

Dynamic Monopolies in Tori *

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Abstract

Let G be a simple connected graph where every node is colored either *black* or *white*. Consider now the following repetitive process on G : each node recolors itself, at each local time step, with the color held by the majority of its neighbors. Depending on the initial assignment of colors to the nodes and on the definition of majority, different *dynamics* can occur. We are interested in *dynamos*; i.e., initial assignments of colours which lead the system to a monochromatic configuration in a finite number of steps. In the context of distributed computing and communication networks, this repetitive process is particularly important in that it describes the impact that a set of initial faults can have in majority-based systems (where black nodes correspond to faulty elements and white to non-faulty ones). In this paper we study two particular forms of dynamos (*irreversible* and *monotone*) in *tori*, focusing on the minimum number of initial black elements needed to reach the fixed point. We derive lower and upper bounds on the size of dynamos for three types of tori, under different assumptions on the majority rule (*simple* and *strong*). These bounds are *tight* within an additive constant. The upper bounds are *constructive*: for each topology and each majority rule, we exhibit a dynamo of the claimed size. For the constructed dynamos, we also analyze their *time complexity*, i.e. the number of steps necessary to reach the monochromatic configuration when the process is synchronous.

*This work has been supported in part by the University of Pisa, by N.S.E.R.C and by F.C.A.R. Preliminary versions of this paper have been presented to EUROPAR 98 [7] and SIROCCO 99[8].

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1 Introduction

In *majority-based* distributed systems and communication networks, faulty elements can induce a faulty behavior in their neighbors. This is for example the situation in distributed systems where majority voting among various copies of crucial data are performed between neighbours at each step [17]: if the majority of its neighbors is faulty (e.g., has corrupted data), a non-faulty element will exhibit a faulty behavior (e.g., its data will become corrupted) and will therefore be indistinguishable from a faulty one. Let G be the graph describing the communication topology of the system; then the dynamics of the system can be viewed as a repetitive process on G : initially, each node is colored either *black* (faulty) or *white* (non-faulty); every node recolors itself at each local time step, according to the color of the “majority” of its neighbors. This process is possibly asynchronous, as the local clocks might not be synchronized.

In this paper, we are interested in the patterns of initial faults which may lead the entire system to a faulty behaviour (e.g., every entity has corrupted data). Such patterns are called *dynamos* and their study has first been introduced by Peleg [18]. If the initial faults are permanent, the dynamo is said to be *irreversible*; if even the initial faults can be mended by the majority rule, the dynamo will be called *reversible*. A dynamo is *monotone* if $B(t) \subseteq B(t+1)$, where $B(t)$ denotes the set of black vertices at (global) time t . Note that irreversible dynamos are always monotone; hence, in the following, we will use the term “monotone” to indicate “reversible monotone”.

The dynamics of majority-based systems have been extensively studied in the context of cellular automata, and much effort has been concentrated on determining the asymptotic behaviors of different majority rules on different graph structures. Most of the existing research has focused, for example, on the study of configurations leading to *periodic* dynamics on finite graphs [9, 20], on the number of fixed points on finite rings [1, 10] and (finite and infinite) lines [13, 14], and on the behaviors of infinite graphs [15].

Although the dynamics of majority rules have been extensively investigated, very little is known about patterns leading to *monochromatic* fixed points, i.e. dynamos. This is quite surprising considering that dynamos describe occurrences of faults which lead the entire system to a faulty behaviour.

Most of the previous results were known in terms of *monopolies*: monotone dynamos which lead the system to an all black state in a single step [4, 2, 17].¹ Several related results were established in the study of *catastrophic fault patterns*; that is, irreversible dynamos for infinite chordal rings based on *directional majority*: a node becomes black if all its neighbours in the same direction are black (e.g., [5, 16, 21]).

Recently, researchers have started to focus directly on dynamos: some general lower and upper bounds on the size of monotone dynamos have been established in [18], and characterizations of irreversible dynamos has been given for chordal rings [6], and

¹Dynamos were introduced as a generalization of monopolies; in fact, the term “dynamo” is a contraction of “dynamic monopolies” [18].

butterfly [12].

In this paper we study *irreversible* and *monotone* dynamos in *tori*, focusing on the minimum number of initial black elements needed to reach the fixed point. The torus is one of the simplest and most natural way of connecting processors in a network. We consider three different types of tori: the *toroidal mesh* (the classical architecture used in VLSI), the *torus cordalis* (a.k.a. double-loop interconnection networks), and the *torus serpentinus* (used as the topology of, e.g., ILLIAC IV). All these graphs are regular with degree four; hence, two different majority rules are possible to decide whether a node becomes black: *simple* (two neighbours) and *strong* (three neighbours) majority, each leading to different dynamics.²

We derive lower and upper bounds on the size of irreversible and monotone dynamos for each of these topologies and for each majority rule. These bounds, summarized in Table 1, are *tight* within an additive constant.

The upper bounds are *constructive*: for each topology and each majority rule, we exhibit a dynamo of the claimed size. For the constructed dynamos, we also analyze their *time complexity*, i.e. the number of steps necessary to reach a fixed point in the synchronous case; the results are shown in Table 2.

Note that although meshes with toroidal connections avoid the border effects which would occur in meshes, our techniques can be easily adapted to meshes.

2 Basic Definitions

Let us consider an $m \times n$ mesh M and denote with $\mu_{i,j}$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, a vertex of M . The differences among the considered topologies consist only in the way that border vertices (i.e., $\mu_{i,0}, \mu_{i,n-1}$ with $0 \leq i \leq m-1$ and $\mu_{0,j}, \mu_{m-1,j}$ with $0 \leq j \leq n-1$) are linked to other processors. The vertices $\mu_{i,n-1}$ on the last column are usually connected either to the opposite ones on the same rows (i.e. to $\mu_{i,0}$), thus forming ring connections in each row, or to the opposite ones on the successive rows (i.e. to $\mu_{i+1,0}$), in a snake-like way. The same linking strategy is applied for the last row.

