Spanners of Additively Weighted Point Sets*

Prosenjit Bose Paz Carmi
Mathieu Couture
Scool of Computer Science, Carleton University, Ottawa, Canada

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Abstract

We study the problem of computing geometric spanners for (additively) weighted point sets. A weighted point set is a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_ip_j| - r_i - r_j$. We show that in the case where all r_i are positive numbers and $|p_ip_j| \geq r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane), a variant of the Yao graph is a $(1 + \epsilon)$-spanner that has a linear number of edges. We also show that the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) has constant spanning ratio. The straight line embedding of the Additively Weighted Delaunay graph may not be a plane graph. We show how to compute a plane embedding that also has a constant spanning ratio.

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1 Introduction

Let G be a complete weighted graph where edges have positive weight. Given two vertices u, v of G, we denote by $\delta_G(u, v)$ the length of a shortest path in G between u and v. A spanning subgraph H of G is a *t-spanner* of G if $\delta_H(u, v) \leq t\delta_G(u, v)$ for all pair of vertices u and v. The smallest t having this property is called the *spanning ratio* of the graph H with respect to G. Thus, a graph with spanning ratio t approximates the $\binom{n}{2}$ distances between the vertices of G within a factor of t. Let P be a set of n points in the plane. A *geometric graph* with vertex set P is an undirected graph whose edges are line segments that are weighted by their length. The problem of constructing t-spanners of geometric graphs with $O(n)$ edges for any given point set has been studied extensively; see the book by Narasimhan and Smid [25] for an overview.

In this paper, we address the problem of computing geometric spanners with additive constraints on the points. More precisely, we define a weighted point set as a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_ip_j| - r_i - r_j$. The problem we address is to compute a spanner of a complete graph on a weighted point set. To the best of our knowledge, the problem of constructing a geometric spanner in this context has not been previously addressed. We show how the Yao graph can be adapted to compute a $(1+\epsilon)$-spanner in the case where all r_i are positive real numbers and $|p_ip_j| \geq r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane). In the same case, we also how the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) provides a plane spanner that has the same spanning ratio as the Delaunay graph of a set of points.

1.1 Motivations

It has been claimed (see [2, 30, 31]) that geometric spanners can be used to address the link selection problem in wireless networks. In most cases, however, two assumptions are made:

1. nodes can be represented as points in the plane and
2. the cost of routing a message is a function of the length of the links that are successively used.

However, these assumptions do not always hold. For example, the first assumption does not hold in the case of wide area mesh networks, where nodes are vast areas such as villages [28]. The second assumption does not take into account the fact that some nodes may have higher energy resources or introduce more delay than others. In such cases, an additional cost must be taken into account for each node. The study of spanners of additively weighted point sets is a first step in addressing some of these issues.

1.2 Paper Organization

The rest of the paper is divided as follows: In Section 2, we review related work. In Section 3, we give a formal definition of our problem and show that it is not solved by a straightforward extension of the Yao graph. However, in Section 4, we show that a minor adjustment to the Yao graph allows to compute a $(1+\epsilon)$-spanner. In Section 5, we develop some tools used in Section 6 to show that the Additively Weighted Delaunay graph has a constant spanning ratio. We conclude in Section 7.

2 Related Work

Well known examples of geometric t-spanners include the Yao graph [33], the θ-graph [29], the Delaunay graph [19], and the Well-Separated Pair Decomposition (WSPD) [8]. Let $\theta < \pi/4$ be an angle such that $2\pi/\theta = k$, where k is an integer. The Yao graph with angle θ is defined as follows. For every point p, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. Then, there is an oriented edge from p to q if and only if q is the closest point to p in some cone $C_{p,i}$. The Yao graph is sometimes confused with the θ-graph, although they are different graphs. The first phase of the construction of the θ-graph using k cones with angle θ and apex p is identical to the construction of the Yao graph. This may be the root of the confusion. However, there is an oriented edge from p to q in the θ-graph if and only if q has the shortest projection on the bisector of the cone containing q. For Yao graphs [33], the spanning ratio is at most $1/(\cos\theta - \sin\theta)$ provided that $\theta < \pi/4$, and for θ-graphs, the spanning ratio is at most $1/(1-2\sin\frac{\theta}{2})$ provided that $\theta < \pi/3$ [29].

Given a set of points in the plane, there is an edge between p and q in the Delaunay graph if and only if there is an empty circle with p and

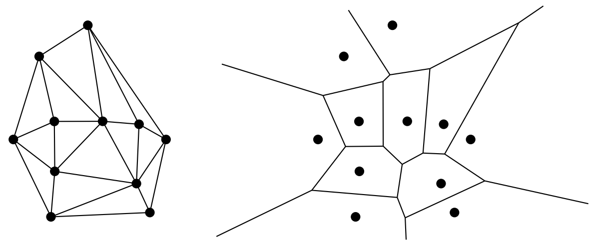

Figure 1: The Delaunay graph and its dual: the Voronoi diagram.

q on its boundary [19]. If no four points are cocircular, then the Delaunay graph is a uniquely defined near-triangulation. Otherwise, four or more co-circular points may create crossings. In that case, removing edges that cause crossings leads to *a* Delaunay triangulation. Since our results hold for any Delaunay triangulation, when we refer to *the* Delaunay triangulation in the case of co-circular points, we mean *any* Delaunay triangulation. Dobkin *et al.* [11] showed that the Delaunay triangulation has a spanning ratio of at most $\frac{1+\sqrt{5}}{2}\pi \approx 5.08$. This result was improved by Keil and Gutwin [19], who showed that the spanning ratio of the Delaunay triangulation is at most $2\pi/(3\cos(\pi/6)) \approx 2.42$. Later, Bose *et al.* [7] showed that the Delaunay triangulation is also a strong t-spanner for the same constant $t = 2\pi/(3\cos(\pi/6))$. Although the exact spanning ratio of the Delaunay triangulation is unknown, it is conjectured that the spanning ratio is $\pi/2$. For the remainder of this paper, we will refer to the spanning ratio of the Delaunay triangulation as the spanning ratio of the *standard* Delaunay triangulation and denote it as SP-DT.

The *Voronoi diagram* [10] of a finite set of points P is a partition of the plane into $|P|$ regions such that each region contains exactly those points having the same nearest neighbor in P. The points in P are also called *sites*. It is well known that the Voronoi diagram of a set of points is the face dual of the Delaunay graph of that set of points [10], i.e. two points have adjacent Voronoi regions if and only if they share an edge in the Delaunay graph (see Figure 1).

Let $s > 0$ be a real number. Two set of points A and B in $\mathbb{R}^d$ are *well-separated with respect to s* if there exists two d-dimensional balls C_A and C_B of same radius r respectively containing the bounding boxes of A and B such that the distance between C_A and C_B is greater than or equal to $s \times r$. The distance between C_A and C_B is defined as the distance between their centers minus $2 \times r$. A *Well-Separated Pair Decomposition with separation ratio s of a set of points P* [8, 25] is a set of unordered pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ of subsets of P that are well-separated with respect to s with the additional property that for every two points $p, q \in P$ there is exactly one pair $\{A_i, B_i\}$ such that $p \in A_i$ and $q \in B_i$. Callahan and Kosaraju [8] showed that for $s > 4$, every point set admits a WSPD with separation ratio s of $O(n)$ size that can be computed in $O(n \log n)$ time. Choosing one edge per pair allows to construct a t-spanner that has $O(n)$ size with $t = (s+4)/(s-4)$.

