Track Layouts of Graphs *

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Abstract

A (k, t)-*track layout* of a graph G consists of a (proper) vertex t-colouring of G, a total order of each vertex colour class, and a (non-proper) edge k-colouring such that between each pair of colour classes no two monochromatic edges cross. This structure has recently arisen in the study of three-dimensional graph drawings. This paper presents the beginnings of a theory of track layouts. Our focus is on methods for the manipulation of track layouts, and the relationship between track layouts and other models of graph layout, namely stack and queue layouts, and geometric thickness. In addition we determine the maximum number of edges in a (k, t)-track layout, and show how to colour the edges given fixed linear orderings of the vertex colour classes.

Keywords: graph layout, graph drawing, track layout, stack layout, queue layout, book embedding, track-number, queue-number, stack-number, page-number, book-thickness, geometric thickness, three-dimensional graph drawing.

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1 Introduction

In its simplest form, a *track layout* of a graph consists of a vertex colouring and a total order on each colour class, such that there is no pair of crossing edges between any two colour classes. The purpose of this paper is to develop the beginnings of a theory of track layouts. Our focus is on methods for the manipulation of track layouts, and the relationship between track layouts and other models of graph layout. We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$.

A *vertex $|I|$-colouring* of a graph G is a partition $\{V_i : i \in I\}$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. The elements of I are *colours*, and each set V_i is a *colour class*. Suppose that $<_i$ is a total order on each colour class V_i. Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : i \in I\}$ is an $|I|$-*track assignment* of G. To ease the notation we denote track assignments by $\{V_i : i \in I\}$ when the ordering on each colour class is implicit.

An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j. An *edge k-colouring* of G is simply a partition $\{E_i : 1 \leq i \leq k\}$ of $E(G)$. A (k,t)-*track layout* of G consists of a t-track assignment of G and an edge k-colouring of G with no monochromatic X-crossing. A graph admitting a (k,t)-track layout is called a (k,t)-*track graph*. The minimum t such that a graph G is a (k,t)-track graph is denoted by $\mathsf{tn}_k(G)$.

$(1,t)$-track layouts (that is, with no X-crossing) are of particular interest due to applications in three-dimensional graph drawing (see below). A $(1,t)$-track layout is called a *t-track layout*. A graph admitting a t-track layout is called a *t-track graph*. The *track-number* of G is $\mathsf{tn}_1(G)$, simply denoted by $\mathsf{tn}(G)$. Dujmović *et al.* [10, 11] first introduced track layouts, although similar structures are implicit in many previous works [17, 23, 24, 31].

The graphs that admit 2-track layouts are easily characterised as follows, where a *caterpillar* is a tree such that deleting the leaves gives a path, as illustrated in Figure 1.

Lemma 1. [21] *A graph has a 2-track layout if and only if it is a forest of caterpillars.*

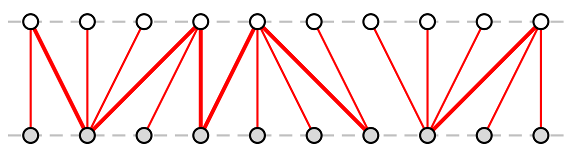

Figure 1: A 2-track layout of a forest of caterpillars.

Table 1 summarises the known bounds on the track-number. Whether planar graphs have bounded track-number is arguably the most important open problem in the field.

Part of the motivation for studying track layouts is a connection with three-dimensional graph drawings. A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *3D drawing*, is a placement of the vertices at distinct points in $\mathbb{Z}^3$ (called *gridpoints*), such that the line-segments representing the edges are pairwise non-crossing. That is, distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. The *bounding box* of a 3D drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. Minimising the volume in 3D drawings is a widely studied problem [2–7, 10, 11, 15–17, 22, 28, 30, 32]. The following general bounds are known for the volume of 3D drawings in terms of the track-number. Other papers to employ track layouts in the production of 3D drawings include [5, 7, 16, 17, 22].

Table 1: Upper bounds on the track-number.

graph family	track-number	reference
n vertices	n	trivial
m edges	$15m^{2/3}$	Dujmović and Wood [16]
m edges, max. degree Δ	$14\sqrt{\Delta m}$	Dujmović and Wood [16]
no K_h-minor	$\mathcal{O}(h^{3/2}n^{1/2})$	Dujmović and Wood [16]
genus γ	$\mathcal{O}(\gamma^{1/2}n^{1/2})$	Dujmović and Wood [16]
tree-width w	$3^w \cdot 6^{(4^w-3w-1)/9}$	Dujmović *et al.* [10, 15]
tree-width w, max. degree Δ	$72\Delta w$	Dujmović *et al.* [10, 15]
queue-number k, star chromatic number c	$c(2k+1)^{c-1}$	Corollary 4
path-width p	$p+1$	Dujmović *et al.* [10, 11]
band-width b	$b+1$	Lemma 14
2-trees	18	Dujmović and Wood [12, 15]
Halin[2]	8	Di Giacomo and Meijer [7]
X-trees[2]	6	Di Giacomo and Meijer [7]
outerplanar[2]	6	Felsner *et al.* [17]
arched levelled planar	5	Di Giacomo and Meijer [7]; Lemma 16
trees	3	Felsner *et al.* [17]

Theorem 1. [10, 11, 16] *Let G be a c-colourable t-track graph with n vertices. Then*

(a) *G has a $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$ straight-line drawing with $\mathcal{O}(t^2 n)$ volume, and*

(b) *G has a $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^4 n)$ straight-line drawing with $\mathcal{O}(c^7 tn)$ volume.*

Moreover, if G has an $X \times Y \times Z$ straight-line drawing then G has track-number $\mathsf{tn}(G) \leq 2XY$.

This paper is organised as follows. In Section 2 we answer the extremal question: what is the maximum number of edges in (k,t)-track layout? Section 3 describes methods for manipulating track layouts. These results suggest a tradeoff between the number of track and number of edge colours in a track layout. In Sections 4 and 5 we explore the relationship between track layouts and other models of graph layout; in particular, stack and queue layouts in Section 4, and geometric thickness in Section 5. Section 6 concludes with some open problems regarding the computational complexity of recognising (k,t)-track graphs. Note that a number of results in this paper are used to prove results in our companion paper on layouts of graph subdivisions [14].

