Flipping your Lid

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Abstract

Given a polygon P, a *flipturn* involves reflecting a pocket p of P through the midpoint of the lid of p. We show that any polygon on n vertices will be convex after any sequence of at most $n(n-3)/2$ flipturns. The best known previous result for this problem is due to Joss and Shannon (published in Grünbaum (1995)) who showed any polygon is convexified after at most $(n-1)!$ flipturns.

1 Introduction

Given a simple polygon P, a *flipturn* involves reflecting a pocket p of the convex hull of P through the midpoint of the convex hull edge defining p. See Fig. 1 for an example.

Flipturns, their inverses, and their relatives have an important application in generating random simple polygons on the integer lattice $\mathbb{Z}^2$, which are used, for example, in the field of polymer chemistry [7, 8, 2]. For such an application, it is important to show that the set of operations (flipturns, their inverses and relatives) are *complete*, i.e., that any polygon can be generated using these operations. One way of showing this is by showing that any polygon can be reduced to a canonical form, in this case a convex polygon. Also important is how many operations are required to achieve this canonical form, since this dictates, in part, how long an algorithm for generating random polygons should be run in order to achieve a sufficient approximation to the uniform distribution.

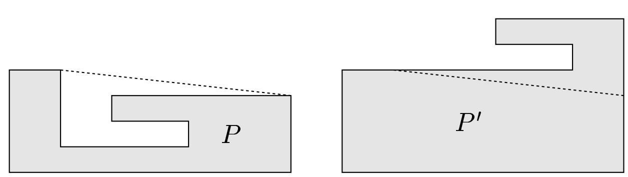

Figure 1: An example of a flipturn.

Motivated by this, the problem we study in this paper is that of determining how many flipturns, in the worst case, are required to convexify any polygon.

Dubins *et al* [3] show that $n-4$ carefully chosen flipturns are always sufficient to convexify any lattice polygon on n vertices. (A lattice polygon is a polygon in which all edges have length 1 and are either horizontal or vertical.)

Surprisingly, the more general case of arbitrary polygons was studied as early as 1973, when Joss and Shannon (see Grünbaum [4]) showed that any simple polygon can be convexified by *any* sequence of at most $(n-1)!$ flipturns. They conjecture that this bound is not tight and that $n^2/4$ flipturns always suffices.

Biedl [1] has found an example where a sequence of $\Omega(n^2)$ carefully chosen flipturns are required to convexify a polygon. However, the same polygon can be convexified using a diffferent sequence of $O(n)$ flipturns.

Grünbaum and Zaks [5] showed that even non-simple polygons can be convexified with a finite sequence of flipturns. For a survey of these and other results on flipping polygons, see the paper by Toussaint [9].

Note that we have distinguished between "carefully chosen" flipturns and arbitrary sequences of flipturns. This is because the bounds for these two cases may be different. For a given polygon P, consider the set X of all polygons that can be derived from P using flipturns. If we consider a directed graph G whose vertices are elements of X and for which the edge (a, b) is present if there is a flipturn taking a to b, then G is a directed acyclic graph with one source s corresponding to P and one sink t corresponding to a convex polygon. Results that use carefully chosen flipturns give bounds on the shortest path from s to t. Results that allow arbitrary sequences of flipturns give bounds on the longest path from s to t.

In this paper we show that any simple polygon P with n vertices will be convexified after *any* sequence of at most $n(n-3)/2$ flipturns. More generally, any polygon for which the slopes of the edges take on at most s different values will be convexified after at most $n(s-1)/2 - s$ flipturns. In Section 2 we give some definitions. Section 3 presents our proof. Section 4 summarizes and concludes with open problems.

2 Preliminaries

Let P be a simple polygon whose vertices in counterclockwise order are $v_0, \ldots, v_{n-1}$, and let the edges of P be oriented counterclockwise so that $e_i = (v_{i-1}, v_i)$.[1] A *pocket* $p = (v_i, \ldots, v_j)$ of P is a subchain of P such that v_i and v_j are on the convex hull of P and v_k is not on the convex hull of P for all $i < k < j$. A *lid* (v_i, v_j) is the line segment joining the two endpoints of a pocket $(v_i, \ldots, v_j)$.

In our proof, there is a special degenerate case that must be treated carefully. Let (v_i, v_j) be a lid of P. Let l be the line containing v_i and v_j and let v_k be the first vertex at or following v_j such that v_{k+1} is not contained in l. Then we call $(v_i, \ldots, v_k)$ a *modified pocket* of P and the segment (v_i, v_k) is called a *modified lid* of P. Modified pockets and lids are equivalent to standard pockets

[1] Here an henceforth, all subscripts will be taken $\bmod n$.

