# Queue Layouts and Three-Dimensional Straight-Line Grid Drawings *

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June 24, 2002

**Abstract**

A famous result due to de Fraysseix, Pach, and Pollack [*Combinatorica*, 1990] and Schnyder [*Order*, 1989] states that every $n$-vertex planar graph has a (two-dimensional) straight-line grid drawing with $O(n^2)$ area. A *three-dimensional straight-line grid drawing* of a graph represents the vertices by grid-points in 3-space and the edges by non-crossing line segments. This research is motivated by the following question of Felsner, Liotta, and Wismath [Graph Drawing '01, *Lecture Notes in Comput. Sci.*, 2002]: *does every planar graph have a three-dimensional straight-line grid drawing with $O(n)$ volume*? A queue layout consists of a linear order $\sigma$ of the vertices of a graph, and a partition of the edges into queues, such that no two edges in the same queue are nested with respect to $\sigma$. Let $G$ be a member of a proper minor-closed family of graphs (such as a planar graph), and let $F(n)$ be a set of functions closed under taking polynomials (such as $O(1)$ or $\operatorname{polylog} n$). We prove that $G$ has a $F(n) \times F(n) \times O(n)$ straight-line grid drawing if and only if $G$ has a queue layout with $F(n)$ queues. Thus the above question is closely related to the open problem of Heath, Leighton, and Rosenberg [*SIAM J. Discrete Math.*, 1992], who ask whether every planar graph has $O(1)$ queuenumber? As a corollary we improve the best known upper bound on the volume of three-dimensional straight-line grid drawings of series-parallel graphs from $O(n \log^2 n)$ to $O(n)$.

**Keywords**: three-dimensional graph drawing, graph layout, queue layout, queuenumber, pathwidth, star colouring, star chromatic number, series-parallel graph

## 1 Introduction

A famous result independently due to de Fraysseix, Pach, and Pollack [7] and Schnyder [28] states that every $n$-vertex planar graph has a (two-dimensional) straight-line grid drawing with $O(n^2)$ area. One might expect that in three dimensions, planar graphs would admit straight-line grid drawings with $o(n^2)$ volume. However, this has remained an elusive open problem. The purpose of this paper is to reduce this question of three-dimensional graph drawing to an existing one-dimensional graph layout problem.

* Technical Report TR-02-02, School of Computer Science, Carleton University, Ottawa, Canada.

## 1.1 Definitions and notation

Throughout this paper all graphs $G$ are undirected, simple and finite with vertex set $V(G)$ and edge set $E(G)$; $n = |V(G)|$ and $m = |E(G)|$ denote the number of vertices and edges of $G$, respectively. For all $A \subseteq V(G)$, the subgraph of $G$ induced by $A$ is denoted by $G[A]$. For all disjoint $A, B \subseteq V(G)$, the bipartite subgraph of $G$ consisting of all edges $vw \in E(G)$ such that $v \in A$ and $w \in B$ is denoted by $G[A, B]$.

A *$k$-tree* is defined recursively as follows. The complete graph $K_k$ is a $k$-tree, and the graph obtained from a $k$-tree by adding a new vertex adjacent to a $k$-clique is a $k$-tree. A *partial $k$-tree* is a subgraph of a $k$-tree. A *tree decomposition* of a graph $G$ is a tree $T$ together with a collection of subsets $T_x$ (called *bags*) of $V(G)$ indexed by the vertices of $T$ such that:

- $\bigcup_{x \in V(T)} T_x = V(G)$,
- for every edge $vw \in E(G)$, there is a vertex $x \in V(T)$ such that the bag $T_x$ contains both $v$ and $w$, and
- for all vertices $x, y, z \in V(T)$, if $y$ is on the path from $x$ to $z$ in $T$, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree decomposition is one less than the maximum cardinality of a bag. A *path decomposition* is a tree decomposition where the tree $T$ is a path. The *pathwidth* (respectively, *treewidth*) of a graph $G$, denoted by $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$), is the minimum width of a path (tree) decomposition of $G$. It is well known that the treewidth of $G$ equals the minimum $k$ such that $G$ is a partial $k$-tree (see [3]).