Before we move on, here are some definitions. The subgraph of a graph G induced by a set of vertices $A \subseteq V(G)$ is denoted by $G[A]$. For all $A, B \subseteq V(G)$, we denote by $G[A,B]$ the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$. A graph H is a *minor* of G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A minor-closed class of graphs is *proper* if it is not the class of all graphs.

A *graph parameter* is a function α that assigns to every graph G a non-negative integer $\alpha(G)$. Let $\mathcal{G}$ be a class of graphs. By $\alpha(\mathcal{G})$ we denote the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum of $\alpha(G)$, taken

[2] A track layout that allows edges between consecutive vertices in a track is called an *improper track layout* in [10]. This concept, in the case of three tracks, was introduced by Felsner *et al.* [17], who proved that every outerplanar graph has an improper 3-track layout. It is easily seen that the tracks can be 'doubled' to obtain a (proper) 6-track layout [10]. Similarly, Di Giacomo and Meijer [7] proved that X-trees have improper 3-track layouts, and Halin graphs have improper 4-track layouts. Thus X-trees have (proper) 6-track layouts, and Halin graphs have (proper) 8-track layouts.

over all n-vertex graphs $G \in \mathcal{G}$. We say $\mathcal{G}$ has *bounded* α if $\alpha(\mathcal{G}) \in \mathcal{O}(1)$. A graph parameter α is *bounded by* a graph parameter β (for some class $\mathcal{G}$), if there exists a *binding* function g such that $\alpha(G) \leq g(\beta(G))$ for every graph G (in $\mathcal{G}$). If α is bounded by β (in $\mathcal{G}$) and β is bounded by α (in $\mathcal{G}$) then α and β are *tied* (in $\mathcal{G}$). Clearly, if α and β are tied then a graph family $\mathcal{G}$ has bounded α if and only if $\mathcal{G}$ has bounded β. These notions were introduced by Gyárfás [20] in relation to near-perfect graph families for which the chromatic number is bounded by the clique-number.

A *vertex ordering* of an n-vertex graph G is a bijection $\sigma : V(G) \to \{1, 2, \dots, n\}$. We write $v <_\sigma w$ to mean that $\sigma(v) < \sigma(w)$. One can thus view $<_\sigma$ as a total order on $V(G)$. We say G (or $V(G)$) is *ordered by* $<_\sigma$. At times, it will be convenient to express σ by the list $(v_1, v_2, \dots, v_n)$, where $v_i <_\sigma v_j$ if and only if $1 \leq i < j \leq n$. These notions extend to subsets of vertices in the natural way. Suppose that $V_1, V_2, \dots, V_k$ are disjoint sets of vertices, such that each V_i is ordered by $<_i$. Then $(V_1, V_2, \dots, V_k)$ denotes the vertex ordering σ such that $v <_\sigma w$ whenever $v \in V_i$ and $w \in V_j$ with $i < j$, or $v \in V_i$, $w \in V_i$, and $v <_i w$. We write $V_1 <_\sigma V_2 <_\sigma \dots <_\sigma V_k$.

2 Extremal Questions

Consider the maximum number of edges in a track layout. It follows from Lemma 1 that an n-vertex 2-track graph has at most $n-1$ edges, which generalises to $(k, 2)$-track graphs as follows.

Lemma 2. *Let $\{A, B\}$ be a $(k, 2)$-track layout of a graph G. Then G has at most $k(|A|+|B|-k)$ edges. Moreover, for all $k \geq 1$ and $n_1, n_2 \geq k$, there exists a $(k, 2)$-track layout with $k(n_1 + n_2 - k)$ edges, and with n_1 vertices in the first track and n_2 vertices in the second track.*

Proof. First we prove the upper bound. Suppose $A = (v_1, v_2, \dots, v_{|A|})$, and $B = (w_1, w_2, \dots, w_{|B|})$. For each edge $v_i w_j$, let $\lambda(v_i w_j) = i + j$. Observe that $2 \leq \lambda(e) \leq |A| + |B|$ for each edge e. If distinct edges e and f have $\lambda(e) = \lambda(f)$ then e and f form an X-crossing. Thus at most k edges have the same λ value. Moreover, for all $1 \leq i \leq k-1$, at most i edges e have $\lambda(e) = i + 1$, and at most i edges e have $\lambda(e) = |A| + |B| + 1 - i$. Thus the number of edges is at most

$$2\sum_{i=1}^{k-1} i \;+\; \big(|A| + |B| - 1 - 2(k-1)\big)k \;\;=\;\; k\big(|A| + |B| - k\big) \;\; .$$

Now we prove the lower bound. Let $A = (v_1, v_2, \dots, v_{n_1})$ and $B = (w_1, w_2, \dots, w_{n_2})$. Construct a graph G with $V(G) = A \cup B$. For each $1 \leq \ell \leq k$, let E_ℓ be the set of edges

$$\big\{v_\ell w_j : 1 \leq j \leq n_2 + 1 - \ell\big\} \bigcup \big\{v_i w_{n_2+1-\ell} : \ell + 1 \leq i \leq n_1\big\} \;\; .$$

Observe that $E_{\ell_1} \cap E_{\ell_2} = \emptyset$ for distinct ℓ_1 and ℓ_2. Let $E(G) = \bigcup_\ell E_\ell$. Clearly no two edges in each E_ℓ form an X-crossing, as illustrated in Figure 2. Thus G has a $(k, 2)$-track layout. The number of edges is

$$\sum_{\ell=1}^{k} \big((n_2 + 1 - \ell) + (n_1 - \ell)\big) \;=\; k(n_1 + n_2) - \sum_{\ell=1}^{k} (2\ell - 1) \;=\; k(n_1 + n_2 - k) \;\; .$$

□

Lemma 2 generalises to (k, t)-track layouts as follows.

Lemma 3. *Every n-vertex (k, t)-track graph has at most $k\big(n(t-1) - k\binom{t}{2}\big)$ edges, and for every $k \geq 1$, $t \geq 2$ and $n \geq kt$ there exists a (k, t)-track graph with n vertices and $k\big(n(t-1) - k\binom{t}{2}\big)$ edges.*

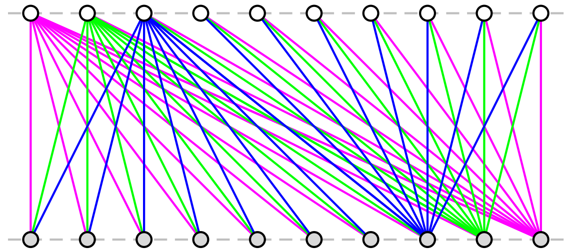

Figure 2: An edge-maximal $(3,2)$-track layout.

