Simultaneous Diagonal Flips in Plane Triangulations

Prosenjit Bose, Jurek Czyzowicz,
Zhicheng Gao, Pat Morin,
David R. Wood

TR-03-05 July 2003

School of Computer Science, Carleton University
Ottawa, Canada, KIS 5B6

Simultaneous Diagonal Flips in Plane Triangulations *

Prosenjit Bose[‡] Jurek Czyzowicz[§] Zhicheng Gao[¶] Pat Morin[‡]
David R. Wood[‡]

July 25, 2003

Abstract

Simultaneous diagonal flips in plane triangulations are investigated. It is proved that every triangulation with at least six vertices has a simultaneous flip into a 4-connected triangulation. This result is used to prove that for any two n-vertex triangulations, there exists a sequence of $O(\log n)$ simultaneous flips to transform one into the other. The total number of individual edges flipped in this sequence of $O(\log n)$ simultaneous flips is $O(n)$. The maximum size of a simultaneous flip is then studied. It is proved that every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges. On the other hand, it is shown that every triangulation has a simultaneous flip of at least $\frac{2}{3}(n-2)$ edges.

1 Introduction

Let $G=(V,E)$ be a *triangulation*; that is, a simple planar graph with a fixed plane (combinatorial) embedding such that every face consists of three edges. Let vw be an edge of G. Let x and y be the vertices such that vw is incident to the faces vwx and wvy. If xy is not an edge of G then vw is *flippable*. Suppose vw is flippable. Let G' be the triangulation obtained from G by deleting vw and adding the edge xy, such that in the cyclic order of edges incident to x (respectively, y), xy is added between xv and xw (yw and yv). We say G is *flipped* into G' by vw. This operation is called a *(diagonal) flip*, and is illustrated in Figure 1.

Diagonal flips in planar triangulations are widely studied [10, 11, 14, 15, 17, 19, 21, 23, 27]. Wagner [27] proved that there is finite sequence of diagonal flips to transform a given triangulation into any other with the same number of vertices. In particular, it is proved that any triangulation can be transformed into the triangulation shown in Figure 2. Komuro [14] proved that in fact $O(n)$ flips suffice, and Gao *et al.* [10] proved that $O(n \log n)$ flips suffice for labelled triangulations. Diagonal

[‡]School of Computer Science, Carleton University, Ottawa, Canada. Email: {jit,morin,davidw}@scs.carleton.ca

[§]Département d'informatique et d'ingénierie, Université du Québec en Outaouais, Gatineau, Canada. Email: jurek@uqo.ca

[¶]School of Mathematics and Statistics, Carleton University, Ottawa, Canada. Email: zgao@math.carleton.ca

*Technical Report TR-2003-05, School of Computer Science, Carleton University, Ottawa, Canada. Research supported by NSERC and a PIV 2001 Grant of Generalitat de Catalunya.

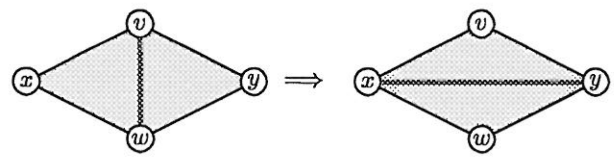


Figure 1: Edge vw is flipped into xy.

flips of triangulations on surfaces other than the sphere have also been widely studied [3, 6, 7, 16, 18–22, 28]

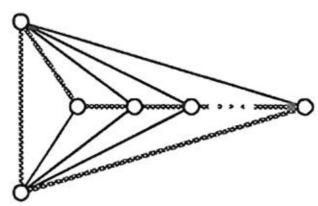


Figure 2: The canonical triangulation and a Hamiltonian cycle.

Two edges of a triangulation that are incident to a common face are *consecutive*. To simultaneously flip two edges in a triangulation G it is necessary that they are not consecutive. That is, the corresponding edges of the dual G^* have no vertex in common. (The *dual* G^* of G is the plane graph with one vertex for every face of G, such that two vertices of G^* are adjacent whenever the corresponding faces in G are incident to a common edge.)

Let S be a set of edges in a triangulation G such that S contains no two consecutive edges, and no parallel edges are produced by flipping every edge in S. Then we say S is *(simultaneously) flippable*. Note that it is possible for S to be flippable, yet S contains non-flippable edges, and it is possible for every edge in S to be flippable, yet S itself is not flippable. The graph obtained from a triangulation G by flipping every edge in a flippable set S is denoted by $G\langle S\rangle$. We say G is *flipped* into $G\langle S\rangle$ by S. This operation is called a *simultaneous (diagonal) flip*. As far as the authors are aware, simultaneous flips have only been studied in the more restrictive context of geometric triangulations of a point set [9]. Sequential flips flips have also been studied in a geometric context [12, 13].

Our first result states that every triangulation can be transformed by one simultaneous flip into a *Hamiltonian* triangulation; that is a triangulation containing a spanning cycle. This result is presented in Section 2. In Section 3 we prove that for any two n-vertex triangulations, there exists a sequence of $O(\log n)$ simultaneous flips to transform one into the other. This result is optimal for many pairs of triangulations. For example, if one triangulation has $\Theta(n)$ maximum degree and the other has $O(1)$ maximum degree, then $\Omega(\log n)$ simultaneous flips are needed, since one simultaneous flip can at most halve the degree of a vertex. This also holds for diameter instead of maximum degree. Finally in Section 4 the maximum size of a simultaneous flip is studied. It is proved that every simultaneous flip has at most $n-2$ edges, and there exist triangulations with a maximum simultaneous flip of $\frac{6}{7}(n-2)$ edges. On the other hand, it is shown that every triangulation has a simultaneous flip of $\frac{2}{3}(n-2)$ edges.

2 Flipping into a Hamiltonian Triangulation

We have the following sufficient condition for a set of edges to be flippable. A cycle C in a triangulation G is *separating* if deleting the vertices of C from G produces a disconnected graph. A 3-cycle in a triangulation is called a *triangle*.

Lemma 1. *Let S be a set of edges in a triangulation G such that no two edges of S appear in a common triangle, and every edge in S is in a separating triangle. Then S is flippable.*

Proof. Since no two edges of S appear in the same triangle, no two edges in S are consecutive. Suppose, for the sake of contradiction, that $G\langle S\rangle$ has parallel edges $e_1 = vw$ and $e_2 = vw$. Let S' be the set of edges in $G\langle S\rangle$ that are not in G. Initially suppose that exactly one of e_1 and e_2, say e_1, is in S'; see Figure 3(a). Let xy be the edge of G flipped to e_1. By assumption, xy is in a separating triangle xyz of G. Then in G, z is either inside the triangle $\{e_2, vx, xw\}$ or outside the triangle $\{vy, yw, e_2\}$. ($z \neq v$ and $z \neq w$ as otherwise xyz would not be a separating triangle.) However, in these cases, z could not be adjacent to y and x, respectively, without breaking planarity. Now suppose that e_1 and e_2 are both in S'; see Figure 3(b). Let xy be the edge of G flipped to e_1, and let rs be the edge of G flipped to e_2. By assumption, each of xy and rs is in a separating triangle of G. Let z and t be the vertices of G such that xyz and rst are separating triangles. For xyz and rst to be separating triangles of G, it must be the case that $z = r$, $t = x$, and $y = s$; or $z = s$, $t = y$, and $x = r$. In either case, xy and rs are in a common triangle in G, contradicting the initial assumption. In each case, we have a contradiction. Hence S is flippable. □

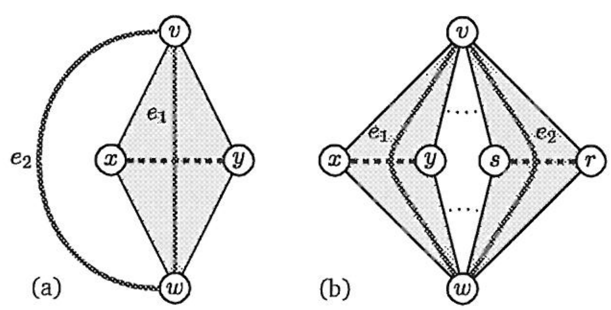


Figure 3: Dashed edges are flipped to create bold parallel edges. Shaded regions are faces.

