New Results in Graph Layout[*]

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Abstract

A *track layout* of a graph consists of a vertex colouring, an edge colouring, and a total order of each vertex colour class such that between each pair of vertex colour classes, there is no monochromatic pair of crossing edges. This paper studies track layouts and their applications to other models of graph layout. In particular, we improve on the results of Enomoto and Miyauchi [*SIAM J. Discrete Math.*, 1999] regarding stack layouts of subdivisions, and give analogous results for queue layouts. We solve open problems due to Felsner, Wismath, and Liotta [*Proc. Graph Drawing*, 2001] and Pach, Thiele, and Toth [*Proc. Graph Drawing*, 1997] concerning three-dimensional straight-line grid drawings. We initiate the study of three-dimensional polyline grid drawings and establish volume bounds within a logarithmic factor of optimal.

Keywords: graph layout, graph drawing, track layout, stack layout, queue layout, book embedding, subdivision, three-dimensional grid drawing, star colouring, star chromatic number, strong star colouring, strong star chromatic number.

Contents

1 Introduction

In its simplest form, a *track layout* of a graph consists of a proper vertex colouring and a total order of each colour class, such that between each pair of colour classes no two edges cross. (See Section 3 for a more formal definition.) Track layouts were introduced in [41, 42] although they are implicit in many previous works. In this paper, we generalise the definition of a track layout to incorporate a (non-proper) edge colouring. It is now required that between any two (vertex) colour classes, no two monochromatic edges cross. This paper studies track layouts and their applications to other models of graph layout, notably stack and queue layouts, and three-dimensional grid drawings.

First some formalities. We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. The subgraph of G induced by a set of vertices $V' \subseteq V(G)$ is denoted by $G[V']$. The spanning subgraph of G induced by a set of edges $E' \subseteq E(G)$ is denoted by $G[E']$. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A minor-closed class of graphs is *proper* if it is not the class of all graphs. The *genus* of a graph G is the minimum γ such that G can be embedded in the orientable surface with γ handles.

A *graph parameter* is a function α that assigns to every graph G a non-negative integer $\alpha(G)$. Let $\mathcal{G}$ be a class of graphs. By $\alpha(\mathcal{G})$ we denote the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum of $\alpha(G)$, taken over all n-vertex graphs $G \in \mathcal{G}$. We say $\mathcal{G}$ has *constant* α if $\alpha(\mathcal{G}) \in \mathcal{O}(1)$. A graph parameter α is *bounded by* a graph parameter β (for some class $\mathcal{G}$), if there exists a function g such that $\alpha(G) \leq g(\beta(G))$ for every graph G (in $\mathcal{G}$). If α is bounded by β and β is bounded by α then α and β are *tied*.

1.1 Stack and Queue Layouts

A *vertex ordering* of a graph G is linear order σ of $V(G)$. Let $V_1, V_2, \ldots, V_k$ be disjoint sets of vertices such that each V_i is ordered by $<_i$. Then $(V_1, V_2, \ldots, V_k)$ denotes the vertex ordering $<_\sigma$ such that $v <_\sigma w$ whenever $v \in V_i$ and $w \in V_j$ with $i < j$, or $v \in V_i$, $w \in V_i$ and $v <_i w$. Let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint such that $L(e) <_\sigma L(f)$. If $L(e) <_\sigma L(f) <_\sigma R(e) <_\sigma R(f)$ then e and f *cross*, and if $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$ then e and f *nest*, and f is *nested inside* e. A *stack* (respectively, *queue*) is a set of edges $E' \subseteq E(G)$ such that no two edges in E' cross (nest). Observe that when traversing the vertex ordering, edges in a stack (queue) appear in LIFO (FIFO) order — hence the names. A queue E' has a linear order $\preceq$, called the *queue order*, such that

$$\forall e, f \in E', \ e \preceq f \iff L(e) \leq_\sigma L(f) \text{ and } R(e) \leq_\sigma R(f) \ . \tag{1}$$

A *k-stack* (*queue*) *layout* of G consists of a vertex ordering σ of G and a partition $\{E_\ell : 1 \leq \ell \leq k\}$ of $E(G)$, such that each E_ℓ is a *stack* (*queue*) in σ. At times we write stack$(e) = \ell$ (or queue$(e) = \ell$ if $e \in E_\ell$. A graph admitting a k-stack (queue) layout is called a *k-stack* (*queue*) *graph*. The *stack-number* of a graph G, denoted by $\mathsf{sn}(G)$, is the minimum k such that G is a k-

stack graph. The *queue-number* of a graph G, denoted by $\mathsf{qn}(G)$, is the minimum k such that G is a k-queue graph. By interpreting a queue layout as a partition of the edges into sets that satisfy (1), the queue-number of a graph is a natural measure of its 'linearity'.

Stack layouts were independently introduced by Bernhart and Kainen [8] and by Cottafava and D'Antona [33]. Queue layouts were introduced by Heath *et al.* [71, 75]. Stack layouts are more commonly called *book embeddings*, and stack-number has been called *book-thickness*, *fixed outer-thickness*, and *page-number*. Applications of stack and queue layouts include sorting permutations [50, 65, 108, 119], VLSI design [30, 110], parallel process scheduling [9], complexity theory [55, 56], compact graph encodings [82, 97], compact routing tables [60], and graph drawing [10, 123–125].

The 1-stack graphs are precisely the outerplanar graphs [8]. 2-stack graphs are characterised as the subgraphs of planar Hamiltonian graphs [30], which implies that it is NP-complete to test if the stack-number of a given graph is at most two. Heath and Rosenberg [75] characterised 1-queue graphs as the 'arched levelled' planar graphs, and proved that it is NP-complete to recognise such graphs. Numerous other aspects of stack and queue layouts have been studied in the literature [11–14, 24, 25, 27, 32, 35, 44, 48, 54, 57, 58, 68–71, 75, 77, 83, 88–90, 94–96, 101, 106, 109, 111, 112, 114–117, 121, 126]. Stack and queue layouts of directed graphs and posets have also been studied [5, 6, 36, 66, 67, 72–74, 87, 100, 118].

Table 1 summarise some of the known bounds on the stack-number and queue-number of various classes of graphs. A blank entry in the table indicates that a more general result provides the best known bound for that family of graphs.

Table 1: Upper bounds on the stack-number and queue-number of graphs.

graph family	stack-number	reference	queue-number	reference
n vertices	$\lceil \frac{n}{2} \rceil$	[30]	$\lfloor \frac{n}{2} \rfloor$	[75]
m edges	$\mathcal{O}(\sqrt{m})$	[90]	$e\sqrt{m}$	Theorem 5
proper minor closed	bounded	[14]		
genus γ	$\mathcal{O}(\sqrt{\gamma})$	[89]		
path-width p	p	[121]	p	[125]
tree-width w	$w+1$	[58]	$3^w \cdot 6^{(4^w-3w-1)/9} - 1$	[42]
tree-width w, max. degree Δ			$36\Delta w$	[125]
band-width b	$b-1$	[117]	$\lceil \frac{b}{2} \rceil$	[75]
toroidal	7	[44]		
planar	4	[126]		
series-parallel	2	[109]	3	[109]
outerplanar	1	[8]	2	[71]
trees	1	[30]	1	[75]

A *subdivision* of a graph G is a graph obtained from G by replacing each edge by a path with at least one edge. Internal vertices on such a path are called *division* vertices. Let G' and G'' be the subdivisions of G with exactly one, respectively two, division vertices per edge. That every graph

has a 3-stack subdivision is a fundamental result observed by many authors [13, 45, 46, 91]. This theorem can be traced to the seminal result by Atneosen [7] that every graph has a topological embedding in a 3-page book. The best known upper bound on the number of division vertices in a 3-stack subdivision is due to Enomoto and Miyauchi [45, 46] who proved that for every $d \geq 2$, every n-vertex graph has a $(d+1)$-stack subdivision with $\mathcal{O}(\log_d n)$ division vertices per edge. Enomoto *et al.* [47] claimed that this upper bound is 'essentially best possible' by proving that every $(d+1)$-stack subdivision of the complete graph K_n has $\Omega(n^2 \log_d n)$ division vertices in total. In this paper we prove the following refinement of the upper bound of Enomoto and Miyauchi [45, 46], in which the number of division vertices per edge depends on the queue- or stack-number of the given graph. An analogous result for queue layouts is also proved.

Theorem 1. *Let G be a graph with queue-number $\mathsf{qn}(G) \leq q$ and stack-number $\mathsf{sn}(G) \leq s$. For every integer $d \geq 2$,*

(a) *G has a d-queue subdivision with $\mathcal{O}(\log_d q)$ division vertices per edge, and*

(b) *G has a $(d+1)$-stack subdivision with $\mathcal{O}(\log_d \min\{q, s\})$ division vertices per edge.*

The proof of Theorem 1 can be found in Theorems 7, 9 and 10. Unfortunately we have not been able to prove the missing component of Theorem 1 that every s-stack graph has 2-queue subdivision with $f(s)$ division vertices per edge for some function f.

1.2 Three-Dimensional Grid Drawings

A *three-dimensional polyline grid drawing* of a graph, henceforth called a *polyline drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$ (called *gridpoints*), and represents each edge as a polyline between its endpoints with bends (if any) also at gridpoints, such that distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. A polyline drawing with at most b bends per edge is called a *b-bend drawing*. A 0-bend drawing is called a *straight-line drawing*. Of course, a b-bend polyline drawing of a graph G is precisely a straight-line drawing of a subdivision of G with at most b division vertices per edge.

In contrast to the case in the plane, it is well known that every graph has a three-dimensional straight-line drawing. We therefore are interested in optimising certain measures of the aesthetic quality of such drawings. The *bounding box* of a polyline drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ polyline drawing with *volume $X \cdot Y \cdot Z$*. That is, the volume of a polyline drawing is the number of gridpoints in the bounding box. This definition is formulated so that two-dimensional drawings have positive volume. We are interested in polyline drawings with small volume. The volume of straight-line drawings has been widely studied [21, 26, 31, 37, 41, 42, 51, 103, 107, 125]. Three-dimensional graph drawings in which the vertices are allowed real coordinates have also been studied [23, 28, 29, 34, 43, 59, 78–81, 93, 102]. Aesthetic criteria besides volume that have been considered include symmetry [78–81], aspect ratio [29, 59], angular resolution [29, 59], edge-separation [29, 59], and convexity [28, 29, 43]. Table 2 summarises the best known upper bounds on the volume of polyline drawings, including those established in this paper.

Table 2: Volume of polyline drawings of graphs with n vertices and $m \geq n$ edges.

graph family	bends per edge	volume	reference
arbitrary	0	$\mathcal{O}(n^3)$	[31]
arbitrary	0	$\mathcal{O}(m^{4/3}n)$	Theorem 13
maximum degree Δ	0	$\mathcal{O}(\Delta mn)$	Theorem 12
maximum degree Δ	0	$\mathcal{O}(\Delta^{15/2}m^{1/2}n)$	Theorem 19
c-colourable	0	$\mathcal{O}(c^2 n^2)$	[103]
c-colourable	0	$\mathcal{O}(c^6 m^{2/3}n)$	Theorem 15
K_h-minor free	0	$\mathcal{O}(h^{17/2} \log^{7/2} h \cdot n^{3/2})$	Theorem 18
genus γ	0	$\mathcal{O}(\gamma^4 n^{3/2})$	Theorem 17
planar	0	$\mathcal{O}(n^{3/2})$	Theorem 16
outerplanar	0	$\mathcal{O}(n)$	[51]
constant tree-width	0	$\mathcal{O}(n)$	[42]
c-colourable q-queue	1	$\mathcal{O}(cqm)$	Theorem 20
arbitrary	1	$\mathcal{O}(nm)$	Theorem 21
q-queue	2	$\mathcal{O}(qn)$	Theorem 22
q-queue (constant $\epsilon > 0$)	$\mathcal{O}(1)$	$\mathcal{O}(mq^{\epsilon})$	Theorem 23
q-queue	$\mathcal{O}(\log q)$	$\mathcal{O}(m \log q)$	Theorem 24

Cohen *et al.* [31] proved that every graph has a straight-line drawing with $\mathcal{O}(n^3)$ volume, and that this bound is asymptotically optimal for complete graphs K_n. Our edge-sensitive bounds of $\mathcal{O}(m^{4/3}n)$ and $\mathcal{O}(\Delta mn)$ are greater than $\mathcal{O}(n^3)$ in the worst case. As discussed in Section 8, it is unknown whether there are edge-sensitive bounds which match the $\mathcal{O}(n^3)$ bound in the case of complete graphs.

Pach *et al.* [103, 104] proved that graphs with constant chromatic number have straight-line drawings with $\mathcal{O}(n^2)$ volume. For c-colourable graphs the actual bound is $\mathcal{O}(c^2 n^2)$. Our edge-sensitive volume bound of $\mathcal{O}(m^{2/3}n)$ is an improvement on this result for graphs with constant chromatic number and $o(n^{3/2})$ edges. Pach *et al.* [103, 104] also proved an $\Omega(n^2)$ lower bound for the volume of straight-line drawings of the complete bipartite graph $K_{n,n}$. This lower bound was generalised to all graphs by Bose *et al.* [21], who proved that straight-line (and polyline) drawings have volume at least $\frac{1}{8}(n + m)$.

Graphs with constant maximum degree have constant chromatic number, and thus, by the result of Pach *et al.* [103, 104], have straight-line drawings with $\mathcal{O}(n^2)$ volume. Pach *et al.* [103, 104] conjectured that graphs with constant maximum degree have straight-line drawings with $o(n^2)$ volume. We verify this conjecture by proving that graphs with constant maximum degree have straight-line drawings with $\mathcal{O}(n^{3/2})$ volume.

The first $\mathcal{O}(n)$ upper bound on the volume of straight-line drawings was established by Felsner *et al.* [51] for outerplanar graphs. This result was generalised by the authors for graphs with constant tree-width [42]. Felsner *et al.* [51] proposed the following inviting open problem: does every planar graph have a straight-line drawing with $\mathcal{O}(n)$ volume? In this paper we provide a

partial solution to this problem, by proving that planar graphs have straight-line drawings with $\mathcal{O}(n^{3/2})$ volume. Note that $\mathcal{O}(n^2)$ is the optimal area for plane straight-line grid drawings, and $\mathcal{O}(n^2)$ was the previous best upper bound on the volume of straight-line drawings of planar graphs. Our result generalises to graphs of constant genus, and graphs with no complete graph minor.

This paper initiates the study of upper bounds on the volume of three-dimensional polyline drawings. In general, we demonstrate a tradeoff between few bends and small volume in such drawings, which is evident in Table 2. By the lower bound of Bose *et al.* [21] discussed above, our upper bound of $\mathcal{O}(m \log q)$ is within a factor of $\mathcal{O}(\log q)$ of being optimal for all q-queue graphs.

This paper is organised as follows. Section 2 introduces so-called strong star colourings. In Section 3 we present results regarding track layouts, which form the basis for many of our results in the remainder of the paper. Sections 4 and 5 present results concerning queue and stack layouts. In addition to proving Theorem 1 above, we present a number of simple proofs of known results about queue layouts. Sections 6 and 7 present our results for straight-line and polyline drawings. We conclude with some open problems in Section 8.

2   Strong Star Colourings

Let G be a graph. A *vertex c-colouring* of G is a partition $\{V_i : 1 \leq i \leq c\}$ of $V(G)$, such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Each $i \in \{1, 2, \ldots, c\}$ is a *colour*, and each set V_i is a *colour class*. At times it will be convenient to write $\mathrm{col}(v) = i$ rather than $v \in V_i$. If G has a vertex c-colouring then G is *c-colourable*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum c such that G is c-colourable.

A vertex colouring is *acyclic* if there is no bichromatic cycle. The *acyclic chromatic number* of a graph G, denoted by $\chi_{\mathrm{a}}(G)$, is the minimum number of colours in an acyclic vertex colouring of G. A *star colouring* of G is a vertex colouring with no bichromatic 4-vertex path; that is, each bichromatic subgraph is a forest of stars. The *star chromatic number* of G, denoted by $\chi_{\mathrm{st}}(G)$, is the minimum number of colours in a star colouring of G. Acyclic and star colourings have been studied extensively [1–3, 16, 16–20, 52, 53, 63, 85, 86, 86, 99, 113]. By definition $\chi_{\mathrm{a}}(G) \leq \chi_{\mathrm{st}}(G)$ for every graph G. Conversely, if $\chi_{\mathrm{a}}(G) \leq c$ then $\chi_{\mathrm{st}}(G) \leq c \cdot 2^{c-1}$ [53]. The star chromatic number is bounded for a wide class of graphs. In particular, Nešetřil and Ossona de Mendez [99] proved that every proper minor-closed graph family has bounded star chromatic number. In fact, the star chromatic number of a graph G is at most a quadratic function of the maximum chromatic number of a minor of G.