## 1.2 Three-dimensional straight-line grid drawing

A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *three-dimensional drawing*, represents the vertices by distinct points in 3-space with integer coordinates (called *grid-points*), and represents each edge as a line-segment between its end-vertices, such that edges only intersect at common end-vertices. In contrast to the case in the plane, it is well known that every graph has a three-dimensional drawing. We therefore are interested in optimising certain measures of the aesthetic quality of a drawing. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ three-dimensional drawing with *volume* $X \cdot Y \cdot Z$. In this paper we study three-dimensional drawings with small volume.

Cohen, Eades, Lin, and Ruskey [6] proved that every graph has a three-dimensional drawing with $O(n^3)$ volume, and this bound is asymptotically tight for the complete graph $K_n$. Calamoneri and Sterbini [5] proved that every 4-colourable graph has a three-dimensional drawing with $O(n^2)$ volume. Generalising this result, Pach, Thiele, and Tóth [24] proved that every $k$-colourable graph, for fixed $k \geq 2$, has a three-dimensional drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. The first linear volume bound was established by Felsner, Wismath, and Liotta [12], who proved that every outerplanar graph has a drawing with $O(n)$ volume. Poranen [26] proved that series-parallel digraphs have upward three-dimensional drawings with $O(n^3)$ volume, and that this bound can be improved to $O(n^2)$ and $O(n)$ in certain special cases. di Giacomo, Liotta, and Wismath [8] proved that series-parallel graphs with maximum degree 3 have three-dimensional drawings with $O(n)$ volume. Dujmović, Morin, and Wood

[11] proved that every graph $G$ has a three-dimensional drawing with $O(n \cdot \mathsf{pw}(G)^2)$ volume. This implies $O(n \log^2 n)$ volume drawings for graphs of bounded treewidth, such as series-parallel graphs.

Since a planar graph $G$ is 4-colourable and has $\mathsf{pw}(G) \in O(\sqrt{n})$, by the results of Calamoneri and Sterbini [5], Pach *et al.* [24], and Dujmović *et al.* [11] discussed above, every planar graph has a three-dimensional drawing with $O(n^2)$ volume. Of course this result also follows from the classical algorithms of de Fraysseix *et al.* [7] and Schnyder [28] for producing plane grid drawings. This paper is motivated by the following open problem.

**Open Problem (Felsner *et al.* [12]).** Does every $n$-vertex planar graph have a three-dimensional drawing with $O(n)$ volume? In fact, any $o(n^2)$ volume bound would be of interest.

The main result of this paper is to reduce this question to an open problem in the theory of queue layouts.

### 1.3 Queue layouts

For a graph $G$, a linear order of $V(G)$ is called a *vertex-ordering* of $G$. A *queue layout* of $G$ consists of a vertex-ordering $\sigma$ of $G$, and a partition of $E(G)$ into *queues*, such that no two edges in the same queue are *nested* with respect to $\sigma$. That is, there are no edges $vw$ and $xy$ in a single queue with $v <_\sigma< x <_\sigma< y <_\sigma< w$. The minimum number of queues in a queue layout of $G$ is called the *queuenumber* of $G$, and is denoted by $\mathsf{qn}(G)$. A similar concept is that of a stack layout, which is also called a *book embedding*. A *stack layout* of a graph $G$ consists of a vertex-ordering of $G$, and a partition of $E(G)$ into *stacks* (or *pages*) such that no two edges in a single stack *cross*. That is, there are no edges $vw$ and $xy$ in a single stack with $v <_\sigma< x <_\sigma< w <_\sigma< y$. The minimum number of stacks in a stack layout of $G$ is called the *stacknumber* of $G$, and is denoted by $\mathsf{sn}(G)$. Stacknumber has also been called *pagenumber* and *book thickness* in the literature. Applications of stack and queue layouts include VLSI layout, fault-tolerant processing, parallel processing, matrix computations, and sorting networks (see [25]). Stack layouts have previously appeared in a graph drawing context in [31].

