Star Colourings of Subdivisions

David R. Wood*

March 9, 2003

Abstract

Let G be a graph with chromatic number $\chi(G)$ and maximum degree $\Delta(G)$. A *star colouring* of G is a function that assigns a colour to each vertex such that adjacent vertices receive distinct colours, and there is no bichromatic 4-vertex path. The *star chromatic number* $\chi_{\mathsf{st}}(G)$ is the minimum number of colours in a star colouring of G. Star colourings of subdivisions of graphs are investigated. Let G' be the graph obtained from G by subdividing each edge once. Bounds on $\chi_{\mathsf{st}}(G')$ in terms of $\chi(G)$ and $\Delta(G)$ are proved. In particular, $\chi_{\mathsf{st}}(G') \leq \max\{\chi(G), 3\}$ and $\chi_{\mathsf{st}}(G') \leq \sqrt{\Delta(G)} + 3$. Furthermore, if $\chi_{\mathsf{st}}(G') \leq k$ then $\chi(G) \leq k \cdot 2^{2k}$. Hence $\chi_{\mathsf{st}}(G')$ is tied to $\chi(G)$. On the other hand, a graph H obtained from G by subdividing each edge at least twice has $\chi_{\mathsf{st}}(H) \leq 4$.

Keywords: graph colouring, star colouring, star chromatic number, subdivision.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The maximum degree of G is denoted by $\Delta(G)$, and the subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$.

A *vertex colouring* of G is a function that assigns a colour to every vertex of G such that adjacent vertices receive distinct colours. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colours in a colouring of G. An *edge colouring* of G is a function that assigns a colour to every edge of G such that edges with a vertex in common receive distinct colours. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colours in an edge colouring of G.

A vertex colouring of G is a *star colouring* if there is no bichromatic 4-vertex path; that is, each bichromatic subgraph is a union of disjoint stars. The *star chromatic number* of G, denoted by $\chi_{\mathsf{st}}(G)$, is the minimum number of colours in a star colouring of G. Star colourings have been investigated in [3–5, 8, 12] for example, and are closely related to acyclic colourings and oriented colourings of graphs. A general result by Nešetřil

*School of Computer Science, Carleton University, Ottawa, Canada. Technical Report TR-2003-02. Research supported by NSERC. E-mail: `davidw@scs.carleton.ca`.

and Ossona de Mendez [12] states that $\chi_{\mathsf{st}}(G)$ is at most a quadratic function of the maximum chromatic number of a minor of G.

A *subdivision* of G is a graph obtained from G by replacing each edge by a path of at least one edge. The vertices of a subdivision of G corresponding to vertices of G are said to be *original* vertices. The remaining vertices are called *division* vertices. The subdivision of G obtained by replacing each edge vw by a 3-vertex path (v, x, w) is denoted by G'. The main results of this paper establish bounds on the star chromatic number of G' in terms of $\chi(G)$ and $\Delta(G)$. These results are presented in Sections 2 and 3, respectively. Section 4 presents bounds on the star chromatic number of subdivisions where each edge is replaced by a path of at least four vertices.

2 Star Colourings of G' and $\chi(G)$

Clearly $\chi(G') \leq 2$ for every graph G. We now relate $\chi_{\mathsf{st}}(G')$ and $\chi(G)$.

Theorem 1. *For every graph G, $\chi_{\mathsf{st}}(G') \leq \max\{\chi(G), 3\}$.*

Proof. Consider a vertex colouring of G with $\chi(G)$ colours. Define a vertex colouring of G' in which each original vertex inherits its colour from G. If $\chi(G) \leq 2$ then let all the division vertices receive one new colour. Otherwise (if $\chi(G) \geq 3$), for each division vertex, choose one of the $\chi(G)$ colours different from the two colours assigned to its two neighbours. A 4-vertex path in G' contains a trichromatic path (v, x, w), where x is the division vertex of the edge vw. Thus G' has a star colouring with $\max\{\chi(G), 3\}$ colours. □

Theorem 1 gives an upper bound on $\chi_{\mathsf{st}}(G')$ in terms of $\chi(G)$. We now prove a corresponding lower bound.