Define a star colouring to be *strong* if between every pair of colour classes, all edges (if any) are incident to a single vertex. That is, each bichromatic subgraph consists of a star and possibly some isolated vertices. The *strong star chromatic number* of a graph G, denoted by $\chi_{\mathrm{sst}}(G)$, is the minimum number of colours in a strong star colouring of G. The motivation for studying this highly restrictive type of colouring is Lemma 9(e) and (f) in Section 3.2. The next result is the main contribution of this section

Theorem 2. *Every graph G with m edges and maximum degree $\Delta \geq 1$ has strong star chromatic number $\chi_{\mathrm{sst}}(G) \leq \lceil 4(\Delta + \sqrt{\Delta(1 + 4m)}) \rceil < 14\sqrt{\Delta m}$.*

To prove Theorem 2 we use the weighted version of the Lovász Local Lemma [49].

Lemma 1. [92, p. 221] *Let $\mathcal{E} = \{A_1, \ldots, A_n\}$ be a set of 'bad' events. Let $0 \leq p \leq \frac{1}{4}$ be a real number, and let $t_1, \ldots, t_n \geq 1$ be integers. Suppose that for all $A_i \in \mathcal{E}$,*

(a) *the probability $\mathbf{P}(A_i) \leq p^{t_i}$,*

(b) *A_i is mutually independent of $\mathcal{E} \setminus (\{A_i\} \cup D_i)$ for some $D_i \subseteq \mathcal{E}$, and*

(c) *$\displaystyle\sum_{A_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$.*

Then with positive probability, no event in $\mathcal{E}$ occurs.

Proof of Theorem 2. Let $c \geq 4$ be a positive integer to be specified later. Let $p = \frac{1}{c}$. Then $0 < p \leq \frac{1}{4}$. For each vertex $v \in V(G)$, randomly and independently choose $\mathrm{col}(v)$ from $\{1, 2, \ldots, c\}$.

For each edge $vw \in E(G)$, let A_{vw} be the *type-I* event that $\mathrm{col}(v) = \mathrm{col}(w)$. Let E' be the set of arcs $E' = \{(v, w), (w, v) : vw \in E(G)\}$. For each pair of arcs $(v, w), (x, y) \in E'$ with no endpoint in common, let $B_{(v,w),(x,y)}$ be the *type-II* event that $\mathrm{col}(v) = \mathrm{col}(x)$ and $\mathrm{col}(w) = \mathrm{col}(y)$.

We will apply Lemma 1 to obtain a colour assignment such that no type-I event and no type-II event occurs. No type-I event implies that we have a (proper) vertex colouring. No type-II event implies that no two disjoint edges share the same pair of colours; that is, we have a strong star colouring.

For each type-I event A, $\mathbf{P}(A) = \frac{1}{c}$. Let $t_A = 1$. Then $\mathbf{P}(A) = p^{t_A}$. For each type-II event B, $\mathbf{P}(B) = \frac{1}{c^2}$. Let $t_B = 2$. Then $\mathbf{P}(B) = p^{t_B}$. Thus condition (a) of Lemma 1 is satisfied.

An event involving a particular set of vertices is dependent only on other events involving at least one of the vertices in that set. Each vertex is involved in at most Δ type-I events, and at most $2\Delta|E'| = 4\Delta m$ type-II events. A type-I event involves two vertices, and is thus mutually independent of all but at most 2Δ type-I events and at most $8\Delta m$ type-II events. A type-II event involves four vertices, and is thus mutually independent of all but at most 4Δ type-I events and at most $16\Delta m$ type-II events.

For condition (c) of Lemma 1 to hold we need $2\Delta \left(\frac{2}{c}\right)^1 + 8\Delta m \left(\frac{2}{c}\right)^2 \leq \frac{1}{2}$ for the type-I events, and $4\Delta \left(\frac{2}{c}\right)^1 + 16\Delta m \left(\frac{2}{c}\right)^2 \leq 1$ for the type-II events. It is a happy coincidence that these two equations are equivalent, and it is easily verified that $c = \lceil 4(\Delta + \sqrt{\Delta(1 + 4m)}\,) \rceil \geq 4$ is a solution.

Thus by Lemma 1, with positive probability no type-I event and no type-II event occurs. Thus for every vertex $v \in V(G)$, there exists $\mathrm{col}(v) \in \{1, \ldots, c\}$ such that no type-I event and no type-II event occurs. As proved above such a colouring is a strong star colouring. Since $\Delta \leq \sqrt{\Delta m}$, the number of colours $c \leq \lceil 4(1 + \sqrt{5})\sqrt{\Delta m} \rceil \leq 14\sqrt{\Delta m}$. □

We have the following corollary of Theorem 2.

Lemma 2. *Every graph G with m edges has strong star chromatic number $\chi_{\mathrm{sst}}(G) \leq 15m^{2/3}$.*

Proof. Let X be the set of vertices of G with degree greater than $\frac{1}{4}m^{1/3}$. Let H be the subgraph $G[V(G) \setminus X]$. Thus H has maximum degree at most $\frac{1}{4}m^{1/3}$. By Theorem 2, H has a strong star colouring at most $14(\frac{1}{4}m^{1/3}m)^{1/2} = 7m^{2/3}$. Now $|X| \leq 2m/(\frac{1}{4}m^{1/3}) = 8m^{2/3}$. By adding each vertex in X to its own colour class we obtain a strong star colouring of G with at most $15m^{2/3}$ colours. □

Note the following lower bounds on the strong star chromatic number.

Lemma 3. *Every graph G with m edges and maximum degree $\Delta \geq 1$ has strong star chromatic number $\chi_{\mathrm{sst}}(G) > \sqrt{2m/\Delta}$. For every $\Delta \geq 1$, there exists infinitely many values of m such that there is a Δ-regular m-edge graph G with $\chi_{\mathrm{sst}}(G) > \sqrt{2m}$.*

Proof. First we prove the lower bound for any graph G. Suppose G has a strong star colouring with c colours. Between each pair of colour classes in a strong star colouring there are at most Δ edges. Thus $m \leq \binom{c}{2}\Delta$ and $c > \sqrt{2m/\Delta}$.

Now we prove the existential lower bound. For any $\Delta \geq 1$ and $k \geq 1$, let G be the graph consisting of k disjoint copies of the complete graph $K_{\Delta+1}$. Then G is Δ-regular and has $m = \frac{1}{2}k\Delta(\Delta + 1)$ edges. Consider a strong star colouring of G with c colours. Between any pair of colour classes all edges are incident on a single vertex. Moreover, there are no edges vx and vy with x and y in the same colour class, since in this case x and y would also be adjacent. Thus there is at most one edge between each pair of colour classes. Therefore $m \leq \binom{c}{2}$ and $c > \sqrt{2m}$. $\qquad\square$

It is an interesting open problem to determine the minimum α such that $\chi_{\mathrm{sst}}(G) \in \mathcal{O}(m^\alpha)$ for every m-edge graph G. (Why this is interesting is described in Section 8). As far as the authors are aware, such results are not known even for acyclic and star colourings. In this direction, we have the following bounds which are proved in a similar fashion to Lemma 2.

Lemma 4. *Every graph G with m edges has star chromatic number $\chi_{\mathrm{st}}(G) \leq 11m^{3/5}$, and acyclic chromatic number $\chi_{\mathrm{a}}(G) \leq 17m^{4/7}$.*

Proof. We first prove the result for star colourings. Let X be the set of vertices of G with degree greater than $\frac{3}{10}m^{2/5}$. Let H be the subgraph $G[V(G) \setminus X]$. Thus H has maximum degree at most $\frac{3}{10}m^{2/5}$. Fertin *et al.* [53] prove that every graph with maximum degree Δ has star chromatic number at most $\lceil 20\Delta^{3/2} \rceil$. Thus H has a star colouring with $\lceil 20(\frac{3}{10}m^{2/5})^{3/2} \rceil < \frac{13}{3}m^{3/5}$ colours. Now $|X| \leq 2m/(\frac{3}{10}m^{2/5}) = \frac{20}{3}m^{3/5}$. Adding each vertex in X to its own colour class we obtain a star colouring of G with at most $11m^{3/5}$ colours.

The proof for acyclic colourings is similar. Let X be the set of vertices of G with degree greater than $\frac{1}{4}m^{3/7}$. Let H be the subgraph $G[V(G) \setminus X]$. Thus H has maximum degree at most $\frac{1}{4}m^{3/7}$. Alon *et al.* [2] prove that every graph with maximum degree Δ has acyclic chromatic number at most $\lceil 50\Delta^{4/3} \rceil$. Thus H has an acyclic colouring with $\lceil 50(\frac{1}{4}m^{3/7})^{4/3} \rceil \leq 9m^{4/7}$ colours. Now $|X| \leq 2m/(\frac{1}{4}m^{3/7}) = 8m^{4/7}$. Adding each vertex in X to its own colour class we obtain a acyclic colouring of G with at most $17m^{4/7}$ colours. $\qquad\square$

3   Track Layouts

Let $\{V_i : 1 \leq i \leq t\}$ be a vertex t-colouring of a graph G. Let $<_i$ be a total order on each colour class V_i. Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : 1 \leq i \leq t\}$ is a *t-track assignment* of G. To ease the notation we denote track assignments by $\{V_i : 1 \leq i \leq t\}$ when the ordering on each colour class is implicit. The *span* of an edge vw in a track assignment $\{V_i : 1 \leq i \leq t\}$ is $|i - j|$ where $v \in V_i$ and $w \in V_j$. That there is a fixed ordering of the tracks in a track assignment is implicit in this definition of span.

An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j. An *edge k-colouring* of G is simply a partition $\{E_i : 1 \le i \le k\}$ of $E(G)$. An edge $vw \in E_i$ is said to be *coloured i*, written $\mathrm{col}(vw) = i$. A *(k,t)-track layout* of G consists of a t-track assignment of G and an edge k-colouring of G with no monochromatic X-crossing. The minimum t such that G has a (k,t)-track layout is denoted by $\mathsf{tn}_k(G)$. A $(1,t)$-track layout (that is, with no X-crossing) is called a *t-track layout*. The *track-number* of G is $\mathsf{tn}_1(G)$, or simply $\mathsf{tn}(G)$.

A track assignment with k pairwise X-crossing edges needs at least k edge colours to be a track layout. In fact the converse holds. This result essentially says that permutation graphs are perfect.

Lemma 5. *A t-track assignment $\{(T_i, <_i) : 1 \le i \le t\}$ of a graph G can be extended into a (k,t)-track layout where k is the maximum number of pairwise X-crossing edges.*

Proof. The colour of each edge vw is determined as follows. Suppose $v \in T_i$ and $w \in T_j$ such that $i < j$. Let ℓ be the maximum integer such that there exist edges $x_1 y_1, x_2 y_2, \ldots, x_\ell y_\ell$ with $x_1 <_i x_2 <_i \cdots <_i x_\ell <_i v$ and $w <_j y_\ell <_j y_{\ell-1} <_j \cdots <_j y_1$. Let $\mathrm{col}(vw) = \ell + 1$.

Suppose edges ab and cd form an X-crossing. Without loss of generality $a <_i c$ and $d <_j b$. Let ℓ be the maximum integer such that there exist edges $x_1 y_1, x_2 y_2, \ldots, x_\ell y_\ell$ with $x_1 <_i x_2 <_i \cdots <_i x_\ell <_i a$ and $b <_j y_\ell <_j y_{\ell-1} <_j \cdots <_j y_1$. Then $\mathrm{col}(ab) = \ell + 1$. The edges $x_1 y_1, x_2 y_2, \ldots, x_\ell y_\ell$ along with ab imply that $\mathrm{col}(cd) \ge \ell + 2$. Thus ab and cd are coloured differently, and there is no monochromatic X-crossing. The maximum edge colour is k. □

3.1 Manipulating a Track Layout

In this section we describe methods for manipulating track layouts. The first result describes how to 'wrap' a track layout, and is a generalisation of a result from [41], which in turn is based on an idea due to Felsner *et al.* [51]. Let $\{V_{i,j} : 0 \le i \le t-1, 1 \le j \le b\}$ be a tb-track assignment of a graph G. Define the *partial span* of an edge $vw \in E(G)$ with $v \in V_{i_1,j_1}$ and $w \in V_{i_2,j_2}$ to be $|i_1 - i_2|$.

Lemma 6. *Let $\{V_{i,j} : 0 \le i \le t-1, 1 \le j \le b\}$ be a (k,tb)-track layout of a graph G with maximum partial span s. Then* (a) $\mathsf{tn}_{2k}(G) \le (s+1)b$, *and* (b) $\mathsf{tn}_k(G) \le (2s+1)b$.

Proof. Let $\{E_\ell : 1 \le \ell \le k\}$ be the edge colouring in the given track layout. We first prove (a). For each $0 \le \alpha \le s$ and $1 \le j \le b$, let

$$W_{\alpha,j} = \bigcup \{V_{i,j} : i \equiv \alpha \pmod{s+1}, 0 \le i \le t-1\} .$$

Order $W_{\alpha,j}$ by

$$(V_{\alpha,j}, V_{\alpha+(s+1),j}, V_{\alpha+2(s+1),j}, \ldots) .$$

Since every edge $vw \in E(G)$ has partial span at most s, if $v \in W_{\alpha_1,j_1}$ and $w \in W_{\alpha_2,j_2}$ then $\alpha_1 \ne \alpha_2$. Hence $\{W_{\alpha,j} : 0 \le \alpha \le s, 1 \le j \le b\}$ is a track assignment of G. For each $1 \le \ell \le k$, let

$$E'_\ell = \{vw \in E_\ell : v \in V_{i_1,j_1} \cap W_{\alpha_1,j_1}, w \in V_{i_2,j_2} \cap W_{\alpha_2,j_2}, i_1 < i_2, \alpha_1 < \alpha_2\}, \text{ and}$$

$$E''_\ell = \{vw \in E_\ell : v \in V_{i_1,j_1} \cap W_{\alpha_1,j_1}, w \in V_{i_2,j_2} \cap W_{\alpha_2,j_2}, i_1 < i_2, \alpha_2 < \alpha_1\} .$$

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An X-crossing between edges both from some E'_ℓ (or both from some E''_ℓ) implies that the same edges form an X-crossing in the original track layout. Thus $\{E'_\ell, E''_\ell : 1 \le \ell \le k\}$ defines an edge $2k$-colouring with no monochromatic X-crossing. Thus we have a $(2k, (s+1)b)$-track layout of G.

We now prove (b). For each $0 \le \alpha \le 2s$ and $1 \le j \le b$, let

$$W_{\alpha,j} = \bigcup \{V_i : i \equiv \alpha \pmod{2s+1}, 0 \le i \le t-1\} \ .$$

Order $W_{\alpha,j}$ by

$$(V_{\alpha,j}, \ V_{\alpha+(2s+1),j}, \ V_{\alpha+2(2s+1),j}, \ \dots) \ .$$

Clearly $\{W_{\alpha,j} : 0 \le \alpha \le 2s, 1 \le j \le b\}$ is a track assignment of G. It remains to prove that there is no monochromatic X-crossing, where edge colours are inherited from the given track layout. Notice that each $E_\ell = E'_\ell \cup E''_\ell$. By the same argument used for part (a), edges in E'_ℓ or in E''_ℓ do not form an X-crossing. In the track layout defined for part (b), edges in E'_ℓ have partial span at most s, and edges in E''_ℓ have partial span at least $s+1$. Thus an edge from E'_ℓ and an edge from E''_ℓ do not form an X-crossing. Hence we have a $(k, 2s+1)$-track layout of G. □

Applying Lemma 6(b) with $b = 1$ we have:

Corollary 1. *Every graph G admitting a (k,t)-track layout with maximum span s has* (a) $\mathsf{tn}_{2k}(G) \le s+1$, *and* (b) $\mathsf{tn}_k(G) \le 2s+1$. □

We now show how to reduce the number of tracks in a track layout, at the expense of increasing the number of edge colours.