Queue layouts of (undirected) graphs have been studied by Heath *et al.* [17], Heath and Rosenberg [21], Pemmaraju [25], Rengarajan and Veni Madhavan [27], and Shahrokhi and Shi [29]. Queue layouts of directed graphs [19, 20, 25] and posets [18, 25] have also been investigated. Heath and Rosenberg [21] characterise graphs admitting 1-queue layouts as the 'arched leveled planar' graphs. They also prove that it is NP-complete to recognise such graphs. This result is in contrast to the situation for stack layouts — the graphs admitting 1-stack layouts are precisely the outerplanar graphs, which can be recognised in linear time [30]. On the other hand, it is NP-hard to minimise the number of stacks in a stack layout which respects a given vertex-ordering [15]. However for queue layouts with a fixed vertex-ordering this problem can be solved as follows. A *k-rainbow* in a vertex-ordering $\sigma$ consists of a matching $\{v_i w_i : 1 \leq i \leq k\}$ such that

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1 \ .$$

A straightforward application of Dilworth's Theorem [10] proves that the minimum number of queues in a queue layout respecting a fixed vertex-ordering is precisely the size of the largest rainbow [21]. (Heath and Rosenberg [21] also prove that such an assignment of edges to queues can be computed in $O(m \log\log n)$ time.) Thus determining $\mathsf{qn}(G)$ can be viewed simply as a vertex layout problem. In particular, $\mathsf{qn}(G)$ is the the minimum, taken over all vertex-orderings $\sigma$ of $G$, of the size of the maximum rainbow in $\sigma$.

A closely related vertex layout problem is that of *vertex separation number* (see [9]). Consider the vertex numbering $(v_1, v_2, \ldots, v_n)$ induced by a vertex-ordering $\sigma$ of a graph $G$. The *vertex cut at position* $i$ in $\sigma$ is defined to be the set of vertices $v_j \in V(G)$ such that there exists an edge $v_j v_k \in E(G)$ with $j \leq i < k$. The *vertex separation number* of $\sigma$ is the maximum size of a vertex cut in $\sigma$, and the *vertex separation number* of $G$ is the minimum vertex separation number of a vertex-ordering of $G$. Clearly a vertex-ordering with vertex separation number $k$ has no rainbow with more than $k$ edges (see Figure 1). Since the vertex-separation number of a graph equals its pathwidth (see [9]), the next result immediately follows.

**Observation 1.** *For every graph $G$,* $\mathsf{qn}(G) \leq \mathsf{pw}(G)$.

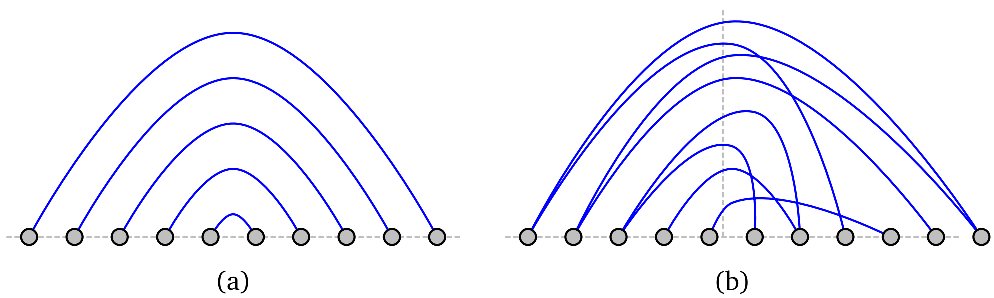

Figure 1: A rainbow (a) and a vertex cut (b) each of size 5.

A similar upper bound is obtained by Heath and Rosenberg [21], who show that every graph $G$ has $\mathsf{qn}(G) \leq \lceil \frac{1}{2}\mathsf{bw}(G) \rceil$, where $\mathsf{bw}(G)$ is the bandwidth of $G$. In many cases this result is weaker than Observation 1 since $\mathsf{pw}(G) \leq \mathsf{bw}(G)$ (see [9]). Furthermore, Observation 1 provides a partial solution to an open problem of Ganley and Heath [14], who ask whether the queuenumber of a graph is bounded by its treewidth. (Ganley and Heath [14] prove that the stacknumber $\mathsf{sn}(G) \leq \mathsf{tw}(G) + 1$.) Since $\mathsf{tw}(G) \in O(\mathsf{pw}(G) \cdot \log n)$ [3], the queuenumber $\mathsf{qn}(G) \in O(\mathsf{tw}(G) \cdot \log n)$.

### 1.4 Our results

While our motivation is for planar graphs, our results apply to any *proper* minor-closed family of graphs; that is, a minor-closed family which is not the class of all graphs. We now state our main result.

**Theorem 1.** *Let $\mathcal{G}$ be a proper minor-closed family of graphs, and let $F(n)$ be a set of functions closed under taking polynomials (for example, $O(1)$ or $\mathrm{polylog}\, n$). Then every $n$-vertex graph $G \in \mathcal{G}$ has a $F(n) \times F(n) \times O(n)$ three-dimensional drawing if and only if $G$ has queue number* $\mathsf{qn}(G) \in F(n)$.