Our work falls in the context of computing spanners for geometric graphs other than the complete Euclidean graph. Typically, variations of the spanner problem arise by either changing the distance function or removing edges from the complete graph. For example, for a set P of points in the plane and a set C of non-intersecting line segments whose endpoints are in P, the *visibility graph* of P with respect to C is the geometric graph with vertex set P and there is an edge (pq) if and only if the segment $\overline{pq}$ is in C or it does not cross any segment in C (in that case, p and q are said to be *visible*). A spanner of the visibility graph should then approximate Euclidean distances for every pair of points that are visible from each other. The constrained Delaunay triangulation (a variation of the Delaunay triangulation) is a 2.42-spanner of the visibility graph [6, 17, 22].

Unit disk graphs [9, 16] received a lot of attention from the wireless community. A *unit disk graph* is a graph whose nodes are points in the plane and edges join two points whose distance is at most one unit. It is well-known that intersecting a unit disk graph with the Delaunay or the Yao graph of the points provides a t-spanner of the unit disk graph [7], where the constant t is the same as the one of the original graph. However, this simple strategy does not work with all spanners. In particular, it does not work with the θ-graph [5]. Unit disk graphs can be seen as intersection graphs of disks of same radius in the plane. The general problem of computing spanners for geometric intersection graphs has been studied by Furer and Kasiviswanathan [13].

Another graph that has been looked at is the *complete k-partite Euclidean graph*. In that case, points are assigned a unique color (which may be thought of as a positive integer) between 1 and k, and there is an edge between two points if and only if they are assigned different colors. Bose *et al.* [4] showed that the WSPD can be adapted to compute a t-spanner of that graph that has $O(n)$ edges for arbitrary values of t strictly greater than 5.

For spanners of arbitrary geometric graphs, much less is known. Althöfer *et al.* [1] have shown that for any $t > 1$, every weighted graph G with n vertices contains a subgraph with $O(n^{1+2/(t-1)})$ edges, which is a t-spanner of G. Observe that this result holds for any weighted graph; in particular, it is valid for any geometric graph. For geometric graphs, a lower bound was given by Gudmundsson and Smid [15]: They proved that for every real number t with $1 < t < \frac{1}{4}\log n$, there exists a geometric graph H with n vertices, such that every t-spanner of H contains $\Omega(n^{1+1/t})$ edges. Thus, if we are looking for spanners with $O(n)$ edges of arbitrary geometric graphs, then the best spanning ratio we can obtain is $\Theta(\log n)$.

In the literature, spanners that use a distance other than the Euclidean distance have also been proposed. For example, in a *power* spanner [3, 14, 24, 30], the distance used to measure the length of an edge is the square of the Euclidean distance between its two end points. This models the fact that in wireless networks, the amount of energy needed to send a packet is proportional to a power (not necessarily the square, however) of the Euclidean distance between the sender and receiver [27]. When reducing the latency is more important than reducing the amount of energy being used, a *hop* spanner [2], which gives an equal weight to every edge, can be used.

In this paper, the Additively Weighted Voronoi diagram (AW-Voronoi diagram) is of particular interest. Its definition is similar to that of the (standard) Voronoi diagram, except that each site p_i is assigned a weight which is a real number r_i. Weights are used to define a weighted distance. More detail about how the weighted distance is used to define the AW-Voronoi diagram is given in Section 6. The Additively Weighted Delaunay graph (AW-Delaunay graph) is defined as the face-dual of the AW-Voronoi diagram. Properties of the AW-Voronoi diagram and its dual have been studied by Lee and Drysdale [23], who showed how to compute it in $O(n \log^2 n)$ time. Later on, Fortune [12] showed how to compute it in $O(n \log n)$ time. The AW-Voronoi diagram may have empty cells. For this reason, one would hope that it is possible to design an algorithm whose running time gets better as the number of empty cells increases. Karavelas and Yvinec [18] provided an $O(nT(h) + h \log h)$ time algorithm to compute the AW-Voronoi diagram where h is the number of non-empty cells and $T(h)$ is the time to locate the nearest neighbor of a query point within a set of h points. Experimental results suggested an $O(n \log h)$ behavior. In 3D, the complexity of the (Additively Weighted) Voronoi diagram is $\Theta(n^2)$ [21]. Aurenhammer [3] showed how to compute it in time $O(n^2)$ using Power Voronoi diagrams. Will [32] gave an $O(n^2 \log n)$ time algorithm with experimental results suggesting an

$O(n \log^2 n)$ time behavior in the expected case. Kim *et al.* [20] showed how to obtain a running time of $O(nm)$, where m is the number of edges.

3 Definitions and Notation

Definition 3.1 *A set $P = \{(p_1, r_1), \ldots, (p_n, r_n)\}$ of ordered pairs, where each p_i is a point in the plane and each r_i is a real number, is called a* weighted point set*. The notation $p_i \in P$ means that there exists an ordered pair (p_i, r_i) such that $(p_i, r_i) \in P$. The* additive distance *from a point $p \notin P$ in the plane to a point $p_i \in P$, noted $d(p, p_i)$, is defined as $|pp_i| - r_i$, where $|pp_i|$ is the Euclidean distance from p to p_i. The* additive distance *between two points $p_i, p_j \in P$, noted $d(p_i, p_j)$, is defined as $|p_ip_j| - r_i - r_j$, where $|p_ip_j|$ is the Euclidean distance from p_i to p_j.*

The problem we address in this paper is the following:

Problem 3.2 *Let P be a weighted point set and let $K(P)$ be the complete weighted graph with vertex set P and edges weighted by the additive distance between their endpoints. Compute a t-spanner with $O(n)$ edges of $K(P)$ for a fixed constant $t > 1$.*

Notice that in the case where all r_i are positive numbers, the pairs (p_i, r_i) can be viewed as disks D_i in the plane. If, for all i, j we also have $d(p_i, p_j) \geq 0$, then the disks are disjoint. In that case, the distance $d(D_i, D_j) = d(p_i, p_j) = |p_ip_j| - r_i - r_j$ is also equal to $\min\{|q_iq_j| : q_i \in D_i \text{ and } q_j \in D_j\}$, where the notation $q_i \in D_i$ means $|p_iq_i| \leq r_i$. To compute a spanner of an additively weighted point set is then equivalent to computing a spanner of a set of disks in the plane. **From now to the end of this paper, it is assumed that all r_i are positive numbers and $d(p_i, p_j) \geq 0$ for all i, j.** If $\mathcal{D}$ is a set of disks in the plane, then a *spanner* of $\mathcal{D}$ is a spanner of the complete graph whose vertex set is $\mathcal{D}$ and whose edges (D_i, D_j) are given weights equal to $d(D_i, D_j)$.

Notice also that the additive distance may not be a metric since the triangle inequality does not necessarily hold (see Figure 2). Although this may seem counter-intuitive, this makes sense in some networks, since a direct communication is not always easier than routing through a common neighbor. For example, in wireless networks, the amount of energy that is needed to transmit a message is a power of the Euclidean distance between the sender

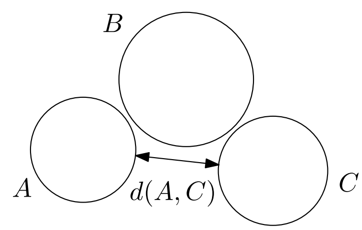

Figure 2: The additively weighted distance is not a metric.

and the receiver. Therefore, using several small hops can be more energy efficient that a direct communication over one long-distance link.