Lemma 7. *Let G be a graph admitting a (k,t)-track layout with maximum span s ($\le t-1$). For every vertex colouring $\{V_i : 1 \le i \le c\}$ of G, there is a $(2sk, c)$-track layout of G with tracks $\{V_i : 1 \le i \le c\}$.*

Proof. Let $\{T_j : 1 \le j \le t\}$ be a (k,t)-track layout of G with maximum span s and edge colouring $\{E_\ell : 1 \le \ell \le k\}$. Given a vertex colouring $\{V_i : 1 \le i \le c\}$ of G, order each V_i by $(V_i \cap T_1, V_i \cap T_2, \dots, V_i \cap T_t)$. Thus $\{V_i : 1 \le i \le c\}$ is a c-track assignment of G. Now we define an edge $2sk$-colouring. For each ℓ and α such that $1 \le \ell \le k$ and $1 \le |\alpha| \le s$, let

$$E_{\ell,\alpha} = \{vw \in E_\ell : v \in V_{i_1} \cap T_{j_1}, \ w \in V_{i_2} \cap T_{j_2}, \ i_1 < i_2, \ j_1 - j_2 = \alpha\} \ .$$

Consider two edges vw and xy in some $E_{\ell,\alpha}$ between a pair of tracks V_{i_1} and V_{i_2}. Without loss of generality $i_1 < i_2$, $v \in V_{i_1} \cap T_{j_1}$, $w \in V_{i_2} \cap T_{j_1+\alpha}$, $x \in V_{i_1} \cap T_{j_2}$, $y \in V_{i_2} \cap T_{j_2+\alpha}$, and $j_1 \le j_2$. If $j_1 = j_2$ then vw and xy are between the same pair of tracks in the given track layout, and the relative order of the vertices is preserved. Thus if vw and xy form an X-crossing in the c-track assignment then they are coloured differently. If $j_1 < j_2$ then $v <_{i_1} x$ and $w <_{i_2} y$, and the edges do not form an X-crossing. Hence vw and xy do not form a monochromatic X-crossing, and we have a $(2sk, c)$-track layout of G. □

Now consider how to reduce the number of edge colours in a track layout, at the expense of increasing the number of tracks. Observe that the underlying vertex colouring in a (monochromatic) t-track layout is acyclic — in fact the subgraph induced by each pair of tracks is a forest of caterpillars [64]. Hence $\chi_a(G) \le \mathsf{tn}(G)$ for every graph G. Since $\chi_a(G) \le c$ implies $\chi_{st}(G) \le c \cdot 2^{c-1}$ [53], star chromatic number is bounded by track number. The following 'converse' result can be easily extracted from the proof of Lemma 7 in [125].

Lemma 8. [125] *Let G be a graph admitting a (k, t)-track layout in which the underlying vertex t-colouring is a star colouring. Then G has track-number $\operatorname{tn}(G) \leq t(k+1)^{t-1}$.*

We have the following corollary of Lemmata 7 and 8.

Corollary 2. *Let G be a graph admitting a (k, t)-track layout with maximum span s. If G has star chromatic number $\chi_{\mathrm{st}}(G) \leq c$ then G has track-number $\operatorname{tn}(G) \leq c(2sk+1)^{c-1}$.* □

3.2 Upper Bounds on the Track Number

We have the following upper bounds on the track-number. Note that the authors also proved that track-number is bounded by (a doubly exponential function of) tree-width [42].

Lemma 9. *Let G be a graph with n vertices, maximum degree Δ, path-width p, tree-width w, genus γ, and with no K_h-minor. Then the track-number of G satisfies:* (a) $\operatorname{tn}(G) \leq p+1$, (b) $\operatorname{tn}(G) \leq 72w\Delta$, (c) $\operatorname{tn}(G) \in \mathcal{O}(\gamma^{1/2}n^{1/2})$, (d) $\operatorname{tn}(G) \in \mathcal{O}(h^{3/2}n^{1/2})$, (e) $\operatorname{tn}(G) \leq 14\sqrt{\Delta m}$, *and* (f) $\operatorname{tn}(G) \leq 15m^{2/3}$.

Proof. Part (a) is by Dujmović *et al.* [41]. Part (b) is by the authors [42]. Gilbert *et al.* [61] and Djidjev [40] independently proved that G has a $\mathcal{O}(\gamma^{1/2}n^{1/2})$-separator, and thus has $\mathcal{O}(\gamma^{1/2}n^{1/2})$ path-width (see [15, Theorem 20(iii)]). Hence (c) follows from (a). Similarly (d) follows from the result by Alon *et al.* [4] that G has a $\mathcal{O}(h^{3/2}n^{1/2})$-separator. With an arbitrary order on each colour class in a strong star colouring, there is no X-crossing. Thus track-number $\operatorname{tn}(G) \leq \chi_{\mathrm{sst}}(G)$ for every graph G, and parts (e) and (f) are immediate corollaries of Theorem 2 and Lemma 2. □

Clearly the track-number of a graph is at most the maximum track-number of its connected components. This idea can be extended to blocks (that is, maximal biconnected components) as follows.

Lemma 10. *For every $k \geq 1$, every graph G has $\operatorname{tn}_k(G) \leq 3 \cdot \max\{\operatorname{tn}_k(B) : B \text{ is a block of } G\}$.*

Proof. Clearly we can assume that G is connected. Let T be the *block-tree* of G. That is, there is one vertex in T for each block of G, and two vertices of T are adjacent if the corresponding blocks share a cut-vertex. T is acyclic, as otherwise a cycle in T would correspond to a single block in G. Suppose we have a (k, t)-track layout of each block of G, where $t = \max\{\operatorname{tn}_k(B)\}$. The following strategy starts with a track layout of T and replaces each track by t tracks containing the track layouts of the corresponding blocks.

Root T at an arbitrary vertex r. For all $i \geq 0$, let D_i be the set of blocks of G whose corresponding vertex in T is at distance i from r. For all $i \geq 1$, each block $B \in D_i$ has exactly one cut-vertex v that is also in some block $B' \in D_{i-1}$. We call v the *parent cut-vertex* of B. B' is the *parent block* of B, and B is a *child block* of B'. Observe that each cut-vertex v is the parent cut-vertex of all but one block containing v. If a vertex v is in only one block B then we say v is *grouped* with B. Otherwise v is a cut-vertex and we say v is *grouped* with the block for which it is not the parent block.

Now order each D_i firstly with respect to the order of the parent blocks in D_{i-1}, and secondly with respect to the order of the parent cut-vertices in the track layouts of the parent blocks. More

formally, for each $i \geq 1$, let $<_i$ be a total order of D_i such that for all blocks $A, B \in D_i$ (with parent blocks $A', B' \in D_{i-1}$) we have $A <_i B$ whenever (1) $A' <_{i-1} B'$, or (2) $A' = B'$, $A \cap A' = \{v\}$, $B \cap B' = \{w\}$, and $v < w$ in some track of the (k,t)-track layout of A'. (If v and w are in different tracks of the (k,t)-track layout of the parent block then the relative order of A and B is not important.)

For each $i \geq 0$ and $1 \leq j \leq t$, let $V_{i,j}$ be the set of vertices v of G in a some block $B \in D_i$ such that v is grouped with B, and v is in the jth track of the track layout of B. Now order each $V_{i,j}$ firstly with respect to the order $<_i$ of the blocks in D_i, and within a block B, by the order of the kth track of the track layout of B. Colour each edge e of G by the same colour assigned to e in the (k,t)-track layout of the block containing e. We claim there is no monochromatic X-crossing.

The parent cut-vertex of a block B is grouped with the parent block of B, and no block and its parent block are in the same D_i. Thus if vw is an edge with $v \in V_{i,j_1}$ and $w \in V_{i,j_2}$ then both v and w are grouped with the block containing vw. Since within each track vertices are ordered primarily by their block, and by assumption there is no monochromatic X-crossing between edges in the same block, there is no monochromatic X-crossing between tracks V_{i,j_1} and V_{i,j_2} for all $i \geq 0$ and $1 \leq j_1, j_2 \leq t$.

If vw is an edge with $v \in V_{i_1,j_1}$ and $w \in V_{i_2,j_2}$ for distinct i_1 and i_2, then without loss of generality, $i_2 = i_1 + 1$ and v is the parent cut-vertex of the block containing vw. Since sibling blocks are ordered with respect to the ordering of their parent cut-vertices, there is no X-crossing amongst edges between tracks V_{i_1,j_1} and V_{i_2,j_2} for all $i_1, i_2 \geq 0$ and $1 \leq j_1, j_2 \leq t$. Thus $\{V_{i,j} : i \geq 0, 1 \leq j \leq t\}$ is a k-edge colour track layout of G such that every edge has a partial span of one. By Lemma 6(b), G has $\mathsf{tn}_k(G) \leq 3t$. $\qquad\square$

3.3 Track Layouts of Subdivisions

We first consider track layouts of G' and G'', the subdivisions of a graph G with one and two division vertices per edge, respectively.

Lemma 11. *For every q-queue graph G, the subdivision G' has $\mathsf{tn}_{q+1}(G') \leq 2$.*

Proof. Let σ be the vertex ordering in a q-queue layout of G with queues $\{E_\ell : 1 \leq \ell \leq q\}$. Recall that $L(e)$ and $R(e)$ denote the left and right endpoints in σ of each edge e. Let $X(e)$ denote the division vertex of e in G'. Let $\prec$ be the total order on $\{X(e) : e \in E(G)\}$ such that $X(e) \prec X(f)$ whenever $L(e) <_\sigma L(f)$, or $L(e) = L(f)$ and $R(e) <_\sigma R(f)$. Let $(V(G), \sigma)$ and $(\{X(e) : e \in E(G)\}, \prec)$ define a 2-track assignment of G'. Colour the edges of G' as follows. For all edges $e \in E_\ell$, let $\mathsf{col}(L(e)X(e)) = 0$ and $\mathsf{col}(X(e)R(e)) = \ell$. Since in $\prec$, division vertices are ordered primarily by the left endpoint of the corresponding edge, no two edges $L(e)X(e)$ and $L(f)X(f)$ form an X-crossing. Suppose $e' = X(e)R(e)$ and $f' = X(f)R(f)$ form an X-crossing. Without loss of generality $R(e) <_\sigma R(f)$ and $X(f) \prec X(e)$. By construction $L(f) <_\sigma L(e)$, and e is nested inside f in σ. Thus e and f are in distinct queues, and $\mathsf{col}(e') \neq \mathsf{col}(f')$. Hence there is no monochromatic X-crossing. The number of edge colours is $q+1$. Therefore we have a $(q+1, 2)$-track layout of G'. $\qquad\square$

Lemma 12. *Every c-colourable q-queue graph G has:*
(a) $\mathsf{tn}_2(G') \leq q + 1$,   (b) $\mathsf{tn}(G') \leq c(q+1)$,   *and*   (c) $\mathsf{tn}(G'') \leq 2q + 1$.

Proof. Let σ be the vertex ordering in a q-queue layout of G with queues $\{E_\ell : 1 \leq \ell \leq q\}$. Let $A(e)$ denote the division vertex of e in G'. Let $A_\ell = \{A(e) : e \in E_\ell\}$ for each $1 \leq \ell \leq q$. Let $<_\ell$ denote the queue order of each E_ℓ. Consider $<_\ell$ to also order A_ℓ. That is, for all edges $e, f \in E_\ell$,

$$A(e) \leq_\ell A(f) \iff L(e) \leq_\sigma L(f) \text{ and } R(e) \leq_\sigma R(f) . \tag{2}$$

First we prove (a). The set $\{(A_\ell, <_\ell) : 1 \leq \ell \leq q\} \cup \{(V(G), \sigma)\}$ defines a $(q+1)$-track assignment of G'. Colour edges $L(e)A(e)$ of G' blue, and colour edges $R(e)A(e)$ of G' red. We claim that there is no monochromatic X-crossing. All edges of G' are between a vertex of G and a division vertex. Thus an X-crossing must involve two division vertices on the same track. Consider two edges e and f with $A(e) <_\ell A(f)$ for some $1 \leq \ell \leq q$. By (2), each of the pairs of edges $\{L(e)A(e), L(f)A(f)\}$ and $\{R(e)A(e), R(f)A(f)\}$ do not form an X-crossing. For each pair of edges $\{L(e)A(e), R(f)A(f)\}$ and $\{R(e)A(e), L(f)A(f)\}$ the edges are coloured differently. Thus there is no monochromatic X-crossing and we have a $(2, q+1)$-track layout of G'.

Now we prove (b). Let $\{V_i : 1 \leq i \leq c\}$ be a vertex c-colouring of G. Let $A_{i,\ell} = \{A(e) : e \in E_\ell, L(e) \in V_i\}$ for all $1 \leq \ell \leq q$ and $1 \leq i \leq c$. Thus $\{(A_{i,\ell}, <_\ell) : 1 \leq i \leq c, 1 \leq \ell \leq q\} \cup \{(V_i, <_\sigma) : 1 \leq i \leq c\}$ defines a $(qc + c)$-track assignment of G'. Consider division vertices $A(e), A(f) \in A_{i,\ell}$ such that $A(e) <_\ell A(f)$. By (2), $L(e) \leq L(f)$ in the ordering on V_i. Thus the pair of edges $\{L(e)A(e), L(f)A(f)\}$ do not form an X-crossing. Since both $R(e)$ and $R(f)$ are not in V_i, the pairs of edges $\{L(e)A(e), R(f)A(f)\}$ and $\{R(e)A(e), L(f)A(f)\}$ do not form an X-crossing. If both $R(e)$ and $R(f)$ are in the same colour class V_j, then $R(e) \leq_j R(f)$ by (2), and the pair of edges $\{R(e)A(e), R(f)A(f)\}$ do not form an X-crossing. Thus we have a $(qc + c)$-track layout of G'.

Finally we prove (c). Let $(L(e), A(e), B(e), R(e))$ be the path replacing each edge e in G''. Define and order B_ℓ as with A_ℓ. The set $\{(A_\ell, <_\ell), (B_\ell, <_\ell) : 1 \leq \ell \leq q\} \cup \{(V(G), \sigma)\}$ define a $(2q+1)$-track assignment of G''. An X-crossing can only be between pairs of edges $\{L(e)A(e), L(f)A(f)\}$, $\{A(e)B(e), A(f)B(f)\}$, or $\{B(e)R(e), B(f)R(f)\}$. By (2), such pairs of edges do not form an X-crossing. Thus we have $(2q+1)$-track layout of G''. $\qquad\square$

The next theorem is the starting point for our results on stack and queue layouts of subdivisions, and our constructions of polyline drawings.

Theorem 3. *For every integer $d \geq 2$, every q-queue graph G has a subdivision H with $1 + 2\lceil \log_d q \rceil$ division vertices per edge, such that H admits a $(d, 2 + \lceil \log_d q \rceil)$-track layout with maximum span one.*

Proof. Let σ be the vertex ordering in a q-queue layout of G. Let $Q = \lceil \log_d q \rceil$. Let H be the subdivision of G obtained by replacing each edge $e = vw$ by the path

$$\begin{aligned} v = &(e, v, 0), (e, v, 1), (e, v, 2), \dots, (e, v, Q), (e, v, Q+1) = (e, w, Q+1), \\ &(e, w, Q), (e, w, Q-1), \dots, (e, w, 1), (e, w, 0) = w , \end{aligned}$$

where each (e, v, i) is a vertex of H. Observe that H has $2Q + 1$ division vertices per edge of G. Label the queues of G by $0, 1, \dots, q - 1$. Let $D(\ell)$ be the d-ary representation of each queue

ℓ including leading zeros, $0 \leq \ell \leq q - 1$. Thus $D(\ell)$ is a string of exactly Q characters over the alphabet $\{0, 1, \ldots, d - 1\}$. Denote by $D_i(\ell)$ the first i characters of $D(\ell)$ for each $1 \leq i \leq Q$. Also define $D_0(\ell) = \emptyset$ for all ℓ. Order each set $\{D_i(\ell) : 0 \leq \ell \leq q - 1\}$ lexicographically. That is, for all $0 \leq \ell_1, \ell_2 \leq q - 1$, $D_i(\ell_1) \prec_{\text{lex}} D_i(\ell_2)$ if for some $1 \leq j \leq i$, $D_{j-1}(\ell_1) = D_{j-1}(\ell_2)$ and the jth character of $D_i(\ell_1)$ is less than the jth character of $D_i(\ell_2)$.

Let $V_i = \{(e, v, i) : e \in E(G), v \in e\}$ for each $0 \leq i \leq Q + 1$. Then $\{V_i : 0 \leq i \leq Q + 1\}$ is a vertex colouring of H with maximum span one. Define a total order $\prec_i$ on each V_i by the following list of prioritised rules:

(a) $D_i(\mathsf{queue}(e)) \prec_{\text{lex}} D_i(\mathsf{queue}(f)) \implies (e, v, i) \prec_i (f, w, i)$
   $\big(\forall\, 0 \leq i \leq Q,\ e, f \in E(G),\ v \in e,\ w \in f\big)$

(b) $v <_\sigma w \implies (e, v, i) \prec_i (f, w, i)$
   $\big(\forall\, 0 \leq i \leq Q,\ e, f \in E(G),\ v \in e,\ w \in f,\ D_i(\mathsf{queue}(e)) = D_i(\mathsf{queue}(f))\big)$

(c) $u <_\sigma w \implies (vu, v, i) \prec_i (vw, v, i)$
   $\big(\forall\, 1 \leq i \leq Q,\ vu, vw \in E(G),\ D_i(\mathsf{queue}(vu)) = D_i(\mathsf{queue}(vw))\big)$

Define a total order $\prec_{Q+1}$ on V_{Q+1} by the following list of prioritised rules, where $<_\ell$ is the queue order for each queue $0 \leq \ell \leq q - 1$ (see (1)):

(d) $\mathsf{queue}(e) < \mathsf{queue}(f) \implies (e, v, Q+1) \prec_{Q+1} (f, w, Q+1)$
   $\big(\forall\, e, f \in E(G),\ v \in e,\ w \in f\big)$

(e) $e <_\ell f \implies (e, v, Q+1) \prec_{Q+1} (f, w, Q+1)$
   $\big(\forall\, e, f \in E(G),\ v \in e,\ w \in f,\ \mathsf{queue}(e) = \mathsf{queue}(f) = \ell\big)$

Thus $\{(V_i, \prec_i) : 0 \leq i \leq Q + 1\}$ is a track-assignment of H. We now colour the edges of H. For all $1 \leq i \leq Q$, $e \in E(G)$, and $v \in e$, let $\mathsf{col}((e, v, i - 1)(e, v, i))$ be the ith character of $D(\mathsf{queue}(e))$. For all $e \in E(G)$, let $\mathsf{col}((e, L(e), Q)(e, L(e), Q + 1)) = 0$, and $\mathsf{col}((e, R(e), Q)(e, R(e), Q + 1)) = 1$. Since $d \geq 2$ the number of colours is d. We claim that there is no monochromatic X-crossing.