In the *toroidal mesh* rings are formed in rows and columns; in the *torus cordalis* there are rings in the columns and snake-like connections in the rows; finally, in *torus serpentinus* there are snake-like connections in rows and columns. Formally, we have:

Definition 1 Toroidal Mesh

A toroidal mesh of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ is connected to the four vertices $\tau_{(i-1) \bmod m, j}$, $\tau_{(i+1) \bmod m, j}$, $\tau_{i, (j-1) \bmod n}$, $\tau_{i, (j+1) \bmod n}$ (*mesh connections*).

²In the terminology of [18], simple majority and strong majority correspond to the “self-not-included, prefer-black” and “self-not-included, prefer-current”, respectively.

Irreversible Dynamos				
	Simple Majority		Strong Majority	
	<i>Lower Bound</i>	<i>Upper Bound</i>	<i>Lower Bound</i>	<i>Upper Bound</i>
<i>Toroidal mesh</i>	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K + 1)$
<i>Torus cordalis</i>	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{m}{3} \rceil (n + 1)$
<i>Torus serpentinus</i>	$\lceil \frac{N}{2} \rceil$	$\lfloor \frac{N}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K + 1)$

Monotone Dynamos				
	Simple Majority		Strong Majority	
	<i>Lower Bound</i>	<i>Upper Bound</i>	<i>Lower Bound</i>	<i>Upper Bound</i>
<i>Toroidal mesh</i>	$m + n - 2$	$m + n - 1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$
<i>Torus cordalis</i>	$n + 1$	$n + 1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$
<i>Torus serpentinus</i>	$N + 1$	$N + 1$	$\lceil \frac{mn+1}{2} \rceil$	$\lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil^*$

Table 1: Bounds on the size of monotone dynamos, for tori of $m \times n$ vertices; $M = \max\{m, n\}$, $N = \min\{m, n\}$, and $H, K = m, n$ or $H, K = n, m$ (choose the alternative that yields stricter bounds). The asterisk denotes a worst case.

Irreversible Dynamos		
	Simple Majority	Strong Majority
<i>Toroidal mesh</i>	$\lceil \frac{m+n}{2} \rceil$	$\frac{N}{2} + 1$
<i>Torus cordalis</i> and <i>Torus serpentinus</i>	$\lfloor \frac{M-3}{2} \rfloor N + 3$	
Monotone Dynamos		
	Simple Majority	Strong Majority
<i>Toroidal mesh</i>	$\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1$	$(m \text{ even}, n \text{ odd}) \ n + \frac{m}{2} - 2$
<i>Torus cordalis</i> and <i>Torus serpentinus</i>	$(M \text{ odd}) \ \lceil \frac{mn}{2} \rceil - N^*$ $(M \text{ even}) \ \frac{mn}{2} - N - 1^*$	$(m, n \text{ even}) \ N + \frac{M}{2} - 2$ $(m, n \text{ odd}) \ N + \frac{M-1}{2} - 2$

Table 2: Time complexity of the monotone dynamos achieving the size upperbounds for tori of $m \times n$ vertices; $M = \max\{m, n\}$, $N = \min\{m, n\}$, the asterisk denotes a worst case.

Definition 2 Torus Cordalis

A torus cordalis of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $0 \leq i \leq m-1, 0 \leq j \leq n-1$ has mesh connections except for the last vertex $\tau_{i,n-1}$ of each row i , which is connected to the first vertex $\tau_{(i+1) \bmod m, 0}$ of row $i+1$.

Notice that this torus can be seen as a chordal ring with one chord, and it is known in the literature also as *double loop network* (e.g., see [3]).

Definition 3 Torus Serpentinus

A torus serpentinus of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $1 \leq i \leq m-1, 0 \leq j \leq n-1$ has mesh connections, except for the last vertex $\tau_{i,n-1}$ of each row i which is connected to the first vertex $\tau_{(i+1) \bmod m, 0}$ of row $i+1$, and for the last vertex

$\tau_{m-1,j}$ of each column j which is connected to the first vertex $\tau_{0,(j-1) \bmod n}$ of column $j - 1$.

Majority will be defined as follows:

Each vertex v takes the color of the majority of its neighbours. In case of tie

- (*simple majority*) it becomes black
- (*strong majority*) it does not change color

In the case of *Irreversible-majority rule*, an initially black node does not change its value, while for the *Reversible-majority rule* also initial black nodes are subject to changes

In the tori, simple (resp., strong) majority asks for the presence of at least two (resp., three) black neighbours in order to color black a vertex; and three (resp., four) white neighbours cause a vertex to become white.

A *simple* (respectively: *strong*) *irreversible dynamo* is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the simple (respectively: strong) irreversible-majority rule.

A *simple* (resp., *strong*) *monotone dynamo* is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the simple (resp., strong) majority rule, and such that no black vertex ever becomes white during the process. Thus, in a monotone dynamo, at any time, the set of black vertices must include all the black sets of the previous steps. In other words, it must never happen that a black vertex has a majority of white neighbours.

Irreversible and monotone dynamos will be treated separately, because they exhibit different properties. First, however, we establish a general concept.

Let T be the set of vertices of a torus. A *simple* (resp., *strong*) *white block* W is a subset of T composed of all white vertices, each of which has at least three (resp., two) neighbours in W .

For example, two adjacent white rows or white columns, form a simple white block in a toroidal mesh, two adjacent white columns form a white block for a torus cordalis (but not for a serpentinus). White vertices placed on a cycle form a strong white block in any torus. Clearly if a simple (resp., strong) white block W is present, the subset $T - W$ of the other vertices cannot contain a simple (resp., strong) dynamo, because W would never turn black.

Let $S \subseteq T$ be a subset of the vertices; S is *connected* if each of its vertices can be reached from any other vertex of S through a sequence of adjacent vertices of S .

3 Irreversible Dynamos with Simple Majority

Irreversible dynamos generate simpler network evolution than reversible ones. Still, the network behaviour changes drastically if we pass from simple to strong majority. We start our study from the former case.