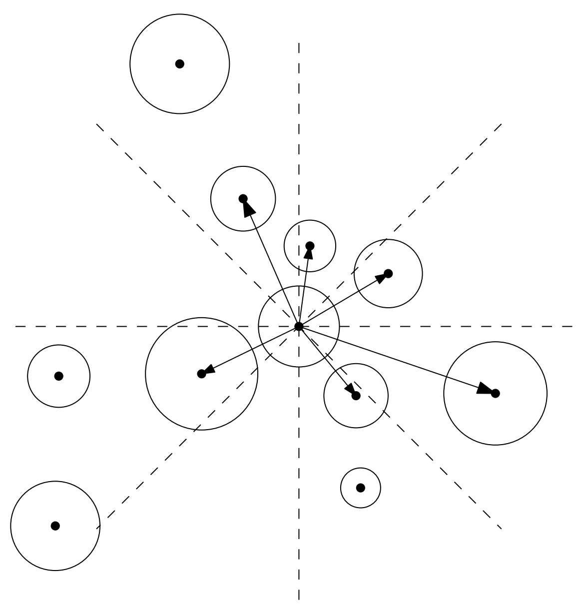

Figure 3: A straightforward generalization of the Yao graph.

Figure 3 shows how the Yao graph can be generalized using the additive distance: every node keeps an outgoing edge with the closest disk that intersects each cone. However, this graph is not a spanner. Figure 4 shows how to construct an example with four disks that has an arbitrarily large spanning ratio. Nonetheless, in Section 4, we see that a minor adjustment to the Yao graph can be made in order to compute a $(1+\epsilon)$-spanner of a set of disjoint disks that has $O(n)$ edges.

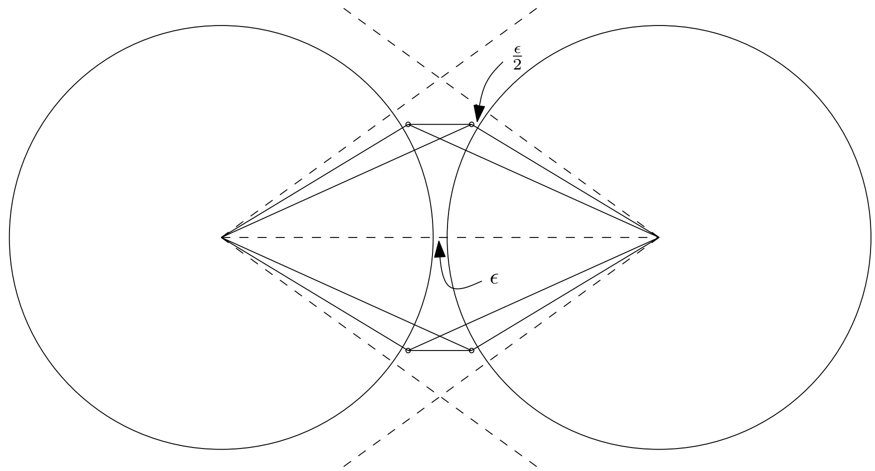

Figure 4: The straightforward generalization of the Yao graph does not have constant spanning ratio.

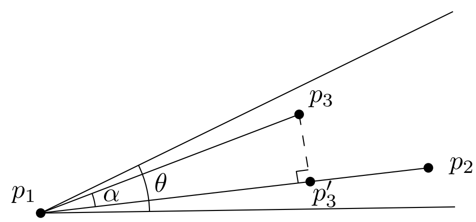

Figure 5: Illustration of the proof of Lemma 4.2.

The Delaunay graph in the additively weighted setting is computable in time $O(n \log n)$ [12]. To the best of our knowledge, its spanning properties have not been previously studied. In the two next sections, we show that it is a spanner and that its spanning ratio is SP-DT (i.e the same as that of the standard Delaunay graph). Finally, we show that although the additively weighted Delaunay graph is not necessarily plane, it contains a plane subgraph that is a spanner with the same spanning ratio.

4 The Additively Weighted Yao Graph

As we saw in the previous section, a straightforward generalization of the Yao graph fails to provide a graph with bounded spanning ratio. In this section, we show how a few subtle modifications to the construction, provide an approach to build a $(1+\epsilon)$-spanner. We define the modified Yao construction below.

Definition 4.1 *Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta \leq 0.228$ be an angle such that $2\pi/\theta = k$, where k is an integer. The* $\text{Yao}(\theta, \mathcal{D})$ *graph is defined as follows. For every disk $D = (p, r)$, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. A disk* blocks *a cone $C_{p,i}$ provided that the disk intersects both rays of $C_{p,i}$. Let $F \in \mathcal{D}$ be a disk different from D with center in $C_{p,j}$. Add an edge from D to F in* $\text{Yao}(\theta, \mathcal{D})$ *if and only if one of the two following conditions is met:*

1. *among all blocking disks that have their center in $C_{p,j}$, F is the one that is the closest to D;*

2. *among all disks that have their center in $C_{p,j}$ and are at a distance of at least r to D, F is the one that is the closest to D.*

Notice that there are two main changes. Within each cone, we now add potentially two edges as opposed to only one edge in the case of unweighted points. Next, in the second condition to add an edge, we do not add an edge to the closest disk within a cone but to the closest disk whose distance is at least r from the disk centered at the apex with radius r. We now prove that these two modifications imply that the resulting graph is a $(1 + \epsilon)$-spanner.

Lemma 4.2 *Let p_1, p_2, p_3 such that the angle $\angle p_3p_1p_2 = \alpha \leq \theta < \pi/4$ and $|p_1p_3| \leq |p_1p_2|$. Then $|p_2p_3| \leq |p_1p_2| - (\cos\theta - \sin\theta)|p_1p_3|$.*

*P*roof: Let p_3' be the projection of p_3 on the line through p_1 and p_2 (see Figure 5). Then

$$\begin{aligned}
|p_2p_3| &\leq |p_2p_3'| + |p_3'p_3| \\
&= |p_1p_2| - |p_1p_3'| + |p_3'p_3| \\
&= |p_1p_2| - |p_1p_3|(\cos\alpha - \sin\alpha) \\
&\leq |p_1p_2| - |p_1p_3|(\cos\theta - \sin\theta)
\end{aligned}$$

□

Theorem 4.3 *Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta \leq 0.228$. Then $Y(\theta, \mathcal{D})$ is a t-spanner of $\mathcal{D}$, where $t = 1/(\cos 2\theta - \sin 2\theta - 2\sin(\theta/2))$.*

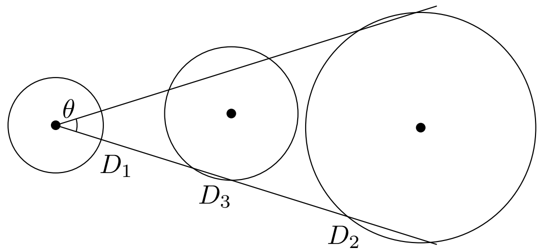

Figure 6: If D_2 blocks the cone but the edge (D_1, D_2) is not in Yao$(\theta, \mathcal{D})$, then there exists D_3 such that $d(D_1, D_3) + d(D_3, D_2) < d(D_1, D_2)$.

*P*roof: We proceed by induction on the rank of the weighted distances between the pairs of disks D_1 and D_2.

Base case: The disks D_1 and D_2 form a closest pair. In that case, the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$.

Induction case: Let $D_1 = (p_1, r_1)$ and $D_2 = (p_2, r_2)$. Without loss of generality, $r_1 \leq r_2$. If the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$, then there is nothing to prove. Otherwise, there are two cases to consider depending on whether or not the shortest path from D_1 to D_2 in the complete graph on $\mathcal{D}$ is the edge (D_1, D_2). If the shortest path is not the edge (D_1, D_2), then all edges on the shortest path must have length less than $d(D_1, D_2)$. By applying the induction hypothesis on each of those edges, we conclude that the distance from D_1 to D_2 in Yao$(\theta, \mathcal{D})$ is at most t times the length of the shortest path D_1 to D_2 in the complete graph on $\mathcal{D}$, as required.

We now consider the case when the edge (D_1, D_2):

1. is not in Yao$(\theta, \mathcal{D})$ and

2. is the shortest path from D_1 to D_2 in the complete graph.