Consider two edges $e' = (e, v, i - 1)(e, v, i)$ and $f' = (f, w, i - 1)(f, w, i)$ of H with no common endpoint for some $1 \leq i \leq Q$. (The case of $i = Q + 1$ will be handled later.) Without loss of generality $(e, v, i - 1) \prec_{i-1} (f, w, i - 1)$.

First suppose that $(e, v, i - 1) \prec_{i-1} (f, w, i - 1)$ by rule (a). That is, $D_{i-1}(\mathsf{queue}(e)) \prec_{\text{lex}} D_{i-1}(\mathsf{queue}(f))$. Then $D_i(\mathsf{queue}(e)) \prec_{\text{lex}} D_i(\mathsf{queue}(f))$, and $(e, v, i) \prec_i (f, w, i)$ by rule (a). Hence e' and f' do not form an X-crossing.

Suppose that $(e, v, i-1) \prec_{i-1} (f, w, i-1)$ by rule (b). That is, $D_{i-1}(\mathsf{queue}(e)) = D_{i-1}(\mathsf{queue}(f))$ and $v <_\sigma w$. If $D_i(\mathsf{queue}(e)) = D_i(\mathsf{queue}(f))$ then $(e, v, i) \prec_i (f, w, i)$ again by rule (b), and e' and f' do not form an X-crossing. Otherwise $D(\mathsf{queue}(e))$ and $D(\mathsf{queue}(f))$ differ in their i^{th} characters. Thus e' and f' are coloured differently and do not form a monochromatic X-crossing.

Suppose that $(e, v, i-1) \prec_{i-1} (f, w, i-1)$ by rule (c). That is, $D_{i-1}(\mathsf{queue}(e)) = D_{i-1}(\mathsf{queue}(f))$, $v = w$, $e = va$, $f = vb$, and $a <_\sigma b$. If $D_i(\mathsf{queue}(e)) = D_i(\mathsf{queue}(f))$ then $(e, v, i) \prec_i (f, w, i)$ again by rule (c), and e' and f' do not form an X-crossing. Otherwise $D(\mathsf{queue}(e))$ and $D(\mathsf{queue}(f))$ differ in their i^{th} characters. Thus e' and f' are coloured differently and do not form a monochromatic X-crossing.

Consider two edges $e' = (e, v, Q)(e, v, Q+1)$ and $f' = (f, w, Q)(f, w, Q+1)$ of H with no common endpoint. Thus $e \neq f$. Without loss of generality $(e, v, Q) \prec_Q (f, w, Q)$. First suppose that $(e, v, Q) \prec_Q (f, w, Q)$ by rule (a). That is, $\mathsf{queue}(e) < \mathsf{queue}(f)$. Then $(e, v, Q+1) \prec_{Q+1} (f, w, Q+1)$ by rule (d). Hence e' and f' do not form an X-crossing. Now suppose that $(e, v, Q) \prec_Q (f, w, Q)$ by rule (b). That is, $v <_\sigma w$ and $\mathsf{queue}(e) = \mathsf{queue}(f) = \ell$ for some queue ℓ. Suppose that e' and f' are monochromatic. That is, $v = L(e)$ and $w = L(f)$, or $v = R(e)$ and $w = R(f)$. By (1), $e <_\ell f$. Hence $(e, v, Q+1) \prec_{Q+1} (f, w, Q+1)$ by rule (e), and e' and f' do not form a monochromatic X-crossing.

In each case we have proved that there is no monochromatic X-crossing. Thus the track assignment is a $(d, Q+2)$-track layout. This completes the proof. $\qquad\square$
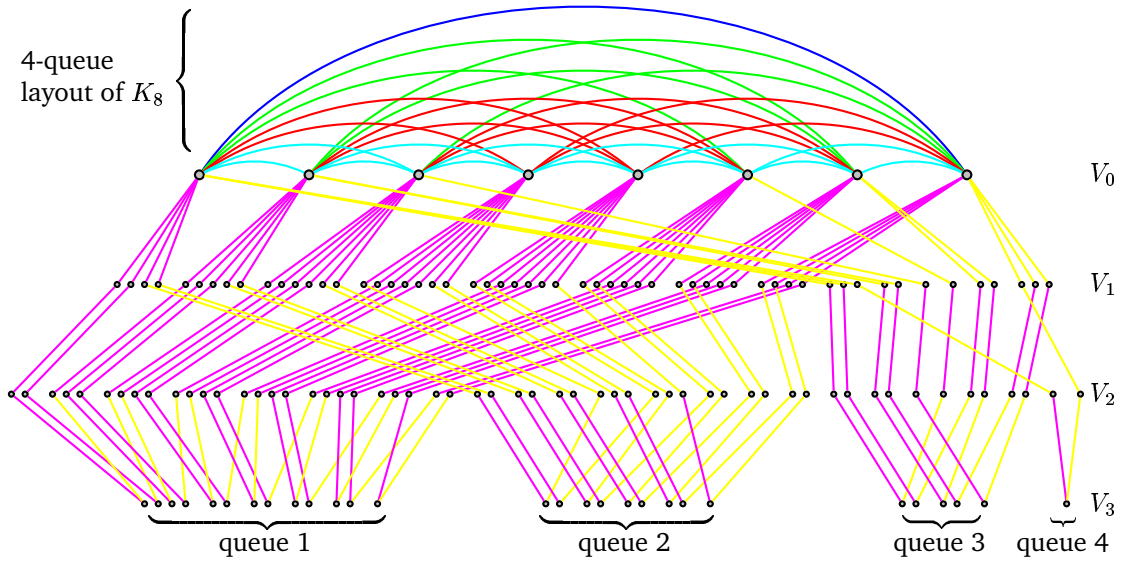

Figure 1: $(2,4)$-track layout of K_8 from Theorem 3

We have the following corollary of Lemma 6 and Theorem 3.

Corollary 3. *For every integer $d \geq 2$, every q-queue graph has a subdivision H with $1 + 2\lceil \log_d q \rceil$ division vertices per edge, such that H admits a $(2d, 2)$-track layout and a $(d, 3)$-track layout.* $\qquad\square$

Observe that the vertex colouring in the track layout of H produced by Theorem 3 is a star colouring. It remains a star colouring when wrapped on to 3 tracks in Corollary 3. Thus by Lemma 8, $\mathsf{tn}(H) \leq 3(d+1)^2$. This result can be improved as follows.

Theorem 4. *For every integer $d \geq 2$, every q-queue graph G has a subdivision H with $3 + 2\lceil \log_d q \rceil$ division vertices per edge, such that the track-number $\mathsf{tn}(H) \leq 3d$.*

Proof. Let $Q = \lceil \log_d q \rceil$. Let H be the subdivision of G defined in Theorem 3, with the exception that H has $3 + 2Q$ division vertices per edge instead of $1 + 2Q$. In particular, each edge $e = vw$ is

replaced by the path

$$v =(e,v,0),(e,v,1),(e,v,2),\ldots,(e,v,Q+1),(e,v,Q+2)=(e,w,Q+2),$$
$$(e,w,Q+1),(e,w,Q),\ldots,(e,w,1),(e,w,0)=w\ ,$$

where each (e,v,i) is a vertex of H. For all $0 \leq i \leq Q$ and $0 \leq \ell \leq q-1$, define $D_i(\ell)$ as in Theorem 3, and in addition let $D_{Q+1}(\ell) = D_Q(\ell)$. For each $0 \leq i \leq Q+2$, let $V_i = \{(e,v,i) : e \in E(G), v \in e\}$. For each $0 \leq i \leq Q+1$, order the vertices in V_i by rules (a), (b) and (c) in Theorem 3. Order the vertices in V_{Q+2} by rules (d) and (e) in Theorem 3 (with $Q+1$ replaced by $Q+2$).

Edges are coloured in a similar way to Theorem 3. In particular, for all $1 \leq i \leq Q$, $e \in E(G)$, and $v \in e$, let $\mathsf{col}((e,v,i-1)(e,v,i))$ be the ith character of $D(\mathsf{queue}(e))$. For all $e \in E(G)$, let $\mathsf{col}((e,L(e),Q)(e,L(e),Q+1)) = 0$, let $\mathsf{col}((e,R(e),Q)(e,R(e),Q+1)) = 1$, let $\mathsf{col}((e,L(e),Q+1)(e,L(e),Q+2)) = 0$, and let $\mathsf{col}((e,R(e),Q+1)(e,R(e),Q+2)) = 0$. Since $d \geq 2$ the number of colours is d.

For all $i \geq 1$, define the *incoming colour* of each vertex (e,v,i) to be $\mathsf{col}((e,v,i)) = \mathsf{col}((e,v,i-1)(e,v,i))$. This is clearly well defined for all $1 \leq i \leq Q+1$, and it is also the case for $i = Q+2$ since $\mathsf{col}((e,L(e),Q+1)(e,L(e),Q+2)) = \mathsf{col}((e,R(e),Q+1)(e,R(e),Q+2))$. For convenience let the *incoming colour* of each non-division vertex $(e,v,0)$ be 0.

For each $1 \leq i \leq Q+2$ and $0 \leq j \leq d-1$, let $V_{i,j} = \{(e,v,i) : e \in E(G), v \in e, \mathsf{col}((e,v,i)) = j\}$. That is, each track V_i is split into 'sub-tracks' with the same incoming colour. All edges between any pair of tracks are monochromatic. Hence, by the same argument used in Theorem 3 to show that there is no monochromatic X-crossing, there is no X-crossing in this construction. Therefore $\{V_{i,j} : 0 \leq i \leq Q+2, 0 \leq j \leq d-1\}$ is a track layout of H such that every edge has a partial span of one. Thus $\mathsf{tn}(H) \leq 3d$ by Lemma 6(b). $\square$

4 Queue Layouts

4.1 Basics

Consider the problem of assigning edges to queues given a fixed vertex ordering σ of a graph G. A *rainbow* in σ is a matching $\{v_i w_i \in E(G) : 1 \leq i \leq k\}$ such that

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1$$

Furthermore, the rainbow $\{v_i w_i : 2 \leq i \leq k\}$ is said to be *inside* $v_1 w_1$. A vertex ordering containing a k-edge rainbow needs at least k queues. In fact, as proved by Heath and Rosenberg [75], the converse also holds. We now give a simple proof of this result.

Lemma 13. *A vertex ordering of a graph G with no $(k+1)$-edge rainbow admits a k-queue layout of G.*

Proof. For every edge $vw \in E(G)$, let $\mathsf{queue}(vw)$ be the maximum number of edges in a rainbow inside vw plus one. If vw is nested inside xy then $\mathsf{queue}(vw) < \mathsf{queue}(xy)$. Hence we have a valid queue assignment. The number of queues is the maximum number of edges in a rainbow. $\square$

The above proof is essentially the application of the easy half of Dilworth's Theorem [39] for partitioning a partially ordered set into k antichains, where k is the maximum size of a chain. See Lemma 5 for a similar proof.

Now consider the extremal problem: what is the maximum number of edges in a q-queue graph? The answer for $q = 1$ was given by Heath and Rosenberg [75] and Pemmaraju [106]. We now give a simple proof for this case. The proof by Heath and Rosenberg [75] is based on the characterisation of 1-queue graphs as the 'arched levelled planar' graphs. The proof by Pemmaraju [106] is based on a relationship between queue layouts and 'staircase covers of matrices'. The observant reader will notice parallels between the following proof and that of the lower bound on the volume of three-dimensional drawings due to Bose *et al.* [21].

Lemma 14. *A queue in a graph with n vertices has at most $2n - 3$ edges.*

Proof. Let E' be a queue in a vertex-ordering $(v_1, v_2, \ldots, v_n)$. For each edge $v_i v_j \in E'$, let $\lambda(v_i v_j) = \frac{1}{2}(i+j)$. Then $\lambda(v_i v_j) \in \{\frac{3}{2}, 2, 2\frac{1}{2}, \ldots, n - \frac{1}{2}\}$. Distinct edges with the same λ value are nested. Since no two edges in E' are nested, and there are $2n-3$ possible values for λ, we have $|E'| \leq 2n-3$. $\square$

An immediate generalisation of Lemma 14 is that every q-queue graph has at most $q(2n - 3)$ edges [75]. The following improved upper bound was first discovered by Pemmaraju [106] with a longer proof. That this bound is tight for all values of n and q is new.

Lemma 15. *Every q-queue graph with n vertices has at most $q(2n - 3) - 2q(q - 1)$ edges. For every q and n, there exists a graph with queue-number q and $q(2n - 3) - 2q(q - 1)$ edges*

Proof. Define $\lambda(e)$ for each edge e as in Lemma 14. Since at most q edges are pairwise nested in a q-queue layout, at most q edges have the same λ value. Moreover, for all integers $1 \leq i \leq q$, at most $i - 1$ edges have $\lambda(e) = i$, and at most $i - 1$ edges have $\lambda(e) = i - \frac{1}{2}$. At the other end of the vertex ordering, for all integers $1 \leq i \leq q - 1$, at most i edges have $\lambda(e) = n - i$, and at most i edges have $\lambda(e) = n - i + \frac{1}{2}$. It follows that the number of edges is at most

$$2 \sum_{i=1}^{q} (i - 1) \; + \; (2n - 4q + 1)q \; + \; 2 \sum_{i=1}^{q-1} i \; = \; q(2n - 3) - 2q(q - 1) \; .$$

Consider the graph $G(n, q)$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_j : 1 \leq |j - i| \leq 2q\}$. Heath and Rosenberg [75] proved that $G(n, q)$ has queue-number q, and Swaminathan *et al.* [117] proved that $G(n, q)$ has $q(2n - 3) - 2q(q - 1)$ edges. (The graph $G(n, q)$ appears in [75, 117] in regard to the relationship between bandwidth and queue- and stack-number, respectively.) $\square$

We now establish a relationship between vertex colourings and queue layouts. An analogous result for stack layouts was proved by Bernhart and Kainen [8] and Cottafava and D'Antona [32].

Lemma 16. *Every q-queue graph G has a vertex $4q$-colouring.*

Proof. Let V' be a set of n' vertices of G. The induced subgraph $G[V']$ has less than $2qn'$ edges by Lemma 15. Hence $G[V']$ has a vertex of degree less than $4q$. Applying the well-known minimum-degree-greedy vertex colouring heuristic the claimed result is obtained (see for example [92]). $\square$

Malitz [90] proved that the stack-number of an m-edge graph is $\mathcal{O}(\sqrt{m})$. The exact bound is in fact $72\sqrt{m}$. Heath and Rosenberg [75] (see also [111]) observed that an analogous method proves that the queue-number of an an m-edge graph is $\mathcal{O}(\sqrt{m})$. We now establish this result using a simpler argument than that of Malitz [90] and with an improved constant.

Theorem 5. *Every graph G with m edges has queue-number* $\mathsf{qn}(G) < \boldsymbol{e}\sqrt{m}$, *where $\boldsymbol{e}$ is the base of the natural logarithm.*

Proof. Let $n = |V(G)|$. Let σ be a random vertex ordering of G. For all positive integers k, let A_k be the event that there exists a k-edge rainbow in σ. Then

$$\mathbf{P}\{A_k\} \;\leq\; \underbrace{\binom{m}{k}}_{(1)} \cdot \underbrace{\binom{n}{2k}}_{(2)} \cdot \underbrace{\frac{2^k \, k!(n-2k)!}{n!}}_{(3)} \;,$$

where:

(1) is an upper bound on the number of k-edge matchings M,

(2) is the number of vertex positions in σ for M, and

(3) is the probability that M with fixed vertex positions is a rainbow.