3.1 Toroidal Mesh

Consider a toroidal mesh with a set T of $m \times n$ vertices, $m, n > 2$. Each vertex has four neighbors, then two black neighbors are enough to color black a white vertex. Let $S \subseteq T$ be a generic subset of vertices, and R_S be the *smallest rectangle* containing S . The size of R_S is $m_S \times n_S$. If S is all black, a *spanning set* for S (if any) is a *connected* black set $\sigma(S) \supseteq S$ derivable from S with consecutive applications of the simple majority rule. We have:

Lemma 1 *Let S be a black set with $m_S < m - 1$ (respectively: $n_S < n - 1$). Then any (non necessarily connected) black set B derivable from S is such that $m_B \leq m_S$ (respectively: $n_B \leq n_S$).*

Proof If $m_S < m - 1$, there are $m - m_S \geq 2$ adjacent white rows forming a simple white block that will never turn black. The reasoning for $n_S < n - 1$ is similar. \square

Notice that the previous lemma holds also in the case of monotone dynamos.

Theorem 1 *Let S be a black set. The existence of a spanning set $\sigma(S)$ implies $|S| \geq \lceil \frac{(m_S + n_S)}{2} \rceil$.*

Proof Let S be a set of minimal cardinality such that $\sigma(S)$ exists. By induction on the size of R_S .

Basis We include four cases:

$$\begin{array}{ll}
 m_S = n_S = 1 & \bullet \\
 m_S = 2, \ n_S = 1 & \begin{array}{c} \bullet \\ \bullet \end{array} \\
 m_S = 1, \ n_S = 2 & \bullet \ \bullet \\
 m_S = n_S = 2 & \begin{array}{c} \bullet \ \bullet \\ \bullet \end{array}
 \end{array}$$

The hypothesis on $|S|$ is verified. Note that in the first three cases $S = \sigma(S)$.

Induction step. $m_S > 2$ and/or $n_S > 2$. First observe that a minimal set S is not connected. In fact, a connected set with $m_S > 2$ and/or $n_S > 2$ should contain a chain of three adjacent black vertices xyz , where y could be eliminated from S (and then reconstructed from xz in $\sigma(S)$) without preventing the construction of $\sigma(S)$. Consider now any algorithm that builds $\sigma(S)$ from S by successive blackenings of vertices. Starting from (the non connected) S , consider the first step in which the blackening of a white vertex z joins all the vertices of S into a single connected black set $\alpha(S)$, with $\alpha(S) \subseteq \sigma(S)$. Having four neighbours, z joins two, or three, or four disjoint black sets.

We then pose $\alpha(S) = \sigma(P) \cup \sigma(Q) \cup \sigma(X) \cup \sigma(Y) \cup \{z\}$, where P, Q, X, Y are disjoint subsets of S , with $P \cup Q \cup X \cup Y = S$; P and Q are nonempty; X and/or Y may be empty; and the spanning sets $\sigma(P), \sigma(Q), \sigma(X), \sigma(Y)$ are also disjoint. We consider three cases:

Case 1. X and Y are empty; there are two black vertices $p \in \sigma(P)$ and $q \in \sigma(Q)$ adjacent to z ; and p, z, q are on the same (horizontal or vertical) straight-line segment. Recall that $\sigma(P) \subseteq R_P$, $\sigma(Q) \subseteq R_Q$. We have: $m_P + m_Q \geq m_S + 1$, $n_P + n_Q \geq n_S - 1$, if p, z, q are on the same horizontal segment; and $m_P + m_Q \geq m_S - 1$, $n_P + n_Q \geq n_S + 1$, if p, z, q are on the same vertical segment. Hence, in both cases:

$$m_P + n_P + m_Q + n_Q \geq m_S + n_S. \quad (1)$$

By the inductive hypothesis we also have:

$$|P| \geq \lceil \frac{(m_P + n_P)}{2} \rceil, |Q| \geq \lceil \frac{(m_Q + n_Q)}{2} \rceil. \quad (2)$$

Combining relations 1, 2, and noting that $|S| = |P| + |Q|$, we immediately obtain:

$$|S| \geq \lceil \frac{(m_P + n_P)}{2} \rceil + \lceil \frac{(m_Q + n_Q)}{2} \rceil \geq \lceil \frac{(m_S + n_S)}{2} \rceil. \quad (3)$$

Case 2. As in case 1, X and Y are empty, and z is adjacent to two black vertices $p \in \sigma(P)$, $q \in \sigma(Q)$, but the chain p, z, q is bent. We now have: $m_P + m_Q \geq m_S$ and $n_P + n_Q \geq n_S$, from which we again derive relation 1. Relations 2 and 3 again follow.

Case 3. X and/or Y are not empty. We have: $m_P + m_Q + m_X + m_Y \geq m_S$ and $n_P + n_Q + n_X + n_Y \geq n_S$. Since $|S| = |P| + |Q| + |X| + |Y|$ we easily obtain by induction:

$$|S| \geq \lceil \frac{(m_P + n_P)}{2} \rceil + \lceil \frac{(m_Q + n_Q)}{2} \rceil + \lceil \frac{(m_X + n_X)}{2} \rceil + \lceil \frac{(m_Y + n_Y)}{2} \rceil \geq \lceil \frac{(m_S + n_S)}{2} \rceil. \quad \square$$

An example of set S , and some phases of the blackening algorithm, are illustrated in figure 1. We now pose:

Definition 4 An alternating chain C is a sequence of adjacent vertices starting and ending with black. The vertices of C are alternating black and white, however, if C has even length there is exactly one pair of consecutive black vertices somewhere.

Theorem 2 Let C be an alternating chain in a toroidal mesh $m \times n$, placed in column 1, rows 2 to m , and in row m , columns 1 to $n - 1$. The set S of the black vertices of C is a simple irreversible dynamo with $|S| = \lceil \frac{(m+n)}{2} \rceil - 1$.

Proof By construction, using the following algorithm.

