SEPARATION OF GRAPHS OF BOUNDED GENUS

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SEPARATION OF GRAPHS OF BOUNDED GENUS

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Abstract. *In this paper we improve some separator theorems for graphs of bounded genus. Let G be an n vertex connected graph of orientable genus g and T be a spanning tree of G. Any simple cycle of G that contains exactly one edge not in T is called a T-cycle of G. First we show that, if G is a triangulation, there exist g+1 T-cycles of G whose removal divides G into components of no more than (2/3)n vertices each. This improves the best previous 2g+1 bound. The tool used to obtain this result is a sparse weighted graph associated with the genus g imbedding of G. Next we prove that a set C called a separator of no more than* $\sqrt{(6g+6)n}$ *vertices of G that divides G into components of no more than (2/3)n vertices each exists. This is an improvement over the best previous* $\sqrt{(12g+6)n}$ *bound on the size of C. Similar results are derived for graphs imbedded on non-orientable surfaces. A lower bound on the size of the smallest separator for toroidal graphs is obtained.*

1. Introduction

Let G be an n-vertex graph of orientable genus g and a spanning tree T. There exists a set C of no more than $\beta\sqrt{(g+1)n}$ vertices of G (called a *separator*) whose removal divides G into components of no more than $(2/3)n$ vertices each [3,6]. In this paper we address the problem of finding upper and lower bounds on the optimal constant that can replace β in the above *separator theorem*. It is known that for any separator C, $|C|=\Omega(\sqrt{(g+1)n})$ [3,6]. Moreover if $g=0$ (i.e. if G is planar), then $\beta\geq 1{,}555$ [2].

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Let us sketch the known proof technique for obtaining the upper bound on the size of the separator. Suppose that G is connected and let T be a spanning tree of G. A *T-cycle* of G will be called any simple cycle of G that contains exactly one edge not in T. If G is a *triangulation* then there exists a set of no more than $2g$ T-cycles of G whose removal leaves a "pseudo-planar" graph (i.e. such that each of the remaining T-cycles divides the surface the modified graph is imbedded on). Thus if the vertices from these cycles are removed from G then one additional cycle suffices to divide the resulting graph into components of no more than $(2/3)n$ vertices each [7]. Any essential improvement on the number $2g+1$ of these cycles will lead to an improvement on the constant β in the separator theorem. It is not known what is the lowest value of $k(g)$ that can replace the constant 2 before g. In the paper we show that $k(g) \leq 1$, i.e. that there exist $g+1$ T-cycles of G whose removal divides G into components of no more than $(2/3)n$ vertices each. Then we prove a separator theorem that reduces the upper bound on the size of the separator from $\sqrt{(12g+6)n}$ to $\sqrt{(6g+6)n}$.

To obtain the claimed improvement we use the structure *separation graph* of G [4], denoted by $S(G)$. We assume that G is imbedded on a surface of minimum orientable genus. Then $S(G)$ has f vertices and $f+2g-1$ edges with non-negative weights, where f is the number of the faces and g is the genus of G (see [9] for general definitions and facts from topological graph theory). Using the separation graph we reduce the problem of partitioning a graph of small genus to the simpler problem of partitioning a sparse graph. For example, if G is planar (i.e. g=0) then $S(G)$ is a tree and if G is toroidal (i.e. g=1) then $S(G)$ is a tree plus two extra edges. The separation graphs have been exploited in [4] to obtain an $O(n)$ algorithm for finding a so-called *ε-separator*, an improvement over the previous $O(n \log n)$ algorithm. For planar graphs Miller [8] uses a similarly defined (non-weighted) tree in his parallel algorithm for finding a simple-cycle separator.

Another result of the paper is developing a technique for establishing a lower bound on the size of the separator for toroidal graphs. For toroidal graphs (g=1) our separator theorem gives the best known upper bound of $\sqrt{12}\sqrt{n}$ on the size of the separator. We show that the lowest constant that can replace the constant $\sqrt{12} \approx 3.4641$ is not smaller than 1.9046. The results of our paper restrict the interval containing the optimal constant for toroidal graphs from the best previous [1.5550,4.2426] to [1.9046,3.4641].