Lemma 1 gives a sufficient condition for a set of edges S to be flippable; namely, every edge in S is in a separating triangle. This condition can be dropped if G is 5-connected.

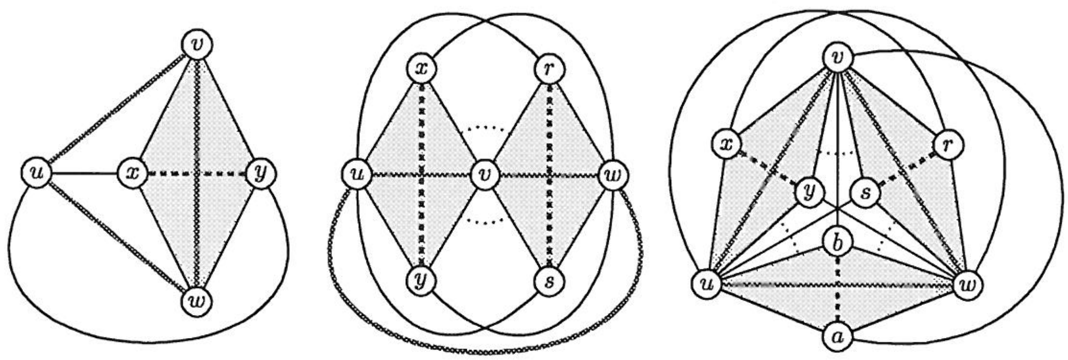
Lemma 2. *Let S be a set of edges in a 5-connected triangulation G. Then S is flippable if and only if no two edges in S are consecutive.*

Proof. If two edges are parallel in $G\langle S\rangle$ then there is a separating 4-cycle in G; see Figure 3(b). □

Lemma 3. *Let G be a triangulation with $n \geq 6$ vertices. Let S be a set of edges in G that satisfy Lemma 1 such that every separating triangle contains an edge in S. Then $G\langle S\rangle$ is 4-connected.*

Proof. Suppose for the sake of contradiction, that $G\langle S\rangle$ contains a separating triangle $T = uvw$. Let S' be the set of edges in $G\langle S\rangle$ that are not in G. We proceed by case-analysis on $|T \cap S'|$ (refer to Figure 4). Since every separating triangle in G has an edge in S, $|T \cap S'| \geq 1$.

Case 1. $|T \cap S'| = 1$: Without loss of generality, $vw \in S'$, $uv \notin S'$, and $uw \notin S'$. Suppose xy was flipped to vw. Then xy is in a separating triangle xyp in G. By an argument similar to that in Lemma 1, $p = u$. Since G has at least six vertices, at least one of the triangles $\{uvx, uvy, uwx, uwy\}$ is a separating triangle. Thus at least one of the edges in these triangles is in S. Since $xy \in S$, and no two edges of S appear in a common triangle, $\{ux, uy, vx, vy, wx, wy\} \cap S = \emptyset$. Thus uv or uw is in S. But then uvw is not a triangle in $G\langle S\rangle$, which is a contradiction.



Case 1 Case 2 Case 3

Figure 4: Dashed edges are flipped to create a bold separating triangle. Shaded regions are faces.

Case 2. $|T \cap S'| = 2$: Without loss of generality, $uv \in S'$, $vw \in S'$, and $uw \notin S'$. Suppose xy was flipped to uv, and rs was flipped to vw. Without loss of generality, y and s are inside uvw in $G\langle S\rangle$. Then in G, xy was in a separating triangle xyz, and rs was in a separating triangle rst. By an argument similar to that in Lemma 1, $z = w$ and $t = u$. But then the subgraph of G induced by $\{u, v, w, x, y, r, s\}$ is not planar, or it contains parallel edges in the case that $x = r$ and $y = s$.

Case 3. $|T \cap S'| = 3$: Suppose xy was flipped to uv, rs was flipped to vw, and ab was flipped to uw. Without loss of generality, y, s and b are inside uvw in $G\langle S\rangle$. In G, xy was in a separating triangle xyz, rs was in a separating triangle rst, and ab was in a separating triangle abc. By an argument similar to that in Lemma 1, $z = w$, $t = u$, and $c = v$. But then the subgraph of G induced by $\{u, v, w, x, y, r, s, a, b\}$ is non-planar, or contains parallel edges in the case that $y = s = b$ and $x = r = a$.

In each case we have derived a contradiction. Therefore $G\langle S\rangle$ has no separating triangle, and thus is 4-connected. $\square$

Observe that the restriction in Lemma 3 to triangulations with at least six vertices is unavoidable. Every triangulation with at most five vertices has a vertex of degree three, and is thus not 4-connected.

We now consider how to determine a set of flippable edges. We will need the following strengthening of Petersen's matching theorem [24] due to Biedl *et al.* [2].

Lemma 4 ([2]). *For every edge e of a 3-regular bridgeless planar graph H with n vertices, there is a perfect matching of H containing e that can be computed in $O(n)$ time.*

The dual G^* of an n-vertex triangulation G is a 3-regular bridgeless planar graph with $O(n)$ vertices. By Lemma 4 we have:

Lemma 5. *Let e with an edge of an n-vertex triangulation G. Then G has a set of edges M that can be computed in $O(n)$ time such that $e \in M$ and every face of G has exactly one edge in M.* $\square$

Note that Lemma 5 only accounts for triangles of G that are faces.

Lemma 6. *Let e be an edge of an n-vertex triangulation G. Then G has a set of edges S such that $e \in S$ and every triangle of G has exactly one edge in S.*

Proof. We proceed by induction on the number of separating triangles. Suppose G is a triangulation with no separating triangles. By Lemma 5, G has a set of edges S such that $e \in S$ and every face of G has exactly one edge in S. Since G has no separating triangles, S has exactly one edge for every triangle of G. Now suppose G has $k > 0$ separating triangles, and the lemma holds for triangulations with less than k separating triangles. Let T be a separating triangle of G. Let the components of $G \setminus T$ have vertex sets V_1 and V_2. Consider the induced subgraphs $G_1 = G[V_1 \cup T]$ and $G_2 = G[V_2 \cup T]$. Without loss of generality, suppose the given edge e is in G_1. Both G_1 and G_2 have less than k separating triangles. By induction G_1 has a set of edges S_1 such that $e \in S_1$, and every triangle of G_1 has exactly one edge in S_1. Let e_2 be the edge in $S_1 \cap T$. By induction, G_2 has a set of edges S_2 such that $e_2 \in S_2$, and every triangle of G_2 has exactly one edge in S_2. Thus $S = S_1 \cup S_2$ is a set of edges of G such that $e \in S$, and every triangle of G has exactly one edge in S. $\square$

By taking as a flip set those edges in the set S from Lemma 6 that are in some separating triangle, Lemma 3 implies that every triangulation with at least six vertices has a simultaneous flip into a 4-connected triangulation. However, it is not obvious how to implement Lemma 6 in $O(n)$ time. In what follows we show how to do this.

First we outline a few properties of separating triangles. Let T be a separating triangle of a triangulation G. Removing the vertices of T from G disconnects the graph into two connected components. One of the two components, called the *inner component*, does not contain any of vertices of the outerface. The other component is called the *outer component*. Denote by $\mathsf{int}(T)$ the set of vertices of the inner component, and by $\mathsf{ext}(T)$ the set of vertices of the outer component. Define a *containment* relation, denoted by $\preceq$, on the set of separating triangles of G as follows. For all separating triangle T_1

and T_2 of G, let $T_1 \preceq T_2$ whenever $\text{int}(T_1) \subseteq \text{int}(T_2)$. Clearly $\preceq$ is a partial order. Let R be a total order on the set of separating triangles of G that is an extension of $\preceq$.