Graphs with constant queuenumber include de Bruijn graphs, FFT and Beneš network graphs [21]. Thus Theorem 1 implies all these graphs have three-dimensional drawings with linear volume. Rengarajan and Veni Madhavan [27] prove that outerplanar graphs have queuenumber at most 2, and 2-trees have queuenumber at most 3. A graph is a 2-tree if and only if every biconnected component is series-parallel (see [3]). We therefore have the following corollary of Theorem 1.

**Corollary 1.** *Every $n$-vertex series-parallel graph has a three-dimensional drawing with $O(n)$ volume.*

Corollary 1 improves and/or generalises the above results for three-dimensional drawings of outerplanar and series-parallel graphs in [8, 11, 12, 26]. Note that the algorithm by Felsner *et al.* [12] closely parallels the construction of 2-queue layouts of outerplanar graphs due to Rengarajan and Veni Madhavan [27], both of which are based on breadth-first search.

## 2 Proof of Theorem 1

Our proof of Theorem 1 depends on the notion of an ordered layering introduced by Dujmović *et al.* [11]. An *ordered $k$-layering* of a graph $G$ consists of a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ into *layers*, and a total order $<_i$ of each $V_i$, such that for every edge $vw$, if $v <_i w$ then there is no vertex $x$ with $v <_i x <_i w$. The *span* of an edge $vw$ is $|i - j|$ where $v \in V_i$ and $w \in V_j$. An *intralayer* edge is an edge with zero span. An *X-crossing* consists of two edges $vw$ and $xy$ such that for distinct layers $i$ and $j$, $v <_i x$ and $y <_j w$.

**Lemma 1 (Dujmović *et al.* [11]).** *Let $F(n)$ be a set of functions closed under taking polynomials. Then an $n$-vertex graph $G$ has a $F(n) \times F(n) \times O(n)$ three-dimensional drawing if and only if $G$ has an ordered $k$-layering with no X-crossing, for some $k \in F(n)$. Furthermore, if $G$ has an ordered layering with no X-crossing and maximum edge span $s$ then $G$ has a $O(s) \times O(s) \times O(n)$ three-dimensional drawing. Refer to Figure 2.*

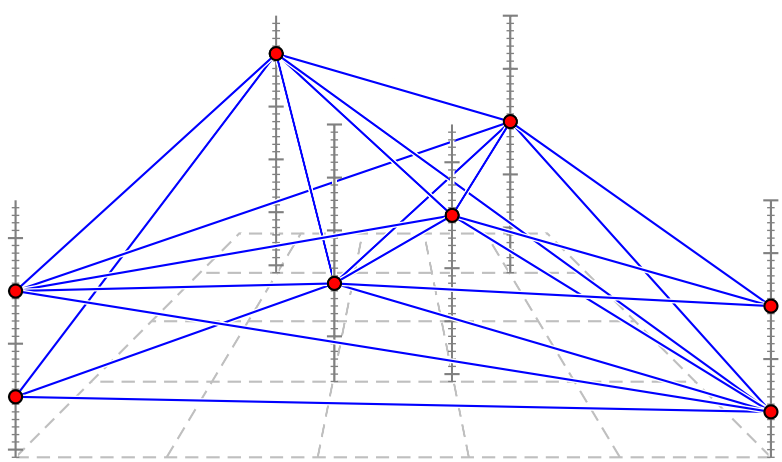

Figure 2: A three-dimensional drawing produced from an ordered layering with no X-crossing; vertices in each layer are placed on a vertical 'rod'.

Dujmović *et al.* [11] prove that every graph $G$ has an ordered $(\mathsf{pw}(G) + 1)$-layering with no X-crossing. That $G$ has a three-dimensional drawing with $O(n \cdot \mathsf{pw}(G)^2)$ volume follows from Lemma 1. The result of Felsner *et al.* [12] also fits into this framework; they prove that an outerplanar graph has an ordered layering with maximum edge span 1.

It follows from Lemma 1 that Theorem 1 can be proved if we show that $G$ has $\mathsf{qn}(G) \in F(n)$ if and only if $G$ has an ordered $k$-layering with no X-crossing, for some $k \in F(n)$. The next lemma highlights the inherent relationship between ordered layerings and queue layouts. It's elementary proof is left as an exercise for the reader (see Figure 3).