1. Color black the whole chain C , thus obtaining a spanning set $\sigma(S)$.

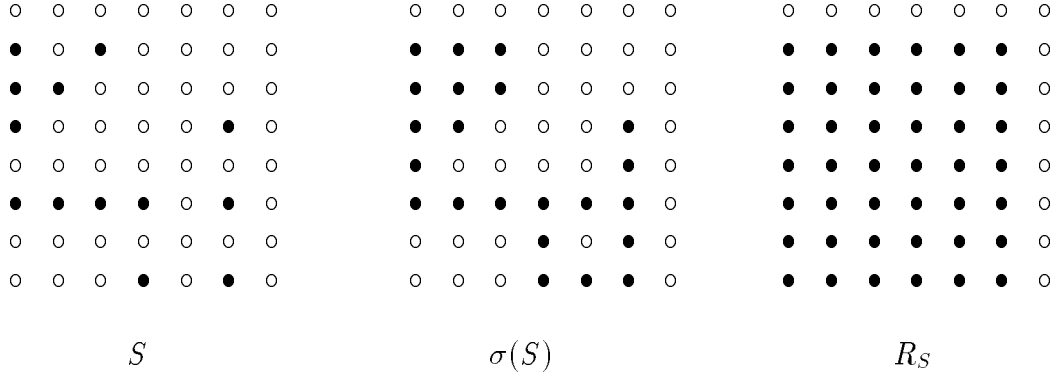


Figure 1: The initial set S ; a spanning set $\sigma(S)$; and the smallest rectangle R_S containing S .

2. Color black R_S as indicated in theorem 1. The whole torus is now black, except for one row r and one column c .
3. Color black the vertices of r and c that are adjacent to two vertices of R_S . In fact all the vertices of r and c are in this condition, except for the one at their crossing.
4. Color black the last vertex.

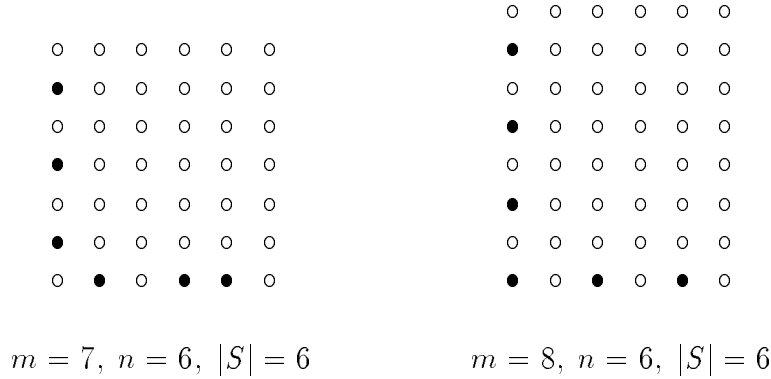


Figure 2: Examples of alternating chains.

Note now that $|C| = m + n - 3$. Refer to figure 2. If m even and n even, or m odd and n odd, we have $|C|$ odd, hence S contains $\lceil \frac{|C|}{2} \rceil = \lceil \frac{m+n}{2} \rceil - 1$ vertices. If m even and n odd, or vice-versa, we have $|C|$ even, hence $|S|$ contains $\frac{|C|}{2} + 1 = \lceil \frac{m+n}{2} \rceil - 1$. \square

Note that the lower bound of theorem 1 matches the upper bound of theorem 2. From the latter theorem we see that a dynamo of minimal cardinality can be built on a proper alternating chain. This is not the only relevant chain. We have:



Figure 3: Examples of dynamos requiring $\lceil \frac{(m+n)}{2} \rceil$ steps.

Lemma 2 *There exists a simple irreversible dynamo S of minimal cardinality starting from which the whole toroidal mesh can be colored black in $\lceil \frac{(m+n)}{2} \rceil$ steps.*

Proof For $m = n$, $|S| = \lceil \frac{(m+n)}{2} \rceil - 1 = m - 1$, let S consist of the vertices of one of the main diagonals, except for one (figure 3, left side). Starting from this configuration, at the first step the two diagonals, adjacent to the main one, become black. The propagation continues on two new diagonals at each step. Therefore in $m - 2$ steps R_S is black. Two more steps are needed to color black the still white vertices on the last row and column.

Consider now $m > n$ (the case $m < n$ is symmetric). Starting from the vertex in row $\lfloor \frac{(m-n)}{2} \rfloor + 1$ and column 1, place $n - 1$ black vertices along the diagonal D heading down and right (figure 3, right side). Starting now from the top vertex of this diagonal place other vertices in column 1 on an alternating chain heading to the top of the mesh, up to row 1. Similarly place other vertices in column $n - 1$ on an alternating chain heading down. A black wave propagates from the two sides of D . The number of steps to color black the whole graph equals the number $\lceil (m + n)/2 \rceil - 2$ of diagonals encountered on each side of D , until R_S is made black. Two more steps are needed to complete the border. \square

3.2 Torus Cordalis and Torus Serpentinus

We now show that any simple irreversible dynamo must have size $\geq \lceil \frac{n}{2} \rceil$, and that there exist dynamos with almost optimal size.

Theorem 3 *Let S be a simple irreversible dynamo for a torus cordalis $m \times n$. We have: $|S| \geq \lceil \frac{n}{2} \rceil$.*

Proof Immediate from the observation that, in a torus cordalis with simple majority, two consecutive white columns form a simple white block (recall that the first and the last columns are consecutive). \square

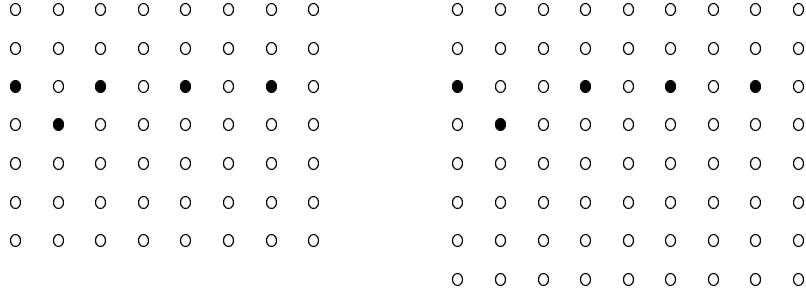


Figure 4: Simple irreversible dynamos of $\lfloor \frac{n}{2} \rfloor + 1$ vertices for torus cordalis and serpentinus, with n even and n odd. Symmetric configurations, exchanging rows and columns, are simple irreverible dynamos for torus serpentinus. In this case we can choose a dynamo of $\lfloor \frac{N}{2} \rfloor + 1$ vertices.

