-1} \sum_{i=1}^n R_i^2 - \frac{1}{n(n-1)} (\sum_{i=1}^n R_i)^2}$$

where $\bar{R} = \frac{1}{n} \sum_{i=1}^n R_i$. In general, there are $n = 252$ trading days for one year, so $\tau = \frac{1}{252}$ and the annualized volatility $\sigma_{annual} = \frac{\sigma_{daily}}{\sqrt{\tau}} = \sigma_{daily} \sqrt{252}$.

Drift

By equation (18), we have the annualized drift = $\frac{\mu}{\tau} - \frac{1}{2}\sigma^2$

5.3 Code for MATLAB

Code for case 1

```
% fix the values needed in VaR equation
% fix the values needed in VaR equation
V0=1000; r=0.05; mu=0.15; sigma=0.2;t=1;
alpha = 0.01; alpha1 = 0.05; alpha2 = 0.10;
beta = 0:0.1:1;
k_alpha=icdf('normal',alpha,0,1);
k_alpha1=icdf('normal',alpha1,0,1);
k_alpha2=icdf('normal',alpha2,0,1);

VaR = V0*(beta*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu*t));
VaR1 = V0*(beta*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha1+mu*t));
VaR2 = V0*(beta*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha2+mu*t));
plot(beta,VaR,beta,VaR1,beta,VaR2,'.-'),legend('\alpha=0.01', '\alpha=0.05', '\alpha=0.10','L
grid on;

fprintf('beta      VaR\n');
y = [beta; VaR];
fprintf('%f      %f\n', y);

fprintf('beta      VaR\n');
y = [beta; VaR1];
fprintf('%f      %f\n', y);
```

```
fprintf('beta      VaR\n');
```

```
y = [beta; VaR2];
```

```
fprintf('%f    %f\n', y);
```

```
betaDesired = interp1(VaR,beta,0,'spline')
```

```
betaDesired1 = interp1(VaR1,beta,0,'spline')
```

```
betaDesired2 = interp1(VaR2,beta,0,'spline')
```

```
beta      VaR
0.000000  -50.000000
0.100000  -17.959217
0.200000  14.081566
0.300000  46.122350
0.400000  78.163133
0.500000  110.203916
0.600000  142.244699
0.700000  174.285482
0.800000  206.326266
0.900000  238.367049
1.000000  270.407832
```

```
beta      VaR
0.000000  -50.000000
0.100000  -28.613038
0.200000  -7.226075
0.300000  14.160887
0.400000  35.547849
0.500000  56.934812
0.600000  78.321774
0.700000  99.708736
0.800000  121.095699
0.900000  142.482661
```

```

1.000000    163.869624
beta        VaR
0.000000    -50.000000
0.100000    -34.914559
0.200000    -19.829118
0.300000    -4.743676
0.400000    10.341765
0.500000    25.427206
0.600000    40.512647
0.700000    55.598088
0.800000    70.683530
0.900000    85.768971
1.000000    100.854412

```

```

betaDesired = 0.1561
betaDesired1 = 0.2338
betaDesired2 = 0.3314

```

```

% fix the values needed in VaR equation

```

```

V0=1000; beta=0.35; r=0.05; mu=0.15; sigma=0.2;t=1;

```

```

beta_H=0.5;

```

```

alpha = 0:0.01:0.1;

```

```

k_alpha=icdf('normal',alpha,0,1);

```

```

% equation of VaR with beta = 0.35

```

```

VaR = V0*(beta*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu*t));

```

```

subplot(2,2,1)

```

```

plot(alpha,VaR,'r'),xlabel('\alpha'),ylabel('VaR'),title('VaR for \beta=0.35'),axis([0.01 0.1 0 100])

```

```

% equation of VaR with beta = 0.5

```

```

VaR_H = V0*(beta_H*(r+1)-r-beta_H*exp(sigma*sqrt(t)*k_alpha+mu*t));

```

```

subplot(2,2,2)

```

```

plot(alpha,VaR_H,'b'),xlabel('\alpha'),ylabel('VaR'),title('VaR for \beta=0.5'),axis([0.01 0.1 0 100])