Observe that the conjunction of those two facts imply that the disk D_2 does not block the cone whose apex is p_1 and contains p_2: If D_2 was blocking the cone, then since (D_1, D_2) is not an edge in Yao$(\theta, \mathcal{D})$, there must be a disk D_3 that is also blocking the cone and is closer to D_1 than D_2. However, this implies that the shortest path from D_1 to D_2 in the complete graph is not the edge (D_1, D_2) (see Figure 6).

The conjunction of the three following facts:

1. $r_1 \leq r_2$;

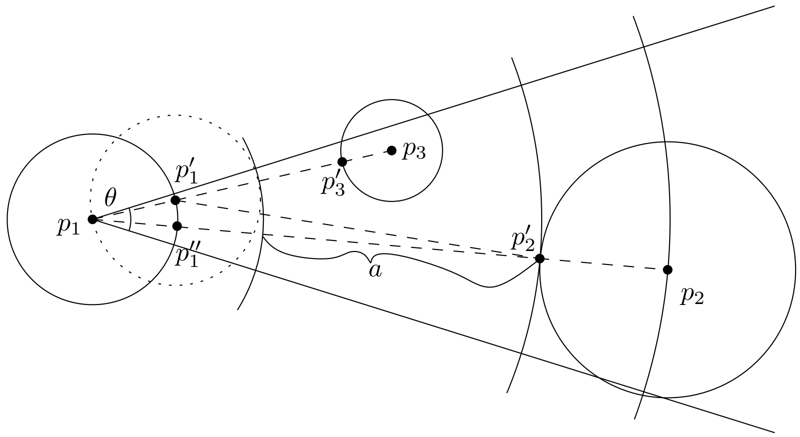

Figure 7: Illustration of the proof of Theorem 4.3.

2. $\theta \leq 0.228 < \sin^{-1}(1/3)$ and

3. D_2 does not block the cone,

imply that $d(D_1, D_2) > r_1$. Since (D_1, D_2) is not an edge, there another disk whose distance is at least r that is closer to D_1. Let $D_3 = (p_3, r_3)$ be the closest disk to D_1 such that p_3 is in the same θ-cone with apex at p_1 as p_2 and $d(D_1, D_3) \geq r_1$. By definition, the edge (D_1, D_3) is in Yao$(\theta, \mathcal{D})$. Observe that $d(D_2, D_3) < d(D_1, D_2)$. To see this, let $a := d(D_1, D_2) - r_1$. We have that

$$d(D_2, D_3) \leq a + 4r_1 \sin(\theta/2) \leq a + 4r_1 \sin(0.114) < a + r_1 = d(D_1, D_2).$$

Let p'_1 be the point of D_1 that is the closest to D_3, p''_1 be the point of D_1 that is the closest to D_2, p'_2 be the point of D_2 that is the closest to D_1, and p'_3 be the point of D_3 that is the closest to D_1 (see Figure 7). Notice that $|p'_1p'_3| \leq |p'_1p'_2|$ and that since $d(D_1, D_2) \geq d(D_1, D_3) \geq r_1$, then the angle $\angle p'_2p'_1p'_3$ is at most $2\theta < \pi/4$. Therefore, we can apply Lemma 4.2 to conclude that

$$|p'_2p'_3| \leq |p'_1p'_2| - (\cos 2\theta - \sin 2\theta)|p'_1p'_3|,$$

which implies that

$$d(D_2, D_3) \leq d(D_1, D_2) + |p'_1p''_1| - (\cos 2\theta - \sin 2\theta)d(D_1, D_3).$$

Also, since $|p'_1p''_1| \leq 2\sin(\theta/2)r_1 \leq 2\sin(\theta/2)d(D_1, D_3)$, we have

$$d(D_2, D_3) \leq d(D_1, D_2) - (\cos 2\theta - \sin 2\theta - 2\sin(\theta/2))d(D_1, D_3).$$

Finally, since $d(D_2, D_3) < d(D_1, D_2)$, the induction hypothesis tells us that Yao$(\theta, \mathcal{D})$ contains a path from D_2 to D_3 whose length is at most $td(D_2, D_3)$. This means that the distance from D_1 to D_2 in Yao$(\theta, \mathcal{D})$ is at most

$$d(D_1, D_3)+td(D_2, D_3) \leq d(D_1, D_3)+t(d(D_1, D_2)-\frac{1}{t}d(D_1, D_3)) = td(D_1, D_2).$$

The value 0.228 is an upper bound on the values of θ such that $t > 0$. □

Corollary 4.4 *For any $\epsilon > 0$ and any set $\mathcal{D}$ of n disjoint disks, it is possible to compute a $(1 + \epsilon)$-spanner of $\mathcal{D}$ that has $O(n)$ edges.*

*P*roof: The bound on the number of edges comes from the fact that each cone contains at most two edges, and the stretch factor of $1 + \epsilon$ comes from the fact that $\lim_{\theta \to 0} 1/(\cos 2\theta - \sin 2\theta - 2\sin(\theta/2)) = 1$. □

5 Quotient Graphs and Quotient Spanners

The main idea in the remainder of this paper is the following: we show how to compute a set of points from each D_i such that the (standard) Delaunay graph of those points is *equivalent* to the Additively Weighted Delaunay graph. By choosing the appropriate equivalence relation as well as the appropriate point set, we can then show that the spanning ratio of the Additively Weighted Delaunay graph is bounded by the spanning ratio of the standard Delaunay graph. The reduction of one graph to another is done by means of a quotient:

Definition 5.1 *Let P_1 and P_2 be non-empty sets of points in the plane. The* distance *between P_1 and P_2, denoted by $|P_1P_2|$, is defined as the minimum $|p_1p_2|$ over all pairs of points such that $p_1 \in P_1$ and $p_2 \in P_2$.*

Definition 5.2 *Let $G = (V, E)$ be a geometric graph and $\mathcal{V}$ be a partition of V. The* quotient graph *of G by $\mathcal{V}$, denoted $G/\mathcal{V}$, is the graph having $\mathcal{V}$ as vertices and there is an edge (U, W) (where U and W are in $\mathcal{V}$) if and only if there exists an edge $(u, w) \in E$ with $u \in U$ and $w \in W$. The weight of the edge (U, W) is equal to $|UW|$.*

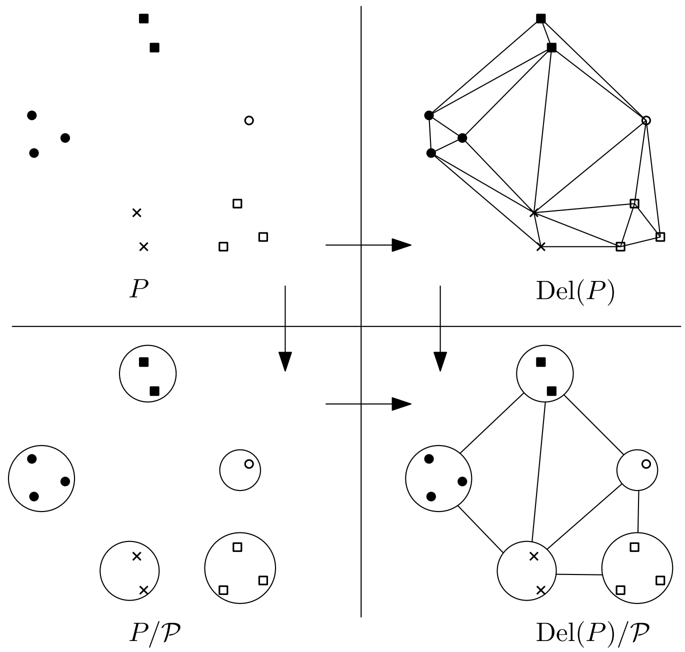

Figure 8: Illustration of Lemma 5.3.