**Lemma 2.** *A bipartite graph $G = (A, B; E)$ has an ordered 2-layering with no X-crossing and no intralayer edges if and only if $G$ has a 1-queue layout such that in the corresponding vertex-ordering, the vertices in $A$ appear before the vertices in $B$.*

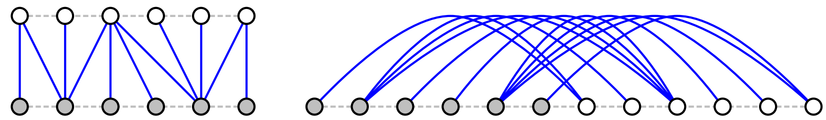

Figure 3: An ordered 2-layering and a 1-queue layout of a bipartite graph.

We now show that a queue layout can be obtained from an ordered layering. This result can be viewed as a generalisation of the construction of a 2-queue layout of an outerplanar graph by Rengarajan and Veni Madhavan [27] (with $s = 1$).

**Lemma 3.** *Let $G$ be a graph with an ordered $k$-layering with no X-crossing and maximum edge span $s$. Then* $\mathsf{qn}(G) \leq \min\{\lfloor \frac{k^2}{4} \rfloor, \frac{s(s+1)}{2}\} + 1$.

*Proof.* Let $\{(V_i, <_i) : 1 \leq i \leq k\}$ be an ordered $k$-layering with no X-crossing and maximum edge span $s$. Let $\sigma$ be the vertex-ordering of $G$ consisting of $V_1$, followed by $V_2$, and so on, up to $V_k$, with each $V_i$ ordered by $<_i$. Partition $\{(i, j) : 1 \leq i < j \leq k\}$ as follows. For all $1 \leq a \leq k$ and $1 \leq b \leq \min\{a, k - a + 1\}$, define

$$I_{a,b} = \Big\{ \big(b, b + a\big), \big(b + a, b + 2a\big), \big(b + 2a, b + 3a\big), \ldots, \big(b + (\lfloor \tfrac{k-b}{a} \rfloor - 1)a, b + \lfloor \tfrac{k-b}{a} \rfloor a\big) \Big\} \quad .$$

For each $a$ and $b$, let $q_{a,b}$ be the union of the edges of $G[V_i, V_j]$ where $(i, j) \in I_{a,b}$. Since there is no X-crossing between $V_i$ and $V_j$, by Lemma 2, there are no nested edges in $q_{a,b}$ with respect to $\sigma$. Using one supplementary queue for all intralayer edges, we obtain a queue layout of $G$. The number of queues is $1 + \sum_{a=1}^{k} \min\{a, k - a + 1\} = 1 + \lfloor \frac{k^2}{4} \rfloor$. Observe that $q_{a,b}$ is empty for $a > s$. Deleting such queues, we obtain a queue layout of $G$ with at most $1 + \sum_{a=1}^{s} a = 1 + \frac{s(s+1)}{2}$ queues. $\square$

We now set out to prove a converse result to Lemma 3. Consider an ordered $k$-layering with no X-crossing and no intralayer edges. The subgraph induced by any two layers is a forest of caterpillars [16]. A slightly weaker family of graphs is a forest of stars. A proper vertex-colouring of a graph is called a *star colouring* if each bichromatic subgraph is a forest of stars; that is, every path on 4 vertices receives at least distinct 3 colours. The minimum number of colours in a star colouring of a graph $G$ is called the *star chromatic number* of $G$, and is denoted by $\chi_{\text{st}}(G)$. Nešetřil and Ossona de Mendez [23] prove that every every planar graph $G$ has $\chi_{\text{st}}(G) \leq 30$. Other graphs with bounded star chromatic number include graphs with bounded acyclic chromatic number [13], graphs embeddable on a fixed surface [2], graphs with bounded maximum degree [1], 1-planar graphs [4], and graphs with bounded treewidth [13]. More generally, Nešetřil and Ossona de Mendez [23] prove that a graph has bounded star chromatic number if and only if it is a member of a proper minor-closed family of graphs $\mathcal{G}$. In this case, the star chromatic number of a graph $G \in \mathcal{G}$ is at most a quadratic function of the maximum chromatic number of a minor of $G$.