Theorem 4 *In a $m \times n$ torus cordalis, the set S of $\lfloor \frac{n}{2} \rfloor + 1$ black vertices shown in figure 4 is a simple irreversible dynamo. Starting from S the whole torus can be colored black in $\lfloor \frac{m-3}{2} \rfloor n + 3$ steps.*

Proof Immediate from a detailed inspection of the configuration S . \square

The bounds for torus serpentinus are similar to the ones for cordalis, but are a little trickier to prove. Consider a *white cross* C , that is a set of white vertices arranged as in Figure 5, with height m and width n . The vertex at the center of the square of nine vertices inside C is the *center* of the cross. Note that the white cross is a simple white block for a torus serpentinus. Let $M = \max\{m, n\}$, $N = \min\{m, n\}$.

Theorem 5 *Let S be a simple irreversible dynamo for a torus serpentinus $m \times n$. We have: $|S| \geq \lceil \frac{n}{2} \rceil$.*

Proof Starting from a white torus, insert a minimum number of black vertices to eliminate all possible white crosses. Let v be the first black vertex inserted, and consider the cross C centered in v . Exactly all the crosses centered in a vertex of C (including C itself) contain v , hence all these crosses are no more white. Repeat the process

blackening new vertices, until no possible white cross remains. Since each new black vertex w eliminates exactly all the crosses centered in a vertex of the cross D centered in w , and D invades two rows and two columns of each side of w , we must include at least $\lceil \frac{N}{2} \rceil$ black vertices to be sure that no white cross remains. \square

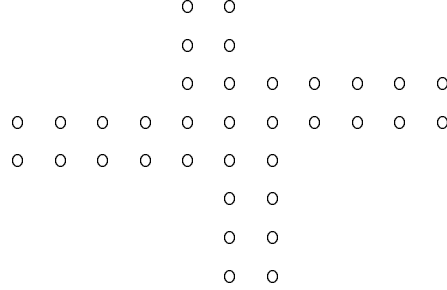


Figure 5: The white cross configuration in a torus serpentinus.

Theorem 6 *In a $m \times n$ torus serpentinus, the set S of $\lfloor \frac{n}{2} \rfloor + 1$ black vertices presented in Figure 4 is a simple irreversible dynamo. Starting from S the whole torus is colored black in $\lfloor \frac{M-3}{2} \rfloor N + 3$ steps.*

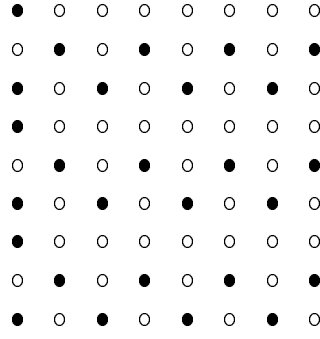
Proof Immediate from inspection of S . \square

4 Irreversible Dynamos with Strong Majority

A strong majority argument allows to derive a significant lower bound valid for the three considered families of tori (simply denoted by tori). Since these tori have a neighbourhood of four, three adjacent vertices are needed to color black a white vertex under strong majority. We have:

Theorem 7 *Let S be a strong irreversible dynamo for a torus $m \times n$. Then $|S| \geq \lceil \frac{mn+1}{3} \rceil$.*

Proof Let T and E be the sets of vertices and edges of the torus; $S \subseteq T$ be the set of black vertices; R be the restriction of the torus to the subset $T - S$ of white vertices. If S is a dynamo, R cannot contain any cycle that would be a strong white block. Then R must be a forest, and its set E_R of edges is such that $|E_R| \leq |T| - |S| - 1$. Note that each edge of the set $E - E_R$ has at least one extreme in S . Since each vertex of T has four neighbours, we have $|E| - |E_R| \leq 4|S|$. We then have: $|E| = 2mn \leq |T| - |S| - 1 + 4|S| = mn + 3|S| - 1$ and the bound follows. \square



$$m = 9, n = 8, |S| = \lceil \frac{m}{3} \rceil (n + 1) = 27$$

Figure 6: A strong irreversible dynamo for toroidal mesh, torus cordalis or torus serpentinus.

We now derive an upper bound also valid for all tori.

Theorem 8 *Any $m \times n$ torus admits a strong irreversible dynamo S with $|S| = \lceil \frac{m}{3} \rceil (n + 1)$. Starting from S the whole torus can be colored black in $\lfloor \frac{n}{2} \rfloor + 1$ steps.*

Proof Let each group of three consecutive rows have the following configuration of colors: in the first row all vertices are white, except for the first one; in the second and in the third row place alternating colors, respectively starting with a white vertex, and a black vertex (see figure 6). If m is not a multiple of three, make the same configuration without the first row, or the first and the second row. We have $\lceil \frac{m}{3} \rceil (n + 1)$ black vertices. It can be easily seen that, after one step, the second and third rows in each group of three, and the first column, become black. At each consecutive step two new columns, adjacent to the columns already colored black, become also black. Therefore the whole torus is colored black in $\lfloor n/2 \rfloor + 1$ steps. \square

For particular values of m and n the bound of theorem 8 can be made stricter for the toroidal mesh and the torus serpentinus. In fact these networks are symmetrical with respect to rows and columns, hence the pattern of black vertices reported in figure 6 can be turned of 90 degrees, still constituting a dynamo. We immediately have:

Corollary 1 *Any $m \times n$ toroidal mesh or torus serpentinus admits a strong irreversible dynamo S with $|S| = \max\{\lceil \frac{m}{3} \rceil (n + 1), \lceil \frac{n}{3} \rceil (m + 1)\}$.*

5 Monotone Dynamos with Simple Majority

As for the irreversible case, the network behaviour change drastically depending on the type of majority applied. In this section we study tori where becoming black is decided

by simple majority; since every vertex has four neighbors, then two black neighbors are enough to color black a white vertex.

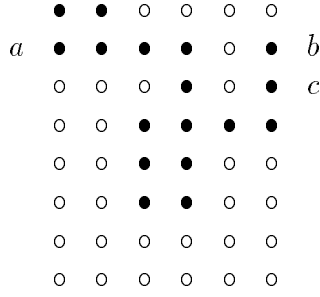


Figure 7: A black compact (black vertices) in a toroidal mesh: note that vertex b is adjacent to two black vertices a, c .