Proof. First we prove the upper bound. Let n_i denote the number of vertices in the i^{th} track. By Lemma 2, the number of edges between the i^{th} and j^{th} tracks is at most $k(n_i+n_j-k)$. Hence the total number of edges is at most

$$\sum_{1\leq i<j\leq t} k(n_i+n_j-k) \;=\; k\Big(\sum_{1\leq i<j\leq t}(n_i+n_j) \;-\; k\tbinom{t}{2}\Big) \;=\; k\Big((t-1)n-k\tbinom{t}{2}\Big)\enspace.$$

Now we prove the lower bound. Given any $k\geq 1$, $t\geq 2$ and $n\geq kt$, arbitrarily partition n into t integers $n=n_1+n_2+\cdots+n_t$ with each $n_i\geq k$. Construct a (k,t)-track layout with n_i vertices in the i^{th} track, and $k(n_i+n_j-k)$ edges between the i^{th} and j^{th} tracks, as in Lemma 2. By the above analysis, the total number of edges is $k\big((t-1)n-k\binom{t}{2}\big)$. □

Since $\binom{t}{2}\geq 1$, Lemma 3 implies the following lower bound on $\mathsf{tn}_k(G)$.

Corollary 1. *For all $k\geq 1$, every graph G with n vertices and $m\geq 1$ edges satisfies $\mathsf{tn}_k(G)\geq\frac{k^2+m}{kn}+1$.* □

3 Manipulating Track Layouts

3.1 The Wrapping Lemma

Consider a track assignment $\{V_i:1\leq i\leq t\}$ with a fixed ordering of the tracks. The *span* of an edge vw in $\{V_i:1\leq i\leq t\}$ is $|i-j|$ where $v\in V_i$ and $w\in V_j$. It will also be useful to consider track layouts whose index set is two-dimensional. Let $\{V_{i,j}:i\geq 0, 1\leq j\leq b_i\}$ be a track assignment of a graph G. Define the *partial span* of an edge $vw\in E(G)$ with $v\in V_{i_1,j_1}$ and $w\in V_{i_2,j_2}$ to be $|i_1-i_2|$.

The following lemma describes how to 'wrap' a track layout, and is a generalisation of a result by Dujmović *et al.* [10, 11], which in turn is based on an idea due to Felsner *et al.* [17].

Lemma 4. *Let $\{V_{i,j}:i\geq 0, 1\leq j\leq b_i\}$ be a (k,t)-track layout of a graph G with maximum partial span s (for some irrelevant value t). For each $0\leq\alpha\leq s$, let $t_\alpha=\max\{b_i:i\equiv\alpha\pmod{s+1}\}$. For each $0\leq\alpha\leq 2s$, let $t'_\alpha=\max\{b_i:i\equiv\alpha\pmod{2s+1}\}$. Then*

$$\text{(a) } \mathsf{tn}_{2k}(G)\leq\sum_{\alpha=0}^{s}t_\alpha\enspace,\ \textit{and}\ \ \text{(b) } \mathsf{tn}_k(G)\leq\sum_{\alpha=0}^{2s}t'_\alpha\enspace.$$

Proof. Let $\{E_\ell:1\leq\ell\leq k\}$ be the edge colouring in the given track layout. First we prove (a). By adding extra empty tracks where necessary, we can assume that the track layout is indexed $\{V_{i,j}:i\geq 0, 1\leq j\leq t_\alpha, \alpha=i \bmod (s+1)\}$. For each $0\leq\alpha\leq s$ and $1\leq j\leq t_\alpha$, let

$$W_{\alpha,j}=\bigcup\{V_{i,j}:i\equiv\alpha\pmod{s+1},\, i\geq 0\}\enspace.$$

Order $W_{\alpha,j}$ by

$$(V_{\alpha,j},\, V_{\alpha+(s+1),j},\, V_{\alpha+2(s+1),j},\, \dots) \enspace .$$

Since every edge $vw \in E(G)$ has partial span at most s, if $v \in W_{\alpha_1,j_1}$ and $w \in W_{\alpha_2,j_2}$ then $\alpha_1 \neq \alpha_2$ or $j_1 \neq j_2$. Hence $\{W_{\alpha,j} : 0 \leq \alpha \leq s, 1 \leq j \leq t_\alpha\}$ is a track assignment of G. For each $1 \leq \ell \leq k$, let

$$E'_\ell = \{vw \in E_\ell \,:\, v \in V_{i_1,j_1} \cap W_{\alpha_1,j_1},\, w \in V_{i_2,j_2} \cap W_{\alpha_2,j_2}, i_1 \leq i_2,\, \alpha_1 \leq \alpha_2\}, \text{ and}$$
$$E''_\ell = \{vw \in E_\ell \,:\, v \in V_{i_1,j_1} \cap W_{\alpha_1,j_1},\, w \in V_{i_2,j_2} \cap W_{\alpha_2,j_2},\, i_1 < i_2,\, \alpha_2 < \alpha_1\} \enspace .$$

An X-crossing between edges both from some E'_ℓ (or both from some E''_ℓ) implies that the same edges form an X-crossing in the original track layout. Thus $\{E'_\ell, E''_\ell : 1 \leq \ell \leq k\}$ defines an edge $2k$-colouring with no monochromatic X-crossing. Thus we have a $(2k, \sum_{\alpha=0}^{s} t_\alpha)$-track layout of G.