The paper is organized as follows. In Section 2 we introduce the notion of a separation graph and establish a relation between the problem of dividing a graph and the problem of dividing its separation graph. In Section 3 we prove a separator theorem for graphs of genus g. In Section 4 the non-orientable case is considered and in Section 5 a lower bound on the smallest size of a separator of toroidal graphs is obtained. Sections 2 and 5 are updated versions of the material covered in [1].

2. Separation graphs

A *graph* G is an ordered pair $(V(G),E(G))$ of sets, where $V(G)$ is a set of *vertices* and $E(G)$ is a set of *edges*. Each edge is an unordered pair of different vertices. For the purposes of our paper we will allow existence in the graphs of more than one edge between a pair of vertices. Other basic definitions and facts from graph theory the interested reader can find in any textbook on graphs.

The *orientable genus* (or shortly the *genus*) of a graph G is the minimum genus of an orientable surface, called the *genus surface* of G, onto which G can be imbedded so that no two edges intersect. Any imbedding $I(G)$ of G on a surface Z divides Z into maximal connected regions called *faces* of $I(G)$. If Z is the genus surface of G, then each face of $I(G)$ is homeomorphic to an open disc. We call $I(G)$ (or G if no ambiguity

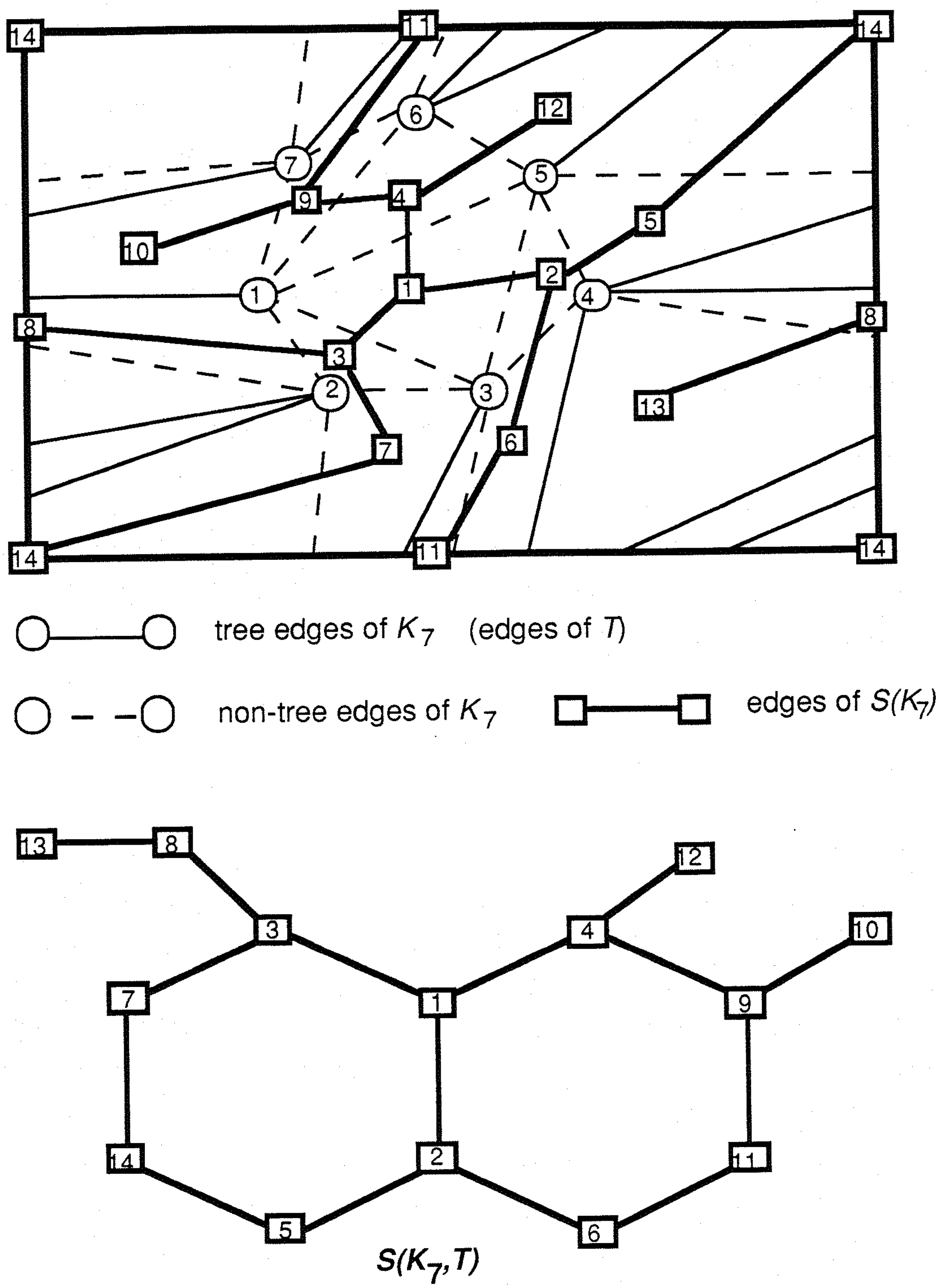

Figure 2.1. Imbedding $I(K_7)$ of K_7 on the torus and the separation graph of $I(K_7)$ regarding a spanning tree T of K_7

arises) a *triangulation* if each face of $I(G)$ is a triangle. G is a *maximal graph of genus* g if the addition of any edge between a pair of non-adjacent vertices increases the genus g of G. The sets of vertices, edges, and faces of $I(G)$ (or G) will be denoted respectively by $V(G)$, $E(G)$, and $F(G)$ hereafter. The most important relation between the numbers of vertices, edges, faces, and genus of $I(G)$ is given by the Euler's formula

$$|V(G)| - |E(G)| + |F(G)| = 2c - 2g ,$$

where c is the number of components of G.