We first show how to compute R in $O(n)$ time. We then show how to use R to compute the set S in Lemma 6 in $O(n)$ time. A useful tool in the computation of R is the *canonical ordering* of de Fraysseix *et al.* [8]. Let G be a plane triangulation with outerface abc. An ordering of the vertices $v_1 = a, v_2 = b, v_3, \ldots, v_n = c$ is *canonical* if the following conditions hold for all $4 \leq i \leq n$:

1. The subgraph G_{i-1} induced by $v_1, v_2, \ldots, v_{i-1}$ is 2-connected, and the boundary of its outerface is a cycle C_{i-1} containing the edge ab;

2. The vertex v_i is in the outerface of G_i, and the set of vertices in G_{i-1} that are adjacent to v_i form a subinterval of the path $C_{i-1} \setminus \{ab\}$ consisting of at least two vertices.

de Fraysseix *et al.* [8] proved that every triangulation has a canonical ordering, and Chrobak and Payne [5] proved that it can be computed in $O(n)$ time. Given a canonical ordering, define the *level* of a separating triangle T, denoted by $\ell(T)$, as the largest index in the canonical order of the vertices of T.

Lemma 7. *Let T_1 and T_2 be separating triangles such that $\ell(T_1) < \ell(T_2)$. Then $T_1 \preceq T_2$ or $\text{int}(T_1) \cap \text{int}(T_2) = \emptyset$.*

Proof. Let $T_1 = abc$ and $T_2 = xyz$. For the sake of a contraction, suppose that $T_2 \preceq T_1$. Then $\text{int}(T_2) \subset \text{int}(T_1)$ since T_1 and T_2 are distinct. Without loss of generality, let c be the vertex of T_1 defining $\ell(T_1) = i$ and z be the vertex of T_2 defining $\ell(T_2) = j$. Since $\ell(T_1) = i < \ell(T_2) = j$, c is distinct from z and the canonical index of c is smaller than that of z. By the canonical ordering, no vertex in $\text{int}(T_1)$ is on the outerface of any G_k for $k \geq i$. Since z occurs after c in the canonical ordering, this implies that all the vertices adjacent to z in $\text{int}(T_2)$ are on the outerface of G_j. But none of these vertices are in $\text{int}(T_1)$, which is the desired contraction. □

Lemma 8. *A total order R of the separating triangles of an n-vertex plane triangulation G that is an extension of $\preceq$ can be computed in $O(n)$ time.*

Proof. First note a canonical ordering can be computed in $O(n)$ time [5]. Lemma 7 implies that if all of the separating triangles of G have different levels, then ordering them by increasing level gives us the total order R. What remains is to order the separating triangles at the same level. The separating triangles at the same level share a common vertex v_i whose canonical index i defines the level of the separating triangles. By definition, the set of vertices of G_{i-1} adjacent to v_i form a path $P = p_1, p_2, \ldots, p_k$ on the boundary of the outerface of G_{i-1}. Every separating triangle of G at level i consists of the vertex v_i and two non-consecutive vertices of P. To establish the containment relation between these triangles, we simply need to look at the indices of the vertices of P. Let $T_1 = v_i p_a p_b$ and $T_2 = v_i p_c p_d$ be distinct separating triangles with $a < b$ and $c < d$. If $a < b \leq c < d$ or $c < d \leq a < b$ then $\text{int}(T_1) \cap \text{int}(T_2) = \emptyset$ by the canonical ordering. It is impossible for $a < c < b < d$ or $c < a < d < b$ since the graph induced on P is outerplanar and this would violate planarity. If $a \leq c < d \leq b$ then

$T_2 \preceq T_1$ and if $c \leq a < b \leq d$ then $T_1 \preceq T_2$. Since we can compute the graph induced on $p_1, p_2, \ldots, p_k$ in $O(k)$ time, all of the separating triangles at level i can be ordered in $O(k)$ time by performing a breadth-first search on the graph induced on P. The result follows since the sum of the degrees of a plane graph is $O(n)$. $\square$

We now turn our attention to computing the set S from Lemma 6 in $O(n)$ time. Denote by **FaceSet**(G, xy) the set M from Lemma 5 such that every face of G has exactly one edge in M, and if xy is specified then $xy \in M$. Let $R = T_1, T_2, \ldots, T_k$ be the ordered separating triangles of G. Note that each T_i is a face of $G \setminus \mathsf{int}(T_i)$ and any separating triangle T_j of G in $G \setminus \mathsf{ext}(T_i)$ has $j < i$.

FlipSet(G,R,S,M)
Input: a triangulation G, the ordered list of separating triangles R, M initially empty, S initially empty.
Output: a set of edges M of G such that every face of G is incident to exactly one edge of M, and a set of edges S of G such that every separating triangle of G has exactly one edge in S and no triangle of G contains two edges of S.

1: **if** $|R| > 0$ **then**
2: Let $T = abc$ be the first triangle in R.
3: FlipSet($G \setminus \mathsf{int}(T), R \setminus t$, S, M)
4: Let xy be the one edge of ab, bc, or ac that is in M
5: Let $S = S \cup \{xy\}$
6: Let $M = M\cup$ **FaceSet**($G \setminus \mathsf{ext}(T), xy$)
7: **else**
8: Let $M = M\cup$ **FaceSet**(G, unspecified)
9: **end if**

We now prove the correctness and running time of the algorithm.

Lemma 9. *Let G be an n-vertex triangulation. A set of edges S can be computed in $O(n)$ time, such that every separating triangle of G is adjacent to one edge of S, and no triangle of G is adjacent to more than one edge of S.*

Proof. We proceed by induction on $|R|$. Suppose that $|R| = 0$. That is, there are no separating triangles and so the set S should be empty. Algorithm **FlipSet**(G, R) correctly computes S and M with a call to **FaceSet**(G). Now assume that for $|R| < k, k > 0$, **FlipSet**(G, R, S, M) correctly computes S and M.

Let $|R| = k$. Let $T = abc$ be the first triangle of R. By the inductive hypothesis, the call to **FlipSet**$(G \setminus \mathsf{int}(T), R \setminus T, S, M)$ correctly computes S and M in $G \setminus \mathsf{int}(T)$, since $|R \setminus t| < k$, and $R \setminus T$ is a total order on all separating triangles in $G \setminus \mathsf{int}(T)$ that extends $\preceq$. What remains is the edge for separating triangle T and the edges in $G \setminus \mathsf{ext}(T)$. The addition of xy to S in line 5, guarantees that S is correct for G since there are no separating triangles in $G \setminus \mathsf{ext}(T)$ by the fact that T is the first separating triangle in R. The call to **FaceSet**$(G \setminus \mathsf{ext}(T), xy)$ guarantees that M is updated correctly.

The running time of the algorithm is described by the recurrence $T(n) = T(n-k) + O(k)$, where k is the size of $\mathsf{int}(T)$. This recurrence solves to $O(n)$. $\square$

Theorem 1. *Every triangulation G with $n \geq 6$ vertices has a simultaneous flip into a 4-connected triangulation that can be computed in $O(n)$ time.*

Proof. By Lemma 9, G has a set of edges S such that every separating triangle of G has exactly one edge in S and no triangle of G contains two edges of S. By Lemma 1, S is flippable. By Lemma 3, $G\langle S\rangle$ is 4-connected. □

We can obtain a stronger result at the expense of a slower algorithm. The following consequence of the 4-colour theorem is essentially a Tait edge-colouring [26].