If P is a (non-weighted) point set and $\mathcal{P}$ is a partition of P, then the notation $P/\mathcal{P}$ designates the quotient of the complete Euclidean graph on P by $\mathcal{P}$. If $\mathcal{S}$ is a set of pairwise disjoint sets of points in the plane such that $P \subseteq \bigcup \mathcal{S}$, then the notation $P/\mathcal{S}$ designates the quotient of the complete Euclidean graph on P by the partition of P induced by $\mathcal{S}$.

Lemma 5.3 *Let $G = (V, E)$ be a complete geometric graph, $\mathcal{V}$ be a partition of V and S be a t-spanner of G. Then $S/\mathcal{V}$ is a t-spanner of $G/\mathcal{V}$.*

*P*roof: Let (U, W) be an edge of $G/\mathcal{V}$ and (u, w) be an edge of G such that $|uw| = |UW|$. Since G is complete, the edge (u, w) is in G, and since S is a t-spanner of G, there is a path $\psi = u_1, \ldots, u_k$ in S such that $u_1 = u, u_k = w$ and the length of ψ is at most $t|uw|$. For each u_i of ψ, let $U_i \in \mathcal{V}$ be such that $u_i \in U_i$. Notice that it is possible that $U_i = U_{i+1}$ for some i. Let Ψ be the subsequence of $U = U_1, \ldots, U_k = W$ that consists in those U_i such that $i < k$ and $U_i \neq U_{i+1}$. By definition, the sequence Ψ is a path in $S/\mathcal{V}$ and it consists of at most $k' \leq k$ nodes. The length of Ψ is at most

$$\sum_{i=1}^{k'-1} |U_i U_{i+1}| \leq \sum_{i=1}^{k-1} |u_i u_{i+1}| \leq t|uw| = t|UW|$$

which means that Ψ is a t-spanning path for (U, W) in $S/\mathcal{V}$. $\square$

6 The Additively Weighted Delaunay Graph

Lee and Drysdale [23] studied a variant of the Voronoi diagram called the Additively Weighted Voronoi diagram, which is defined as follows: Let P be a weighted point set. The *Additively Weighted Voronoi diagram* of P is a partition of the plane into $|P|$ regions such that each region contains exactly the points in the plane having the same closest neighbor in P according to the additive distance. In other words, the Voronoi cell of a pair (p_i, r_i) contains the points p such that $d(p, p_i)$ is minimum over all other pairs in P. The *Additively Weighted Delaunay graph* (AW-Delaunay graph) is defined as the face-dual of the Additively Weighted Voronoi diagram.

Alternatively, if all r_i are positive and for all i, j, we have $|p_i p_j| \geq r_i + r_j$, then the pairs (p_i, r_i) can be seen as disks D_i of radius r_i centered at p_i and $d(p, D_i)$ is the minimum $|pq|$ over all $q \in D_i$. For a set $\mathcal{D}$ of disks in the plane, we denote the AW-Delaunay graph computed from $\mathcal{D}$ as $\mathrm{Del}(\mathcal{D})$. When no two disks intersect, the AW-Delaunay graph is a natural generalization of the Delaunay graph of a set of points. We say that two disks A and B *properly* intersect if $|A \cap B| > 1$.

Proposition 6.1 *Let $\mathcal{D}$ be a set of disjoint disks in the plane, and $A, B \in \mathcal{D}$. The edge (A, B) is in $\mathrm{Del}(\mathcal{D})$ if and only if there is a disk C that is tangent to both A and B and does not properly intersect any other disk in $\mathcal{D}$.*

*P*roof: Suppose (A, B) is in $\mathrm{Del}(\mathcal{D})$, and let c be a point on the boundary of the Voronoi cells of A and B and r be the distance from c to A. Since c is equidistant from A and B, it is also at distance r from B. This means that the disk C centered at c is tangent to both A and B. This disk cannot properly intersect any other disk of $\mathcal{D}$, since this would contradict the fact that c is in the Voronoi cells of A and B. Similarly, if there is a disk that is

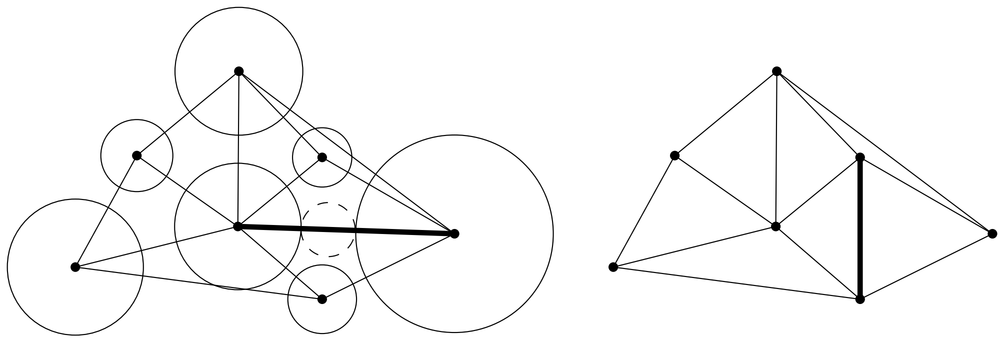

Figure 9: The Additively Weighted Delaunay graph compared with the Delaunay graph of the disks centers.

tangent to both A and B but does not properly intersect any other disk of $\mathcal{D}$, then A and B are Voronoi neighbors. $\square$

Note that the Additively Weighted Delaunay graph is not necessarily isomorphic to the Delaunay graph of the centers of the disks (see Figure 9). When all radii are equal, however, the two graphs coincide. We now show that if $\mathcal{D}$ is a set of disks in the plane, then $\mathrm{Del}(\mathcal{D})$ is a spanner of $\mathcal{D}$. The intuition behind the proof is the following: we show the existence of a finite set of points P such that $K(P)/\mathcal{D}$ (where $K(P)$ is the complete graph with vertex set P) is isomorphic to the complete graph on $\mathcal{D}$ and $\mathrm{Del}(P)/\mathcal{D}$ is a subgraph of $\mathrm{Del}(\mathcal{D})$. Then, we use Lemma 5.3 to prove that $\mathrm{Del}(P)/\mathcal{D}$ is a spanner of $\mathcal{D}$, which implies that $\mathrm{Del}(\mathcal{D})$ is a spanner of $\mathcal{D}$.

Definition 6.2 *Let A, B be disjoint disks and S a set of points such that $A \cap S = \emptyset$ and $B \cap S = \emptyset$. A set of points R* represents *S with respect to A and B if for every disk F that is tangent to both A and B, we have $F \cap S \neq \emptyset \Rightarrow F \cap R \neq \emptyset$. If $\mathcal{D}$ is a set of disjoint disks, then a set of points $\mathcal{R}$* represents *$\mathcal{D}$ if for all $A, B, C \in \mathcal{D}$, there is a subset of $\mathcal{R}$ that represents C with respect to A and B.*

From here to the end of the proof of Lemma 6.6, unless stated otherwise, let

1. A, B be two disjoint disks in the plane having their center on the x-axis;
2. $D(y)$ be the disk that is tangent to both A and B and whose center has y-coordinate equal to y;

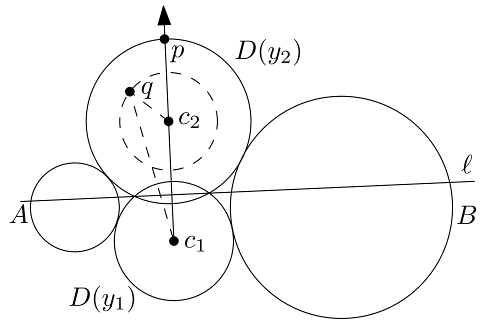

Figure 10: Illustration of the proof of Lemma 6.3.