Let G be a triangulation with non-negative vertex weights $wt(v)$ and a spanning tree T. Suppose that G is imbedded on a surface Z of genus g such that each face is a triangle. For any connected subgraph K of G, by $wt(K)$ we denote the sum of the weights of the vertices of K.

Let $v \in V(G)$. Denote by $E(v,0)$ the set of all non-tree edges of G incident to v and by $E(v,1)$ the set of all non-tree edges of G whose both endpoints are adjacent to v. As the set of vertices at distance 1 from v form a simple cycle and there are no simple cycles in any spanning tree of G, then $E(v,1) \neq \emptyset$.

Definition: A *separation graph* of $I(G)$ regarding the spanning tree T is a graph $S=S(G,T)$ with weights on the edges, where

$V(S) = F(G)$,

$E(S) = \{(f_1,f_2) : f_1, f_2 \in F(G), f_1$ and f_2 share an edge$\}$ (Figure 2.1),

and weights are assigned by Algorithm 2.1 below.

Algorithm 2.1 {Assigning weights $wt(e)$ on the edges of S}

{We first define for each non-tree edge of G (or equivalently for each edge of S) a set $vert(e)$ of vertices of G such that the sets $vert(e)$, $e \in E(G) \setminus E(T)$, form a partition of $V(G)$.}

1. {Initialize} **for** each $e \in E(S)$ **do** $vert(e) := \emptyset$.

2. For each $v \in V(G)$ **do**

if $E(v,0) \neq \emptyset$ **then**

e_1:= any edge in $E(v,0)$;

$vert(e_1) := vert(e_1) \cup \{v\}$.

else $\{E(v,1) \neq \emptyset\}$

e_2:= any edge in $E(v,1)$;

$vert(e_2) := vert(e_2) \cup \{v\}$.

3. For each $e \in E(S)$ **do**

$$wt(e) := \sum_{v \in vert(e)} wt(v).$$

For ease of notations we will call S a *separation graph of* G instead of a separation graph of $I(G)$, if no ambiguity arises.

Notice that if the pairs of vertices of S corresponding to edges of T are added to $E(S)$, the *dual* graph of G will be obtained. Since the imbedding of G is a triangulation, the separation graph S is connected and has degree three. Obviously S can be constructed in $O(|V(G)|+|E(G)|)$ time given the imbedding of G. As shown below, if g is small in comparison with $|V(G)|$, then S is much simpler (sparser) than G. Moreover, if weights are assigned to the edges of S in a proper way, we can find separators consisting of T-cycles by considering S instead of G. The sparsity of S easily follows from the Euler's formula.