Lemma 10. *Every n-vertex planar graph G has an edge 3-colouring that can be computed in $O(n^2)$ time, such that every triangle receives three distinct colours.*

Proof. Robertson *et al.* [25] prove that G has a proper vertex 4-colouring that can be computed in $O(n^2)$ time. Let the colours be $\{1, 2, 3, 4\}$. Colour an edge *red* if its endpoints are coloured 1 and 2, or 3 and 4. Colour an edge *blue* if its endpoints are coloured 1 and 3, or 2 and 4. Colour an edge *green* if its endpoints are coloured 1 and 4, or 2 and 3. Since each triangle T of G receives three distinct vertex colours, T also receives three distinct edge colours. □

Theorem 2. *Let G be a triangulation with $n \geq 6$ vertices. Then G has three pairwise disjoint flippable sets of edges S_1, S_2, S_3 that can be computed in $O(n^2)$ time, such that each $G\langle S_i\rangle$ is 4-connected.*

Proof. By Lemma 10, G has an edge 3-colouring such that every triangle receives three distinct colours. For any of the three colours, let S be the set of edges receiving that colour and in a separating triangle of G. By Lemma 1, S is flippable. By Lemma 3, $G\langle S\rangle$ is 4-connected. □

We have the following corollary of Theorems 1 and 2, since every triangulation on at most five vertices (that is, K_3, K_4 or $K_5 \setminus e$) is Hamiltonian, and every 4-connected triangulation has a Hamiltonian cycle [29] that can be computed in $O(n)$ time [4].

Corollary 1. *Every n-vertex triangulation G has a simultaneous flip into a Hamiltonian triangulation that can be computed in $O(n)$ time. Furthermore, G has three disjoint simultaneous flips that can be computed in $O(n^2)$ time, such that each transforms G into a Hamiltonian triangulation.* □

3 Simultaneous Flips Between Given Triangulations

In this section we prove the following theorem.

Theorem 3. *Let G_1 and G_2 be triangulations on n vertices. There is a sequence of $O(\log n)$ simultaneous flips to transform G_1 into G_2.*

The main idea in the proof of Theorem 3 is that G_1 and G_2 can each be transformed by $O(\log n)$ simultaneous flips into the canonical triangulation shown in Figure 2. Thus to transform G_1 into G_2 we first transform G_1 into the canonical triangulation, and then apply the corresponding flips for G_2

in reverse order. To transform a given triangulation into the canonical triangulation we first apply Corollary 1 to obtain a Hamiltonian triangulation with one simultaneous flip. We thus have two outerplanar graphs whose intersection is the Hamiltonian cycle. With $O(\log n)$ simultaneous flips we transform each of these outerplanar graphs so that the internal edges of each graph form a star rooted at adjacent vertices. This is the canonical triangulation. Since outerplanar graphs play an important role in the proof of Theorem 3, we begin by outlining some of their properties.

3.1 Outerplanar graphs

An outerplanar graph is a plane graph where every vertex lies on a specified face, called the *outerface*. An outerplanar graph is *maximal* if every face is a triangle, except for possibly the outerface. In the remainder of this section, all outerplanar graphs considered are maximal. An edge on the outerface of an outerplanar graph is called *external*, and an edge that is not on the outerface is *internal*. It is well known that a maximal outerplanar graph with n vertices has $2n-3$ edges, and is 3-colourable.

Let $G=(V,E)$ be a maximal outerplanar graph. Let G^* denote the *weak dual* of G. That is, the dual of G without the vertex corresponding to the outerface. Observe that G^* is a tree with maximum degree at most three.

The notions of *diagonal flip* and *flippable set* for triangulations are extended to maximal outerplanar graphs in the natural way, except that only internal edges are allowed to be flipped. (It is not clear what it means to flip an edge of the outerface since for $n>3$, the outerface is not a triangle.)

Lemma 11. *Every internal edge of a maximal outerplanar graph G is flippable.*

Proof. Suppose that an internal edge e is not flippable; that is, flipping e to an edge e' results in a parallel edge. Then e' is already in G. But this is a contradiction, since the outerface along with e and e' form a subdivision of K_4, and it is well known that an outerplanar graph has no subdivision of K_4. □

The next result follows immediately from Lemma 11.

Lemma 12. *Let S be a set of internal edges in a maximal outerplanar graph G. Then S is flippable if and only if the corresponding dual edges S^* form a matching in G^*.* □

Lemma 13. *Any set S of internal edges in a maximal outerplanar graph G can be removed from G with three simultaneous flips.*

Proof. By the construction used in Lemma 10 we obtain an edge 3-colouring of G such that each internal face is trichromatic. (Note that the 4-colour theorem is not needed here since G is 3-colourable.) Suppose the edge colours are red, blue, and green. Each colour class corresponds to a matching in the dual. By Lemma 12, the red edges in S are flippable. Let G' be the maximal outerplanar graph obtained by flipping the red edges in S. Blue edges might be consecutive in G', and similarly for green edges. However, the subgraph of the dual induced by the blue and green edges has maximum degree at most two (since G^* has maximum degree at most three), and thus consists of a set of disjoint paths (since G^* is a tree). Thus two simultaneous flips are sufficient to flip the red and blue edges in S. Clearly no edge in S is flipped back into the maximal outerplanar graph obtained. $\square$

Note that in Lemma 13, the resulting graph may be different depending on the choice of colouring. Recall that our aim is to transform a maximal outerplanar graph into one in which the internal edges form a star. The next lemma shows how to do this.

Lemma 14. *Let v be a vertex of a maximal outerplanar graph G whose dual tree has diameter k. Then G can be transformed by at most k simultaneous flips into a maximal outerplanar graph in which v is adjacent to every other vertex.*

Proof. Let P be the set of internal faces incident with v in G. In the dual G^*, the corresponding vertices of P forms a path P^*. Define the *distance* of each vertex x in G^* as the minimum number of edges in a path from x to a vertex in P^*. Since the diameter of the dual tree is k, every vertex in G^* has distance at most k. No two vertices in G^* both with distance one are adjacent, as otherwise G^* would contain a cycle. Each vertex of P^* is adjacent to at most one vertex at distance one, since the endpoints of P^* correspond to faces with an edge on the outerface. Therefore the set of edges incident to P^* but not in P^* forms a matching between the vertices at distance one and the vertices of P^*, such that all vertices at distance one are matched. By Lemma 12, these edges can be flipped simultaneously. After doing so, the distance of each vertex not adjacent to P^* is reduced by one. Thus, by induction, at most k simultaneous flips are required to reduce the distance of every vertex in G^* to zero, in which case v is adjacent to every other vertex of G. $\square$

Lemma 14 suggests that reducing the diameter of the dual tree is a good strategy. In what follows all logarithms have base 2.

Lemma 15. *Every maximal outerplanar graph with n vertices can be transformed by a sequence of at most $3\log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph whose dual tree has diameter at most $2\log n \,/\, (\log 3 - 1)$.*

Proof. We proceed by induction on n. The lemma holds trivially for $n = 3$. Assume the lemma holds for all outerplanar graphs with less than n vertices. Let G be a maximal outerplanar graph with n vertices. Since G is 3-colourable, G has an independent set I of at least $n/3$ vertices. Let S be the set of internal edges incident to any vertex in I. By Lemma 13, G can be transformed by three simultaneous flips into a maximal outerplanar graph G' containing no edge in S. Since I is an independent set, each vertex in I has degree two in G'. Let $G'' = G' \setminus I$. Then G'' is maximal outerplanar with at most $2n/3$ vertices. By the inductive hypothesis, G'' can be transformed by $3(\log \frac{2n}{3}) \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph G''' whose dual has diameter at most $2(\log \frac{2n}{3}) \,/\, (\log 3 - 1)$. The total number of simultaneous flips is $3 + 3(\log \frac{2n}{3}) \,/\, (\log 3 - 1) = 3\log n \,/\, (\log 3 - 1)$ as desired. Adding the vertices in I, which have degree two, back into G''' increases the diameter of the dual tree by at most two, since degree two vertices correspond to leaves in the dual tree. Therefore, the diameter is $2 + 2(\log \frac{2n}{3}) \,/\, (\log 3 - 1) = 2\log n \,/\, (\log 3 - 1)$ as desired. $\square$