A *black compact* C is a connected subset of black vertices, each of which has at least two neighbours in C . Notice that, under both simple and strong majority rules, the vertices of a black compact will never become white. An example of black compact is shown in Figure 7. Other examples of black compacts are: a black column in a toroidal mesh or in a torus cordalis; a black row, or a black column in a toroidal mesh etc.

By definition of black compacts and white block, for all three tori we have:

Lemma 3

Let S be a simple (strong) monotone dynamo. Then:

- (i) S is a collection of black compacts;
- (ii) $T - S$ does not contain any simple (strong) white block.

5.1 Toroidal Mesh

Consider a toroidal mesh with a set T of $m \times n$ vertices, ($m, n > 2$). Let $S \subseteq T$ and R_S be the *smallest rectangle* containing S ; we denote by $m_S \times n_S$ the size of R_S . If S is all black, a *spanning set* for S (if any) is a *connected* black set $\sigma(S) \supseteq S$ derivable from S with consecutive applications of the simple majority rule.

Notice that, lemma 1 holds also in this case

We define a *black gallows* G as any subset of T composed of all black vertices lying in a whole row and in a whole column of the mesh. A black gallows G is also a black compact, and we have $|G| = m + n - 1$.

To establish a lower bound to the cardinality of a monotone dynamo, we first prove an elementary property of black compacts.

Lemma 4

Let C be a black compact of a toroidal mesh of size $m \times n$. Then:

Case 1. One of the black compacts, say C_1 , has $m_{C_1} = m$ and/or $n_{C_1} = n$. If $m_{C_1} + n_{C_1} \geq m+n-1$ we have $|C_1| \geq m+n-2$ by lemma 4 (i), and the theorem is proved. Now let $m_{C_1} = m$, $n_{C_1} < n-1$ (the case $m_{C_1} < m-1$, $n_{C_1} = n$ is symmetric). Other black compacts C_2, \dots, C_k must provide black vertices in some of the $n - n_{C_1}$ columns outside R_{C_1} , so that $n_S \geq n-1$ (point (i) of this theorem), and no two adjacent such columns are white. It can be immediately verified that, in a set of minimal cardinality with this property, each C_i must have $m_{C_i} < m$, $2 \leq i \leq k$. In fact, each C_i has $n_{C_i} < n-1$, otherwise C_i would intersect C_1 ; and $m_{C_i} \geq 2$, otherwise C_i would not be a compact. By lemma 4 (ii) we have: $|C_2| + \dots + |C_k| \geq m_{C_2} + \dots + m_{C_k} + n_{C_2} + \dots + n_{C_k} \geq 2(k-1) + n_{C_2} + \dots + n_{C_k}$, and we must minimize this function with the condition $n_{C_2} + \dots + n_{C_k} \geq n - n_{C_1} - k$, where the term $-k$ denotes the existence of at most one white column between any pair of black compacts. We then have: $|C_2| + \dots + |C_k| \geq n - n_{C_1} + k - 2$, that is minimized for $k = 2$ (i.e., only two black compacts C_1, C_2 are present). In conclusion we have: $|S| = |C_1| + |C_2| \geq m + n_{C_1} - 1 + n - n_{C_1} = m + n - 1$.

Case 2. All the black compacts C_i have $m_{C_i} < m$ and $n_{C_i} < n$. If one such compact, say C_1 , has $m_{C_1} = m-1$ and $n_{C_1} = n-1$, then $|C_1| \geq m+n-2$ by lemma 4 (ii), and the theorem is proved. Otherwise S must contain more than one such compact. The proof is by induction on the size of the sets built from S by successive applications of the reversible majority rule. Consider the first step in which a unique connected set $\alpha(S) \supset S$ is generated, with a white vertex z turning black to join several disjoint black compacts. It can be easily verified that the possible black neighbours of any vertex cannot belong to more than two black compacts. z joins exactly two disjoint black subsets, say $\sigma(P)$, $\sigma(Q)$, with $P = C_1 \cup \dots \cup C_h$, $Q = C_{h+1} \cup \dots \cup C_k$. We have: $m_P + n_P + m_Q + n_Q \geq m_S + n_S$. Inductively assume: $|P| \geq m_P + n_P$, $|Q| \geq m_Q + n_Q$, where the basis of the induction for a single C_i is given by lemma 4 (ii). From the two relations above we have: $|S| = |P| + |Q| \geq m_S + n_S$, and from point (i) of this theorem we finally derive: $|S| \geq m + n - 2$. \square

To build a monotone dynamo whose size is close to the bound of theorem 9, we first establish the following:

Lemma 5

Let S be a collection of black compacts, such that a spanning set $\sigma(S)$ exists. Then a black rectangle R_S can be built from S .

Proof If $\sigma(S) = R_S$ we are done (this includes the case $m_S = 1$ and/or $n_S = 1$). Otherwise, there must exist a white vertex $p \in R_S - \sigma(S)$ in a concave corner of $\sigma(S)$, that is p has two adjacent black vertices (this is inevitable because $\sigma(S)$ spans R_S). Color p black and iterate on the white vertices. \square

From this lemma, with an easy computation of the number of blackening steps, we immediately have:

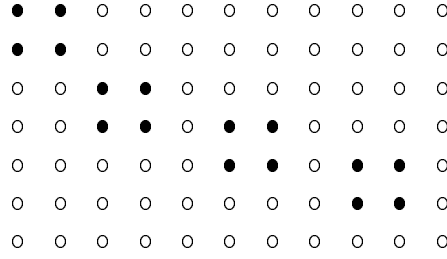


Figure 9: A collection C_0, C_1, C_2, C_3 of black compacts in a 7×11 toroidal mesh. $C = C_0 \cup C_1 \cup C_2 \cup C_3$ is a monotone dynamo with $|C| = m + n - 2$.

Theorem 10

In a $m \times n$ toroidal mesh, a black gallows G is a simple monotone dynamo of cardinality $m + n - 1$. Starting from G , the whole torus is colored black in $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1$ steps.