**Lemma 4.** *Let $G$ be a graph with star chromatic number $\chi_{\mathsf{st}}(G) \le c$, and queuenumber $\mathsf{qn(G)} \le k$. Then $G$ has an ordered $t$-layering with no X-crossing where*

$$t \;\le\; c\big(2(c-1)k+1\big)^{c-1} \;.$$

*Proof.* Let $V_1, \dots, V_c$ be the colour classes of a star colouring of $G$. Pemmaraju [25] (see also [18]) proved that a $k$-queue graph layout can be 'separated' by a vertex $c$-colouring to produce a $2(c-1)k$-queue layout with the vertices in each colour class consecutive in the vertex-ordering. Applying this result to the given queue layout and star colouring, we obtain a $2(c-1)k$-queue layout of $G$ with vertex-ordering $\sigma$ consisting of vertices in $V_1$, followed by vertices in $V_2$, and so on, up to $V_c$. Let $k' = 2(c-1)k$ and let $q_1, \dots, q_{k'}$ be the queues in this queue layout.

For every vertex $v \in V_i$, $1 \le i \le c$, and $j \in \{1, \dots, c\} \setminus \{i\}$, let $d_j(v)$ be the degree of $v$ in $G[V_i, V_j]$. Define the $j^{\text{th}}$ *label* of $v$, denoted by $\phi_j(v)$, as follows. If $d_j(v) \ge 2$ then let $\phi_j(v) =$ 'r'; in this case $v$ is the root of a star in $G[V_i, V_j]$. If $d_j(v) = 1$ then let $\phi_j(v) = q_a$ where the edge in $G[V_i, V_j]$ incident to $v$ is in the queue $q_a$, $1 \le a \le k'$. If $d_j(v) = 0$ then let $\phi_j(v)$ be some arbitrary queue $q_a$. Let the *label* of $v \in V_i$ be $\phi(v) = (\phi_1(v), \dots, \phi_{i-1}(v), \phi_{i+1}(v), \dots, \phi_c(v))$. Let $S_i$ be the set of possible labels for a vertex in $V_i$. Then $|S_i| = (k'+1)^{c-1}$. We now group the vertices with the same colour and the same label. Let $V_{i,L} = \{v \in V_i : \phi(v) = L\}$ for all labels $L \in S_i$ and $1 \le i \le c$, and consider each $V_{i,L}$ to be ordered by $\sigma$. Thus $\{V_{i,L} : 1 \le i \le c, L \in S_i\}$ is an ordered layering of $G$. We denote the $j^{\text{th}}$ label of $L \in S_i$ by $L[j]$.

Consider a subgraph $G[V_{i,P}, V_{j,Q}]$ for some $1 \le i < j \le c$ and labels $P \in S_i$ and $Q \in S_j$. We claim that all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. If $P[j] =$ 'r' and $Q[i] =$ 'r' then $G[V_{i,P}, V_{j,Q}]$ has no edges. If $P[j] =$ 'r' and $Q[i] = q_a$ for some queue $q_a$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are in $q_a$. Similarly, if $Q[i] =$ 'r' and $P[j] = q_a$ for some queue $q_a$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are in $q_a$. Finally, consider the case in which $P[j] = q_a$ and $Q[i] = q_b$ for some queues $q_a$ and $q_b$. If $a \ne b$ then there are no edges in $G[V_{i,P}, V_{j,Q}]$, and if $a = b$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are in queue $q_a (= q_b)$. In each case, all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. By Lemma 2, $V_{i,P}$ and $V_{j,Q}$ form an ordered 2-layering of $G[V_{i,P}, V_{j,Q}]$ with no X-crossing. In general, $\{V_{i,L} : 1 \le i \le c, L \in S_i\}$ is an ordered layering of $G$ with no X-crossing and $c(k'+1)^{c-1} = c(2(c-1)k+1)^{c-1}$ layers. $\square$

We now prove the main result.