Lemma 2.1. Let n and m denote the number of vertices and the number of edges of S respectively. Then $m-n = 2g-1$.

Proof: Let m', n' and f' be the number of edges, vertices, and faces of G. By the Euler formula we have $n'-m'+f' = 2-2g$. By the definition of S it holds $n = f'$ and $m = m'-(n'-1)$. Thus $m-n = m'-n'+1-f' = 2g-1$. □

For instance, if g=0, then by Lemma 2.1 S is a tree. In the general case S is a tree with plus $2g$ non-tree edges.

Furthermore, from the description of Algorithm 2.1 it follows directly:

Lemma 2.2. $\sum_{e \in E(S)} wt(e) = \sum_{v \in V(G)} wt(v).$

The next theorem shows that for the purposes of graph separation we can use $S(G)$ instead of G.

Theorem 2.1. Let S be a separation graph corresponding to a triangulation imbedding of a graph G on a surface Z. Let M be a set of edges that divide S into components $S_1,\ldots,S_k$. Then the corresponding set, $C(M)$, of cycles of G divides G into (possibly some empty) components $G_1,\ldots,G_k$ such that

(2.1) $\qquad wt(G_i) \le wt(S_i) \qquad , i=1,..,k.$

Proof: Let the curves corresponding to the set of cycles $C(M)$ divide Z into maximal connected open regions $Z_1,\ldots,Z_{k'}$. Denote by E'_i, $i=1,\ldots,k'$, the set of all non-tree edges of G imbedded inside Z_i and the corresponding sets of edges in S by S'_i, $i = 1,\ldots,k'$, (Figure 2.2). We are going to show that $k'=k$ and $\{S'_1,\ldots,S'_{k'}\} = \{E(S_1),\ldots,E(S_k)\}$.

Obviously S'_i induces a connected graph. Moreover, for $i \neq j$ and any $e' \in S'_i$ and $e'' \in S'_j$ the edges e' and e'' share no common endpoint in S since $Z_i \cap Z_j = \emptyset$. Therefore S'_i induces a component of S. We showed that

$$M \cup S'_1 \cup \ldots \cup S'_{k'} = E(S) = M \cup E(S_1) \cup \ldots \cup E(S_k),$$

$$S'_i \cap S'_j = \emptyset, \quad E(S_i) \cap E(S_j) = \emptyset \ (i \neq j),$$

and that S'_i induces a component of S. Then $k' = k$ and $S'_i = E(S_{perm(i)})$, $i=1,\ldots,k$, where $perm$ is a permutation of $1,\ldots,k$. Let V_i be the set of all endpoints of edges in G that correspond to edges in $S_{perm(i)}$ minus the set of vertices on cycles from $C(M)$. Denote by G_i the subgraph of G induced by V_i. Informally, G_i is the subgraph of G contained in the interior of Z_i. We will prove that (2.1) holds for G_i.