The upper bound of Theorem 10 differs only by 1 from the lower bound of Theorem 9. However, an upper bound of $m + n - 2$ can be derived for an infinite subfamily of toroidal meshes with special dimensions m, n . Namely, let: $m + n = 6 + 4k$ ($k = 0, 1, \dots$), and $|m - n| = 2r$ ($0 \leq r \leq k$). Proper values are, for example, $m = 42$ and $n = 60$, for which we have $m + n = 6 + 4 \cdot 24$, $|m - n| = 2 \cdot 9$. Build a collection of black compacts C_0, C_1, \dots, C_k , each composed of four black vertices arranged in a square of side 2. For $0 \leq i \leq r$, place each C_i in rows $2i, 2i + 1$, and in columns $2i, 2i + 1$. For $m > n$ (hence $r < k$), and for $1 \leq j \leq k - r$, place C_{r+j} in rows $2r + 3j, 2r + 1 + 3j$, and in columns $2r + j, 2r + 1 + j$. For $n > m$ (hence $r < k$), and for $1 \leq j \leq k - r$, place C_{r+j} in rows $2r + j, 2r + 1 + j$, and in columns $2r + 3j, 2r + 1 + 3j$. A collection of black compacts C_0, \dots, C_3 for $m = 7, n = 11$, is shown in figure 9, where we have $m + n = 18$, $k = 3$, $r = 2$, $n > m$. The reader may immediately discover that $C = \bigcup_{i=0}^k C_i$ is a monotone dynamo, with $|C| = 4 \cdot (k + 1) = m + n - 2$.

5.2 Torus Cordalis and Torus Serpentinus

For the tori cordalis and serpentinus we can establish lower and upper bounds that match exactly.

We have:

Theorem 11

Let S be a monotone dynamo for a torus cordalis of size $m \times n$. Then we have: $|S| \geq n + 1$.

Proof From lemma 3, let $S = C_1 + \dots + C_k$ be a collection of black compacts, and let n_{C_i} denote the number of columns where the vertices of C_i are placed. If one such

a compact C_i has $n_{C_i} = n$, we have $|S| \geq |C_i| \geq n + 1$ because a single black row, or a "snake-like" black row containing the vertices $\tau_{i,j} \dots \tau_{i,n-1} \tau_{(i+1) \bmod m, 0} \dots \tau_{(i+1) \bmod m, j-1}$, is not a compact in this torus, unless another black vertex is added to close a cycle. If $n_{C_i} < n$, for all i , we must have $n_{C_1} + \dots + n_{C_k} \geq n - k$, to prevent the existence of a simple white block composed of two adjacent white columns. For each C_i to be a compact we must have $|C_i| \geq n_{C_i} + 2$ (as in the proof of lemma 4). Then we have: $|C| = |C_1| + \dots + |C_k| \geq n_{C_1} + \dots + n_{C_k} + 2k \geq n + k$. \square

The matching upper bound is the following:

Theorem 12

In a torus cordalis of size $m \times n$, the set S of $n + 1$ black vertices placed in a whole row i , and in row $(i + 1) \bmod m$, column 0, is a simple monotone dynamo. Starting from S the whole torus is colored black in $\lceil \frac{mn}{2} \rceil - n$ steps for m odd, or in $\lceil \frac{mn}{2} \rceil - n - 1$ steps for m even.

Proof From immediate inspection of the proposed configuration of black vertices, S is recognized as a simple monotone dynamo (note that S is a black compact). The number of steps to color the torus black also easily follows. \square

Let $M = \max\{m, n\}$, $N = \min\{m, n\}$. In a similar way we can prove that:

Theorem 13

Let S be a monotone dynamo for a torus serpentinus of size $m \times n$. We have: $|S| \geq N + 1$.

Theorem 14 *In a torus serpentinus $m \times n$, the set S of $N + 1$ black vertices placed in a whole row i , and in row $(i + 1) \bmod m$, column 0, for $N = n$; or placed in a whole column j , and in column $(j + 1) \bmod n$, row 0, for $N = m$; is a simple monotone dynamo. Starting from S the whole torus is colored black in $\lceil \frac{mn}{2} \rceil - N$ steps for M odd, or in $\frac{mn}{2} - N - 1$ steps for M even.*

6 Monotone Dynamos with Strong Majority

In this section we study tori where becoming black is decided by strong majority. In this case, to rule out the presence of strong white blocks we must impose that the restriction of the torus to the subset of initial white vertices is a forest. Moreover, in order to ensure monotonicity, the initial black vertices must form a collection of black compacts.

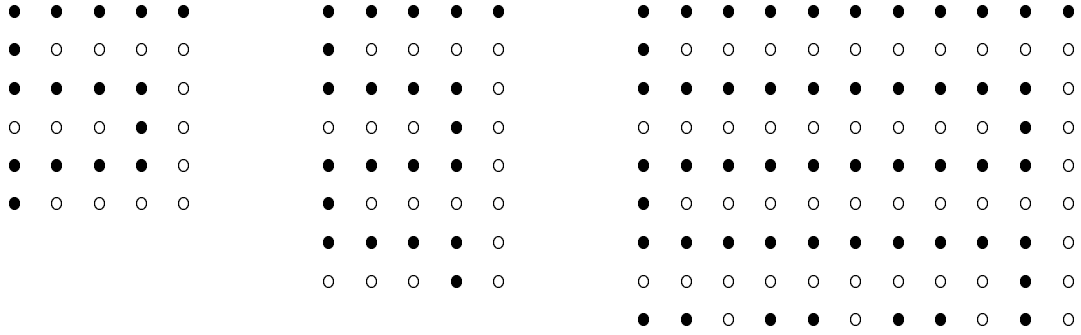
A single argument yields a significant lower bound for the three families of tori. We have:

Theorem 15

Let S be a strong monotone dynamo for a torus of size $m \times n$. Then $|S| \geq \lceil \frac{mn+1}{2} \rceil$.