We now prove (b). Again by adding extra empty tracks where necessary, we can assume that the track layout is indexed $\{V_{i,j} : i \geq 0, 1 \leq j \leq t'_\alpha, \alpha = i \bmod (2s+1)\}$. For each $0 \leq \alpha \leq 2s$ and $1 \leq j \leq t'_\alpha$, let

$$W_{\alpha,j} = \bigcup \{V_{i,j} : i \equiv \alpha \pmod{2s+1},\, i \geq 0\} \enspace .$$

Order $W_{\alpha,j}$ by

$$(V_{\alpha,j},\, V_{\alpha+(2s+1),j},\, V_{\alpha+2(2s+1),j},\, \dots) \enspace .$$

Clearly $\{W_{\alpha,j} : 0 \leq \alpha \leq 2s, 1 \leq j \leq t'_\alpha\}$ is a track assignment of G. It remains to prove that there is no monochromatic X-crossing, where edge colours are inherited from the given track layout. Notice that each $E_\ell = E'_\ell \cup E''_\ell$. As in part (a), edges in E'_ℓ or in E''_ℓ do not form an X-crossing. In the track layout defined for part (b), edges in E'_ℓ have partial span at most s, and edges in E''_ℓ have partial span at least $s+1$. Thus an edge from E'_ℓ and an edge from E''_ℓ do not form an X-crossing. Hence we have a $(k, \sum_{\alpha=0}^{2s} t'_\alpha)$-track layout of G. □

The full generality of Lemma 4 is used in our companion paper [14]. For other applications the following two special cases suffice. By Lemma 4 with $b_i = b$ for all $i \geq 0$, we have:

Lemma 5. *Let $\{V_{i,j} : i \geq 0, 1 \leq j \leq b\}$ be a (k,t)-track layout of a graph G with maximum partial span s. Then* (a) $\mathsf{tn}_{2k}(G) \leq (s+1)b$, *and* (b) $\mathsf{tn}_k(G) \leq (2s+1)b$. □

The next special case is Lemma 5 with $b = 1$. Lemma 6(b) with $k = 1$ was proved by Dujmović *et al.* [10, 11].

Lemma 6. *Let G be a (k,t)-track graph with maximum span s. Then* (a) $\mathsf{tn}_{2k}(G) \leq s+1$, *and* (b) $\mathsf{tn}_k(G) \leq 2s+1$. □

3.2 Biconnected Components

Clearly the track-number of a graph is at most the maximum track-number of its connected components. We now prove a similar result for maximal biconnected components (*blocks*).

Lemma 7. *For every $k \geq 1$, every graph G satisfies:*

(a) $\mathsf{tn}_{2k}(G) \leq 2 \cdot \max\{\mathsf{tn}_k(B) : B$ *is a block of* $G\}$, *and*

(b) $\mathsf{tn}_k(G) \leq 3 \cdot \max\{\mathsf{tn}_k(B) : B$ *is a block of* $G\}$.

Proof. Suppose we have a (k,t)-track layout of each block of G, where $t = \max\{\mathsf{tn}_k(B) : B \text{ is a block of } G\}$. Clearly we can assume that G is connected. Let T be the *block-tree* of G. That is, there is one vertex in T for each block of G, and two vertices of T are adjacent if the corresponding blocks share a cut-vertex. T is acyclic, as otherwise a cycle in T would correspond to a single block in G. Root T at an arbitrary node r.

For all $i \geq 0$, let D_i be the set of blocks of G whose corresponding node in T is at distance i from r. For all $i \geq 1$, each block $B \in D_i$ has exactly one cut-vertex v that is also in some block $B' \in D_{i-1}$. We call v the *parent cut-vertex* of B, B' is the *parent block* of B, and B is a *child block* of B'. Observe that each cut-vertex v is the parent cut-vertex of all but one block containing v. If a vertex v is in only one block B then we say v is *grouped* with B. Otherwise v is a cut-vertex and we say v is *grouped* with the block for which it is not the parent block.

Now order each D_i firstly with respect to the order of the parent blocks in D_{i-1}, and secondly with respect to the order of the parent cut-vertices in the track layouts of the parent blocks. More formally, for each $i \geq 1$, let $<_i$ be a total order of D_i such that for all blocks $A, B \in D_i$ (with parent blocks $A', B' \in D_{i-1}$) we have $A <_i B$ whenever (1) $A' <_{i-1} B'$, or (2) $A' = B'$, $A \cap A' = \{v\}$, $B \cap B' = \{w\}$, and $v < w$ in some track of the (k,t)-track layout of A'. (If v and w are in different tracks of the (k,t)-track layout of the parent block then the relative order of A and B is not important.)

For each $i \geq 0$ and $1 \leq j \leq t$, let $V_{i,j}$ be the set of vertices v of G in a some block $B \in D_i$ such that v is grouped with B, and v is in the j^{th} track of the track layout of B. Now order each $V_{i,j}$ firstly with respect to the order $<_i$ of the blocks in D_i, and within a block B, by the order of the j^{th} track of the track layout of B. Colour each edge e of G by the same colour assigned to e in the (k,t)-track layout of the block containing e. We claim there is no monochromatic X-crossing.

The parent cut-vertex of a block B is grouped with the parent block of B, and no block and its parent block are in the same D_i. Thus if vw is an edge with $v \in V_{i,j_1}$ and $w \in V_{i,j_2}$ then both v and w are grouped with the block containing vw. Since within each track vertices are ordered primarily by their block, and by assumption there is no monochromatic X-crossing between edges in the same block, there is no monochromatic X-crossing between tracks V_{i,j_1} and V_{i,j_2} for all $i \geq 0$ and $1 \leq j_1, j_2 \leq t$.

If vw is an edge with $v \in V_{i_1,j_1}$ and $w \in V_{i_2,j_2}$ for distinct i_1 and i_2, then without loss of generality, $i_2 = i_1 + 1$ and v is the parent cut-vertex of the block containing vw. Since sibling blocks are ordered with respect to the ordering of their parent cut-vertices, there is no X-crossing amongst edges between tracks V_{i_1,j_1} and V_{i_2,j_2} for all $i_1, i_2 \geq 0$ and $1 \leq j_1, j_2 \leq t$. Thus $\{V_{i,j} : i \geq 0, 1 \leq j \leq t\}$ is a k-edge colour track layout of G such that every edge has a partial span of one. By Lemma 5, G has $\mathsf{tn}_{2k}(G) \leq 2t$, and G has $\mathsf{tn}_k(G) \leq 3t$. $\square$

3.3 Tracks vs. Colours

We now show how to reduce the number of tracks in a track layout, at the expense of increasing the number of edge colours.