```

```

subplot(2,1,2)
plot(alpha,VaR,alpha,VaR_H,'.-'),legend('\beta=0.35', '\beta=0.5'), xlabel('\alpha'),ylabel('VaR')
fprintf('alpha      VaR\n');
y = [alpha; VaR];
fprintf('%f      %f\n', y);

fprintf('alpha      VaR\n');
y = [alpha; VaR_H];
fprintf('%f      %f\n', y);

```

alpha	VaR
0.000000	317.500000
0.010000	62.142741
0.020000	47.834211
0.030000	38.342976
0.040000	30.983553
0.050000	24.854368
0.060000	19.534283
0.070000	14.790092
0.080000	10.478173
0.090000	6.503348
0.100000	2.799044

alpha	VaR
0.000000	475.000000
0.010000	110.203916
0.020000	89.763159
0.030000	76.204252
0.040000	65.690790
0.050000	56.934812
0.060000	49.334690
0.070000	42.557274

```

0.080000    36.397390
0.090000    30.719068
0.100000    25.427206

```

Code for case 2

```

alpha = 0.01:0.01:0.1;
k_alpha=icdf('normal',alpha,0,1);
var = @(V0,beta,c,r,t,k_alpha,mu,sigma) V0*((beta+c)*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu);
VaR = var(1000,0.01,0.05,0.05,1,k_alpha,0.15,0.2);
VaR_H = var(1000,0.02,0.10,0.05,1,k_alpha,0.15,0.2);

subplot(2,2,1)
plot(alpha,VaR,'r'),xlabel('\alpha'),ylabel('VaR'),title('VaR for \beta=0.01, c=0.05'),axis([0

subplot(2,2,2)
plot(alpha,VaR_H,'b'),xlabel('\alpha'),ylabel('VaR'),title('VaR for \beta=0.02, c=0.10'),axis([0

subplot(2,1,2)
plot(alpha,VaR,alpha,VaR_H,'.-'),legend('\beta=0.01,c=0.05', '\beta=0.02,c=0.10'), xlabel('\alpha')

fprintf('alpha      VaR\n');
y = [alpha; VaR];
fprintf('%f    %f\n', y);
fprintf('alpha      VaR_H\n');
y = [alpha; VaR_H];
fprintf('%f    %f\n', y);

```

```

alpha      VaR
0.010000   5.704078
0.020000   5.295263
0.030000   5.024085
0.040000   4.813816

```

0.050000	4.638696
0.060000	4.486694
0.070000	4.351145
0.080000	4.227948
0.090000	4.114381
0.100000	4.008544
alpha	VaR_H
0.010000	61.408157
0.020000	60.590526
0.030000	60.048170
0.040000	59.627632
0.050000	59.277392
0.060000	58.973388
0.070000	58.702291
0.080000	58.455896
0.090000	58.228763
0.100000	58.017088

Code for example

```
alpha = 0:0.01:0.1;
k_alpha=icdf('normal',1-alpha,0,1);
k_alpha1=icdf('normal',alpha,0,1);
Phi_z = normpdf(k_alpha,0,1);

var = @(V0,mu,sigma,k_alpha1) -(k_alpha1.*V0*sigma+mu*V0);
cvar = @(V0,mu,sigma,Phi_z,alpha) -V0*mu+V0*sigma*Phi_z./alpha;

VaR = var(1000,0.15,0.2,k_alpha1);
CVaR = cvar(1000,0.15,0.2,Phi_z,alpha);

plot(alpha, CVaR, alpha, VaR, '.-'), xlabel('\alpha'), ylabel('CVaR & VaR'), legend('CVaR', 'VaR'), axis
```

```
fprintf('alpha      CVaR\n');
y = [alpha; CVaR];
fprintf('%f      %f\n', y);
```

```
fprintf('alpha      VaR\n');
y = [alpha; VaR];
fprintf('%f      %f\n', y);
```

alpha	CVaR
0.000000	NaN
0.010000	383.042844
0.020000	334.181359
0.030000	303.613009
0.040000	280.868870
0.050000	262.542562
0.060000	247.076551
0.070000	233.622613
0.080000	221.665566
0.090000	210.868062
0.100000	200.996664
alpha	VaR
0.000000	Inf
0.010000	315.269575
0.020000	260.749782
0.030000	226.158722
0.040000	200.137214
0.050000	178.970725
0.060000	160.954719
0.070000	145.158206
0.080000	131.014312
0.090000	118.151007
0.100000	106.310313