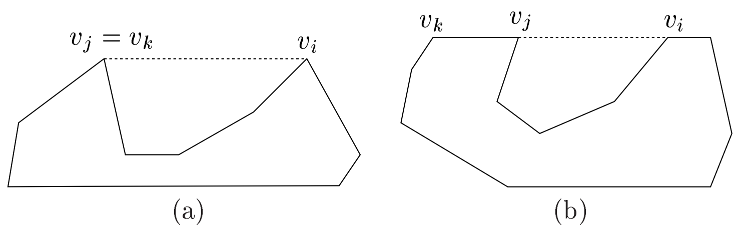

Figure 2: The pocket $v_i, \ldots, v_j$ and modified pocket $v_i, \ldots, v_k$ in (a) a non-degenerate case and (b) a degenerate case.

and lids except when convex hull edges have the same slope as edges of P. Fig. 2 illustrates modified pockets.

Let $p = (v_i, \ldots, v_k)$ be a modified pocket of P. Then a *flipturn* $f_{i,k}(P)$ of the polygon P transforms P into a new polygon P' by reflecting all edges of p through the midpoint of the modified lid (v_i, v_k). Equivalently, $f_{i,k}(P)$ rotates the modified pocket $p = (v_i, \ldots, v_k)$ 180 degrees about the midpoint of the lid (v_i, v_k).

Let $dir(e_i)$ be the direction of an edge of P, measured as the angle, in radians, between a right oriented horizontal ray and e_i. Let $S = \bigcup_{i=0}^{n-1} \{dir(e_i), -dir(e_i)\}$, i.e., the set of all directions and their negations used by edges of P. We will label the directions in S as $d_0, \ldots, d_{m-1}$ in increasing order. For two directions d_i and d_j in S we define the *discrete angle* between d_i and d_j, as $\overline{\angle} d_i d_j = (j-i) \bmod m$, i.e., one plus the number of other directions in S between d_i and d_j as we rotate d_i in the counterclockwise direction.

For a vertex v_i of P incident on edges e_i and e_{i+1} we define the *weight* of v_i as

$$w(v_i) = \begin{cases} \overline{\angle} dir(e_i) dir(e_{i+1}) & \text{if } v_i \text{ is convex} \\ \overline{\angle} dir(e_{i+1}) dir(e_i) & \text{if } v_i \text{ is reflex} \end{cases} .$$

We define the weight of P as $w(P) = \sum_{i=0}^{n-1} w(v_i)$. See Fig. 3 for an example.

For ease of notation, we define the variable s as $|S/2|$, which is exactly the number of distinct slopes used by supporting lines of edges of P. From these definitions, it is clear that $w(v_i) \leq s - 1$ and therefore $w(P) \leq n(s-1)$.

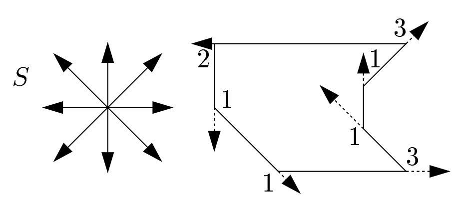

Figure 3: A polygon for which $|S| = 8$ labelled with its vertex weights.

3 Proof of the Main Theorem

In this section we prove our main theorem by showing that the weight of P decreases by at least 2 after every flipturn. We start with the following simple lemma.

Lemma 1. *For any convex polygon P, we have $w(P) = 2s$.*

Proof. Consider the circle of all directions. The weight of a vertex v_i is the number of elements in S contained in the circular interval $I_i = [dir(e_{i-1}), dir(e_i))$. Since P is a polygon, $\bigcup_{i=0}^{n-1} I_i$ is the interval $[0, 2\pi)$. Therefore, each element of S contributes at least one to $w(P)$ so $w(P) \geq 2s$. Since P is convex, $e_0, \ldots, e_{n-1}$ are ordered in decreasing order of direction, therefore no two intervals I_i and I_j, $i \neq j$ overlap. Thus, each element of S contributes at most one to $w(P)$, so $w(P) \leq 2s$. □

Consider a modified pocket p of P, and without loss of generality assume that the modified lid of p is parallel to the x-axis. Let v_i and v_j be the left and right vertices of the modified lid of p. Let r and b be the weight of v_i and v_j, respectively, before performing a flipturn on p and let r' and b' be the weight of the v_i and v_j, respectively, after performing the flipturn.

Lemma 2. $r + b - r' - b' \geq 2$

Proof. Let $d_w = dir(e_{i-1})$, $d_x = dir(e_i)$, $d_y = dir(e_{j-1})$, and $d_z = dir(e_j)$. To aid in understanding the problem, we place v_i and v_j at the same point and draw the four edges incident on v_i and v_j along with their extensions. There are now four cases to consider, depending on the order of d_w, d_x, d_y, and d_z. These four cases are illustrated in Fig. 4.