Lemma 8. *Let G be a (k,t)-track graph with maximum span s $(\leq t-1)$. For every vertex colouring $\{V_i : 1 \leq i \leq c\}$ of G, there is a $(2sk, c)$-track layout of G with tracks $\{V_i : 1 \leq i \leq c\}$.*

Proof. Let $\{T_j : 1 \leq j \leq t\}$ be a (k,t)-track layout of G with maximum span s and edge colouring $\{E_\ell : 1 \leq \ell \leq k\}$. Given a vertex colouring $\{V_i : 1 \leq i \leq c\}$ of G, order each V_i by $(V_i \cap T_1, V_i \cap T_2, \dots, V_i \cap T_t)$. Thus $\{V_i : 1 \leq i \leq c\}$ is a c-track assignment of G. Now we define an edge $2sk$-colouring. For each ℓ and α such that $1 \leq \ell \leq k$ and $1 \leq |\alpha| \leq s$, let

$$E_{\ell,\alpha} = \{vw \in E_\ell : v \in V_{i_1} \cap T_{j_1},\, w \in V_{i_2} \cap T_{j_2},\, i_1 < i_2,\, j_1 - j_2 = \alpha\} \enspace .$$

Consider two edges vw and xy in some $E_{\ell,\alpha}$ between a pair of tracks V_{i_1} and V_{i_2}. Without loss of generality $i_1 < i_2$, $v \in V_{i_1} \cap T_{j_1}$, $w \in V_{i_2} \cap T_{j_1+\alpha}$, $x \in V_{i_1} \cap T_{j_2}$, $y \in V_{i_2} \cap T_{j_2+\alpha}$, and $j_1 \leq j_2$. If $j_1 = j_2$ then vw and

xy are between the same pair of tracks in the given track layout, and the relative order of the vertices is preserved. Thus if vw and xy form an X-crossing in the c-track assignment then they are coloured differently. If $j_1 < j_2$ then $v <_{i_1} x$ and $w <_{i_2} y$, and the edges do not form an X-crossing. Hence vw and xy do not form a monochromatic X-crossing, and we have a $(2sk, c)$-track layout of G. □

We now show how to reduce the number of edge colours in a track layout, at the expense of increasing the number of tracks. A vertex colouring is *acyclic* if there is no bichromatic cycle [19]. The *acyclic chromatic number* of a graph G, denoted by $\chi_{\mathsf{a}}(G)$, is the minimum number of colours in an acyclic vertex colouring of G. A *star colouring* of G is a vertex colouring with no bichromatic 4-vertex path [18, 27]; that is, each bichromatic subgraph is a forest of stars. The *star chromatic number* of G, denoted by $\chi_{\mathsf{st}}(G)$, is the minimum number of colours in a star colouring of G.

By definition, $\chi_{\mathsf{a}}(G) \leq \chi_{\mathsf{st}}(G)$ for every graph G. Conversely, $\chi_{\mathsf{a}}(G) \leq c$ implies $\chi_{\mathsf{st}}(G) \leq c \cdot 2^{c-1}$ [18]. Thus acyclic chromatic number and star chromatic number are tied. By Lemma 1, the underlying vertex colouring in a (monochromatic) track layout is acyclic. Thus $\chi_{\mathsf{a}}(G) \leq \mathsf{tn}(G)$, and star chromatic number is bounded by track-number. The following 'converse' result is implicit in results by Dujmović *et al.* [10, 32]. We include the proof for completeness.

Lemma 9. [10, 32] *Let G be a (k, t)-track graph in which the underlying vertex t-colouring is a star colouring. Then G has track-number $\mathsf{tn}(G) \leq t(k+1)^{t-1}$.*

Proof. Let $\{V_i : 1 \leq i \leq t\}$ be a (k, t)-track layout of G with edge colouring $\{E_\ell : 1 \leq \ell \leq k\}$. For every vertex $v \in V_i$, $1 \leq i \leq t$, and $j \in \{1, \dots, t\} \setminus \{i\}$, let $d_j(v)$ be the degree of v in $G[V_i, V_j]$. Define the j^{th} *label* of v, denoted by $\phi_j(v)$, as follows. If $d_j(v) \neq 1$ then let $\phi_j(v) =$ 'r'. (In this case v is the root of a star, or is isolated in $G[V_i, V_j]$.) If $d_j(v) = 1$ then let $\phi_j(v) = \ell$, where the single edge in $G[V_i, V_j]$ incident to v is in E_ℓ. Let the *label* of $v \in V_i$ be $\phi(v) = (\phi_1(v), \dots, \phi_{i-1}(v), \phi_{i+1}(v), \dots, \phi_t(v))$. Let S_i be the set of possible labels for a vertex in V_i. Then $|S_i| = (k+1)^{t-1}$.

Now group the vertices by colour and label, as illustrated in Figure 3. Let $V_{i,L} = \{v \in V_i : \phi(v) = L\}$ for all labels $L \in S_i$ and $1 \leq i \leq t$, and consider each $V_{i,L}$ to be ordered as in V_i. Thus $\{V_{i,L} : 1 \leq i \leq t, L \in S_i\}$ is a track assignment of G. We denote the j^{th} label of $L \in S_i$ by $L[j]$.

Consider a subgraph $G[V_{i,P}, V_{j,Q}]$ for some $1 \leq i < j \leq t$ and labels $P \in S_i$ and $Q \in S_j$. We claim that all edges in $G[V_{i,P}, V_{j,Q}]$ are monochromatic. If $P[j] =$ 'r' and $Q[i] =$ 'r' then $G[V_{i,P}, V_{j,Q}]$ has no edges. If $P[j] =$ 'r' and $Q[i] = \ell$ for some $1 \leq \ell \leq k$, then all edges in $G[V_{i,P}, V_{j,Q}]$ are coloured ℓ. Similarly, if $Q[i] =$ 'r' and $P[j] = \ell$. Finally, consider the case in which $P[j] = \ell_1$ and $Q[i] = \ell_2$ for some $1 \leq \ell_1, \ell_2 \leq k$. If $\ell_1 \neq \ell_2$ then there are no edges in $G[V_{i,P}, V_{j,Q}]$, and if $\ell_1 = \ell_2$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are coloured $\ell_1 (= \ell_2)$. In each case, all edges in $G[V_{i,P}, V_{j,Q}]$ are monochromatic. Thus $\{V_{i,L} : 1 \leq i \leq t, L \in S_i\}$ is a track layout of G with $t(k+1)^{t-1}$ tracks. □

To apply Lemma 9, we use the following general bound on the star chromatic number due to Nešetřil and Ossona de Mendez [27].