Theorem 2. *For every graph G, if $\chi_{\mathsf{st}}(G') \leq k$ then $\chi(G) \leq k \cdot 2^{2k}$.*

Proof. Arbitrarily orient the edges of G, and consider a star colouring of G' with k colours. Denote by $\mathsf{col}(u)$ the colour assigned to each vertex u of G'. For each edge $e \in E(G)$, denote by x_e the division vertex of e in G'. For each vertex v of G, let

$$S_v^+ = \big\{(i, +) : e = \overrightarrow{vw} \in E(G), \mathsf{col}(x_e) = i\big\}, \text{ and}$$
$$S_v^- = \big\{(i, -) : e = \overrightarrow{wv} \in E(G), \mathsf{col}(x_e) = i\big\} \ .$$

Colour each vertex v of G by the triple $(\mathsf{col}(v), S_v^+, S_v^-)$. We claim that adjacent vertices in G receive distinct colours. Suppose that $e = \overrightarrow{vw} \in E(G)$, and v and w receive the same colour. Thus $\mathsf{col}(v) = \mathsf{col}(w)$, $S_v^+ = S_w^+$, and $S_v^- = S_w^-$. Let $i = \mathsf{col}(x_e)$. Thus $(i, +) \in S_v^+$ and $(i, -) \in S_w^-$, which implies that $(i, -) \in S_v^-$. That is, there is an edge $f = \overrightarrow{uv} \in E(G)$ such that $\mathsf{col}(x_f) = i$. However this implies that (x_f, v, x_e, w) is a bichromatic 4-vertex path in G', which is a contradiction. (In fact, there is a bichromatic 5-vertex path in G'.) Thus G has a (proper) vertex colouring with at most $k \cdot 2^{2k}$ colours. □

Theorems 1 and 2 imply that $\chi_{\mathsf{st}}(G')$ is tied to $\chi(G)$.

3 Star Colourings of G' and $\Delta(G)$

By Brooks' Theorem [6] and Theorem 1, $\chi_{\mathsf{st}}(G') \leq \max\{\Delta(G), 3\}$ unless G is a complete graph. However, a stronger bound on $\chi_{\mathsf{st}}(G')$ in terms of $\Delta(G)$ is possible.

Theorem 3. *For every graph G, $\chi_{\mathsf{st}}(G') \leq \sqrt{\Delta(G)} + 3$.*

To prove Theorem 3 we use the following lemma.

Lemma 1. *Let G be a graph. Suppose that $V(G)$ has a partition $\{V_1, V_2, \dots, V_k\}$ such that the maximum degree $\Delta(G[V_i]) \leq d$ for all i, $1 \leq i \leq k$. Then $\chi_{\mathsf{st}}(G') \leq \max\{k+1, d+3\}$, and $\chi_{\mathsf{st}}(G') \leq \max\{k+1, 3\}$ if $d = 1$.*

Proof. For each original vertex v of G', let $\mathsf{col}(v) = i-1$ where $v \in V_i$. Thus $0 \leq \mathsf{col}(v) \leq k-1$. By Vizing's Theorem [13], each induced subgraph $G[V_i]$ has an edge colouring with d' colours where $d \leq d' \leq d+1$. If $d = 1$ then we can take $d' = 1$. Let the colours in the edge colouring of each $G[V_i]$ be $\{1, 2, \dots, d'\}$. Define $m = \max\{k, d'+1\}$. Consider an edge vw of G with division vertex x. If $\mathsf{col}(v) = \mathsf{col}(w) = i$, then let $\mathsf{col}(x) = (i + j) \bmod m$, where j is the colour of vw. Since $m > j \geq 1$, $\mathsf{col}(x) \in \{0, 1, \dots, m-1\} \setminus \{i\}$. Hence x is coloured differently from both of its neighbours. If $\mathsf{col}(v) \neq \mathsf{col}(w)$ then let $\mathsf{col}(x) = m$. We thus have a (proper) vertex colouring of G'.

Let $P = (v, x, w, y)$ be a 4-vertex path in G'. Without loss of generality, x is the division vertex of the edge vw, and y is the division vertex of some edge wu. If $\mathsf{col}(v) = \mathsf{col}(w) = \mathsf{col}(u)$ then $\mathsf{col}(x) \neq \mathsf{col}(y)$ since the edge colours of vw and wu are distinct, and thus P is not bichromatic. If $\mathsf{col}(v) = \mathsf{col}(w) \neq \mathsf{col}(u)$ then $0 \leq \mathsf{col}(x) \leq m-1$ and $\mathsf{col}(y) = m$, and thus P is not bichromatic. If $\mathsf{col}(v) \neq \mathsf{col}(w)$ then $\mathsf{col}(x) = m$, and P is not bichromatic. Therefore G' has star colouring with $m + 1 = \max\{k+1, d'+2\}$ colours. □

Consider the following result by Lovász [11] (see [1, 2, 7, 9, 10] for related work).