Thus

$$\mathbf{P}\{A_k\} \;\leq\; \frac{m^k}{k!} \cdot \frac{n!}{(2k)!(n-2k)!} \cdot \frac{2^k \, k!(n-2k)!}{n!} \;=\; \frac{(2m)^k}{(2k)!} \;.$$

By Stirling's formula, $\mathbf{P}\{A_k\} < \left(\frac{e^2 m}{2k^2}\right)^k$. Let $k_0 = \lceil \boldsymbol{e}\sqrt{m} \rceil$. Thus, $\mathbf{P}\{\overline{A_{k_0}}\} > 1 - \left(\frac{1}{2}\right)^{\lceil \boldsymbol{e}\sqrt{m} \rceil} > 0$. That is, with positive probability a random vertex ordering has no k_0-edge rainbow. Hence there exists a vertex ordering with no k_0-edge rainbow. By Lemma 13, G has a queue layout with $k_0 - 1 < \boldsymbol{e}\sqrt{m}$ queues. $\qquad\square$

4.2 Queue and Track Layouts

In the following series of lemmata we explore the relationship between track and queue layouts. We first consider how to convert a queue layout into a track layout.

Lemma 17. *For every vertex colouring $\{V_i : 1 \leq i \leq c\}$ of a q-queue graph G, there is a $(2q, c)$-track layout of G with tracks $\{V_i : 1 \leq i \leq c\}$.*

Proof. Let σ be the vertex ordering in a q-queue layout of G with queues $\{E_\ell : 1 \leq \ell \leq q\}$. Let $\{(V_i, \sigma) : 1 \leq i \leq c\}$ be a c-track assignment of G, and for each $1 \leq \ell \leq q$, let

$$E'_\ell = \{vw \in E_\ell : v \in V_i, w \in V_j, i < j, v <_\sigma w\}, \text{ and}$$
$$E''_\ell = \{vw \in E_\ell : v \in V_i, w \in V_j, i < j, w <_\sigma v\} \;.$$

An X-crossing in the track assignment between edges both from some E'_ℓ (or both from some E''_ℓ) implies that these edges are nested in σ. Since no two edges in E_ℓ are nested in σ, the set $\{E'_\ell, E''_\ell : 1 \leq \ell \leq q\}$ defines an edge $2q$-colouring with no monochromatic X-crossing in the track assignment. Thus we have a $(2q, c)$-track layout of G. $\qquad\square$

Lemma 17 is similar in spirit to a result by Pemmaraju [106] which says that a queue layout can be 'separated' by a vertex colouring. We have the following corollary of Lemmata 16 and 17.

Corollary 4. *Every q-queue graph has a $(2q, 4q)$-track layout.* □

The next result follows from Lemmata 8 and 17, and slightly improves an analogous result in [125].

Corollary 5. *Every q-queue graph G with $\chi_{\text{st}}(G) \leq c$ has track-number $\text{tn}(G) \leq c(2q+1)^{c-1}$.* □

We now consider how to convert a track layout into a queue layout. First we give a simple proof of a result from [125].

Lemma 18. *Every graph G admitting a (k, t)-track layout with maximum span s $(\leq t - 1)$ has a ks-queue layout.* □

Proof. Let $\{V_i : 1 \leq i \leq t\}$ be a (k, t)-track layout of G with maximum span s and edge colouring $\{E_\ell : 1 \leq \ell \leq k\}$. Let σ be the vertex ordering $(V_1, V_2, \ldots, V_t)$ of G. Let $E_{\ell,\alpha}$ be the set of edges in E_ℓ with span α in the given track layout. Two edges from the same pair of tracks are nested in σ if and only if they form an X-crossing in the track layout. Since no two edges in E_ℓ form an X-crossing in the track layout, no two edges in E_ℓ and between the same pair of tracks are nested in σ. If two edges not from the same pair of tracks have the same span then they are not nested in σ. (This idea is due to Heath and Rosenberg [75].) Thus no two edges are nested in each $E_{\ell,\alpha}$, and we have a ks-queue layout of G. □

The next result from [125] summarises the relationship between queue layouts and track layouts. It follows from Corollary 5, Lemma 18, and the result by Nešetřil and Ossona de Mendez [99] concerning star chromatic number that is discussed in Section 2.

Theorem 6. [125] *Let $\mathcal{G}$ be a proper minor closed graph family. Then queue-number $\text{qn}(\mathcal{G}) \in \mathcal{F}(n)$ if and only if track-number $\text{tn}(\mathcal{G}) \in \mathcal{F}(n)$, for any family $\mathcal{F}$ of functions closed under multiplication, such as $\mathcal{O}(1)$ or $\mathcal{O}(\text{polylog}\, n)$.*

4.3 Queue Layouts of Subdivisions

In this section we study the relationship between the queue-number of a graph G and the queue-number of a subdivision of G. First note that Lemma 11 and Lemma 18 imply:

Lemma 19. *The subdivision G' of a q-queue graph G has a $(q+1)$-queue layout.* □

The following corollary of Theorem 3 and Lemma 18 (with $s = 1$) is our main contribution regarding queue layouts.

Theorem 7. *For every integer $d \geq 2$, every q-queue graph has a d-queue subdivision with $1 + 2\lceil \log_d q \rceil$ division vertices per edge.* □

We now prove that Theorem 7 is optimal up to a constant factor.

Theorem 8. *Let G be a graph with queue-number $\text{qn}(G) = q$. For all $d \geq 2$, if H is a d-queue subdivision of G with at most k division vertices per edge, then $k \geq \frac{1}{6} \log_d q$.*

To prove Theorem 8 we use the following series of lemmata. Recall that G' is the subdivision of G with one division vertex per edge.

Lemma 20. *For every graph G, if G' has a q-queue layout then G has a $q(2q+1)$-queue layout using the same vertex ordering.*

Proof. Let σ be the vertex ordering in a q-queue layout of G'. Let X be the set of division vertices of G'. The vertex sets $V(G)$ and X partition $V(G')$ and define a vertex 2-coloring of G'. By Lemma 17, G' has a $(2q, 2)$-track layout with tracks $(V(G), \sigma)$ and (X, σ). Let $1 \leq \mathrm{col}(e) \leq 2q$ be the colour assigned to each edge e of G'. Consider a layout of G in which the vertices are ordered by σ and the edges are partitioned into queues as follows. For each edge $vw \in E(G)$ divided by vertex x in G', let $\mathrm{queue}(vw) = \{\mathrm{col}(vx), \mathrm{col}(wx)\}$. We now prove that this layout is a queue layout of G. Say vw is nested inside ab in G and without loss of generality $a <_\sigma v <_\sigma w <_\sigma b$. Let vw be divided by x in G', and let ab be divided by c in G'. First suppose that $x <_\sigma c$. Then each of xw and xv form an X-crossing with ac. Thus $\mathrm{col}(xw) \neq \mathrm{col}(ac)$ and $\mathrm{col}(xv) \neq \mathrm{col}(ac)$. Hence $\mathrm{queue}(vw) \neq \mathrm{queue}(ab)$. Now suppose $c <_\sigma x$. Then bc forms an X-crossing with each of xw and xv. Thus $\mathrm{col}(bc) \neq \mathrm{col}(xw)$ and $\mathrm{col}(bc) \neq \mathrm{col}(xv)$. Hence $\mathrm{queue}(vw) \neq \mathrm{queue}(ab)$. The number of queues in the queue layout of G is $q(2q+1)$. $\qquad\square$

We now prove a slightly more general result than Lemma 20. Here we start with a subdivision with at most one division vertex per edge rather than exactly one division vertex per edge.

Lemma 21. *Let H be a q-queue subdivision of a graph G with at most one division vertex per edge. Then G has a $2q(q+1)$-queue layout.*

Proof. Let σ be the vertex ordering in a q-queue layout of H. Let A be the set of edges of G that are subdivided in H, and let B the set of edges of G that are not subdivided in H. By Lemma 20, $G[A]$ has a $q(2q+1)$-queue layout with vertex ordering σ. By assumption, $G[B]$ has a q-queue layout with vertex ordering σ. Thus G has a $2q(q+1)$-queue layout with vertex ordering σ. $\qquad\square$

Lemma 22. *Let H be a q-queue subdivision of a graph G with at most k division vertices per edge. Then G has a $(\frac{1}{2}(2q+2)^{2k} - 1)$-queue layout.*

Proof. Let $q_i = \frac{1}{2}(2q+2)^{2^i} - 1$, and $k_i = k/2^i$. We proceed by induction on $i \geq 0$ with the hypothesis: *there exists a subdivision H_i of G with at most k_i division vertices per edge, and H_i has a q_i-queue layout.* Consider the base case with $i = 0$. Let $H_0 = H$. Then H_0 is a subdivision of G with $k_0 = k$ division vertices per edge, and H_0 has a q_0-queue layout, since $q_0 = q$.

Suppose there exists a subdivision H_i of G with at most k_i division vertices per edge, and H_i has a q_i-queue layout. By contracting every second division vertex on the path representing each edge of G in H_i, we obtain a graph H_{i+1} such that H_i is a subdivision of H_{i+1} with at most one division vertex per edge, and H_{i+1} is a subdivision of G with at most $k_i/2$ division vertices per edge. By Lemma 21, H_{i+1} has a $2q_i(q_i + 1)$-queue layout. Now $k_i/2 = k_{i+1}$, and $2q_i(q_i + 1) \leq 2(q_i + 1)^2 - 1 = \frac{1}{2}(2q+2)^{2^{i+1}} - 1 = q_{i+1}$. Thus the inductive hypothesis holds for all i.

With $i^* = \lfloor \log_2 k \rfloor + 1$, we have $k_{i^*} < 1$. The only subdivision of G with less than one division vertex per edge is G itself. Thus G has a q_{i^*}-queue layout, and $q_{i^*} = \frac{1}{2}(2q+2)^{(2^{\lfloor \log_2 k \rfloor + 1})} - 1 \leq \frac{1}{2}(2q+2)^{(2^{1+\log_2 k})} - 1 \leq \frac{1}{2}(2q+2)^{2k} - 1$. $\qquad\square$

Proof of Theorem 8. By Lemma 22, G has $(\frac{1}{2}(2d+2)^{2k}-1)$-queue layout. Thus $q \le \frac{1}{2}(2d+2)^{2k}-1$, and $q \le \frac{1}{2}(3d)^{2k}-1$ since $d \ge 2$. That is, $k \ge \frac{1}{2}\log_{3d}2(q+1) = \frac{1}{2}(\log_{3d}d)(\log_d 2(q+1)) \ge \frac{1}{6}\log_d 2(q+1)$ since $d \ge 2$. Therefore $k \ge \frac{1}{6}\log_d q$. $\qquad\square$

Note that $\log_{3d}d \to 1$ for large d, and the lower bound on k in Theorem 8 tends to $\frac{1}{2}\log_d 2(q+1)$.

5 Stack Layouts

The relationship between stack and track layouts is not as pronounced as that between queue and track layouts. However, we have the following results.

Lemma 23. *Every bipartite graph G admitting a (k,t)-track layout with maximum span s $(\le t-1)$ has a $2ks$-stack layout.*

Proof. Let $\{A,B\}$ be the bipartition of G. By Lemma 7, G has a $(2ks,2)$-track layout with tracks $(A,<_A)$ and $(B,<_B)$. Let σ be the vertex ordering of G where the vertices in A are ordered by $<_A$, the vertices in B are ordered by the reverse of $<_B$, and all the vertices of A appear before the vertices of B. Crossing edges in σ form an X-crossing in the $(2ks,2)$-track layout. Thus each of the $2ks$ edge colour classes is a stack in σ, and we have a $2ks$-stack layout of G. $\qquad\square$

We know of no result analogous to Lemma 23 for non-bipartite graphs. However there is a similar result for subdivisions.

Lemma 24. *Let G be a graph admitting a (k,t)-track layout with maximum span s. Then the subdivision G' of G with one division vertex per edge has a $s(k+1)$-stack layout.*

Proof. Let $\{(V_i,<_i) : 1 \le i \le t\}$ be a (k,t)-track layout of G with maximum span s, and with edge colouring $\{E_\ell : 1 \le \ell \le k\}$. Denote by $L(e)$ and $R(e)$ the endpoints of each edge e of G where $L(e) \in V_i$ and $R(e) \in V_j$ with $i < j$. Denote by $X(e)$ the division vertex in G' of e. For each $1 \le i \le t-1$ and $1 \le \alpha \le s$, let

$$X_{i,\alpha} \;=\; \{X(e) \,:\, e \in E(G),\ L(e) \in V_i,\ R(e) \in V_{i+\alpha}\} \ .$$

Since the maximum span is s, every division vertex of G' is in some $X_{i,\alpha}$. Order each $X_{i,\alpha}$ such that for all $X(e), X(f) \in X_{i,\alpha}$, we have $X(e) < X(f)$ whenever $L(f) <_i L(e)$, or $L(e) = L(f)$ and $R(f) <_{i+\alpha} R(e)$. Let σ be the vertex ordering of G' defined by

$$\big(V_1, X_{1,s}, X_{1,s-1}, \ldots, X_{1,1};\ V_2, X_{2,s}, X_{2,s-1}, \ldots, X_{2,1};\ \ldots;\ V_t\big) \ .$$

Note that $L(e) <_\sigma X(e) <_\sigma R(e)$ for every edge e of G. For all $1 \le \alpha \le s$ let

$$E_\alpha = \{L(e)X(e) \,:\, L(e) \in V_i,\ X(e) \in X_{i,\alpha}\} \ .$$

For all $1 \le \ell \le k$ and $0 \le \beta \le s-1$, let

$$E_{\ell,\beta} = \{X(e)R(e) \,:\, e \in E_\ell,\ L(e) \in V_i,\ i \equiv \beta \pmod{s}\} \ .$$

This partitions the edges of G' into $s(k+1)$ sets. We claim that no two edges in a single set cross in σ. Consider two edges e and f of G with no endpoint in common. Say $L(e) \in V_{i_1}$ and $L(f) \in V_{i_2}$.

Consider edges $L(e)X(e)$ and $L(f)X(f)$ both in some E_α. Without loss of generality $i_1 \le i_2$. If $i_1 < i_2$ then $L(e) <_\sigma X(e) <_\sigma L(f) <_\sigma X(f)$, and $L(e)X(e)$ and $L(f)X(f)$ do not cross. If $i_1 = i_2$ then without loss of generality $L(e) <_\sigma L(f)$. Since $L(e)X(e)$ and $L(f)X(f)$ are in E_α, both $X(e)$ and $X(f)$ are in $X_{i_1,\alpha}$. Thus $L(e) <_\sigma L(f) <_\sigma X(f) <_\sigma X(e)$, and $L(f)X(f)$ is nested inside $L(e)X(e)$. Thus each set E_α is a valid stack in σ.

Now suppose the edges $X(e)R(e)$ and $X(f)R(f)$ cross in σ. Without loss of generality $X(e) <_\sigma X(f) <_\sigma R(e) <_\sigma R(f)$. Say $R(e) \in V_{i_3}$ and $R(f) \in V_{i_4}$. Then $i_1 \le i_2 < i_3 \le i_4$. If $i_1 < i_2$ then $i_2 - i_1 < i_3 - i_1 \le s$. Thus $i_1 \not\equiv i_2 \pmod{s}$, and $X(e)R(e)$ and $X(f)R(f)$ are not in the same $E_{\ell,\beta}$. Now suppose $i_1 = i_2$. Since $X(e) <_\sigma X(f)$, we have $i_3 = i_4$ and $L(f) <_{i_1} L(e)$. Furthermore $R(e) <_{i_3} R(f)$ since $R(f) <_\sigma R(e)$. That is, e and f form an X-crossing in the track layout, and are thus coloured differently. Hence $X(e)R(e)$ and $X(f)R(f)$ are not in the same $E_{\ell,\beta}$.

Thus each E_α and each $E_{\ell,\beta}$ is a valid stack, and G' has a $s(k+1)$-stack layout. $\qquad\square$

By Theorem 3 and Lemma 24 with $s = 1$ we have:

Theorem 9. *For every integer $d \ge 2$, every q-queue graph has a $(d+1)$-stack subdivision with $3 + 4\lceil \log_d q \rceil$ division vertices per edge.* $\qquad\square$

The proof of the next result is similar to that of Theorem 3 and Lemma 24.