Code for Expected Return

```
V0=1000; r=0.05; mu=0.15; t=1;
beta=0:0.1:1;
E_R = V0*(r-beta.*(r+1-exp(mu*t)));
plot(beta,E_R),xlabel('\beta'),ylabel('Expected Return'),axis([0 1 50 180])
grid on;
```

```
fprintf('beta      E_R\n');
y = [beta; E_R];
fprintf('%f    %f\n', y);
```

beta	E_R
0.000000	50.000000
0.100000	61.183424
0.200000	72.366849
0.300000	83.550273
0.400000	94.733697
0.500000	105.917121
0.600000	117.100546
0.700000	128.283970
0.800000	139.467394
0.900000	150.650818
1.000000	161.834243

Code for different holding periods

```
alpha = 0:0.01:0.1;
k_alpha=icdf('normal',alpha,0,1);
z = k_alpha-sigma*sqrt(t);
Phi_z = normcdf(z,0,1);
```

```
var = @(V0,beta,c,r,t,k_alpha,mu,sigma) V0*((beta+c)*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu*t));
VaR = var(1000,0.15,0.2,0.05,1,k_alpha,0.15,0.2);
```

```
VaR1 = var(1000,0.15,0.2,0.1025,2,k_alpha,0.15,0.2);
VaR2 = var(1000,0.15,0.2,0.1576,3,k_alpha,0.15,0.2);
```

```
plot(alpha,VaR,alpha,VaR1,alpha,VaR2,'.-'),xlabel('\alpha'),ylabel('VaR'),legend('t=1','t=2','t=3')
```

```
fprintf('VaR      VaR1\n');
y = [VaR; VaR1];
fprintf('%f      %f\n', y);
fprintf('%f      %f\n', VaR2);
```

VaR	VaR1
317.500000	283.375000
208.061175	178.513047
201.928948	170.108055
197.861276	164.429366
194.707237	159.970633
192.080444	156.220812
189.800407	152.939492
187.767182	149.992818
185.919217	147.297981
184.215720	144.799888
182.628162	142.459873
247.560000	142.475504
132.068631	124.937654
119.284557	114.494527
110.276803	106.468829
102.969700	99.712088
96.648615	

Code for CVaRcase1

```
V0=1000; mu = 0.15; sigma = 0.2; t=1; r=0.05;
beta=0.35; beta_H = 0.5;
```

```

alpha = 0:0.01:0.1;
k_alpha=icdf('normal',alpha,0,1);
z = k_alpha-sigma*sqrt(t);
Phi_z = normcdf(z,0,1);
sum = Phi_z;

VaR = V0*(beta*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu*t));
VaR_H = V0*(beta_H*(r+1)-r-beta_H*exp(sigma*sqrt(t)*k_alpha+mu*t));

CVaR = -sum.*(beta./alpha)*V0*exp(mu*t)*exp((sigma^2)*t/2)-V0*r+beta*V0*(r+1);
subplot(2,2,1);
plot(alpha,CVaR,alpha,VaR,'.-'),xlabel('\alpha'),ylabel('CVaR & VaR'),legend('CVaR','VaR'),title('CVaR & VaR');

CVaR_H = -sum.*(beta_H./alpha)*V0*exp(mu*t)*exp((sigma^2)*t/2)-V0*r+beta_H*V0*(r+1);
subplot(2,2,2);
plot(alpha,CVaR_H,alpha,VaR_H,'.-'),xlabel('\alpha'),ylabel('CVaR & VaR'),legend('CVaR','VaR'),title('CVaR_H & VaR_H');

subplot(2,1,2)
plot(alpha,CVaR,alpha,CVaR_H,'.-'),legend('\beta=0.35', '\beta=0.50'), xlabel('\alpha'),ylabel('CVaR & CVaR_H');

fprintf('VaR          CVaR\n');
y = [VaR; CVaR];
fprintf('%f    %f\n', y);

fprintf('VaR_H          CVaR_H\n');
y = [VaR_H; CVaR_H];
fprintf('%f    %f\n', y);

VaR          CVaR
317.500000   NaN
62.142741   78.427851
47.834211   66.387568