Lemma 10. [27] *The star chromatic number of a graph G is at most a quadratic function of the maximum chromatic number of a minor of G. Hence, every proper minor-closed graph family has bounded star chromatic number.*

Lemmata 8 and 9 imply:

Corollary 2. *Let G be a (k, t)-track graph with maximum span s $(\leq t-1)$. If G has star chromatic number $\chi_{\mathsf{st}}(G) \leq c$ then G has track-number $\mathsf{tn}(G) \leq c(2sk+1)^{c-1}$.* □

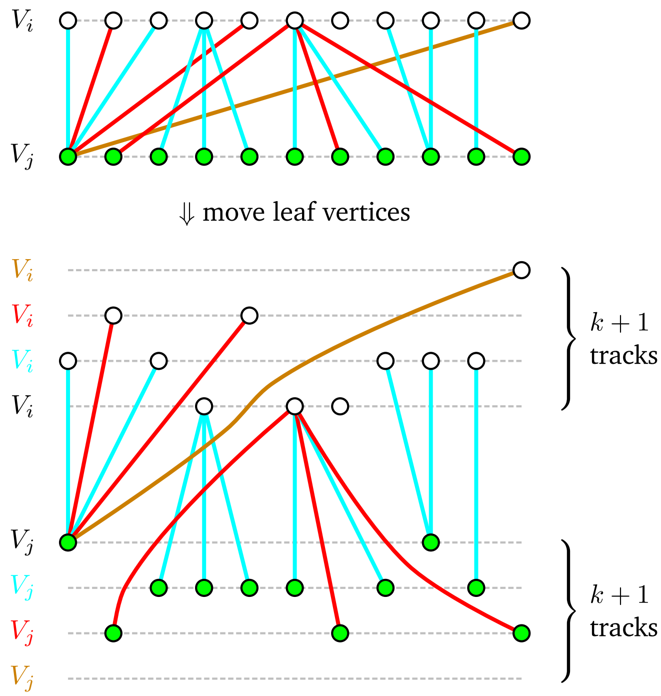

Figure 3: The relative expansion of two tracks in the proof of Lemma 9.

4 Queue and Stack Layouts

In a vertex ordering σ of a graph G, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint such that $L(e) <_\sigma L(f)$. If $L(e) <_\sigma L(f) <_\sigma R(e) <_\sigma R(f)$ then e and f *cross*, and if $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$ then e and f *nest*, and f is *nested inside* e. A *stack* (respectively, *queue*) is a set of edges $E' \subseteq E(G)$ such that no two edges in E' cross (nest). A k-*stack* (*queue*) *layout* of G consists of a vertex ordering σ of G and a partition $\{E_\ell : 1 \leq \ell \leq k\}$ of $E(G)$, such that each E_ℓ is a stack (queue) in σ. A graph admitting a k-stack (queue) layout is called a k-*stack* (*queue*) *graph*. The *stack-number* of a graph G, denoted by $\mathsf{sn}(G)$, is the minimum k such that G is a k-stack graph. The *queue-number* of a graph G, denoted by $\mathsf{qn}(G)$, is the minimum k such that G is a k-queue graph. See our companion paper [13] for a list of references and applications of stack and queue layouts.

The following lemma highlights the fundamental relationship between track layouts, and queue and stack layouts. Its proof follows immediately from the definitions, and is illustrated in Figure 4 for $k = 1$.

Lemma 11. *Let $\{A, B\}$ be a track assignment of a bipartite graph G. Then the following are equivalent:*

(a) *$\{A, B\}$ admits a $(k, 2)$-track layout of G,*

(b) *the vertex ordering (A, B) admits a k-queue layout of G, and*

(c) *the vertex ordering $(A, \overleftarrow{B})$ admits a k-stack layout of G,*

where $\overleftarrow{B}$ denotes the reverse vertex ordering of B. □

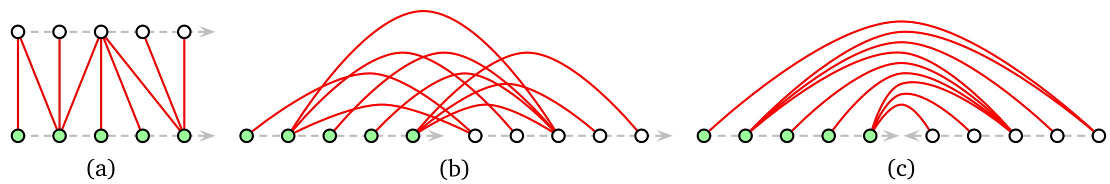

Figure 4: Layouts of a caterpillar: (a) 2-track, (b) 1-queue, (c) 1-stack.

4.1 Fixed Track Assignment

Consider how to colour the edges in a track assignment so that no monochromatic edges form an X-crossing. A track assignment with k pairwise X-crossing edges needs at least k edge colours to be a track layout. The following converse result is a generalisation of the fact that permutation graphs are perfect.

Lemma 12. *A t-track assignment $\{V_i : 1 \leq i \leq t\}$ of a graph G can be extended into a (k,t)-track layout, where k is the maximum number of pairwise X-crossing edges.*

Proof. Clearly we can consider each pair of tracks V_i and V_j separately. Consider the vertex ordering $\sigma = (V_i, V_j)$ of $G[V_i, V_j]$. Two edges in $G[V_i, V_j]$ form an X-crossing in the track assignment if and only if they are nested in σ. Thus at most k edges are pairwise nested in σ. By a result of Heath and Rosenberg [24], σ admits a k-queue layout of G. (See our companion paper [13] for a simple proof). This queue assignment determines an edge k-colouring of $G[V_i, V_j]$ with no monochromatic X-crossing by Lemma 11. □

4.2 Queues into Tracks

Now consider how to convert a vertex ordering into a track layout. The proof of the next result follows immediately from the definitions, and is illustrated in Figure 5.