Theorem 10. *For every integer $d \ge 2$, every s-stack graph G has a $(d+1)$-stack subdivision with $4\lceil \log_d s \rceil - 2$ division vertices per edge.*

Proof. Let σ be the vertex ordering in an s-stack layout of G. Let $S = \lceil \log_d s \rceil$. Let H be the subdivision of G obtained by replacing each edge $e = vw$ by the path

$$v = (e,v,1), (e,v,2), \ldots, (e,v,2S), (e,w,2S), (e,w,2S-1), (e,w,2S-2), \ldots, (e,w,1) = w \ .$$

where each (e,v,i) is a vertex of H. Observe that H has $4S - 2$ division vertices per edge. Label the stacks of G by $0, 1, \ldots, s-1$. Let $D(\ell)$ be the d-ary representation of each stack ℓ including leading zeros, $0 \le \ell \le s-1$. Thus $D(\ell)$ is a string of exactly S characters over the alphabet $\{0, 1, \ldots, d-1\}$. Denote by $D_i(\ell)$ the first i characters of $D(\ell)$ for each $1 \le i \le S$. Also define $D_0(\ell) = \emptyset$ for all ℓ. Order each set $\{D_i(\ell) : 0 \le \ell \le s-1\}$ lexicographically. That is, for all $0 \le \ell_1, \ell_2 \le s-1$, $D_i(\ell_1) \prec_{\mathsf{lex}} D_i(\ell_2)$ if for some $1 \le j \le i$, $D_{j-1}(\ell_1) = D_{j-1}(\ell_2)$ and the jth character of $D_i(\ell_1)$ is less than the jth character of $D_i(\ell_2)$. Let π be the vertex ordering of H defined by the following list of prioritised rules:

(a) $i < j \implies (e,v,i) <_\pi (f,w,j)$
    $\big(\forall\, 1 \le i,j \le 2S,\ e,f \in E(G),\ v \in e,\ w \in f\big)$

(b) $v <_\sigma w \implies w = (f,w,1) <_\pi (e,v,1) = v$
    $\big(\forall\, e,f \in E(G),\ v \in e,\ w \in f\big)$

(c) $D_i(\mathsf{stack}(e)) \prec_{\mathsf{lex}} D_i(\mathsf{stack}(f)) \implies (e,v,2i) <_\pi (f,w,2i)$
    $\big(\forall\, 1 \le i \le S,\ e,f \in E(G),\ v \in e,\ w \in f\big)$

(d) $v <_\sigma w \implies (e,v,2i) <_\pi (f,w,2i)$
    $\big(\forall\, 1 \le i \le S,\ e,f \in E(G),\ v \in e,\ w \in f,\ D_i(\mathsf{stack}(e)) = D_i(\mathsf{stack}(f))\big)$

(e) $u <_\sigma v <_\sigma w$ or $w <_\sigma u <_\sigma v$ or $v <_\sigma w <_\sigma u \implies (vu, v, 2i) <_\pi (vw, v, 2i)$
$\left(\forall\, 1 \leq i \leq S,\ vu, vw \in E(G),\ D_i(\mathsf{stack}(vu)) = D_i(\mathsf{stack}(vw)) \right)$

(f) $(e, v, 2i) <_\pi (f, w, 2i) \implies (f, w, 2i + 1) <_\pi (e, v, 2i + 1)$
$\left(\forall\, 1 \leq i \leq S - 1,\ e, f \in E(G),\ v \in e,\ w \in f \right)$

For each $1 \leq i \leq 2s$, let $V_i = \{(e, v, i) : e \in E(G), v \in e\}$. Rule (a) implies that $\pi = (V_1, V_2, \ldots, V_{2s})$. By rule (b), V_1 (which equals $V(G)$) is ordered in the reverse order to σ. Rules (c), (d) and (e) order each V_{2i} ($1 \leq i \leq S$) firstly with respect to the lexicographical ordering of $\{D_i(\ell) : 0 \leq \ell \leq s - 1\}$, secondly by σ, and thirdly by a particular tie breaking rule for edges with a common endpoint, which is designed so that edges of the form $(e, v, 2S)(e, w, 2S)$ do not cross. Rule (f) orders each V_{2i+1} ($1 \leq i \leq S - 1$) with respect to the reverse order of V_{2i}.

We now colour the edges of H. For all $1 \leq i \leq S - 1$, $e \in E(G)$, and $v \in e$, let $\mathsf{col}((e, v, 2i)(e, v, 2i + 1)) = d$. For all edges $vw \in E(G)$, let $\mathsf{col}((e, v, 2S)(e, w, 2S)) = d$. For all $1 \leq i \leq S$, $e \in E(G)$, and $v \in e$, let $\mathsf{col}((e, v, 2i - 1)(e, v, 2i))$ be the ith character of $D(\mathsf{stack}(e))$. Thus $0 \leq \mathsf{col}((e, v, 2i - 1)(e, v, 2i)) \leq d - 1$. This construction in the case of $d = 2$ is illustrated in Figure 2.

We claim that no two monochromatic edges cross in π, and hence each colour class is a stack. First consider edges coloured d (Cases 1 and 2 below). Such edges are of the form $(e, v, 2S)(e, w, 2S)$, or $(e, v, 2i)(e, v, 2i + 1)$ with $1 \leq i \leq S - 1$. By rule (a), edges of the first type do not cross edges of the second type. Thus it suffices to consider each case separately.

Case 1: Consider edges $e' = (e, v, 2i)(e, v, 2i + 1)$ and $f' = (f, w, 2j)(f, w, 2j + 1)$ for some $1 \leq i, j \leq S - 1$. Without loss of generality $i \leq j$. If $i < j$ then by rule (a), $(e, v, 2i) <_\pi (e, v, 2i + 1) <_\pi (f, w, 2j) <_\pi (f, w, 2j + 1)$, and e' and f' do not cross. If $i = j$ then without loss of generality $(e, v, 2i) <_\pi (f, w, 2i)$. By rule (f), $(f, w, 2i + 1) <_\pi (e, v, 2i + 1)$, and f' is nested inside e' in π.

Case 2: Consider edges $e' = (e, v, 2S)(e, w, 2S)$ and $f' = (f, x, 2S)(f, y, 2S)$. Without loss of generality $\mathsf{stack}(e) \leq \mathsf{stack}(f)$, $v <_\sigma w$, and $x <_\sigma y$. If $\mathsf{stack}(e) < \mathsf{stack}(f)$ then $(e, v, 2S) <_\pi (e, w, 2S) <_\pi (f, x, 2S) <_\pi (f, y, 2S)$ by rule (c), and e' and f' do not cross in π. Otherwise $\mathsf{stack}(e) = \mathsf{stack}(f)$. Thus e and f do not cross in σ. Suppose that e and f have no common endpoint. Then by rule (d), the ordering of the endpoints of e and f in σ is preserved by the ordering of the endpoints of e' and f' in π. Since e and f do not cross in σ, e' and f' do not cross in π. Now consider the case that e and f have a common endpoint. Without loss of generality, either (i) $v = x <_\sigma w <_\sigma y$, (ii) $v <_\sigma w = x <_\sigma y$, or (iii) $v <_\sigma x <_\sigma y = w$. In these cases, by rules (d) and (e) we have (i) $(f, x, 2S) <_\pi (e, v, 2S) <_\pi (e, w, 2S) <_\pi (f, y, 2S)$, (ii) $(e, v, 2S) <_\pi (e, w, 2S) <_\pi (f, x, 2S) <_\pi (f, y, 2S)$, or (iii) $(e, v, 2S) <_\pi (f, x, 2S) <_\pi (f, y, 2S) <_\pi (e, w, 2S)$. In each case e' and f' do not cross in π.

We now prove that for each colour $0 \leq r \leq d - 1$, edges coloured r do not cross. Two edges colored r have the form $(e, v, 2i - 1)(e, v, 2i)$ and $(f, w, 2j - 1)(f, w, 2j)$. By rule (a) these edges do not cross if $i \neq j$. Thus it suffices to prove that two edges $e' = (e, v, 2i - 1)(e, v, 2i)$ and $f' = (f, w, 2i - 1)(f, w, 2i)$ with $\mathsf{col}(e') = \mathsf{col}(f')$ do not cross. To do so we distinguish the cases $i = 1$ and $2 \leq i \leq S$.
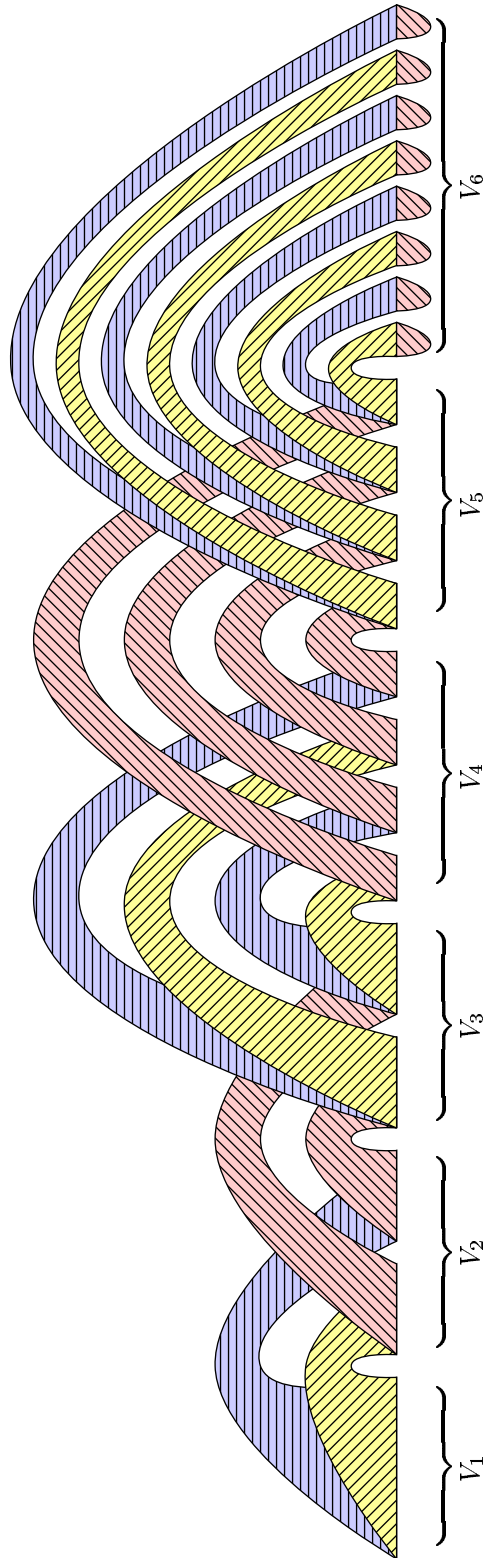
Figure 2: 3-stack subdivision of an 8-stack graph

Case 3: Consider edges $e' = (e, v, 1)(e, v, 2)$ and $f' = (f, w, 1)(f, w, 2)$. Then $v = (e, v, 1)$ and $w = (f, w, 1)$. If $v = w$ then e' and f' do not cross. Assume $v \neq w$. Without loss of generality $v <_\sigma w$. Thus $(f, w, 1) <_\pi (e, v, 1)$ by rule (b). Since $\text{col}(e') = \text{col}(f')$, we have $D_1(\text{stack}(e)) = D_1(\text{stack}(f))$, which implies $(e, v, 2) <_\pi (f, w, 2)$ by rule (d). Thus e' is nested inside f'.

Case 4: Now consider edges $e' = (e, v, 2i - 1)(e, v, 2i)$ and $f' = (f, w, 2i - 1)(f, w, 2i)$ for some $2 \leq i \leq S$. Without loss of generality $(f, w, 2i - 1) <_\pi (e, v, 2i - 1)$. By rule (f), $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$. That $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$ was determined by an application of either rule (c), (d) or (e). First suppose $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$ by rule (c). Then $D_{i-1}(\text{stack}(e)) \prec_{\text{lex}} D_{i-1}(\text{stack}(f))$, and thus $D_i(\text{stack}(e)) \prec_{\text{lex}} D_i(\text{stack}(f))$, which implies $(e, v, 2i) <_\pi (f, w, 2i)$ by rule (c). Thus e' is nested inside f' in π. Now suppose $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$ by rules (d) or (e). Then, $D_{i-1}(\text{stack}(e)) = D_{i-1}(\text{stack}(f))$. Since $\text{col}(e') = \text{col}(f')$, $D_i(\text{stack}(e)) = D_i(\text{stack}(f))$. If $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$ by rule (d), then $v <_\sigma w$ and $(e, v, 2i) <_\pi (f, w, 2i)$ again by rule (d). If $(e, v, 2i - 2) <_\pi (f, w, 2i - 2)$ by rule (e), then $v = w$ and $(e, v, 2i) <_\pi (f, w, 2i)$ again by rule (e). Thus in both cases e' is nested inside f' in π.

In all cases either e' and f' do not cross in π or they are coloured differently. Therefore each colour class is a valid stack in π. The number of colours is $d + 1$. Therefore we have a $(d + 1)$-stack layout of H. $\qquad\square$

Observe that Theorems 7, 9 and 10 together prove Theorem 1 in Section 1. Each of Theorems 9 and 10 with $d = 2$ prove the well known result that every graph has a 3-stack subdivision [13, 45, 46, 91]. A graph has a 2-stack layout if and only if it is a subgraph of a Hamiltonian planar graph [30]. Subdividing edges preserves planarity and non-planarity. Thus a non-planar graph does not have a 2-stack subdivision. It is therefore of interest to ask how many division vertices are needed for any planar graph to have a 2-stack subdivision? Pach and Wenger [105] prove that the subdivision G'' of a planar graph G with two division vertices per edge is the subgraph of a Hamiltonian planar graph, and hence has a 2-stack layout. This result is improved as follows. (Note that Pach and Wenger [105] were more interested in the number of vertices in the Hamiltonian supergraph rather than the number of division vertices per edge.)

Lemma 25. *For every planar graph G, the subdivision G' of G with one division vertex per edge is the subgraph of a Hamiltonian planar graph, and hence has a 2-stack layout.*

Proof. Without loss of generality G is a triangulation. Otherwise we can add edges to G so that every face is a 3-cycle. Let $V = V(G)$. Now subdivide every edge once. Let X be the set of these division vertices. Finally add a single vertex to each face adjacent to the six vertices on that face. Let Y be the set of these vertices. We obtain a planar triangulation H. Observe that $\{V, X, Y\}$ is a vertex 3-colouring of H. Thus every triangle of H contains one vertex from each of V, X and Y. Every such triangle forms a face of H. Therefore every triangle in H is a face, and H has no separating triangles. Since H is a triangulation, by the classical result of Whitney [122], H is Hamiltonian. The subgraph of H induced by $V \cup X$ is G'. $\qquad\square$

We conclude this section with the following elementary observation, which is analogous to Lemma 19 for queue layouts.

Lemma 26. *The subdivision G' of an s-stack graph G has an $(s + 1)$-stack layout.*

Proof. Consider an s-stack layout of G with vertex ordering σ. Denote the subdivision vertex of e in G' by $X(e)$. We now create a stack layout of G'. For each vertex v of G, let $e_1, e_2, \ldots, e_d$ be all the edges incident to v such that each $L(e_i) = v$, and $R(e_d) <_\sigma R(e_{d-1}) <_\sigma \cdots <_\sigma R(e_1)$. Add the division vertices $X(e_1), X(e_2), \ldots, X(e_d)$ immediately to the right of v in this order. Clearly for all edges e and f of G, the edges $L(e)X(e)$ and $L(e)X(f)$ of G' do not cross. Thus all these 'left' edges can be assigned to a single stack. Each 'right' edge $X(e)R(e)$ of G' inherits the stack assigned to e in G. Clearly no two right edges in the same stack cross. Thus G' has a $(s+1)$-stack layout. $\qquad\square$

6 Three-Dimensional Straight-Line Drawings

Vertex colourings and track layouts have previously been used to produce three-dimensional drawings with small volume. Pach *et al.* [103] proved that a c-colourable graph has a straight-line drawing with $\mathcal{O}(c^2 n^2)$ volume, and Dujmović *et al.* [41] proved that a graph with a t-track layout has a straight-line drawing with $\mathcal{O}(t^2 n)$ volume. In fact, track layouts and three-dimensional drawings are inherently related.

Theorem 11. [41] *Every graph G with track-number $\operatorname{tn}(G) \leq t$ has a $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(t^2 n)$ volume. Moreover, for every graph family $\mathcal{G}$, every n-vertex graph $G \in \mathcal{G}$ has a $\mathcal{F}(n) \times \mathcal{F}(n) \times \mathcal{O}(n)$ straight-line drawing if and only if the track-number $\operatorname{tn}(\mathcal{G}) \in \mathcal{F}(n)$, where $\mathcal{F}(n)$ is any family of functions closed under multiplication, such as $\mathcal{O}(1)$ or $\mathcal{O}(\operatorname{polylog} n)$.*

By Lemma 9(e) and Theorem 11 we have:

Theorem 12. *Every graph with n vertices, m edges and maximum degree Δ has a $\mathcal{O}((\Delta m)^{1/2}) \times \mathcal{O}((\Delta m)^{1/2}) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(\Delta m n)$ volume.* $\qquad\square$

By Lemma 9(f) and Theorem 11 we have:

Theorem 13. *Every graph with n vertices and m edges has a $\mathcal{O}(m^{2/3}) \times \mathcal{O}(m^{2/3}) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(m^{4/3} n)$ volume.* $\qquad\square$

In the following sequence of results we combine vertex colourings and track layouts to reduce the quadratic dependence on t in Theorem 11 to linear. This comes at the expense of a higher dependence on the number of colours c. However, in the intended applications c will be constant, or at least will be independent of the size of the graph. The proof of the next lemma is a further generalisation of the 'moment curve' method for producing three-dimensional graph drawings [31, 41, 103].