When viewed this way, it is clear that in each of the four cases $r+b-r'-b' = 2\alpha$, where $\alpha = \min\{d_w, d_y\} - \max\{d_x, d_z\}$. Since the discrete angles between edges of P are non-negative integers, all that remains to show is that $\alpha \neq 0$. In order to have $\alpha = 0$, the two edges defining α must both be pointing in the same direction in P before performing the flipturn. Thus, with the condition $\alpha = 0$ we obtain one of the four situations depicted in Fig. 5. However, in each of these situations, (v_i, v_j) is not a modified lid. We conclude that $\alpha \neq 0$. □

Theorem 1. *Any simple polygon on n vertices is convexified after any sequence of at most $n(s-1)/2 - s$ flipturns.*

Proof. This follows immediately from the following three facts. (1) Initially, the weight of P is at most $n(s-1)$. (2) The weight of P once it is convexified will be $2s$. (3) During a flipturn, the only weights that change are the weights of the two verices of the modified lid being flipped. Therefore, by Lemma 2 the weight of P decreases by at least 2 after every flipturn. □

Strengthening the result of Joss and Shannon [4], we immediately obtain the following corollary by taking $s = n$.

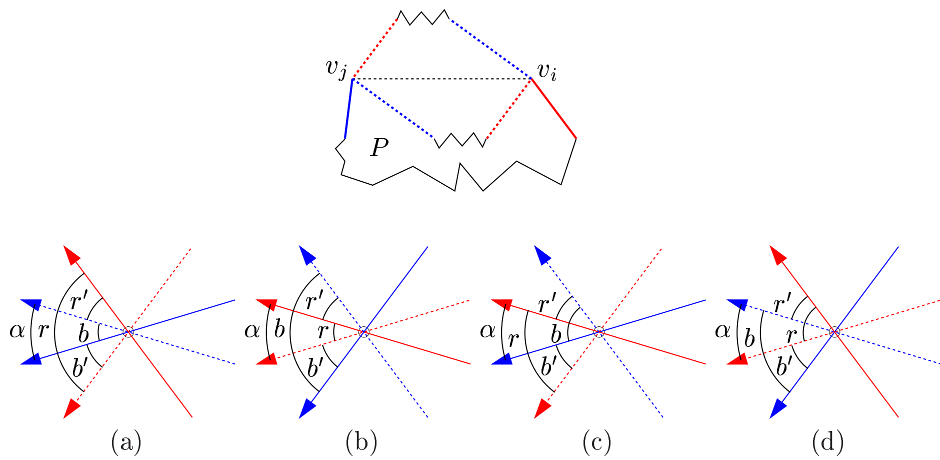

Figure 4: Four cases in the proof of Lemma 2. Arrows indicate the directions of the edges in P before performing the flipturn.

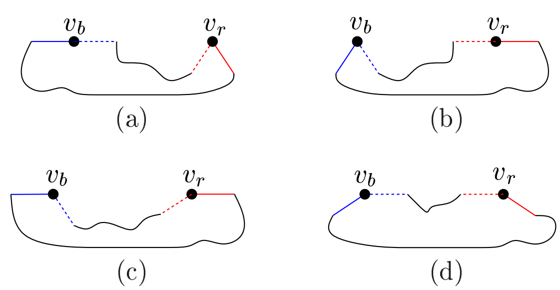

Figure 5: Four situations corresponding to cases in the proof of Lemma 2.

	Arbitrary Polygons		Lattice polygons	
	SP	LP	SP	LP
PLB	$n/2-2$	$\Omega(n^2)$ [1]	$n/2-2$ [3]	$n/2-2$ [3]
PUB	$(n-1)!$ [4]	$(n-1)!$ [4]	$n-4$ [3]	2.6382^n [4, 6]
NUB	$n(n-3)/2$	$n(n-3)/2$	$n/2-2$	$n/2-2$

Table 1: Summary of previous and new results.

Corollary 1. *Any simple polygon on n vertices is convexified after any sequence of at most $n(n-3)/2$ flipturns.*

As for the result of Dubins *et al* [3] we take $s = 2$ and obtain the following.

Corollary 2. *Any simple lattice polygon on n vertices is convexified after* any *sequence of at most $n/2-2$ flipturns.*

Indeed, Corollary 2 is the best bound possible. This is because the weight of any vertex in a lattice polygon P is at most 1, thus the decrease in the weight of P during a flipturn is at most 2. Therefore $n/2-2$ flipturns are necessary to convexify any simple lattice polygon with n corners.

4 Conclusions

Table 1 summarizes the results obtained in this paper and compares them to the previous best known results. Referring to the discussion in the introduction, the columns SP and LP denote the shortest, respectively longest, path in G from s to t. The first row of the table shows the previously known lower bounds, the second row shows the previously known upper bounds and the third row shows the new upper bounds obtained in this work.

In looking at this table, an obvious open problem is that of closing the gap between the linear lower bound and the quadratic upper bound on the length of the shortest path from s to t in G.

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