Lemma 13. *Let σ be a vertex ordering of a graph G. Let $\{V_i : 1 \leq i \leq c\}$ be a vertex colouring of G. For all $1 \leq i, j \leq c$, a pair of edges $vw, xy \in E(G[V_i, V_j])$ form an X-crossing in the track assignment $\{(V_i, \sigma) : 1 \leq i \leq c\}$ if and only if:*

- *vw and xy are nested in σ* (Figures 5(a)-(b))*, or*
- *vw and xy cross in σ with $v <_\sigma y <_\sigma w <_\sigma x$, and $v, x \in V_i$ and $w, y \in V_j$* (Figure 5(d)). □

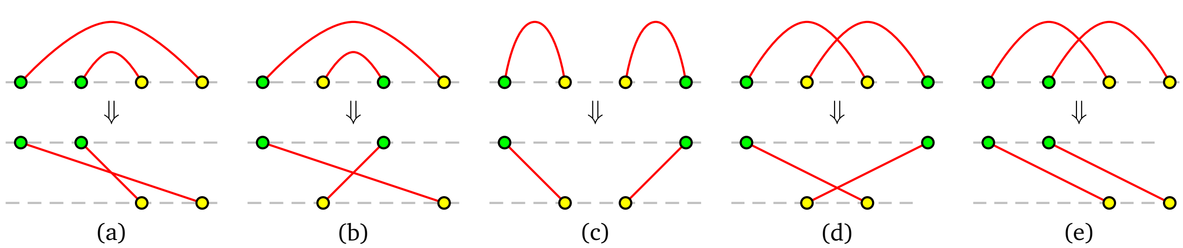

Figure 5: From a linear layout to a track layout: (a)-(b) nested, (c) disjoint, (d)-(e) crossing.

Consider a vertex ordering $\sigma = (v_1, v_2, \dots, v_n)$ of a graph G. For each edge $v_iv_j \in E(G)$, let the *width* of v_iv_j in σ be $|i - j|$. The *band-width* of σ is the maximum width of an edge of G in σ. The *band-width* of G, denoted by $\mathsf{bw}(G)$, is the minimum band-width of a vertex ordering of G.

Lemma 14. *Every graph G with band-width* $\mathsf{bw}(G)$ *has track-number* $\mathsf{tn}(G) \leq \mathsf{bw}(G) + 1$.

Proof. Let $\sigma = (v_0, v_1, \dots, v_{n-1})$ be a vertex ordering of G with band-width $b = \mathsf{bw}(G)$. For each $0 \leq \ell \leq b$, let $V_\ell = \{v_i : i \equiv \ell \bmod (b+1)\}$. Not only is $\{V_\ell : 0 \leq \ell \leq b\}$ a vertex colouring of G, but for every edge vw, if there is a vertex x with $v <_\sigma x <_\sigma w$, then all three vertices are in distinct colour classes. Thus, it follows from Lemma 13 that with each V_ℓ ordered by σ, there is no X-crossing. □

Note that Lemma 14 is in fact weaker than the bound due to Dujmovi'c *et al.* [10, 11] that track-number is at most one more than the path-width. However, we consider that this particularly simple proof deserves mention.

Lemma 15. *For every vertex colouring $\{V_i : 1 \leq i \leq c\}$ of a q-queue graph G, there is a $(2q, c)$-track layout of G with tracks $\{V_i : 1 \leq i \leq c\}$.*

Proof. Let σ be the vertex ordering in a q-queue layout of G with queues $\{E_\ell : 1 \leq \ell \leq q\}$. Let $\{(V_i, \sigma) : 1 \leq i \leq c\}$ be a c-track assignment of G, and for each $1 \leq \ell \leq q$, let

$$E'_\ell = \{vw \in E_\ell : v \in V_i, w \in V_j, i < j, v <_\sigma w\}, \text{ and}$$
$$E''_\ell = \{vw \in E_\ell : v \in V_i, w \in V_j, i < j, w <_\sigma v\} \ .$$

By Lemma 13, an X-crossing in the track assignment between edges both from some E'_ℓ (or both from some E''_ℓ) implies that these edges are nested in σ. Since no two edges in E_ℓ are nested in σ, the set $\{E'_\ell, E''_\ell : 1 \leq \ell \leq q\}$ defines an edge $2q$-colouring with no monochromatic X-crossing in the track assignment. Thus we have a $(2q, c)$-track layout of G. □

Lemma 15 is similar to a result by Pemmaraju [29] which says that a queue layout can be 'separated' by a vertex colouring, although the proof by Pemmaraju, which is based on the characterisation of 1-queue graphs as 'arched levelled planar', is much longer. Pemmaraju used separated queue layouts to prove the next result, which follows from Lemmata 11 and 15.

Theorem 2. [29] *Stack-number is bounded by queue-number for bipartite graphs. In particular,* $\mathsf{sn}(G) \leq 2\,\mathsf{qn}(G)$ *for every bipartite graph G.* □

In a companion paper [13], we prove that every q-queue graph is $4q$-colourable. Thus Lemma 15 implies:

Corollary 3. *Every q-queue graph has a $(2q, 4q)$-track layout.* □

The next corollary of Lemmata 9 and 15 slightly improves an analogous result by Dujmovi'c *et al.* [10, 32].

Corollary 4. *Every q-queue graph G with $\chi_{\mathsf{st}}(G) \leq c$ has track-number* $\mathsf{tn}(G) \leq c(2q+1)^{c-1}$. □

In the case of 1-queue graphs, much improved bounds can be obtained. Di Giacomo and Meijer [7] proved that every 1-queue graph has a 5-track layout, and that there exists a 1-queue graph with track-number at least 4. We now give an alternative proof of the upper bound.

Lemma 16. *Every 1-queue graph has a $(2, 3)$-track layout and a 5-track layout.*

Proof. Let σ be the vertex ordering in a 1-queue layout of a graph G. Partition the vertices into independent sets $V_1, V_2, \dots, V_k$ such that $\sigma = (V_1, V_2, \dots, V_k)$, and for all $1 \leq i \leq k-1$, there exists an edge in $G[V_i, V_{i+1}]$. Such a partition can be computed by starting with each vertex in its own set, and repeatedly merging consecutive sets that have no edge between them. Since σ has no nested edges, by Lemma 13, $\{(V_i, \sigma) : 1 \leq i \leq k\}$ is a track layout with no X-crossing. For all $s \geq 3$, there is no edge in any $G[V_i, V_{i+s}]$, as otherwise it would be nested in σ with an edge in $G[V_{i+1}, V_{i+2}]$. Thus the track layout has span at most two. By Lemma 6, G has a $(2, 3)$-track layout and a 5-track layout. □

4.3 Tracks into Queues

Now consider how to convert a track layout into a queue layout. First we give a simple proof of a generalisation of a result by Dujmovi'c *et al.* [10, 32]. Note that Lemma 17 with $t = 2$ is nothing more than Lemma 11(b).