3. $y(D)$ be the y-coordinate of the center of a disk D;

4. ℓ_1, ℓ_2 be the two lines that are outer-tangent to both A and B (respectively, from below and above);

5. y_1, y_2 be such that $y_1 < y_2$ and $D(y_1) \cap D(y_2) \neq \emptyset$;

6. ℓ be the line through the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ (if $D(y_1)$ and $D(y_2)$ are tangent, then ℓ is the unique line that is tangent to both $D(y_1)$ and $D(y_2)$);

7. $T(A, B)$ denote the region below ℓ_2, above ℓ_1 and between A and B; and

8. l^+ (l^-) be the closed half-plane above (below) a non-vertical line l.

Throughout this section, it is implicitly assumed that $D(\infty) = \ell_2^+$ and $D(-\infty) = \ell_1^-$.

Lemma 6.3 *Given $y_1 < y_2$ and $D(y_1) \cap D(y_2) \neq \emptyset$, we have $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$ and $D(y_2) \cap \ell^- \subset D(y_1) \cap \ell^-$ (see Figure 10).*

*P*roof: Notice that either $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$ or $D(y_2) \cap \ell^+ \subset D(y_1) \cap \ell^+$. Therefore, all we need to show is that $(D(y_2) \cap \ell^+) \setminus (D(y_1) \cap \ell^+)$ is not empty. Let c_1, c_2 be the respective centers of $D(y_1)$ and $D(y_2)$, and p be the intersection point of the infinite ray from c_1 through c_2 with the boundary of $D(y_1) \cup D(y_2)$.

We show by contradiction that p is not in $D(y_1)$. If that was the case, then $D(y_2)$ would be completely contained in $D(y_1)$. The reason for this is

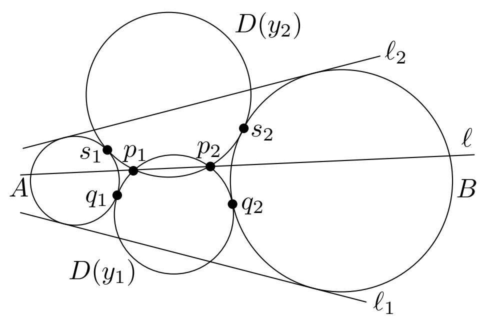

Figure 11: Illustration of the proof of Lemma 6.4.

that there is no point of $D(y_2)$ that is farther from c_1 than p. Let q be a point of $D(y_2)$. Then $|qc_1| \leq |qc_2| + |c_2c_1| \leq |pc_2| + |c_2c_1| = |pc_1|$. But the fact that $D(y_2)$ is completely contained in $D(y_1)$ contradicts the fact that they are both tangent to A and B.

Therefore, since $p \in \ell^+$, we have $p \in (D(y_2) \cap \ell^+) \setminus (D(y_1) \cap \ell^+)$, which imply that $D(y_1) \cap \ell^+ \subset D(y_2) \cap \ell^+$. Similarly, $D(y_2) \cap \ell^- \subset D(y_1) \cap \ell^-$. □

Lemma 6.4 *Let p_1, p_2 be the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ (if $D(y_1)$ and $D(y_2)$ are tangent, then $p_1 = p_2$). Then p_1 and p_2 are in ℓ_2^- and in ℓ_1^+ (see Figure 11).*

*P*roof: Let q_1, q_2 be the tangency points of $D(y_1)$ with A and B and s_1, s_2 be the tangency points of $D(y_2)$ with A and B. By Lemma 6.3, q_1, q_2 are below ℓ and s_1, s_2 are above ℓ. Since ℓ is above q_1 and q_2, which are in turn above ℓ_1, it follows that p_1 and p_2 are above ℓ_1. By a symmetric argument, p_1 and p_2 are below ℓ_2. □

Lemma 6.5 *The following are true:*

1. *For all $p \in \ell_2^+$, there exists a line $y = y_0$ such that for all disk E that is tangent to both A and B, if the center of E is above y_0 then $p \in E$.*

2. *For all $p \in \ell_1^-$, there exists a line $y = y_1$ such that for all disk E that is tangent to both A and B, if the center of E is below y_1 then $p \in E$.*

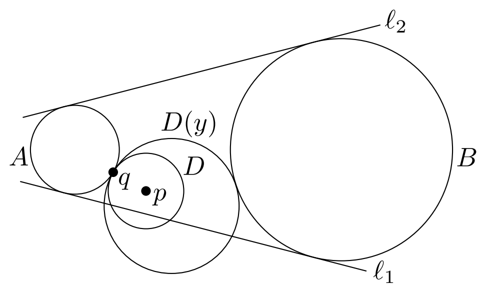

Figure 12: Illustration of the proof of Lemma 6.5 (3) (first part).

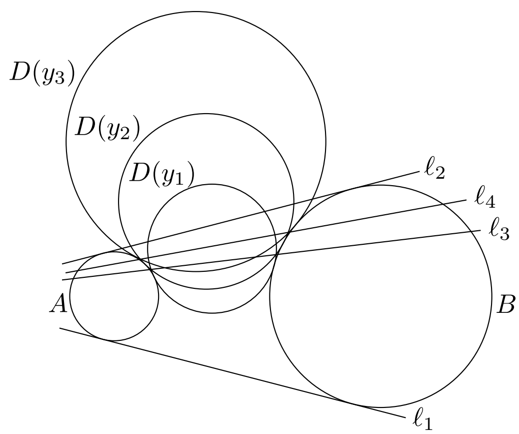

Figure 13: Illustration of the proof of Lemma 6.5 (3) (second part).

3. *For all p in $T(A,B)$, there exists two lines $y = y_0$ and $y = y_1$ such that for all disk E that is tangent to both A and B, $p \in E$ if and only if the center of E is between y_0 and y_1.*

*P*roof: For (1), the existence of y_0 is guaranteed by the fact that $\lim_{y\to\infty} D(y) = \ell_2^+$. Now, let y_0 be such that $p \in D(y_0)$ and $y' > y_0$. Let $L(y_0)$ and $L(y')$ be the lunes respectively defined by the intersection of $D(y_0)$ and $D(y')$ with the half-plane above ℓ_2. By Lemma 6.4, the two points where the boundaries of $D(y_0)$ and $D(y')$ intersect are below ℓ_2. Therefore, we have either $L(y_0) \subset L(y')$ or $L(y') \subset L(y_0)$. But since $y' > y_0$, by Lemma 6.3 we have $L(y_0) \subset L(y')$ and therefore $p \in L(y')$. The proof of (2) is symmetric.

For (3), the existence is easy to show. Without loss of generality, assume $d(p,A) \leq d(p,B)$. Let D be the disk centered at p that is tangent to A and let q be the tangency point of A and D see Figure 12. Since $q \in T(A,B)$,

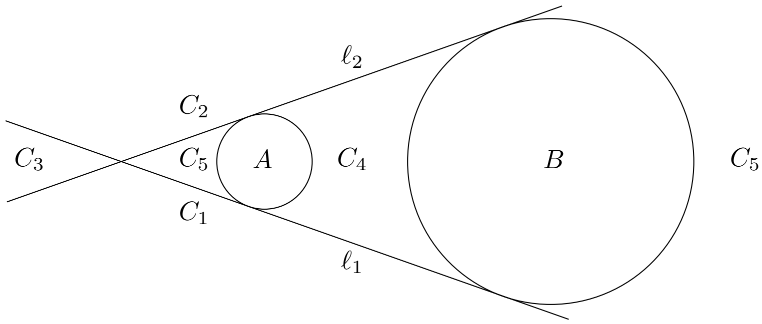

Figure 14: The five regions for Lemma 6.6.

there exists y such that $D(y) \cap A = q$. Since $D \subseteq D(y)$, there exists a disk that is tangent to both A and B and contains p.