Lemmata 14 and 15 imply:

Lemma 16. *Let v be a vertex of a maximal outerplanar graph G with n vertices. Then G can be transformed by a sequence of at most $5 \log n \,/\, (\log 3 - 1) \approx 8.5 \log n$ simultaneous flips into a maximal outerplanar graph in which v is adjacent to every other vertex.* □

3.2 Transforming one triangulation into another

We now describe how to transform a given n-vertex triangulation into any other n-vertex triangulation by a sequence of $O(\log n)$ simultaneous flips. As described at the start of Section 3, it suffices to prove that every triangulation G can be transformed by $O(\log n)$ flips into the canonical triangulation. The first step is to transform G into a Hamiltonian triangulation with one simultaneous flip by Corollary 1. Thus, our starting point is an n-vertex triangulation $G = (V, E)$ containing a Hamiltonian cycle H. The cycle H naturally partitions G into two subgraphs $G_I = (V, E_I)$ and $G_E = (V, E_O)$ such that $E_I \cap E_O = \emptyset$ and both $G_I \cup H$ and $G_O \cup H$ are maximal outerplanar.

At this point, it is tempting to apply Lemma 16 twice, once on $G_I \cup H$ and once on $G_O \cup H$ to reach the canonical triangulation. However, the lemma cannot be applied directly since we need to take into consideration the interaction between these two outerplanar subgraphs. The main problem is that an edge e in G_I may not be flippable since its corresponding flipped edge e' may already be in G_O.

Lemma 17. *Suppose that an edge e in E_I can be flipped in $G_I \cup H$ to an edge e', but e cannot be flipped in G because e' is already in G. Then e' can be flipped in G, provided that $n \geq 5$.*

Proof. The edge e' must be in $G_O \cup H$. By Lemma 11, e' can be flipped to an edge e'' in $G_O \cup H$. The only way that e' is not flippable in G is if e'' is in $G_I \cup H$. But as in Lemma 11, e'' cannot be in $G_I \cup H$. Therefore, e' is flippable in G. □

Lemma 18. *A simultaneous flip S in G_I can be implemented by four simultaneous flips in G (ignoring the effect in G_O).*

Proof. Let S' be the set of edges in S that are flippable in G. Let $S'' = S \setminus S'$. Let S''_O be the edges in G_O blocking the edges in S''. By Lemma 17, each edge in S''_O is flippable. By Lemma 13, the edges in S''_O can be removed with three simultaneous flips. Observe that S' can be simultaneously flipped with one of the flips for S''_O. Once all of the edges in S''_O are removed from G, the edges in S'' can be simultaneously flipped. In total we have four simultaneous flips. □

Lemmata 16 and 18 imply:

Lemma 19. *Let v be a vertex of G. Then $G_I \cup H$ can be transformed by a sequence of at most $20 \log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which v is adjacent to every other vertex.* □

Lemma 20. *Let G be a triangulation with n vertices. Then G can be transformed by a sequence of $25 \log n \,/\, (\log 3 - 1)$ simultaneous flips into the canonical triangulation.*

Proof. Apply one simultaneous flip to G to obtain a Hamiltonian triangulation (Corollary 1). Define H, G_I and G_O as above. Let vw be an edge of the Hamiltonian cycle. By Lemma 19, G_I can be transformed by a sequence of at most $20 \log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which v is adjacent to every other vertex. By Lemma 16, G_O can be transformed by a sequence of at most $5 \log n \,/\, (\log 3 - 1)$ simultaneous flips into a maximal outerplanar graph in which w is adjacent to every other vertex. Observe that the edges in G_I do not interfere with the flips in G_O since every internal edge in G_I is incident to v, and hence there are no edges incident to v in E_O. The triangulation obtained is the canonical triangulation. The total number of flips is $1 + 25 \log n \,/\, (\log 3 - 1)$. □

Proof of Theorem 3. By Lemma 20, each of G_1 and G_2 can be transformed by $1 + 25 \log n \,/\, (\log 3 - 1)$ simultaneous flips into the canonical triangulation. To transform G_1 into G_2 first transform G_1 into the canonical transformation, and then apply the flips for G_2 in reverse order to transform the canonical triangulation into G_2. The total number of simultaneous flips is $2 + 50 \log n \,/\, (\log 3 - 1) \approx 87 \log n$. □

Note that although there are $O(\log n)$ simultaneous flips in Theorem 3, each of which may involve a linear number of edges, the total number of individual flips is $O(n)$.

Theorem 4. *Let G_1 and G_2 be triangulations on n vertices. There is a sequence of $O(\log n)$ simultaneous flips to transform G_1 into G_2, that use $O(n)$ individual flips in total.*

Proof. It suffices to prove that there are $O(n)$ flips in Lemmata 14 and 15, since at most n edges are flipped to make the graph Hamiltonian, and there are constant times as many flips in Theorem 3 as there are in Lemmata 14 and 15. For Lemma 14, a vertex is at distance 1 only once, and only the vertices at distance 1 are involved in one of the parallel flips. Therefore, we flip at most n edges since there are at most n vertices. For Lemma 15, $O(n)$ edges are flipped to obtain a triangulation on $\frac{2n}{3}$ vertices. Therefore, the number of edges flipped $T(n)$ satisfies the recurrence $T(n) = T(\frac{2n}{3}) + O(n)$, which solves to $O(n)$. Thus, the total number of edges flipped is $O(n)$. □

Therefore Theorem 4 provides an alternative proof of the result by Komuro [14] mentioned in Section 1 that $O(n)$ individual flips suffice to transform one n-vertex triangulation into any other.

4 Large Simultaneous Flips

In this section we prove bounds on the size of a maximum simultaneous flip in a triangulation. For any triangulation G, let $\mathsf{msf}(G)$ be the maximum size of a flippable set of edges in G. Note that Gao *et al.* [10] prove that every triangulation has at least $n - 2$ (individually) flippable edges, and every triangulation with minimum degree four has at least $2n + 3$ (individually) flippable edges.

Lemma 21. *For every triangulation G with n vertices, $\mathsf{msf}(G) \leq n - 2$.*

Proof. Let S be a flippable set of edges of G. Every edge in S is incident to two distinct faces, and no other edge on each of these faces is in S. (Otherwise there would be two consecutive edges in S.) There are $2n - 4$ faces in a triangulation. Thus $|S| \leq n - 2$. □

Lemma 22. *For every 5-connected triangulation G with n vertices,* $\mathsf{msf}(G) = n - 2$.

Proof. By Lemma 21, $\mathsf{msf}(G) \leq n - 2$. By Lemma 6, G has a set of edges S such that every triangle of G has exactly one edge in S. By the argument employed in Lemma 21, $|S| = n - 2$. By Lemma 2, S is flippable. □

Theorem 5. *For every triangulation $G = (V, E)$ with n vertices,* $\mathsf{msf}(G) \geq \frac{2}{3}(n - 2)$.

Proof. Let $V_4 = \{v \in V : \deg(v) = 4\}$. By $G[V_4]$ we denote the subgraph of G induced by V_4. It is well known (see for example [1]) that the maximum degree of $G[V_4]$ is at most two, unless G is the octahedron (that is, the 4-regular 6-vertex triangulation). The octahedron has a flippable set of $3 > \frac{2}{3}(n-2)$ edges. Henceforth assume that G is not the octahedron. Thus each connected component of $G[V_4]$ is a cycle or a path. Let C be a connected component of $G[V_4]$ that is a cycle. Then either C is a face of G, in which case we say C is a *facial component* of $G[V_4]$, or $G \setminus C$ consists of two vertices v and w, each of which is adjacent to every vertex in C. In the latter case, as illustrated in Figure 5, G has a flippable set of $n - 3 \geq \frac{2}{3}(n - 2)$ edges (since $n \geq |C \cup \{v, w\}| \geq 5$).