Lemma 27. *Let G be a graph with a vertex c-colouring $\{V_i : 0 \leq i \leq c-1\}$, and a track layout $\{T_{i,j} : 0 \leq i \leq c-1, 1 \leq j \leq t_i\}$, such that each $T_{i,j} \subseteq V_i$. Then G has a $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^5 t n')$ straight-line drawing, where $t = \max_i t_i$ and $n' = \max_{i,j} |T_{i,j}|$.*

Proof. Let p be the minimum prime such that $p \geq c$. Then $p < 2c$ by Bertrand's postulate. Let $v(i,j,k)$ denote the kth vertex in track $T_{i,j}$. Define

$$Y(i,j) = p(2\,i\,t + j) + (i^2 \bmod p), \text{ and}$$
$$Z(i,j,k) = p(20\,c\,i\,n' \cdot Y(i,j) + k) + (i^3 \bmod p) \ .$$

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Position each vertex $v(i, j, k)$ at the gridpoint $(i, Y(i, j), Z(i, j, k))$, and draw each edge as a line-segment between its endpoints. Since $Y(i, j) \in \mathcal{O}(c^2 t)$ and $Z(i, j, k) \in \mathcal{O}(c^3 n' \cdot Y(i, j))$, the drawing is $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^5 tn')$.

Observe that the tracks from a single colour class are within a distinct YZ-plane, each track occupies a distinct vertical line, and the Z-coordinates of the vertices within a track preserve the given ordering of that track. In addition, the Y-coordinates satisfy the following property.

Claim 1. *For all distinct colours i_1 and i_2 and for all $1 \le j_1, j_2 \le t$, we have $2c \,|\, (Y(i_1, j_1) - Y(i_2, j_2)) \,|$ is greater than the Y-coordinate of any vertex.*

Proof. Without loss of generality $i_1 > i_2$. Observe that every Y-coordinate is less than $p(2(c-1)t + t) + p = p(2\,ct - t + 1) \le 2cpt$. Now $2c \,|\, Y(i_1, j_1) - Y(i_2, j_2) \,| \; > \; 2c \,|\, p(2\,i_1\,t + 1) - p(2\,i_2\,t + t + 1) \,| \; \ge \; 2cp \,|\, 2(i_2 + 1)t - (2i_2 t + t) \,| \; = \; 2cpt$. $\qquad \square$

We first prove that the only vertices each edge intersects are its own endpoints. It suffices to prove that if three tracks are collinear in the XY-plane then they are all from the same colour class. Loosely speaking, an edge does not pass through any track. Clearly two tracks from the same colour class are not collinear (in the XY-plane) with a third track from a distinct colour class. Thus we need only consider tracks $\{T(i_\alpha, j_\alpha) : 1 \le i_\alpha \le 3\}$ from three distinct colour classes $\{i_1, i_2, i_3\}$. Let R be the determinant,

$$R = \begin{vmatrix} 1 & i_1 & Y(i_1, j_1) \\ 1 & i_2 & Y(i_2, j_2) \\ 1 & i_3 & Y(i_3, j_3) \end{vmatrix} .$$

If the tracks $\{T(i_\alpha, j_\alpha) : 1 \le i_\alpha \le 3\}$ are collinear in the XY-plane then $R = 0$. However $Y(i, j) \equiv i^2 \pmod{p}$, and thus

$$R \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} = \prod_{1 \le \alpha < \beta \le 3} (i_\alpha - i_\beta) \; \not\equiv 0 \pmod{p} .$$

since $i_\alpha \ne i_\beta$, and p is a prime greater than any $i_\alpha - i_\beta$. Thus $R \ne 0$, and the tracks $\{T(i_\alpha, j_\alpha) : 1 \le i_\alpha \le 3\}$ are not collinear in the XY-plane. Hence the only vertices that an edge intersects are its own endpoints.

It remains to prove that there are no edge crossings. Consider two edges e and e' with distinct endpoints $v(i_\alpha, j_\alpha, k_\alpha)$, $1 \le \alpha \le 4$. (Clearly edges with a common endpoint do not cross.) Let $Y_\alpha = Y(i_\alpha, j_\alpha)$. Consider the following determinant

$$D = \begin{vmatrix} 1 & i_1 & Y_1 & Z(i_1, j_1, k_1) \\ 1 & i_2 & Y_2 & Z(i_2, j_2, k_2) \\ 1 & i_3 & Y_3 & Z(i_3, j_3, k_3) \\ 1 & i_4 & Y_4 & Z(i_4, j_4, k_4) \end{vmatrix} .$$

If e and e' cross then their endpoints are coplanar, and $D = 0$. Thus it suffices to prove that $D \ne 0$. We proceed by considering the number $N = |\{i_1, i_2, i_3, i_4\}|$ of distinct colours assigned to the four endpoints of e and e'. Clearly $N \in \{2, 3, 4\}$.

Case $N = 4$: Since $Y_\alpha \equiv i_\alpha^2 \pmod{p}$ and $Z(i_\alpha, j_\alpha, k_\alpha) \equiv i_\alpha^3 \pmod{p}$,

$$D \equiv \begin{vmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{vmatrix} = \prod_{1 \leq \alpha < \beta \leq 4} (i_\alpha - i_\beta) \not\equiv 0 \pmod{p} ,$$

since $i_\alpha \neq i_\beta$, and p is a prime greater than any $i_\alpha - i_\beta$. Thus $D \neq 0$.

Case $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ where

$$
\begin{aligned}
S_0 &= 4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4) \\
S_1 &= p(Y_2 - Y_1)\big(k_3(i_4 - i_1) - k_4(i_3 - i_1)\big) \\
S_2 &= p(i_4 - i_3)(k_2 Y_1 - k_1 Y_2) \\
S_3 &= p(k_2 - k_1)(Y_4(i_3 - i_1) - Y_3(i_4 - i_1)) \\
S_4 &= (Y_2 - Y_1)\big((i_3 - i_4)(i_1^3 \bmod p) - (i_3 - i_1)(i_4^3 \bmod p) + (i_4 - i_1)(i_3^3 \bmod p)\big) .
\end{aligned}
$$

If $Y_1 = Y_2$ then e and e' do not cross, since no three tracks from distinct colour classes are collinear in the XY-plane. Assume $Y_1 \neq Y_2$. If $i_3 < i_1 < i_4$ or $i_4 < i_1 < i_3$ then e and e' do not cross, simply by considering the projection in the XY-plane. Thus $i_1 < i_3, i_4$ or $i_1 > i_3, i_4$, which implies

$$(i_4 - i_1)(i_3 - i_1) > |i_4 - i_3| . \tag{3}$$

Claim 2. *If* $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ *and* $|S_0| \geq |S_4|$ *then* $D \neq 0$.

Proof. To prove that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ is nonzero it suffices to show that $D' = \pm 5|S_0| \pm |S_1| \pm |S_2| \pm |S_3| \pm |S_4|$ is nonzero for all combinations of pluses and minuses. Consider $X = \pm |S_1| \pm |S_2| \pm |S_3| \pm |S_4|$ for some combination of pluses and minuses. Since $|S_1| \leq |S_0|, |S_2| \leq |S_0|, |S_3| \leq |S_0|$, and $|S_4| \leq |S_0|$, we have $-4|S_0| \leq X \leq 4|S_0|$. Since $S_0 \neq 0$, we have $5|S_0| + X \neq 0$ and $-5|S_0| + X \neq 0$. That is, all values of D' are nonzero. Therefore $D \neq 0$. $\qquad\square$

Therefore, to prove that $D \neq 0$ it suffices to show that $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ and $|S_0| \geq |S_4|$. We will use the following elementary facts regarding absolute values:

$$
\begin{aligned}
\forall a_1, \ldots, a_k \in \mathbb{R} \quad |a_1 a_2 \ldots a_k| &= |a_1||a_2| \cdots |a_k| \\
|a_1 + a_2 + \cdots + a_k| &\leq |a_1| + |a_2| + \cdots + |a_k| \leq k \cdot \max\{|a_1|, |a_2|, \ldots, |a_k|\} .
\end{aligned}
$$

• First we prove that $|S_0| \geq |S_1|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(Y_2 - Y_1)\big(k_3(i_4 - i_1) - k_4(i_3 - i_1)\big)| .$$

Hence,

$$
\begin{aligned}
|S_0| > |S_1| &\Longleftarrow 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \geq |k_3(i_4 - i_1) - k_4(i_3 - i_1)| . \\
&\Longleftarrow 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \geq 2 \cdot \max\{|k_4(i_3 - i_1)|, |k_3(i_4 - i_1)|\} .
\end{aligned}
$$

Since $n' \geq k_3, k_4$ and $|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_1| \iff |i_3 - i_1||i_4 - i_1| \geq \max\{|i_3 - i_1|, |i_4 - i_1|\} \ .$$

Thus $|S_0| \geq |S_1|$ since $|i_3 - i_1| \geq 1$ and $|i_4 - i_1| \geq 1$.

• Now we prove that $|S_0| \geq |S_2|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(i_4 - i_3)(k_2Y_1 - k_1Y_2)| \ .$$

By (3) and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_2| \iff |4cn'(Y_3 - Y_4)| \geq |k_2Y_1 - k_1Y_2| \ .$$
$$\iff |2cn'(Y_3 - Y_4)| \geq \max\{|k_2Y_1|, |k_1Y_2|\} \ ,$$

which holds since $n' \geq k_1, k_2$ and $|2c(Y_3 - Y_4)| \geq \max\{Y_1, Y_2\}$ by Claim 1.

• Now we prove that $|S_0| \geq |S_3|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(k_2 - k_1)\big(Y_4(i_3 - i_1) - Y_3(i_4 - i_1)\big)| \ .$$

Since $n' \geq |k_2 - k_1|$ and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_3| \iff |4c(i_3 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \geq |Y_4(i_3 - i_1) - Y_3(i_4 - i_1)| \ .$$
$$\iff |2c(i_3 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \geq \max\{|Y_4(i_3 - i_1)|, |Y_3(i_4 - i_1)|\} \ ,$$

which holds since $|2c(Y_3 - Y_4)| \geq \max\{Y_1, Y_2\}$ by Claim 1.

• Finally we prove that $|S_0| \geq |S_4|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq$$
$$|(Y_2 - Y_1)\big((i_3 - i_4)(i_1^3 \bmod p) - (i_3 - i_1)(i_4^3 \bmod p) + (i_4 - i_1)(i_3^3 \bmod p)\big)| \ .$$

Since $cn'|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_4| \iff |3p(i_3 - i_1)(i_4 - i_1)| \geq$$
$$|(i_3 - i_4)(i_1^3 \bmod p) - (i_3 - i_1)(i_4^3 \bmod p) + (i_4 - i_1)(i_3^3 \bmod p)|$$
$$\iff |3p(i_3 - i_1)(i_4 - i_1)| \geq$$
$$3 \cdot \max\{|(i_3 - i_4)(i_1^3 \bmod p)|, |(i_3 - i_1)(i_4^3 \bmod p)|, |(i_4 - i_1)(i_3^3 \bmod p)|\}$$
$$\iff |(i_3 - i_1)(i_4 - i_1)| \geq \max\{|i_3 - i_4|, |i_3 - i_1|, |i_4 - i_1|\} \ ,$$

which holds by (3).

Case $N = 2$: Without loss of generality $i_1 = i_2 \neq i_3 = i_4$. If $Y_1 = Y_2$ and $Y_3 = Y_4$ then e and e' do not cross as otherwise there would be an X-crossing in the track layout. If $Y_1 = Y_2$ and $Y_3 \neq Y_4$ (or $Y_1 \neq Y_2$ and $Y_3 = Y_4$) then e and e' do not cross, by considering the projection in the XY-plane. Thus we can assume that $Y_1 \neq Y_2$ and $Y_3 \neq Y_4$. It follows that

$$D = p(i_1 - i_3)\big(5 \cdot 4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1) + (k_1 - k_2)(Y_4 - Y_3) + (k_4 - k_3)(Y_2 - Y_1)\big) \ .$$

As in Claim 2, to show that $D \neq 0$ it suffices to show that

$$\left| 4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1) \right| \geq \left| (k_1 - k_2)(Y_4 - Y_3) \right| , \tag{4}$$

and

$$\left| 4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1) \right| \geq \left| (k_4 - k_3)(Y_2 - Y_1) \right| . \tag{5}$$

Inequalities (4) and (5) hold since $n' > |k_1 - k_2|$ and $n' > |k_4 - k_3|$. $\qquad\square$

Note that the constant 20 in the definition of $Z(i,j,k)$ in the proof of Lemma 27 is chosen to enable a relatively simple proof. It is easily seen that it can be reduced. The proof of the next lemma is based on an idea of Pach *et al.* [103] for balancing the size of the colour classes in a vertex colouring.

Lemma 28. *Let G be an n-vertex graph with a c-colouring $\{V_i : 0 \leq i \leq c - 1\}$, and a track layout $\{T_{i,j} : 0 \leq i \leq c - 1, 1 \leq j \leq t_i\}$, such that each $T_{i,j} \subseteq V_i$. Let $k = \sum_i t_i$ be the total number of tracks. Then G has a $\mathcal{O}(c) \times \mathcal{O}(ck) \times \mathcal{O}(c^4 n)$ straight-line drawing.*

Proof. Replace each track by tracks of size exactly $\lceil \frac{n}{k} \rceil$, except for at most one track of size at most $\lceil \frac{n}{k} \rceil$. Order the vertices within each track according to the original track, and consider the new tracks to belong to the same colour class as the original. Clearly no X-crossing is created. Within V_i there are now at most $t_i + |V_i| / \lceil \frac{n}{k} \rceil$ tracks. The total number of tracks is $\sum_i (t_i + |V_i| / \lceil \frac{n}{k} \rceil) \leq 2k$. For each colour class V_i, partition the set of tracks in V_i into sets of size exactly $\lceil \frac{2k}{c} \rceil$, except for one set of size at most $\lceil \frac{2k}{c} \rceil$. Consider each set to correspond to a colour. The number of colours is now at most $c + 2k / \lceil \frac{2k}{c} \rceil \leq 2c$. Applying Lemma 27 with $2c$ colours, $n' = \lceil \frac{n}{k} \rceil$, and $t = \lceil \frac{2k}{c} \rceil$, we obtain the desired drawing. $\qquad\square$

Theorem 14. *Every c-colourable graph G with n vertices and track-number $\mathsf{tn}(G) \leq t$ has a $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^4 n)$ straight-line drawing.*

Proof. Let $\{V_i : 0 \leq i \leq c - 1\}$ be a c-colouring of G. Let $\{T_j : 1 \leq j \leq t\}$ be a t-track layout of G. For all $0 \leq i \leq c - 1$ and $1 \leq j \leq t$, let $T_{i,j} = V_i \cap T_j$. Then $\{V_i : 0 \leq i \leq c - 1\}$ and $\{T_{i,j} : 0 \leq i \leq c-1, 1 \leq j \leq t\}$ satisfy Lemma 28 with $k = ct$. Thus G has the desired drawing. $\qquad\square$

In the case of bipartite graphs we have a simple proof of Theorem 14 with improved constants.

Lemma 29. *Every n-vertex bipartite graph G with track-number $\mathsf{tn}(G) \leq t$ has a $2 \times t \times n$ straight-line drawing.*

Proof. Let $\{A, B\}$ be the bipartition of $V(G)$. Let $\{T_i : 1 \leq i \leq t\}$ be a t-track layout of G. For each $1 \leq i \leq t$, let $A_i = T_i \cap A$ and $B_i = T_i \cap B$. Order each A_i and B_i as in T_i. Place the jth vertex in A_i at $(0, i, j + \sum_{k=1}^{i-1} |A_k|)$. Place the jth vertex in B_i at $(1, t - i + 1, j + \sum_{k=1}^{i-1} |B_k|)$. The drawing is thus $2 \times t \times n$. Let $A_i B_j$ be the set of edges with one endpoint in A_i and the other in B_j. There is no crossing between edges in $A_i B_j$ and $A_i B_j$ as otherwise there would be an X-crossing in the track layout. Clearly there is no crossing between edges in $A_i B_j$ and $A_i B_k$ for $j \neq k$. Suppose there is a crossing between edges in $A_i B_j$ and $A_k B_\ell$ with $i \neq k$ and $j \neq \ell$, and without loss of generality $i < k$. Then the projections of the edges in the XY-plane also cross, and thus $j < \ell$. This implies that the projections of the edges in the XZ-plane do not cross, and thus the edges do not cross. $\qquad\square$

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Lemma 9(f) and Theorem 14 imply:

Theorem 15. *Every c-colourable graph with n vertices and m edges has a $\mathcal{O}(c) \times \mathcal{O}(c^2 m^{2/3}) \times \mathcal{O}(c^4 n)$ drawing with $\mathcal{O}(c^6 m^{2/3} n)$ volume.* □

The next result is one of the main contributions of this paper.