Lemma 2. [11] *Let G be a graph. For all non-negative integers $d_1, d_2, \dots, d_k$ such that $\sum_{i=1}^{k} d_i = \Delta(G) - k + 1$, there is a partition $\{V_1, V_2, \dots, V_k\}$ of $V(G)$ such that $\Delta(G[V_i]) \leq d_i$ for all i, $1 \leq i \leq k$.*

Proof of Theorem 3. Let $\Delta = \Delta(G)$ and $k = \lceil \sqrt{\Delta} \rceil$. Let $d_1, d_2, \dots, d_k$ be non-negative integers such that $\sum_{i=1}^{k} d_i = \Delta - k + 1$, and each $d_i \in \{\lfloor \frac{\Delta-k+1}{k} \rfloor, \lceil \frac{\Delta-k+1}{k} \rceil\}$. Apply Lemma 2 to obtain a partition $\{V_1, V_2, \dots, V_k\}$ of $V(G)$ such that $\Delta(G[V_i]) \leq d_i \leq \lceil \frac{\Delta-k+1}{k} \rceil \leq \frac{\Delta}{k} \leq \sqrt{\Delta}$ for all i, $1 \leq i \leq k$. By Lemma 1, $\chi_{\mathsf{st}}(G') \leq \max\{\lceil \sqrt{\Delta} \rceil + 1, \sqrt{\Delta} + 3\} \leq \sqrt{\Delta} + 3$. □

We now prove that for complete graphs, Theorem 3 is tight up to the additive constant.

Theorem 4. *For every n, $\chi_{\mathsf{st}}(K'_n) \geq \sqrt{n} > \sqrt{\Delta(K_n)}$.*

To prove Theorem 4 we use the following lemma.

Lemma 3. *For every graph G, the minimum number of colours in a star colouring of G' such that the original vertices are monochromatic is $\chi'(G) + 1$.*

Proof. Given an edge colouring of G, transfer the colour from each edge to the corresponding division vertex, and colour all of the original vertices with a new colour. Let $P = (v, x, w, y)$ be a 4-vertex path of G'. Without loss of generality, x is the division vertex of the edge vw, and y is the division vertex of some edge wu. In the edge colouring, vw and wu receive distinct colours. Hence x and y receive distinct colours, and P is not bichromatic. Thus G' has a star colouring with $\chi'(G) + 1$ colours such that the original vertices are monochromatic.

Consider a star colouring of G' with k colours such that the original vertices are monochromatic. No division vertex can receive this colour, otherwise it is not a vertex colouring. For all edges of G with a vertex in common, the corresponding division vertices receive distinct colours, otherwise there is a bichromatic 5-vertex path in G'. Transferring the colour from each division vertex of G' to the corresponding edge of G, we obtain a edge colouring of G with $k - 1$ colours. □

Proof of Theorem 4. Consider a star colouring of K'_n with $k = \chi_{\mathsf{st}}(K'_n)$ colours. There are at least $p = \lceil \frac{n}{k} \rceil$ monochromatic original vertices. Let K'_p be the subgraph of K'_n formed by these p vertices, along with the subdivided edges between them. By Lemma 3, the number of distinct colours assigned to the division vertices of K'_p is at least $\chi'(K_p) = p - 1$. Thus $k \geq p \geq \frac{n}{k}$, and hence $\chi_{\mathsf{st}}(K'_n) = k \geq \sqrt{n}$. □

4 Star Colourings of Large Subdivisions

We now consider star colourings of subdivisions other than G'.

Lemma 4. *Let H be a subdivision of a graph G such that for every edge vw of G, for some $k \geq 4$ with $k \neq 6$, vw is replaced by a k-vertex path in H. Then $\chi_{\mathsf{st}}(H) \leq 3$, and $\chi_{\mathsf{st}}(H) = 3$ if $E(G) \neq \emptyset$.*

Proof. Let the colour of each original vertex be 2. Orient each edge of G arbitrarily. Suppose that the edge $\overrightarrow{vw}$ of G is replaced by the k-vertex path $P = (v, x_0, x_1, \dots, x_{k-3}, w)$.

Case 1. $k \equiv 0 \pmod 3$ and $k \neq 6$: Let $\mathsf{col}(x_i) = i \bmod 3$ for all i, $0 \leq i \leq k - 6$. Let $\mathsf{col}(x_{k-5}) = 2$, $\mathsf{col}(x_{k-4}) = 1$, and $\mathsf{col}(x_{k-3}) = 0$. Hence P is coloured $(2, 012, 012, \dots, 012, 0, 210, 2)$.

Case 2. $k \equiv 1 \pmod 3$: Let $\mathsf{col}(x_i) = i \bmod 3$ for all i, $0 \leq i \leq k - 5$. Let $\mathsf{col}(x_{k-4}) = 1$ and $\mathsf{col}(x_{k-3}) = 0$. Hence P is coloured $(2, 012, 012, \dots, 012, 10, 2)$.