*Proof of Theorem 1.* Let $G$ be an $n$-vertex graph in some proper minor-closed family of graphs. Nešetřil and Ossona de Mendez [23] prove that $\chi_{\mathsf{st}}(G) \le c$ for some constant $c$. Let $F(n)$ be a set of functions closed under taking polynomials. If $G$ has a $F(n) \times F(n) \times O(n)$ three-dimensional drawing then, by Lemma 1, $G$ has an ordered $k$-layering with no X-crossing, for some $k \in F(n)$. By Lemma 3, $G$ has queuenumber at most $\lfloor \frac{k^2}{4} \rfloor + 1 \in F(n)$. Conversely, if $G$ has queuenumber $k \in F(n)$ then, by Lemma 4, $G$ has an ordered $t$-layering with no X-crossing, where $t \le c(2(c-1)k+1)^{c-1} \in F(n)$. By Lemma 1, $G$ has a $F(n) \times F(n) \times O(n)$ three-dimensional drawing. $\square$

# 3 Conclusion

Let us return to the question that motivated this research. That is, does every planar graph have a three-dimensional drawing with linear volume? By Theorem 1, this question has an affirmative answer if every planar graph has $O(1)$ queuenumber. In 1992, Heath and Rosenberg [21] and Heath *et al.* [17] conjectured that every planar graph *does* have $O(1)$ queuenumber. Perhaps this conjecture was motivated by the desire to match the celebrated result of Yannakakis [32] that planar graphs have stacknumber at most 4.

More recently, Pemmaraju [25] provided 'evidence' that the planar graph obtained by repeated stellation of $K_3$ (that is, by adding a degree 3 vertex to every face) has non-constant queuenumber. This graph does have $O(\log n)$ queuenumber [25]. Pemmaraju [25] and Heath [private communication, 2002] conjecture that every planar graph has $O(\log n)$ queuenumber. By Theorem 1, this conjecture would imply that every planar graph has a three-dimensional drawing with $O(n \operatorname{polylog} n)$ volume.

Corollary 1 states that every series-parallel graph has a three-dimensional drawing with $O(n)$ volume. However the constant in this $O(n)$ bound is at least $6(2 \cdot 5 \cdot 3 + 1)^5 \approx 1.7 \times 10^8$. (Fertin *et al.* [13] prove a tight bound of $\chi_{\text{st}}(G) \leq 6$ for 2-trees $G$.) It is an interesting open problem to determine if a linear volume bound with a smaller constant can be achieved.


## Acknowledgements

Thanks to Prosenjit Bose, Jurek Czyzowicz, Hubert de Fraysseiz, Patrice Ossona de Mendez, and Pat Morin for helpful discussions.


## References


[1] N. ALON, C. MCDIARMID, AND B. REED, Acyclic coloring of graphs. *Random Structures Algorithms*, **2(3)**:277–288, 1991.

[2] N. ALON, B. MOHAR, AND D. P. SANDERS, On acyclic colorings of graphs on surfaces. *Israel J. Math.*, **94**:273–283, 1996.

[3] H. L. BODLAENDER, A partial $k$-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, **209(1-2)**:1–45, 1998.

[4] O. V. BORODIN, A. V. KOSTOCHKA, A. RASPAUD, AND E. SOPENA, Acyclic colouring of 1-planar graphs. *Discrete Appl. Math.*, **114(1-3)**:29–41, 2001.

[5] T. CALAMONERI AND A. STERBINI, 3D straight-line grid drawing of 4-colorable graphs. *Inform. Process. Lett.*, **63(2)**:97–102, 1997.

[6] R. F. COHEN, P. EADES, T. LIN, AND F. RUSKEY, Three-dimensional graph drawing. *Algorithmica*, **17(2)**:199–208, 1996.

[7] H. DE FRAYSSEIX, J. PACH, AND R. POLLACK, How to draw a planar graph on a grid. *Combinatorica*, **10(1)**:41–51, 1990.

[8] E. DI GIACOMO, G. LIOTTA, AND S. WISMATH, Drawing series-parallel graphs on a box. In S. WISMATH, ed., *Proc. 14th Canadian Conf. on Computational Geometry (CCCG '02)*, The University of Lethbridge, Canada, to appear.

[9] J. DÍAZ, J. PETIT, AND M. SERNA, A survey of graph layout problems. *ACM Comput. Surveys*, to appear.

[10] R. P. DILWORTH, A decomposition theorem for partially ordered sets. *Ann. of Math. (2)*, **51**:161–166, 1950.

[11] V. DUJMOVIĆ, P. MORIN, AND D. R. WOOD, Pathwidth and three-dimensional straight-line grid drawings of graphs. In M. GOODRICH, ed., *Proc. 10th International Symp. on Graph Drawing (GD '02)*, Lecture Notes in Comput. Sci., Springer, to appear.