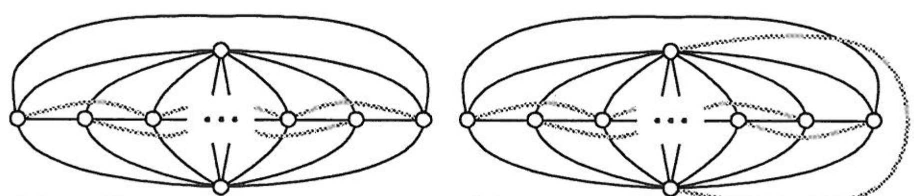


Figure 5: The triangulation with a non-facial cycle of degree four vertices has a flippable set of $n - 3$ edges.

We henceforth assume that every cycle in $G[V_4]$ is a facial component. Let $\mathcal{C}$ be the set of facial components of $G[V_4]$. Let $P = (v_1, v_2, \ldots, v_k)$ be a connected component of $G[V_4]$ that is a k-vertex path. We call P a *k-path component* of $G[V_4]$. It is easily seen that there exist two vertices in G, each of which is adjacent to every vertex in P, as illustrated in Figure 6. Let $\mathcal{P}_k$ be the set of k-vertex paths in $G[V_4]$.

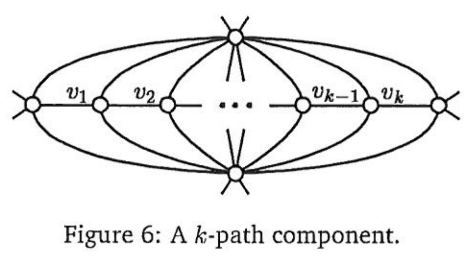


Figure 6: A k-path component.

For every edge vw of G, let pvw and qvw be the faces incident to vw. We say that the vertices p and q *see* vw. Edges vw and xy are a *bad pair* if vw and xy are seen by the same pair of vertices. An edge is *bad* if it is a member of a bad pair. An edge is *good* if it is not bad. Observe that every edge incident to a vertex of degree four is bad.

Construct an auxiliary graph G' from G as follows. First, delete each facial component of $G[V_4]$. (Note that this operation and those to follow are *not* reiterated on the produced graph.) Now consider a k-path component $P = (v_1, v_2, \ldots, v_k)$ in $G[V_4]$ for some $k \geq 1$. Let the neighbours of P be the 4-cycle (a, b, c, d) such that bv_i and dv_i are edges of G for all i, $1 \leq i \leq k$. We say P is *ac-connected* (respectively, *bd-connected*) if ac (bd) is an edge of G, or there exists an edge e not incident to a vertex in P such that e is seen by a and c (b and d). Note that P is not both ac-connected and bd-connected, as otherwise there would be a subdivision of K_5. If P is ac-connected then in G', delete P and merge b and d into a single vertex x_v, as illustrated in Figure 7(a). If P is bd-connected then in G', delete P and merge a and c into a single vertex x_v, as illustrated in Figure 7(b). In the case that P is neither ac-connected or bd-connected, apply either reduction.

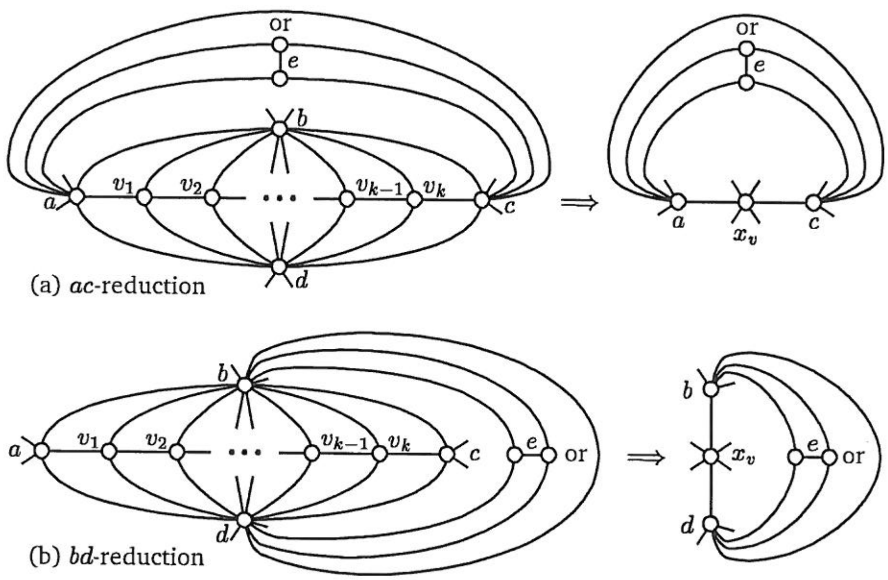


Figure 7: Reducing a path component.

In each merge operation, replace parallel edges on a single face of G' by a single edge. Thus G' is a plane multigraph with each face a triangle. Note that good edges of G are preserved in G'. Let n' be the number of vertices of G'. Then

$$n' = n - 3|\mathcal{C}| - \sum_{k \geq 1} (k+1)|\mathcal{P}_k| \ . \tag{1}$$

Let $\{E'_1, E'_2, E'_3\}$ be the colour classes of a Tait edge 3-colouring of G' (see Lemma 10). For each $i \in \{1, 2, 3\}$, let S'_i be the subset of E'_i consisting of those edges corresponding to good edges in G.

If two bad edges are consecutive in G then their common endpoint has degree four. Since all the degree four vertices of G are not in G', no two bad edges of G appear on a single face of G'. Thus the number of bad edges of G also in G' is at most $n'-2$. Thus the number of good edges of G also in G' is at least $2(n'-2)$. By (1) we have

$$|S'_1| + |S'_2| + |S'_3| \geq 2(n'-2) \geq 2(n-2) - 6|\mathcal{C}| - 2\sum_{k\geq 1}(k+1)|\mathcal{P}_k| \ . \tag{2}$$

We claim that each S'_i is a flippable set of edges in G. Suppose that this is not the case. That is, there exist parallel edges e and f in $G\langle S'_i\rangle$. It is not the case that both e and f are edges of G, as otherwise they would form a bad pair, and all edges in S'_i are good. It is not the case that both e and f are not edges of G, as otherwise they would have been flipped from a bad pair of edges of G in S'_i, and all edges in S'_i are good. The remaining case is that one of e and f is in G and the other is not in G. Say $e = vw$ is in G. Let $pq \in S'_i$ be the edge of G that was flipped into f. Observe that pv is consecutive with pq, pw is consecutive with pq, and vwp is a triangle. Thus vw and pq receive the same colour in the Tait colouring. As proved in Lemma 1, vw is good. Thus $vw \in S'_i$, and vw is not in $G\langle S_i\rangle$, which is the desired contradiction. Thus each set S'_i is flippable in G.

Fix $i \in \{1,2,3\}$, and initialise S_i to be the edges of G corresponding to the set S'_i. For each component of $G[V_4]$, we now add edges to S_i so that it remains a flippable set of edges.

Consider a vertex v of degree 4 in G that is isolated in $G[V_4]$. Let (p,q,r,s) be the neighbours of v in G in cyclic order defined by the embedding, such that q and s are merged into a vertex x in G'. For all $i,j \in \{1,2,3\}$, let A^{ij} be the set of isolated vertices in $G[V_4]$ such that the corresponding edges $xp \in E_i$ and $xr \in E_j$ (or $xr \in E'_i$ and $xp \in E'_j$). By the choice of reduction rule for v, qs is not an edge of G.