We now show that $y_1 < y_2 < y_3$ implies $D(y_1) \cap D(y_3) \subseteq D(y_2)$ (see Figure 13). Let ℓ_3 be the line through the intersection points of the boundaries of $D(y_1)$ and $D(y_2)$ and let ℓ_4 be the line through the intersection points of the boundaries of $D(y_2)$ and $D(y_3)$. Let $p \in D(y_1) \cap D(y_3)$. Since ℓ_4 is above ℓ_3 in $D(y_1) \cap D(y_3)$, p is either above ℓ_3, below ℓ_4 or both. If $p \in \ell_3^+$, then since $y_1 < y_2$, by Lemma 6.3 we have that $D(y_1) \cap \ell_3^+ \subseteq D(y_2) \cap \ell_3^+$ and $p \in D(y_1) \cap D(y_2)$. Similarly, if $p \in \ell_4^-$, then since $y_2 < y_3$, by Lemma 6.3 we have that $D(y_3) \cap \ell_4^- \subseteq D(y_2) \cap \ell_4^-$ and $p \in D(y_3) \cap D(y_2)$. In either case, $p \in D(y_2)$, which completes the proof. $\square$

Lemma 6.6 *Let C be a disk that is disjoint of both A and B. There exists a set of at most six points that represents C with respect to A and B.*

*P*roof: Let

$$\begin{aligned}
C_1 &:= (C \cap \ell_1^-) \setminus \ell_2^+ \\
C_2 &:= (C \cap \ell_2^+) \setminus \ell_1^- \\
C_3 &:= C \cap \ell_1^- \cap \ell_2^+ \\
C_4 &:= C \cap T(A, B) \\
C_5 &:= (C \cap \ell_1^+ \cap \ell_2^-) \setminus T(A, B)
\end{aligned}$$

These five regions partition the disk C (see Figure 14). We show that for each region, there is a finite set of points that represents it. The cardinality of the union of the sets is no more than six.

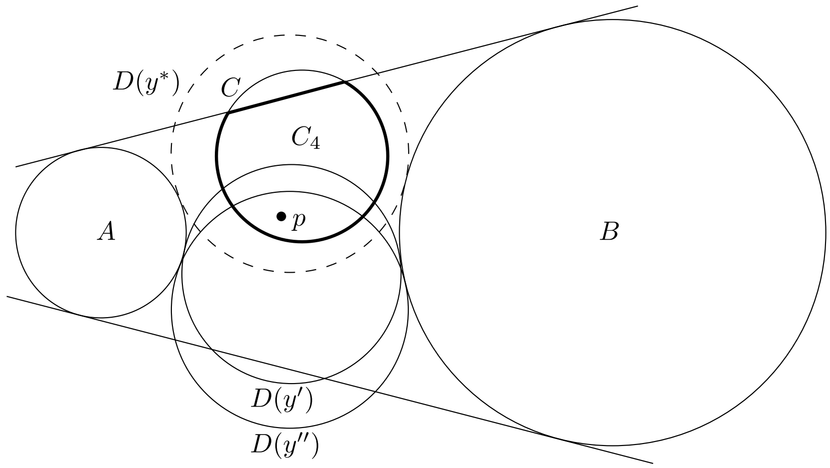

Figure 15: Case C_4 of the proof of Lemma 6.6.

If $C_1 \neq \emptyset$, then let y_0 be the minimum y such that $D(y)$ intersects C_1. Let $p_1 \in C_1 \cap D(y_0)$. By definition of y_0, for any disk E that is tangent to both A and B and intersects C_1, we have $y(E) \geq y_0$, and by Lemma 6.5, we have $p_1 \in E$.

Similarly, if $C_2 \neq \emptyset$, then let y_1 be the maximum y such that $D(y)$ intersects C_2. Let $p_2 \in C_2 \cap D(y_1)$. By definition of y_1, for any disk E that is tangent to both A and B and intersects C_2, we have $y(E) \leq y_1$, and by Lemma 6.5, we have $p_2 \in E$.

If $C_3 \neq \emptyset$, then let y_0 be the minimum $y > 0$ such that $D(y)$ intersects C_3 and y_1 as the maximum $y < 0$ such that $D(y)$ intersects C_3. Let $p_3 \in C_3 \cap D(y_0)$ and $p_4 \in C_3 \cap D(y_1)$. By definition of y_0, for any disk E with $y(E) > 0$ that is tangent to both A and B and intersects C_3, we have $y(E) \geq y_0$, and by Lemma 6.5, we have $p_3 \in E$. The same reasoning applies to p_4 when $y(E) < 0$.

If $C_4 \neq \emptyset$, then let y_0 be the minimum y such that $D(y)$ intersects C_4 and y_1 as the maximum y such that $D(y)$ intersects C_4. Let $p_5 \in C_4 \cap D(y_0)$ and $p_6 \in C_4 \cap D(y_1)$. Let y^* be such that $C \subseteq D(y^*)$ (see Figure 15). Let E be a disk that is tangent to both A and B and intersects C_4. We show that $y(E) \leq y^* \implies p_5 \in E$ (and similarly, $y(E) \geq y^* \implies p_6 \in E$). It is sufficient to show that $y'' < y' < y^* \implies C \cap D(y'') \subset C \cap D(y')$. Let $p \in D(y'') \cap C$. By Lemma 6.5, $\exists y_0(p), y_1(p)$ such that $\forall$ disk E tangent to both A and B, we have $y_0(p) \leq y(E) \leq y_1(p) \Leftrightarrow p \in E$. Therefore, the

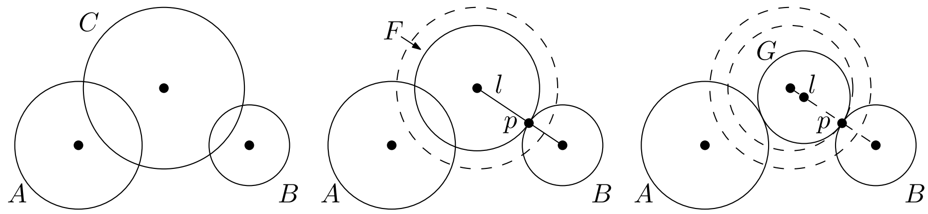

Figure 16: Proof of Lemma 6.8.

following hold:

$$\begin{aligned} y_0(p) \leq y^* &\leq y_1(p) \\ y_0(p) \leq y'' &\leq y_1(p) \end{aligned}$$

But since $y'' < y' < y^*$, we have $y'' < y' < y^*$, which imply that $p \in C \cap D(y')$.

Finally, since $C_5 \cap E = \emptyset$ for any disk E that is tangent to both A and B, there is no need to select representative points for C_5. □

Careful analysis of the proof of Lemma 6.6 allows us to observe that in fact, only two points are necessary to represent a disk C with respect to two other disks A and B. First, note that $C_4 \neq \emptyset \implies C_3 = \emptyset$ and $C_3 \neq \emptyset \implies C_4 = \emptyset$. This reduces to four the number of points that are necessary. Also, if $C_1 \neq \emptyset$ and $C_4 \neq \emptyset$, then p_6 is on ℓ_2 and is not required since any disk that contains it also intersects C_1 and therefore contains p_1. Similarly, if $C_2 \neq \emptyset$ and $C_4 \neq \emptyset$, then p_5 is not required since any disk that contains it also intersects C_2 and therefore contains p_2. Therefore, if $C_4 \neq \emptyset$, then the number of points that are necessary is at most two. A similar argument applies to the case where $C_3 \neq \emptyset$. Finally, if both C_3 and C_4 are empty, then only p_1 and p_2 may be required. Therefore, we have the following corollary:

Corollary 6.7 *Let $\mathcal{D}$ be a set of n disjoint disks. There exists a set of at most $2\binom{n}{3}$ points that represents $\mathcal{D}$.*

Lemma 6.8 *Let A and B be two disjoint disks and C be a disk intersecting both of them. Then there exists a disk G inside C that is tangent to both A and B.*

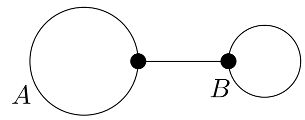

Figure 17: The distance points of A and B.