[12] S. FELSNER, S. WISMATH, AND G. LIOTTA, Straight-line drawings on restricted integer grids in two and three dimensions. In [22], pp. 328–342.

[13] G. FERTIN, A. RASPAUD, AND B. REED, On star coloring of graphs. In A. BRANSTÄDT AND V. B. LE, eds., *Proc. 27th International Workshop on Graph-Theoretic Concepts in Computer Science (WG '01)*, vol. 2204 of *Lecture Notes in Comput. Sci.*, pp. 140–153, Springer, 2001.

[14] J. L. GANLEY AND L. S. HEATH, The pagenumber of $k$-trees is $O(k)$. *Discrete Appl. Math.*, **109(3)**:215–221, 2001.

[15] M. R. GAREY, D. S. JOHNSON, G. L. MILLER, AND C. H. PAPADIMITRIOU, The complexity of coloring circular arcs and chords. *SIAM J. Algebraic Discrete Methods*, **1(2)**:216–227, 1980.

[16] F. HARARY AND A. SCHWENK, A new crossing number for bipartite graphs. *Utilitas Math.*, **1**:203–209, 1972.

[17] L. S. HEATH, F. T. LEIGHTON, AND A. L. ROSENBERG, Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, **5(3)**:398–412, 1992.

[18] L. S. HEATH AND S. V. PEMMARAJU, Stack and queue layouts of posets. *SIAM J. Discrete Math.*, **10(4)**:599–625, 1997.

[19] L. S. HEATH AND S. V. PEMMARAJU, Stack and queue layouts of directed acyclic graphs. II. *SIAM J. Comput.*, **28(5)**:1588–1626, 1999.

[20] L. S. HEATH, S. V. PEMMARAJU, AND A. N. TRENK, Stack and queue layouts of directed acyclic graphs. I. *SIAM J. Comput.*, **28(4)**:1510–1539, 1999.

[21] L. S. HEATH AND A. L. ROSENBERG, Laying out graphs using queues. *SIAM J. Comput.*, **21(5)**:927–958, 1992.

[22] P. MUTZEL, M. JÜNGER, AND S. LEIPERT, eds., *Proc. 9th International Symp. on Graph Drawing (GD '01)*, vol. 2265 of *Lecture Notes in Comput. Sci.*, Springer, 2002.

[23] J. NEŠETŘIL AND P. OSSONA DE MENDEZ, Colorings and homomorphisms of minor closed classes. Tech. Rep. 2001-025, Institut Teoretické Informatiky, Universita Karlova v Praze, Czech Republic, 2001.

[24] J. PACH, T. THIELE, AND G. TÓTH, Three-dimensional grid drawings of graphs. In G. DI BATTISTA, ed., *Proc. 5th International Symp. on Graph Drawing (GD '97)*, vol. 1353 of *Lecture Notes in Comput. Sci.*, pp. 47–51, Springer, 1998.

[25] S. V. PEMMARAJU, *Exploring the Powers of Stacks and Queues via Graph Layouts*. Ph.D. thesis, Virginia Polytechnic Institute and State University, Virginia, U.S.A., 1992.

[26] T. PORANEN, A new algorithm for drawing series-parallel digraphs in 3D. Tech. Rep. A-2000-16, Dept. of Computer and Information Sciences, University of Tampere, Finland, 2000.

[27] S. RENGARAJAN AND C. E. VENI MADHAVAN, Stack and queue number of 2-trees. In D. DING-ZHU AND L. MING, eds., *Proc. 1st Annual International Conf. on Computing and Combinatorics (COCOON '95)*, vol. 959 of *Lecture Notes in Comput. Sci.*, pp. 203–212, Springer, 1995.

[28] W. SCHNYDER, Planar graphs and poset dimension. *Order*, **5(4)**:323–343, 1989.

[29] F. SHAHROKHI AND W. SHI, On crossing sets, disjoint sets, and pagenumber. *J. Algorithms*, **34(1)**:40–53, 2000.

[30] M. M. SYSŁO, Characterizations of outerplanar graphs. *Discrete Math.*, **26(1)**:47–53, 1979.

[31] D. R. WOOD, Bounded degree book embeddings and three-dimensional orthogonal graph drawing. In [22], pp. 312–327.

[32] M. YANNAKAKIS, Embedding planar graphs in four pages. *J. Comput. System Sci.*, **38**:36–67, 1986.