Proof Let T and E be the sets of vertices and edges of the torus. S and $T - S$ are the subsets of initial black and white vertices, respectively. Let E_{WW} and E_{WB} be the subset of edges connecting two white vertices and a white vertex with a black vertex respectively. Since each white vertex has four leaving edges, and the edges of E_{WW} connect two white vertices, we immediately have: $|E_{WB}| = 4|T - S| - 2|E_{WW}|$. Now the restriction of the torus to the subset of white vertices must be a forest, hence we have: $|E_{WW}| \leq |T - S| - 1$, hence $|E_{WB}| \geq 4|T - S| - 2(|T - S| - 1) = 2|T| - 2|S| + 2$. Since S must be a collection of compacts, each of its vertices can have at most two leaving edges that belong to E_{WB} , hence $|E_{WB}| \leq 2|S|$. Combining the two inequalities for $|E_{WB}|$ we obtain $2|S| \geq |T| + 1$ that proves the theorem. \square

We now derive an upper bound valid for all three families of tori. As before, let $M = \max\{m, n\}$, $N = \min\{m, n\}$.



m and/or n even

m and n odd

Figure 10: Strong monotone dynamos for the three families of tori.

Theorem 16

Any $m \times n$ torus admits a strong monotone dynamo S , with $|S| = \frac{mn}{2} + 1$ for m and/or n even, or $|S| = \lceil \frac{mn}{2} + \frac{N}{6} + \frac{2}{3} \rceil$ for m and n odd. Starting from S , the whole torus can be colored black in $N + \frac{M}{2} - 2$ steps for m and n even; or $n + \frac{m}{2} - 2$ steps for m even and n odd; or $N + \frac{M-1}{2} - 2$ steps for m and n odd.

Proof Assign the vertices of S as shown in Figure 10. Note that the black vertices form a compact and the white vertices form a tree. The theorem easily follows from inspection of the figure and all its significant symmetries. \square

7 Concluding Remarks

In this paper, we have established upper and lower bounds on the size of irreversible and monotone dynamos in tori for different types of majority rules.

It is interesting to compare these results; in both monotone and irreversible dynamos the bounds are tight within an additive constant. Hence, the information contained in tables 1 provides a very accurate quantitative description of the size of dynamos in tori. It also provides the first precise look on the relationship between irreversible and reversible monotone dynamos.

As indicated in the table, in the case of *simple majority*, the bounds for irreversible dynamos are smaller by a factor of two than the ones for reversible monotone dynamos. In the case of *strong majority* the gap is narrower: the constant becomes $3/2$.

This raises the intriguing question of whether it is possible to always transform an irreversible dynamo into a monotone one using at most twice the number of initial black nodes (at most $3/2$ in the case of strong majority). Should this not be the case, the next question would be whether it can be done for some constant k .

References

- [1] Z. Agur, A. S. Fraenkel, S. T. Klein. The number of fixed points of the majority rule. *Discrete Mathematics*, 70:295–302, 1988.
- [2] J-C Bermond, J. Bond, D. Peleg, S. Perennes. Tight bounds on the size of 2-monopolies In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity* (SIROCCO 96), Siena, 170-179, 1996.
- [3] J.-C. Bermond, F. Comellas, D.F. Hsu. Distributed loop computer networks: a survey. *Journal of Parallel and Distributed Computing*, 24:2-10, 1995
- [4] J-C Bermond, D. Peleg. The power of small coalitions in graphs. In *Proc. of 2nd Colloquium on Structural Information and Communication Complexity* (SIROCCO 95), Olympia, 173-184, 1995.
- [5] R. De Prisco, A. Monti, L. Pagli. Efficient testing and reconfiguration of VLSI linear arrays. *Theoretical Computer Science*, vol. 197, 171-188, 1998.
- [6] P. Flocchini, F. Geurts, N. Santoro. Irreversible dynamos in chordal rings. *Proc. 25th International Workshop on Graph-Theoretic Concepts in Computer Science* (WG 99), Ascona, 1999.
- [7] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro. Irreversible dynamos in tori. *Proc. EUROPAR 98*, Southampton, 554-562, 1998.
- [8] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro. Monotone dynamos in tori. *Proc. of 6th Colloquium on Structural Information and Communication Complexity* (SIROCCO 99), 152-165, 1999.
- [9] E. Goles, J. Olivos. Periodic behavior of generalized threshold functions. *Discrete Mathematics*, 30:187–189, 1980.

- [10] A. Granville. On a paper by Agur, Fraenkel and Klein. *Discrete Mathematics*, 94:147–151, 1991.
- [11] N. Linial, D. Peleg, Y. Rabinovich, M. Sachs. Sphere packing and local majority in graphs. In *Proc. of 2nd ISTCS*, IEEE Comp. Soc. Press, 141-149, 1993.
- [12] F. Luccio, L. Pagli, H. Sanossian. Irreversible dynamos in butterflies. *Proc. of 6th Colloquium on Structural Information and Communication Complexity* (SIROCCO 99), 204-218, 1999.
- [13] G. Moran. Parametrization for stationary patterns of the r -majority operators on 0–1 sequences. *Discrete Mathematics*, 132:175–195, 1994.
- [14] G. Moran. The r -majority vote action on 0–1 sequences. *Discrete Mathematics*, 132:145–174, 1994.
- [15] G. Moran. On the period-two-property of the majority operator in infinite graphs. *Transactions of the American Mathematical Society*, 347(5):1649–1667, 1995.
- [16] A. Nayak, N. Santoro, R. Tan. Fault intolerance of reconfigurable systolic arrays. *Proc. 20th International Symposium on Fault-Tolerant Computing*, 202-209, 1990.
- [17] D. Peleg. Local majority voting, small coalitions and controlling monopolies in graphs: A review. In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity* (SIROCCO 96), Siena, 152-169, 1996.
- [18] D. Peleg. Size bounds for dynamic monopolies. In *Proc. of 4th Colloquium on Structural Information and Communication Complexity* (SIROCCO 97), Ascona, 151-161, 1997.
- [19] S. Poljak. Transformations on graphs and convexity. *Complex Systems*, 1:1021–1033, 1987.
- [20] S. Poljak, M. Štúr. On periodical behavior in societies with symmetric influences. *Combinatorica*, 1:119–121, 1983.
- [21] N. Santoro, J. Ren, A. Nayak. On the complexity of testing for catastrophic faults. In *Proc. of 6th International Symposium on Algorithms and Computation* (ISAAC 95), 188-197, 1995.