Grid Drawings of k-Colourable Graphs*

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Abstract

It is proved that every k-colourable graph on n vertices has a grid drawing with $\mathcal{O}(kn)$ area, and that this bound is best possible. This result can be viewed as a generalisation of the no-three-in-line problem. A further area bound is established that includes the aspect ratio as a parameter.

Keywords: graph drawing, grid drawing, no-three-in-line problem, area, aspect ratio.

1 Introduction

Let $G = (V, E)$ be a graph. All graphs considered are simple, finite and undirected. A *grid drawing* of G is an injective mapping $\theta : V \to \mathbb{Z}^2$ such that for all edges $vw \in E$ and vertices $x \in V$, $\theta(x) \in \overline{\theta(v)\theta(w)}$ implies that $x = v$ or $x = w$, where $\overline{ab}$ denotes the line-segment with endpoints a and b. That is, a grid drawing of a graph represents each vertex by a distinct gridpoint in the plane, and each edge by a line-segment between its endpoints, such that the only vertices an edge intersects are its own endpoints. Let θ be a grid drawing of a graph $G = (V, E)$ such that $\theta(v) = (X(v), Y(v))$ for all vertices $v \in V$. If $X(u) - X(v) + 1 \leq w$ and $Y(u) - Y(v) + 1 \leq h$ for all vertices $u, v \in V$, then θ is a $w \times h$ grid drawing with *area* wh and *aspect ratio* $\max\{w, h\}/\min\{w, h\}$.

This paper studies grid drawings with small area, and with small aspect ratio as a secondary criterion. Minimising the area and aspect ratio are important considerations in graph visualisation for example [2]. Obviously to view a graph drawing with good resolution on a computer screen (which itself has fixed aspect ratio) requires that the area and aspect ratio be small.

A *k-colouring* of a graph $G = (V, E)$ is a partition of V into *colour classes* $V_0, V_1, \ldots, V_{k-1}$ such that for every edge $vw \in E$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. A graph admitting a k-colouring is *k-colourable*. A *complete k-partite* graph is a k-colourable graph such that each colour class is not empty and there is an edge between every pair of bichromatic vertices. A complete k-partite graph is *balanced* if every colour class has the same number of vertices. Let $K(t, k)$ denote the balanced complete k-partite graph with t vertices in each colour class.

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2 Results

Theorem 1. *For all $k \geq 1$ and $t \geq 1$, the balanced complete k-partite graph $K(t,k)$ has a $k \times pt$ grid drawing, where p is the minimum prime such that $p \geq k$.*

Proof. Let $V_0, V_1, \ldots, V_{k-1}$ be the k-colouring of $K(t,k)$. For each $0 \leq i \leq k-1$, let $V_i = \{v_{i,0}, v_{i,1}, \ldots, v_{i,t-1}\}$, and for each $0 \leq j \leq t-1$, let $\theta(v_{i,j}) = (i, pj + (i^2 \bmod p))$. If an edge intersects a vertex other than its endpoints then the three vertices are collinear. Since the vertices in each V_i are positioned in the $X = i$ line, to prove that θ is a valid grid drawing, it suffices to prove that any three vertices from distinct colour classes are not collinear. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0 \ .$$

For vertices v_{i_1,j_1}, v_{i_2,j_2}, and v_{i_3,j_3} from distinct colour classes we have

$$\begin{vmatrix} 1 & i_1 & pj_1 + (i_1^2 \bmod p) \\ 1 & i_2 & pj_2 + (i_2^2 \bmod p) \\ 1 & i_3 & pj_3 + (i_3^2 \bmod p) \end{vmatrix} \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} \equiv (i_1 - i_2)(i_1 - i_3)(i_2 - i_3) \pmod{p} \ ,$$

which is nonzero since p is a prime and $1 \leq i_\alpha - i_\beta \leq k-1 \leq p-1$ for all $1 \leq \alpha < \beta \leq 3$. Thus v_{i_1,j_1}, v_{i_2,j_2} and v_{i_3,j_3} are not collinear. Therefore the only vertices an edge intersects are its own endpoints, and θ is a valid grid drawing of $K(t,k)$. For every vertex v, $0 \leq X(v) \leq k-1$ and $0 \leq Y(v) \leq p(t-1) + (p-1)$. Thus the drawing is $k \times tp$. □

By Bertrand's Postulate and the Prime Number Theorem we have the following corollary of Theorem 1.