Theorem 16. *Every planar graph with n vertices has a $\mathcal{O}(1) \times \mathcal{O}(n^{1/2}) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(n^{3/2})$ volume.*

Proof. Planar graphs have $\mathcal{O}(n^{1/2})$ path-width (see [15]), and thus have $\mathcal{O}(n^{1/2})$ track-number by Lemma 9(a). The result follows from Theorem 14 since planar graphs are 4-colourable. □

The following generalisation of Theorem 16 for graphs G with genus γ follows from Lemma 9(c), Theorem 14, and the classical result of Heawood [76] that $\chi(G) \in \mathcal{O}(\gamma^{1/2})$.

Theorem 17. *Every n-vertex graph with genus γ has a $\mathcal{O}(\gamma^{1/2}) \times \mathcal{O}(\gamma^{3/2} n^{1/2}) \times \mathcal{O}(\gamma^2 n)$ straight-line drawing with $\mathcal{O}(\gamma^4 n^{3/2})$ volume.* □

The next generalisation of Theorem 16 for graphs with no K_h-minor follows from Lemma 9(d), Theorem 14, and the result independently due to Kostochka [84] and Thomason [120] that $\chi(G) \in \mathcal{O}(h \log^{1/2} h)$ (see [38]).

Theorem 18. *Every n-vertex graph with no K_h-minor has a $\mathcal{O}(h \log^{1/2} h) \times \mathcal{O}(h^{7/2} \log h \cdot n^{1/2}) \times \mathcal{O}(h^4 \log^2 h \cdot n)$ straight-line drawing with volume $\mathcal{O}(h^{17/2} \log^{7/2} h \cdot n^{3/2})$.* □

Finally we consider the maximum degree as a parameter. By the sequential greedy algorithm, G is $(\Delta + 1)$-colourable. Thus by Lemma 9(e) and Theorem 14 we have:

Theorem 19. *Every graph with n vertices, m edges, and maximum degree Δ has a $\mathcal{O}(\Delta) \times \mathcal{O}(\Delta^{5/2} m^{1/2}) \times \mathcal{O}(\Delta^4 n)$ straight-line drawing with $\mathcal{O}(\Delta^{15/2} m^{1/2} n)$ volume.* □

By Theorems 17, 18 and 19 and since graphs with constant maximum degree have $\mathcal{O}(n)$ edges we have:

Corollary 6. *Every n-vertex graph with constant genus, or with no K_h-minor for some constant h, or with constant maximum degree has a $\mathcal{O}(1) \times \mathcal{O}(n^{1/2}) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(n^{3/2})$ volume.* □

7 Three-Dimensional Polyline Drawings

We first prove results for 1-bend drawings.

Theorem 20. *Every c-colourable q-queue graph G with n vertices and m edges has a $2 \times c(q+1) \times (n+m)$ polyline drawing with one bend per edge. The volume is $2c(q+1)(n+m)$.*

Proof. The subdivision G' of G with one division vertex per edge is bipartite and has $n+m$ vertices. By Lemma 12(b) and Lemma 29, G' has a $2 \times c(q+1) \times (n+m)$ straight-line drawing, which is the desired polyline drawing of G. □

The next result applies a construction of Calamoneri and Sterbini [26].

Theorem 21. *Every graph G with n vertices and m edges has an $n \times m \times 2$ polyline drawing with one bend per edge.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be an arbitrary vertex ordering of G. Let $(x_1, x_2, \ldots, x_m)$ be an arbitrary ordering of the division vertices of G'. Place each v_i at $(i, 0, 0)$ and each x_j at $(0, j, 1)$. Clearly the endpoints of any two disjoint edges of G' are not coplanar (see [26]). Thus no two edges cross, and we have an $n \times m \times 2$ straight-line drawing of G', which is a 1-bend polyline drawing of G. $\square$

Now consider 2-bend drawings. For every q-queue graph G, the subdivision G'' is obviously 3-colourable. Thus by Lemma 12(c) and Theorem 14, G has a $\mathcal{O}(1) \times \mathcal{O}(q) \times \mathcal{O}(n + m)$ polyline drawing with two bends per edge. This result can be improved as follows.

Theorem 22. *Every q-queue graph G with n vertices and m edges has a $2 \times 6q \times n$ polyline drawing with two bends per edge. The volume is $12qn$, which is in $\mathcal{O}(n\sqrt{m})$.*

Proof. Let $\sigma = (v_1, v_2, \ldots, v_n)$ be the vertex ordering in a q-queue layout of G. Denote by $(L(e), A(e), B(e), R(e))$ the path replacing e in G'', where $L(e) <_\sigma R(e)$. Partition each queue into queues of n edges except for one queue of at most n edges. We obtain at most $\frac{m}{n} + q$ queues. Since $m < 2qn$ by Lemma 15, the number of queues is at most $3q$. Let $\{E_\ell : 1 \le \ell \le 3q\}$ be the queues. Order the edges in each queue E_ℓ according to the queue order (see (1)). Put each vertex v_i at $(0, 0, i)$. If e is the jth edge in the ordering of E_ℓ, put the division vertices $A(e)$ at $(1, 2\ell, j)$ and $B(e)$ at $(1, 2\ell + 1, j)$. Observe that the projection of the drawing onto the XY-plane is planar. Thus the only possible crossings occur between edges contained in a plane parallel with the Z-axis. Thus an X-crossing could only occur between pairs of edges $\{L(e)A(e), L(f)A(f)\}$, $\{A(e)B(e), A(f)B(f)\}$, or $\{B(e)R(e), B(f)R(f)\}$, where e and f are in a single queue E_ℓ. Suppose $e <_\ell f$. Then the Z-coordinates satisfy: $Z(L(e)) < Z(L(f))$, $Z(R(e)) < Z(R(f))$, $Z(A(e)) < Z(A(e))$, and $Z(B(e)) < Z(B(f))$. Thus there is no crossing. The drawing is at most $2 \times 6q \times n$ with volume $12qn$, which is in $\mathcal{O}(n\sqrt{m})$ by Theorem 5. $\square$

Theorem 23. *Let G be a q-queue graph with n vertices and m edges. For every $\epsilon > 0$, G has a $2 \times 3\lceil q^\epsilon \rceil \times (n + (5 + \frac{2}{\epsilon})m)$ polyline drawing with at most $5 + \frac{2}{\epsilon}$ bends per edge. The volume is $\mathcal{O}(q^\epsilon(n + \frac{m}{\epsilon}))$. For constant ϵ there are $\mathcal{O}(1)$ bends per edge and the volume is $\mathcal{O}(q^\epsilon(n + m))$, which is in $\mathcal{O}(n^\epsilon(n + m))$.*

Proof. Let $d = \lceil q^\epsilon \rceil$. By Theorem 4, G has a subdivision H with $3 + 2\lceil \log_d q \rceil$ division vertices per edge such that the track-number $\mathsf{tn}(H) \le 3d$. Since the number of division vertices per edge is odd, every cycle in H is even, and H is bipartite. Now $\log_d q \le \frac{1}{\epsilon}$. Thus H has at most $3 + 2\lceil \frac{1}{\epsilon} \rceil < 5 + \frac{2}{\epsilon}$ division vertices per edge, and $\mathsf{tn}(H) \le 3\lceil q^\epsilon \rceil$. The number of vertices of H is at most $n + (5 + \frac{2}{\epsilon})m$. By Lemma 29, H has a $2 \times 3\lceil q^\epsilon \rceil \times (n + (5 + \frac{2}{\epsilon})m)$ straight-line drawing, which is the desired polyline drawing of G. The other claims immediately follow since $q \le n$. $\square$

Theorem 24. *Every q-queue graph G with n vertices and m edges has a $2 \times 6 \times (n + m(3 + 2\lceil \log_2 q \rceil))$ polyline drawing with $\mathcal{O}(\log q)$ bends per edge. The volume is $\mathcal{O}(n + m \log q)$ volume, which is in $\mathcal{O}(n + m \log n)$.*

Proof. By Theorem 4, G has a subdivision H with $3 + 2\lceil \log_2 q \rceil$ division vertices per edge, and track-number $\mathsf{tn}(H) \le 6$. Since the number of division vertices per edge is odd, every cycle in H is even, and H is bipartite. The number of vertices of H is $n + m(3 + 2\lceil \log_2 q \rceil)$. By Lemma 29, H has a $2 \times 6 \times (n + (3 + 2\lceil \log_2 q \rceil)m)$ straight-line drawing, which is the desired drawing of G. The volume is $\mathcal{O}(n + m \log n)$ since $q \le n$. (Since the number of tracks is constant, Theorem 11 could also be applied here to obtain the same asymptotic volume bound.) $\qquad\square$

8 Open Problems

We conclude by discussing some open problems.

• Does every graph have a straight-line drawing with $\mathcal{O}(nm)$ volume? Does every graph with constant chromatic number have a straight-line drawing with $\mathcal{O}(n\sqrt{m})$ volume? These bounds match the lower bounds for K_n and $K_{n,n}$, and would make edge-sensitive improvements to the existing upper bounds of $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2)$, respectively. These edge-sensitive bounds would be implied by Theorems 11 and 14 should every graph have $\mathcal{O}(\sqrt{m})$ track-number. In turn, this bound on track-number would be implied if strong star chromatic number is $\mathcal{O}(\sqrt{m})$.

• As mentioned in Section 1, Felsner *et al.* [51] ask whether every planar graph has a three-dimensional straight-line drawing with $O(n)$ volume. By Theorem 11 or Theorem 14, this question has an affirmative answer if every planar graph has $\mathcal{O}(1)$ track-number. Whether every planar graph has $\mathcal{O}(1)$ track-number is an open problem due to H. de Fraysseix [private communication, 2000], and by Theorem 6 is equivalent to the following open problem due to Heath *et al.* [71, 75]. Do planar graphs have constant queue-number? We make the following contribution to the study of this problem.

Lemma 30. *Let $\mathcal{F}(n)$ be the family of functions $\mathcal{O}(1)$ or $\mathcal{O}(\mathrm{polylog}\, n)$. Then every n-vertex planar graph has queue-number in $\mathcal{F}(n)$ if and only if every n-vertex bipartite Hamiltonian planar graph has queue-number in $\mathcal{F}(n)$.*

Proof. The ($\Longrightarrow$) direction is immediate. Suppose that every n-vertex bipartite Hamiltonian planar graph has queue-number at most some function $q(n) \in \mathcal{F}(n)$. Let G be an arbitrary planar graph with n vertices. Without loss of generality G is a triangulation. Let H be the Hamiltonian planar triangulation defined in Lemma 25, where G' is a subgraph of H. Let C be a Hamiltonian cycle of H. We now construct a bipartite Hamiltonian planar graph W from H such that G' is a subgraph of W. Consider a face f of G'. Let x be the vertex adjacent to every vertex of f in H. Exactly two edges incident to x are in C. Say $xv, xw \in C$, where $v, w \in f$. Delete all the edges incident to x except xv and xw. Clearly the resulting graph remains Hamiltonian. In the case that the distance from v to w along the boundary of f is odd, subdivide the edge xv. The resulting graph W is clearly Hamiltonian. It is easily verified that each face of W is an even cycle. Thus W is bipartite. Observe that W has $n + (3n - 6) + 2(2n - 4) < 8n$ vertices. Let $q' = q(8n)$. By assumption, the queue-number $\mathsf{qn}(W) \le q'$, and since G' is a subgraph of W, we have $\mathsf{qn}(G') \le q'$. By Lemma 20, G has queue-number at most $q'(2q' + 1) \in \mathcal{F}(n)$. $\qquad\square$

• Since every planar graph has a 4-stack layout [126], the above question is a special case of a more general question by Heath *et al.* [71]: is queue-number bounded by stack-number? For this problem we make the following contribution which is analogous to Lemma 30 for planar graphs.

Lemma 31. *Queue-number is bounded by stack-number if and only if every bipartite 3-stack graph has constant queue-number.*

Proof. The ($\Longrightarrow$) direction is immediate. Suppose that every 3-stack graph has queue-number at most some constant q. Let G be a graph with stack-number s. By Theorem 10 with $d = 2$, G has a 3-stack subdivision H with $4\lceil \log_2 s \rceil - 2$ division vertices per edge. Each edge of G has an even number of division vertices in H, and thus H is not necessarily bipartite. To make H bipartite, for each edge e of G, replace the edge $(e, v, 2S)(e, w, 2S)$ of H (which is coloured d) by $(e, v, 2S)(e, v, 2S + 1)(e, w, 2S)$ where $v <_\sigma w$, and place $(e, v, 2S + 1)$ immediately to the right of $(e, v, 2S)$ in π. Let the colour of both edges $(e, v, 2S)(e, v, 2S + 1)$ and $(e, v, 2S + 1)(e, w, 2S)$ be d. Since $(e, v, 2S)$ and $(e, v, 2S + 1)$ are consecutive in π, the edge $(e, v, 2S)(e, v, 2S + 1)$ does not cross any other edge, and by the same argument used in Case 2 of Theorem 10, each edge $(e, v, 2S + 1)(e, w, 2S)$ does not cross any other edge coloured d. We obtain a 3-stack subdivision of G with $4\lceil \log_2 s \rceil - 1$ division vertices per edge, which is thus bipartite, and by assumption, has queue-number at most q. By Lemma 22, G has queue-number at most $\frac{1}{2}(2q + 2)^{2(4\lceil \log_2 s \rceil - 1)} - 1$. Since q is constant, queue-number is bounded by stack-number. $\square$

The proof of Lemma 31 illustrates the significance of Theorem 10 in comparison with previous results on 3-stack subdivisions [45, 46, 91]. For Lemma 31 to hold, it is essential that the number of division vertices per edge in H is some function of the stack-number of G.

• Heath *et al.* [71] also asked whether stack-number is bounded by queue-number. Note that there is an infinite class of graphs $\mathcal{G}$, such that $\mathsf{sn}(\mathcal{G}) \in \Omega(3^{\mathsf{qn}(\mathcal{G})})$ [71]. We make the following contribution to this problem. Define the *sub-track-number* of a graph G, denoted by $\mathsf{xn}(G)$, to be the minimum k such that G' has a $(k, 2)$-track layout. This is well-defined since G' is bipartite.

Lemma 32. *Stack-number is bounded by queue-number if and only if stack-number is bounded by X-number.*

Proof. ($\Longrightarrow$) Suppose that stack-number is bounded by queue-number. That is, there exists a function f such that $\mathsf{sn}(G) \leq f(\mathsf{qn}(G))$ for every graph G. Let $k = \mathsf{xn}(G)$. That is, G' has a $(k, 2)$-track layout. By Lemma 18, G' has a k-queue layout, and by Lemma 20, G has a $k(2k+1)$-queue layout. Thus $\mathsf{sn}(G) \leq f(k(2k+1))$. Therefore stack-number is bounded by X-number.

($\Longleftarrow$) Suppose that stack-number is bounded by X-number. That is, there exists a function f such that $\mathsf{sn}(G) \leq f(\mathsf{xn}(G))$ for every graph G. Let $q = \mathsf{qn}(G)$. By Lemma 12(a), G' has a $(q+1, 2)$-track layout. That is, $\mathsf{xn}(G) \leq q + 1$. By assumption, $\mathsf{sn}(G) \leq f(q + 1)$. Therefore stack-number is bounded by queue-number. $\square$

The following conjecture is equivalent to a conjecture of Blankenship and Oporowski [13].

Conjecture 1. There exists a function f such that for any s-stack subdivision of a graph G with at most one division vertex per edge, G has a $f(s)$-stack layout.

Blankenship and Oporowski [13] proved Conjecture 1 for complete graphs G using an elegant Ramsey-theoretic argument. We now prove that an affirmative answer to this conjecture would imply that stack number is bounded by queue-number.

Lemma 33. *Conjecture 1 implies that stack-number is bounded by queue-number.*

Proof. Conjecture 1 implies there exists a function f^* such that for any s-stack subdivision of a graph G with at most k division vertices per edge, G has a $f^*(s, k)$-stack layout. Let G be a graph with $q = \mathsf{qn}(G)$. By Theorem 9, G has a 3-stack subdivision with $3 + 4\lceil \log_2 q \rceil$ division vertices per edge. Thus $\mathsf{sn}(G) \leq f^*(3, 3 + 4\lceil \log_2 q \rceil)$, and stack-number is bounded by queue-number. $\qquad\square$

Acknowledgements

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