```

38.342976	58.540908
30.983553	52.539317
24.854368	47.599070
19.534283	43.355635
14.790092	39.607849
10.478173	36.232196
6.503348	33.147039
2.799044	30.295424
VaR_H	CVaR_H
475.000000	NaN
110.203916	133.468358
89.763159	116.267954
76.204252	105.058439
65.690790	96.484739
56.934812	89.427242
49.334690	83.365193
42.557274	78.011213
36.397390	73.188852
30.719068	68.781484
25.427206	64.707749

Code for CVaRcase2

```

alpha = 0:0.01:0.1;
k_alpha=icdf('normal',alpha,0,1);
z = k_alpha-sigma*sqrt(t);
Phi_z = normcdf(z,0,1);

var = @(V0,beta,c,r,t,k_alpha,mu,sigma) V0*((beta+c)*(r+1)-r-beta*exp(sigma*sqrt(t)*k_alpha+mu*t));
VaR = var(1000,0.048,0.048,0.05,1,k_alpha,0.15,0.2);
VaR_H = var(1000,0.15,0.2,0.05,1,k_alpha,0.15,0.2);

cvar = @(beta,c,V0,mu,Phi_z,alpha,z,sigma,t,r) -Phi_z.*(beta./alpha)*V0*exp(mu*t)*exp((sigma^2*t));

```

```

CVaR = cvar(0.048,0.048,1000,0.15,Phi_z,alpha,z,0.2,1,0.05);
CVaR_H = cvar(0.15,0.2,1000,0.15,Phi_z,alpha,z,0.2,1,0.05);

subplot(2,2,1)
plot(alpha,CVaR,alpha,Var,'.-'),xlabel('\alpha'),ylabel('CVaR & VaR'),legend('CVaR','VaR'),title('CVaR & VaR (beta=0.048, c=0.048)')

subplot(2,2,2)
plot(alpha,CVaR_H,alpha,Var_H,'.-'),xlabel('\alpha'),ylabel('CVaR & VaR'),legend('CVaR_H','VaR_H'),title('CVaR & VaR (beta=0.15, c=0.20)')

subplot(2,1,2)
plot(alpha,CVaR,alpha,CVaR_H,'.-'),legend('\beta=0.048,c=0.048', '\beta=0.15,c=0.20'), xlabel('\alpha'), ylabel('CVaR'), title('CVaR vs alpha')

fprintf('CVaR          VaR\n');
y = [CVaR; VaR];
fprintf('%f    %f\n', y);

fprintf('CVaR_H          VaR_H\n');
y = [CVaR_H; VaR_H];
fprintf('%f    %f\n', y);

CVaR          VaR
NaN    50.800000
18.012962    15.779576
16.361724    13.817263
15.285610    12.515608
14.462535    11.506316
13.785015    10.665742
13.203058    9.936130
12.689076    9.285498
12.226130    8.694149
11.803022    8.149031
11.411944    7.641012

```

CVaR_H	VaR_H
NaN	317.500000
215.040507	208.061175
209.880386	201.928948
206.517532	197.861276
203.945422	194.707237
201.828173	192.080444
200.009558	189.800407
198.403364	187.767182
196.956655	185.919217
195.634445	184.215720
194.412325	182.628162

Optimization code

```
clc
clear
fun=@(x)-x(1).^0.8+x(2)-x(2);
x0 = [0.02,0.10];
A = [1050,213.87
     1,1
     -1 -1];
b = [200
     1
     -0.2338];
[x,fval] = fmincon(fun,x0,A,b);
fprintf('%f %f\n', x)
```

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in feasible directions, to within the default value of the optimality tolerance, and constraints are satisfied to within the default value of the constraint tolerance

0.179394 0.054408

Optimization plot

```
v = 0:0.001:1; % plotting range from 0 to 1
[x,y] = meshgrid(v); %get 2-D mesh for x and y
cond1 = 1050*x+320.408*y < 200; % check conditions for these values
cond2 = x+y < 1;
cond3 = -x-y < -0.24;
cond1 = double(cond1); % convert to double for plotting
cond2 = double(cond2);
cond3 = double(cond3);
cond1(cond1 == 0) = NaN; % set the 0s to NaN so they are not plotted
cond2(cond2 == 0) = NaN;
cond3(cond3 == 0) = NaN;
cond = cond1.*cond2.*cond3; % multiply the two condaces to keep only the common points
surf(x,y,cond)
view(0,90),xlabel('consumption'),ylabel('\beta')% change to top view
```

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