Case 3. $k \equiv 2 \pmod 3$: Let $\mathsf{col}(x_i) = i \bmod 3$ for all i, $0 \leq i \leq k - 4$. Let $\mathsf{col}(x_{k-3}) = 0$. Hence P is coloured $(2, 012, 012, \dots, 012, 01, 0, 2)$.

If Q is a 4-vertex path in H with at least two original vertices then $Q = (v, x_0, x_1, w)$, where Q replaced an edge $\overrightarrow{vw}$ of G, and by Case 2 with $k = 4$, Q is coloured $(2, 1, 0, 2)$, and is thus not bichromatic.

If the edge $\overrightarrow{vw}$ of G is replaced by the path $(v, x_0, x_1, \dots, x_{k-3}, w)$, then the subpaths (v, x_0, x_1) and (w, x_{k-3}, x_{k-2}) are trichromatic. (This is not the case if $k = 6$.) Thus a 4-vertex path containing exactly one original vertex is not bichromatic.

The case-analysis above shows that there is no bichromatic 4-vertex path with no original vertex. Thus there is no bichromatic 4-vertex path in H. Therefore $\chi_{\mathsf{st}}(H) \leq 3$. A graph admitting a star colouring with two colours is a union of disjoint stars, and H is not the union of disjoint stars, unless $E(G) = \emptyset$. Thus $\chi_{\mathsf{st}}(H) = 3$. □

Let H be a subdivision of a graph G such that every edge of G is replaced in H by a path with at least four vertices. In the proof of Lemma 4, the only obstruction to H having a star colouring with three colours is an edge $\overrightarrow{vw}$ of G that is replaced in H by a 6-vertex path $P = (v, x_0, x_1, x_2, x_3, w)$. In this case we introduce a fourth colour, and P can be coloured $(2, 0, 1, 3, 0, 2)$. The following result is easily obtained.

Lemma 5. *Let H be a subdivision of a graph G such that every edge of G is replaced in H by a path with at least four vertices. Then $\chi_{\mathsf{st}}(H) \leq 4$.*

Let G'' be the subdivision of G with every edge of G replaced by a 4-vertex path. Every cycle in G'' is odd. Thus $\chi(G'') \leq 2$ if and only if G is a forest. If G contains a cycle then $\chi(G'') = \chi_{\mathsf{st}}(G'') = 3$. This provides an infinite family of graphs for which the chromatic number and star chromatic number coincide.

References

[1] C. BERNARDI, On a theorem about vertex colorings of graphs. *Discrete Math.*, **64(1)**:95–96, 1987.

[2] B. BOLLOBÁS AND B. MANVEL, Optimal vertex partitions. *Bull. London Math. Soc.*, **11(2)**:113–116, 1979.

[3] O. V. BORODIN, On acyclic colorings of planar graphs. *Discrete Math.*, **25(3)**:211–236, 1979.

[4] O. V. BORODIN, A. V. KOSTOCHKA, A. RASPAUD, AND È. SOPENA, Acyclic k-strong coloring of maps on surfaces. *Mat. Zametki*, **67(1)**:36–45, 2000.

[5] O. V. BORODIN, A. V. KOSTOCHKA, AND D. R. WOODALL, Acyclic colourings of planar graphs with large girth. *J. London Math. Soc. (2)*, **60(2)**:344–352, 1999.

[6] R. L. BROOKS, On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, **37**:194–197, 1941.

[7] V. CHVÁTAL, Remark on a paper of Lovász. *Comment. Math. Univ. Carolinae*, **9**:47–50, 1968.

[8] G. FERTIN, A. RASPAUD, AND B. REED, On star coloring of graphs. In A. BRANSTÄDT AND V. B. LE, eds., *Proc. 27th International Workshop on Graph-Theoretic Concepts in Computer Science (WG '01)*, vol. 2204 of *Lecture Notes in Comput. Sci.*, pp. 140–153, Springer, 2001.

[9] L. GERENCSÉR, On coloring problems. *Mat. Lapok*, **16**:274–277, 1965.

[10] J. LAWRENCE, Covering the vertex set of a graph with subgraphs of smaller degree. *Discrete Math.*, **21(1)**:61–68, 1978.

[11] L. LOVÁSZ, On decomposition of graphs. *Studia Sci. Math. Hungar.*, **1**:237–238, 1966.

[12] J. NEŠETŘIL AND P. OSSONA DE MENDEZ, Colorings and homomorphisms of minor closed classes. Tech. Rep. 2001-025, Institut Teoretické Informatiky, Universita Karlova v Praze, Czech Republic, 2001.

[13] V. G. VIZING, On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, **3**:25–30, 1964.