Corollary 1. *For all $k \geq 1$ and $t \geq 1$, the balanced complete k-partite graph $K(t,k)$ on $n = kt$ vertices has a $k \times 2n$ grid drawing. For all $\epsilon > 0$, there exists k_ϵ such that for all $k \geq k_\epsilon$ and $t \geq 1$, $K(t,k)$ has a $k \times (1+\epsilon)n$ grid drawing.* □

We now prove that the upper bound in Theorem 1 is asymptotically optimal.

Theorem 2. *Every grid drawing of $K(k,t)$ has area at least $\frac{1}{4}k^2 t = \frac{1}{4}kn$.*

Proof. Consider a $w \times h$ grid drawing of $K(t,k)$. Let the *y-row* be the set of vertices with a Y-coordinate of y, and the *x-column* be the set of vertices with an X-coordinate of x. For each colour $0 \leq i \leq k-1$, let r_i be the number of rows containing a vertex coloured i, and let c_i be the number of columns containing a vertex coloured i. Then the arithmetic and harmonic means of $\{c_i : 0 \leq i \leq k-1\}$ satisfy the following (see for example [1]):

$$\left(\frac{1}{k}\sum_i c_i\right)\left(\frac{1}{k}\sum_i \frac{1}{c_i}\right) \geq 1 \ .$$

Clearly $t \leq c_i r_i$ for each $0 \leq i \leq k-1$. Thus $\frac{1}{c_i} \leq \frac{r_i}{t}$, and

$$\left(\sum_i c_i\right)\left(\sum_i r_i\right) \geq k^2 t \ .$$

In each row and column there is at most two distinct colours, as otherwise there would be 3-cycle contained in that row or column. Hence $\sum_i c_i \leq 2w$ and $\sum_i r_i \leq 2h$, which implies that $4wh \geq k^2 t$, and the area $wh \geq \frac{1}{4}k^2 t$. □

In the following result we generalise Theorem 1 for arbitrary k-colourable graphs, and introduce the aspect ratio as a parameter. This result suggests a trade-off between small area and small aspect ratio.

Theorem 3. *Let G be a k-colourable graph with n vertices. For every integer r such that $1 \leq r \leq \frac{n}{k}$, G has a $\frac{2n}{r} \times 4n$ grid drawing, which has area $\frac{8n^2}{r}$ and aspect ratio $2r$.*

Proof. Consider a k-colouring of G. Partition each colour class into sets each with exactly r vertices except for one set with at most r vertices. There are at most $\frac{n}{r}$ sets of size r, and at most k smaller sets, one for each colour class. Since $r \leq \frac{n}{k}$, the total number of sets is at most $\frac{2n}{r}$. Thus we have a $\lfloor \frac{2n}{r} \rfloor$-colouring of G such that each colour class has at most r vertices. Hence G is a subgraph of $K(r, \lfloor \frac{2n}{r} \rfloor)$, and by Corollary 1, G has a $\frac{2n}{r} \times 4n$ grid drawing. $\square$

Observe that with $r = \lfloor \frac{n}{k} \rfloor$ the drawing in Theorem 3 is $\mathcal{O}(k) \times \mathcal{O}(n)$ with area $\mathcal{O}(kn)$.

3 Conclusion

We conclude with some bibliographic remarks and conjectures. Note that a number of ideas in the proofs of Theorems 1 and 3 are from results by Pach *et al.* [6] and Dujmović *et al.* [4] regarding three-dimensional grid drawings (with no crossings). In turn, these proofs date to the seminal construction by Erdös [5] for the no-three-in-line problem. This problem introduced in 1917 by Dudeney [3] asks, what is the maximum number of points in the $n \times n$ grid with no three points collinear? Clearly θ is a grid drawing of a complete graph $K_n = (V, E)$ if and only if $\{\theta(v) : v \in V\}$ is a set of gridpoints with no three collinear. Thus the problem of producing a grid drawing with small area for any given graph can be viewed as a generalisation of the no-three-in-line problem. Note that Theorem 1 applied to a complete graph produces the no-three-in-line construction of Erdös [5].

Conjecture 1. The lower bound in Theorem 2 can be improved to $\frac{1}{2}kn$. (This is clearly the minimum area for a grid drawing of the balanced complete bipartite graph $K(\frac{n}{2}, 2)$.)

Conjecture 2. Every grid drawing of *any* complete k-partite graph with n vertices has area $\Omega(kn)$.

Conjecture 3. Every grid drawing of an n-vertex $K(k, t)$ with aspect ratio r has area $\Omega(\frac{n^2}{r})$.

Conjecture 3 would establish a trade-off between small area and small aspect ratio.

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