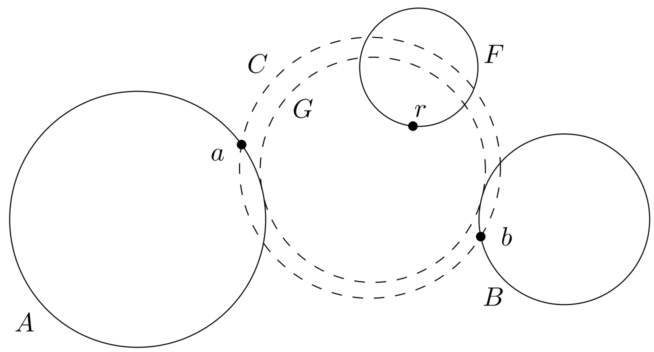

Figure 18: Illustration of the proof of Theorem 6.10.

Proof: We show how to construct G. Let a, b, c and r_A, r_B, r_C respectively be the centers and radii of A, B and C. Without loss of generality, assume $|ac| - r_C \leq |bc| - r_B$. Let F be the disk centered at c and having radius $r_F = |bc| - r_B$ (see Figure 16). The disk F is tangent to B. If F is also tangent to A, then let $G = F$ and we are done. Otherwise, F is properly intersecting A. In that case, let p be the tangency point of F and B, l be the line through b and c, and G be the disk through p having its center on l and tangent to A. The result follows from the fact that G is tangent to B and inside C. $\square$

Definition 6.9 *Let A and B be two disks in the plane. The* distance points *of A and B are the two ends of the shortest line segment between A and B (see Figure 17). If $\mathcal{D}$ is a set of disjoint disks, then the set of* distance points *of $\mathcal{D}$ is the set containing the distance points of every pair of disks in $\mathcal{D}$.*

Theorem 6.10 *Let $\mathcal{D}$ be a set of n disjoint disks. Then* $\mathrm{Del}(\mathcal{D})$ *is a t-spanner of $\mathcal{D}$, where t is the spanning ratio of the Delaunay triangulation of a set of points.*

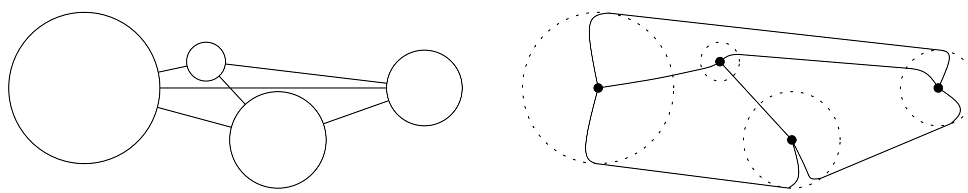

Figure 19: Even if the embedding of the AW-Delaunay graph that consists of straight line segments between the centers of the disks is not necessarily a plane graph, it is planar.

*P*roof: By Corollary 6.7, let R be a set of size at most $2\binom{n}{3}$ that represents $\mathcal{D}$, let S be the set of distance points of $\mathcal{D}$, and let $P = R \cup S$. Since $\mathrm{Del}(P)$ is a t-spanner of P, by Lemma 5.3, we have $\mathrm{Del}(P)/\mathcal{D}$ is a t-spanner of $K(P)/\mathcal{D}$, where $K(P)$ is the complete graph with vertex set P. Since P contains the distance points of $\mathcal{D}$, $K(P)/\mathcal{D}$ is isomorphic to the complete graph defined on $\mathcal{D}$. We show that each edge (A, B) of $\mathrm{Del}(P)/\mathcal{D}$ is in $\mathrm{Del}(\mathcal{D})$. Let (A, B) be an edge of $\mathrm{Del}(P)/\mathcal{D}$. This means that in P, there are two points a and b with $a \in A, b \in B$ such that there is an empty circle C through a and b. By Lemma 6.8, C contains a disk G that is tangent to both A and B. The disk G is a witness of the presence of the edge (A, B) in $\mathrm{Del}(\mathcal{D})$. If that was not the case, this would mean that there exists a disk $F \in \mathcal{D}$ such that $G \cap F \neq \emptyset$. By definition of R, this implies that $G \cap R \neq \emptyset$ and thus $C \cap P \neq \emptyset$, which contradicts the fact that C is an empty circle. Therefore, the edge (A, B) is in $\mathrm{Del}(\mathcal{D})$. Since $\mathrm{Del}(P)/\mathcal{D}$ is a t-spanner of $\mathcal{D}$ and a subgraph of $\mathrm{Del}(\mathcal{D})$, we conclude that $\mathrm{Del}(\mathcal{D})$ is a t-spanner of $\mathcal{D}$. □

Note that the embedding of the AW-Delaunay graph that consists of straight line segments between the centers of the disks is not necessarily a plane graph (see Figure 19). However, the Voronoi diagram of a set of disks $\mathcal{D}$, denoted $\mathrm{Vor}(\mathcal{D})$, is planar [26]. Since $\mathrm{Del}(\mathcal{D})$ is the face-dual of $\mathrm{Vor}(\mathcal{D})$, it is also planar. An important characteristic of the Delaunay graph of a set of points regarded as a spanner is that it is a plane graph. Therefore, a natural question is whether $\mathrm{Del}(\mathcal{D})$ has a plane embedding that is also a spanner.

The proof of Theorem 6.10 suggests the existence of an algorithm allowing to compute such an embedding: compute the Delaunay triangulation of the set P that contains the distance points and the representative of $\mathcal{D}$. The graph $\mathrm{Del}(P)$ can be regarded as a multigraph whose vertex set is $\mathcal{D}$. Then, for each pair of disks that share one or more edges, just keep the shortest of those edges. This simple algorithm allows to compute a plane embedding of Del($\mathcal{D}$) that is also a spanner of $\mathcal{D}$. However, its running time is $O(n^3 \log n)$. Whether or not it is possible to compute a plane embedding of Del($\mathcal{D}$) that is also a spanner of $\mathcal{D}$ in a better running time remains a open question.

7 Conclusion

In this paper, we showed how, given a weighted point set where weights are positive and $|p_i p_j| \geq r_i + r_j$ for all $i \neq j$, it is possible to compute a $(1+\epsilon)$-spanner of that point set that has a linear number of edges. We also showed that the Additively Weighted Delaunay graph is a t-spanner of an additively weighted point set in the same case. The constant t is the same as for the Delaunay triangulation of a point set (the best current value is 2.42 [19]). We could not see how the Well-Separated Pair Decomposition (WSPD) can be adapted to solve that problem. The first difficulty resides in the fact that it is not even clear that, given a weighted point set, a WSPD of that point set always exists. Other obvious open questions are whether our results still hold when some weights are negative or $|p_i p_j| < r_i + r_j$ for some $i \neq j$. Also, we did not verify whether our variant of the Yao graph can be computed in time $O(n \log n)$. Finally, another problem that could be explored is whether it is possible to compute t-spanners for multiplicatively weighted point sets.

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