Lemma 17. *Queue-number is bounded by track-number. In particular, every (k,t)-track graph with maximum span s $(\leq t-1)$ has a ks-queue layout.*

Proof. Let $\{V_i : 1 \leq i \leq t\}$ be a (k,t)-track layout of a graph G with maximum span s and edge colouring $\{E_\ell : 1 \leq \ell \leq k\}$. Let σ be the vertex ordering $(V_1, V_2, \dots, V_t)$ of G. Let $E_{\ell,\alpha}$ be the set of edges in E_ℓ with span α in the given track layout. Two edges from the same pair of tracks are nested in σ if and only if they form an X-crossing in the track layout. Since no two edges in E_ℓ form an X-crossing in the track layout, no two edges in E_ℓ and between the same pair of tracks are nested in σ. If two edges not from the same pair of tracks have the same span then they are not nested in σ. (This idea is due to Heath and Rosenberg [24].) Thus no two edges are nested in each $E_{\ell,\alpha}$, and we have a ks-queue layout of G. □

Note that Lemmata 8 and 11 imply an analogous result to Lemma 17 for stack layouts of bipartite graphs.

Lemma 18. *Every bipartite (k,t)-track graph with maximum span s $(\leq t-1)$ has a $2ks$-stack layout.* □

Observe that Corollary 4 and Lemmata 10 and 17 prove the following relationship between queue layouts and track layouts due to Dujmović *et al.* [10, 32].

Theorem 3. [10, 32] *Queue-number is bounded by track-number (for all graphs). Queue-number and track-number are tied for any proper minor-closed graph family.*

Note that an affirmative solution to the following open problem would imply that queue-number and track-number are tied (for all graphs).

Open Problem 1. Is star chromatic number bounded by queue-number?

5 Geometric Thickness

The *geometric thickness* of a graph G, denoted by $\overline{\theta}(G)$, is the minimum number of colours such that G can be drawn in the plane with edges as coloured straight-line segments, such that monochromatic edges do not cross [8, 25]. Stack-number (when viewed as book-thickness) is equivalent to geometric thickness with the additional requirement that the vertices are in convex position [1]. Thus $\overline{\theta}(G) \leq \mathsf{sn}(G)$ for every graph G. While it is an open problem whether stack number is bounded by track-number or by queue-number (see our companion paper [14]), we prove the weaker results that geometric thickness is bounded by track-number, and geometric thickness is bounded by queue-number.

Theorem 4. *Geometric thickness is bounded by track-number. In particular, every (k,t)-track graph G has geometric thickness $\overline{\theta}(G) \leq k\lceil \frac{t}{2} \rceil \lfloor \frac{t}{2} \rfloor$.*

Proof. Let $p \geq t$ be a prime. Position the j^{th} vertex in the i^{th} track at $(i, pj + (i^2 \bmod p))$. Wood [33] proved that in this layout no three vertices are collinear, unless all three are in a single track. Since a track is an independent set, the only vertices that an edge intersects are its own endpoints. The vertices in each track are positioned on a line parallel to the Y-axis, in the order defined by the track layout. Thus monochromatic edges between any pair of tracks do not cross. If we let each pair of tracks use a distinct palette of k edge colours, then we obtain a drawing of G with $k\binom{t}{2}$ edge colours, such that monochromatic edges do not cross. That is, $\overline{\theta}(G) \leq k\binom{t}{2}$.

This bound can be improved by partitioning the pairs of tracks into sets that can use the same palette of k colours. This amounts to edge-colouring the complete graph K_t with a fixed vertex ordering $(v_1, v_2, \ldots, v_t)$, so that overlapping edges receive distinct colours. To this end, define a partial order on $E(K_t)$ as follows. For all edges v_iv_j and v_av_b (with $i < j$ and $a < b$), let $v_iv_j \prec v_av_b$ if $j \leq a$. Clearly $\preceq$ is a partial order on $E(K_t)$, such that distinct edges are overlapping if and only if they are incomparable under $\preceq$. By Dilworth's Theorem [9], there is a partition of $E(K_t)$ into r sets, each pairwise non-overlapping, where r is the largest set of pairwise overlapping edges. Clearly $r = \lceil \frac{t}{2} \rceil \lfloor \frac{t}{2} \rfloor$. For each such set, assign a distinct palette of k colours to the edges between pairs of tracks corresponding to edges of K_t in this set. In total we have kr edge colours, and $\overline{\theta}(G) \leq kr = k\lceil \frac{t}{2} \rceil \lfloor \frac{t}{2} \rfloor$. □

Theorem 4 and Lemma 15 imply:

Corollary 5. *Every q-queue c-colourable graph G has geometric thickness $\overline{\theta}(G) \leq 2q\lceil \frac{c}{2} \rceil \lfloor \frac{c}{2} \rfloor$.* □

Theorem 4 and Corollary 3 imply:

Corollary 6. *Geometric thickness is bounded by queue-number. In particular, every graph G has geometric thickness $\overline{\theta}(G) \leq 8\,\mathsf{qn}(G)^3$.* □

6 Computational Complexity

We conclude with some open problems regarding the computational complexity of determining whether a given graph admits a particular type of track layout. Note that there is a simple linear time algorithm to recognise 2-track graphs. Is it $\mathcal{NP}$-complete to recognise $(2,2)$-track graphs? Is it $\mathcal{NP}$-complete to recognise 3-track graphs? Given a vertex ordering σ of a graph G, is it $\mathcal{NP}$-complete to test if G has a 3-colouring $\{V_1, V_2, V_3\}$ such that $\{(V_1, \sigma), (V_2, \sigma), (V_3, \sigma)\}$ is a track layout?

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