For each $v \in A^{ii}$, replace xp and xr in S'_i by $\{pq, qr, rs, sp\}$ in S_i. As illustrated in Figure 8(a), these four edges are simultaneously flippable in G, since the degrees of p and r are both at least 5. For each $v \in A^{ij}$ with $j \neq i$, replace xp in S'_i by $\{pq, ps, qs\}$ in S_i. Since qs is not an edge of G, these three edges are simultaneously flippable in G, as illustrated in Figure 8(b). For each $v \in A^{j\ell}$ with $j \neq i$ and $\ell \neq i$, add qs to S_i. Since qs is not an edge of G, qs is flippable in G, as illustrated in Figure 8(c).

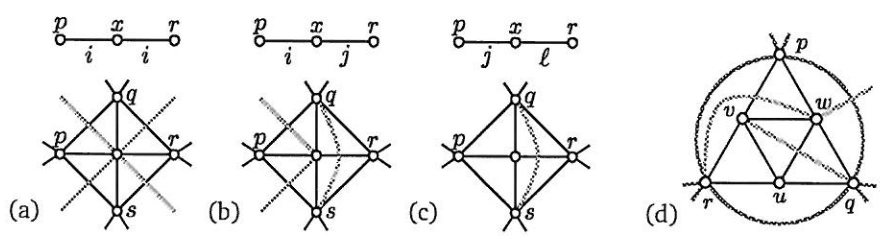
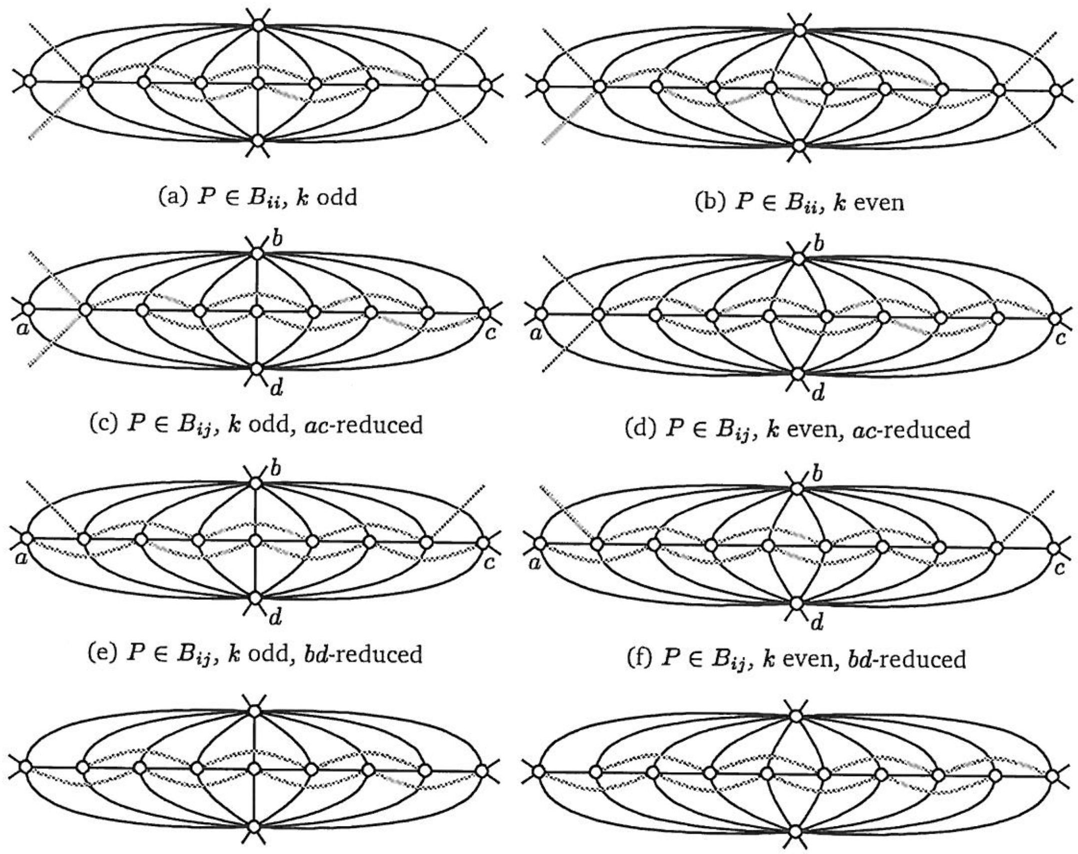


Figure 8: Adding to S_i: (a)-(c) for an isolated degree 4 vertex, and (d) for a facial component.

Consider a face-component $\{u,v,w\}$ of $G[V_4]$, and let $\{p,q,r\}$ be the vertices adjacent to u, v, and w, such that $up, vq, wr \notin E$. Then pqr is a separating triangle of G, and thus each of the edges $\{pq, pr, qr\}$ are good. Every triangle of G receives three distinct colours in the Tait edge 3-colouring.

Thus exactly one of $\{pq, pr, qr\}$ is in S_i'. Suppose without loss of generality that $pq \in S_i'$. Then $\{vq, wr\}$ added to S_i forms a flippable set of edges in G, as illustrated in Figure 8(d).

Let B_{ij} be the set of k-vertex path components P of $G[V_4]$ with $k \geq 2$, such that if P is replaced by the 2-edge path pxr in G' then $xp \in E_i'$ and $xr \in E_j'$ (or $xr \in E_i'$ and $xp \in E_j'$). As illustrated in Figure 9, For every k-path component of $G[V_4]$ with $k \geq 2$, k edges can be added to S_i.



(a) $P \in B_{ii}$, k odd

(b) $P \in B_{ii}$, k even

(c) $P \in B_{ij}$, k odd, ac-reduced

(d) $P \in B_{ij}$, k even, ac-reduced

(e) $P \in B_{ij}$, k odd, bd-reduced

(f) $P \in B_{ij}$, k even, bd-reduced

(g) $P \in B_{j\ell}$, k odd

(h) $P \in B_{j\ell}$, k even

Figure 9: Enlarging the flip set for a path component P.

Let $\{j, \ell\} = \{1, 2, 3\} \setminus \{i\}$. Then

$$|S_i| = |S_i'| + 2|\mathcal{C}| + 2|A^{ii}| + 2|A^{ij}| + 2|A^{i\ell}| + |A^{j\ell}| + \sum_{k \geq 2} k \cdot |\mathcal{P}_k| \ .$$

Thus,

$$
\begin{aligned}
&|S_1| + |S_2| + |S_3| \\
&= |S_1'| + |S_2'| + |S_3'| + 6|\mathcal{C}| + 4(|A^{11}| + |A^{22}| + |A^{33}|) + 5(|A^{12}| + |A^{13}| + |A^{23}|) + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \\
&\geq |S_1'| + |S_2'| + |S_3'| + 6|\mathcal{C}| + 4|P_1| + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \ .
\end{aligned}
$$

By (2),

$$
\begin{aligned}
|S_1| + |S_2| + |S_3| \geq\ & 2(n-2) - 6|\mathcal{C}| - 2\sum_{k\geq 1}(k+1)|\mathcal{P}_k| + 6|\mathcal{C}| + 4|P_1| + 3\sum_{k\geq 2} k \cdot |\mathcal{P}_k| \\
\geq\ & 2(n-2) + \sum_{k\geq 2}(k-2)|\mathcal{P}_k| \\
\geq\ & 2(n-2) \ .
\end{aligned}
$$

Thus one of $\{S_1, S_2, S_3\}$ is a flippable set of at least $\frac{2}{3}(n-2)$ edges in G. □

Lemma 23. *There exists an infinite family $\mathcal{F}$ of triangulations such that* $\mathsf{msf}(G) = \frac{6}{7}(n-2)$ *for every n-vertex triangulation $G \in \mathcal{F}$.*

Proof. Let G_0 be an arbitrary triangulation with n_0 vertices. Let G be the triangulation obtained from G_0 by adding a triangle of three vertices inside each face uvw of G, each of which is adjacent to two of $\{u, v, w\}$. Let n be the number of vertices of G. Then $n - 2 = n_0 + 3(2n_0 - 4) - 2 = 7(n_0 - 2)$. Let S be a flippable set of edges of G. For every face of G_0, at least one of the corresponding seven faces of G does not have an edge in S, as illustrated in Figure 10.

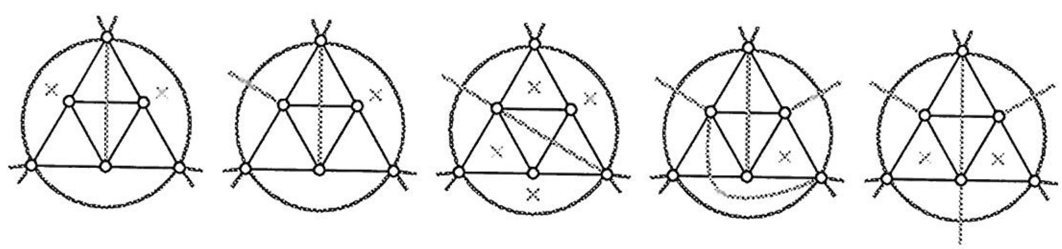


Figure 10: For any number of flips in the outer triangle, at least one internal face does not have an edge in S.

Thus at least $2(n_0 - 2) = \frac{2}{7}(n-2)$ faces of G do not have an edge in S. Every face of G has at most one edge in S. Thus $|S| \leq \frac{1}{2}(2(n-2) - \frac{2}{7}(n-2)) = \frac{6}{7}(n-2)$.

It remains to construct a flippable set of $\frac{6}{7}(n-2)$ edges in G. For each face of G_0, add the edges shown in Figure 11 to a set S. Clearly S is flippable. In every face of G_0, exactly one of the corresponding seven faces of G does not have an edge in S, and the remaining six faces each have exactly one edge in S. By the above analysis, $|S| = \frac{6}{7}(n-2)$. □

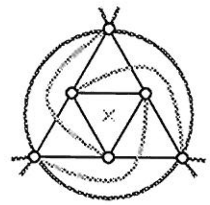


Figure 11: How to construct a flip set for G.

An obvious open problem is to close the gap between the lower bound of $\frac{2}{3}(n-2)$ and the upper bound of $\frac{6}{7}(n-2)$ in the above results.

Acknowledgements

The research of Prosenjit Bose and David Wood was partially completed at the Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain. Many thanks to Ferran Hurtado for graciously hosting our visit.

References

[1] T. BIEDL AND D. F. WILKINSON, Bounded-degree independent sets in planar graphs. In P. BOSE AND P. MORIN, eds., *Proc. 13th International Conf. on Algorithms and Computation (ISAAC '02)*, vol. 2518 of *Lecture Notes in Comput. Sci.*, pp. 416–427, Springer, 2002.

[2] T. C. BIEDL, P. BOSE, E. D. DEMAINE, AND A. LUBIW, Efficient algorithms for Petersen's matching theorem. *J. Algorithms*, **38(1)**:110–134, 2001.

[3] R. BRUNET, A. NAKAMOTO, AND S. NEGAMI, Diagonal flips of triangulations on closed surfaces preserving specified properties. *J. Combin. Theory Ser. B*, **68(2)**:295–309, 1996.

[4] N. CHIBA AND T. NISHIZEKI, The Hamiltonian cycle problem is linear-time solvable for 4-connected planar graphs. *J. Algorithms*, **10(2)**:187–211, 1989.

[5] M. CHROBAK AND T. H. PAYNE, A linear-time algorithm for drawing a planar graph on a grid. *Inform. Process. Lett.*, **54(4)**:241–246, 1995.

[6] C. CORTÉS, C. GRIMA, A. MARQUEZ, AND A. NAKAMOTO, Diagonal flips in outer-triangulations on closed surfaces. *Discrete Math.*, **254(1-3)**:63–74, 2002.

[7] C. CORTÉS AND A. NAKAMOTO, Diagonal flips in outer-torus triangulations. *Discrete Math.*, **216(1-3)**:71–83, 2000.

[8] H. DE FRAYSSEIX, J. PACH, AND R. POLLACK, How to draw a planar graph on a grid. *Combinatorica*, **10(1)**:41–51, 1990.

[9] J. GALTIER, F. HURTADO, M. NOY, S. PÉRENNES, AND J. URRUTIA, Simultaneous edge flipping in triangulations. *Internat. J. Comput. Geom. Appl.*, **13(2)**:113–133, 2003.

[10] Z. GAO, J. URRUTIA, AND J. WANG, Diagonal flips in labelled planar triangulations. *Graphs Combin.*, **17(4)**:647–657, 2001.

[11] Z. GAO AND J. WANG, Enumeration of rooted planar triangulations with respect to diagonal flips. *J. Combin. Theory Ser. A*, **88(2)**:276–296, 1999.

[12] F. HURTADO AND M. NOY, Graph of triangulations of a convex polygon and tree of triangulations. *Comput. Geom.*, **13(3)**:179–188, 1999.

[13] F. HURTADO, M. NOY, AND J. URRUTIA, Flipping edges in triangulations. *Discrete Comput. Geom.*, **22(3)**:333–346, 1999.

[14] H. KOMURO, The diagonal flips of triangulations on the sphere. *Yokohama Math. J.*, **44(2)**:115–122, 1997.

[15] H. KOMURO AND K. ANDO, Diagonal flips of pseudo triangulations on the sphere. *Ars Combin.*, **59**:225–239, 2001.

[16] H. KOMURO, A. NAKAMOTO, AND S. NEGAMI, Diagonal flips in triangulations on closed surfaces with minimum degree at least 4. *J. Combin. Theory Ser. B*, **76(1)**:68–92, 1999.

[17] T. NAKAMIGAWA, A generalization of diagonal flips in a convex polygon. *Theoret. Comput. Sci.*, **235(2)**:271–282, 2000.

[18] A. NAKAMOTO AND S. NEGAMI, Diagonal flips in graphs on closed surfaces with specified face size distributions. *Yokohama Math. J.*, **49(2)**:171–180, 2002.

[19] S. NEGAMI, Diagonal flips in triangulations of surfaces. *Discrete Math.*, **135(1-3)**:225–232, 1994.

[20] S. NEGAMI, Diagonal flips in triangulations on closed surfaces, estimating upper bounds. *Yokohama Math. J.*, **45(2)**:113–124, 1998.

[21] S. NEGAMI, Diagonal flips of triangulations on surfaces, a survey. *Yokohama Math. J.*, **47**:1–40, 1999.

[22] S. NEGAMI AND A. NAKAMOTO, Diagonal transformations of graphs on closed surfaces. *Sci. Rep. Yokohama Nat. Univ. Sect. I Math. Phys. Chem.*, **(40)**:71–97, 1993.

[23] J. PALLO, An efficient upper bound of the rotation distance of binary trees. *Inform. Process. Lett.*, **73(3-4)**:87–92, 2000.

[24] J. PETERSEN, Die Theorie der regulären Graphen. *Acta. Math.*, **15**:193–220, 1891.

[25] N. ROBERTSON, D. SANDERS, P. SEYMOUR, AND R. THOMAS, The four-colour theorem. *J. Combin. Theory Ser. B*, **70(1)**:2–44, 1997.

[26] P. G. TAIT, Note on a theorem in geometry of position. *Trans. Roy. Soc. Edinburgh*, **29**:657–660, 1880.

[27] K. WAGNER, Bemerkung zum Vierfarbenproblem. *Jber. Deutsch. Math.-Verein.*, **46**:26–32, 1936.

[28] T. WATANABE AND S. NEGAMI, Diagonal flips in pseudo-triangulations on closed surfaces without loops. *Yokohama Math. J.*, **47**:213–223, 1999.

[29] H. WHITNEY, A theorem on graphs. *Ann. of Math. (2)*, **32(2)**:378–390, 1931.