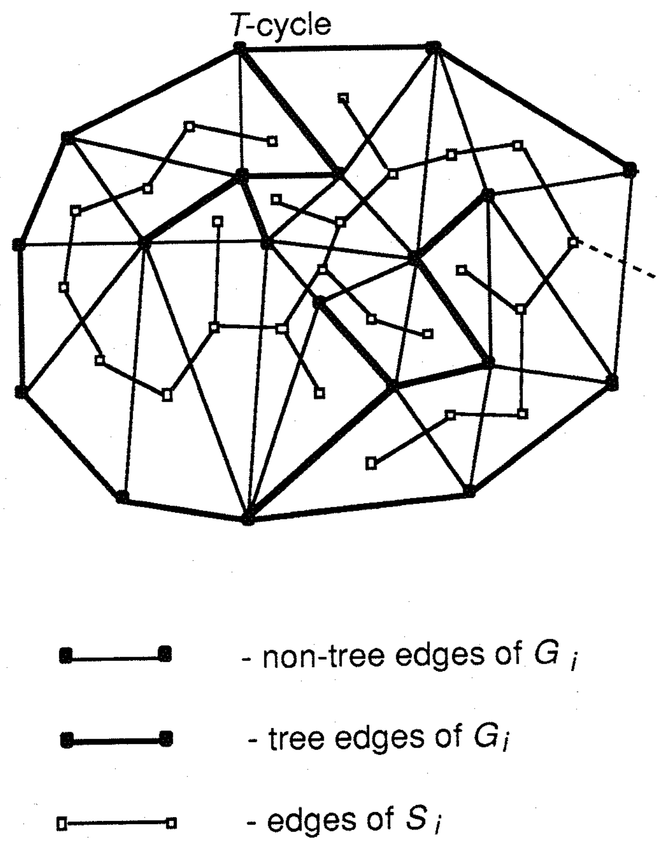

Figure 2.2. A T-cycle of G that determines the region Z_i and the corresponding components G_i and S_i

Let $v \in V(G_i)$ and e be the edge of $E(S)$ such that $v \in vert(e)$. Then the edge in $E(G) \setminus E(T)$ corresponding to e is either incident, or has endpoints adjacent to v. Furthermore, from the definition of S_i, $e \in S_i \cup M$. On the other hand it is not possible that $e \in M$ because this would lead to the contradiction that v belongs to a cycle from $C(M)$. Thus $e \in S_i$ and (2.1) follows from the definition of $wt(S_i)$. □

3. A separator theorem for graphs of genus g

Known proofs of separator theorems for graphs of genus g [3,6,4] consist of the following basic steps. (The graph is assumed to be imbedded on a surface of genus g and the imbedding be a triangulation.)

1. Divide vertices of the graph into the *levels* with regard to their distance from a fixed vertex.

2. Reduce the radius of the graph by deleting and contracting sets of vertices on certain levels.

3. Make the graph pseudo-planar by deleting the sets of vertices of certain $2g$ T-cycles, where T is a spanning tree of the original graph.

4. Construct a separator for the resulting graph by deleting the vertices on one additional T-cycle.

Thus, we need a total of $2g+1$ T-cycles to be deleted in order to divide the graph of reduced radius. Our first goal in this section is to show, using the notion of a separation graph and the lemma below, that $g+1$ T-cycles suffice to divide the graph.

Let G be a graph, $v \in V(G)$, $e \in E(G)$, $v' \notin V(G)$, $e' \notin E(G)$, and the endpoints of e' belong to $V(G)$. The graph G-v is obtained by removing v and all edges incident to v from G. Furthermore G-e, $G+v'$, $G+e'$ are defined by

$$V(G\text{-}e) = V(G), \quad E(G\text{-}e) = E(G)\setminus\{e\},$$

$$V(G+v') = V(G)\cup\{v'\}, \quad E(G+v') = E(G),$$

$$V(G+e') = V(G), \quad E(G+e') = E(G)\cup\{e'\}.$$

Lemma 3.1. Let S be an n-vertex, m-edge connected graph of degree three with weights $wt(\cdot)$ on its edges. Then there exists a set M of $\lceil (m-n+3)/2 \rceil$ edges whose deletion divides S into components each of weight not exceeding $(2/3)wt(S)$.

Proof: If there is an edge such that $wt(e) \geq (1/3)wt(S)$ then we can choose $M = \{e\}$. Assume that all edges of S have weights less than $(1/3)wt(S)$. Let $g = \lceil (m-n+1)/2 \rceil$. We will prove the lemma by induction on g.

If $g=0$, then S has n vertices and n-1 edges and therefore S is a binary tree. It is well known that each binary tree can be partitioned in the above fashion by removing a single edge.

Assume that the proposition of the lemma is true for any graph S_1 of degree 3 such that either $\lceil (|E(S_1)| - |V(S_1)| + 1)/2 \rceil < g$ or $\lceil (|E(S_1)| - |V(S_1)| + 1)/2 \rceil = g$ and $|V(S_1)| < n$.

If there is no vertex in S of degree three then S is a path or a cycle and the lemma holds. Assume that there exists a vertex w of degree three. We consider the following two cases:

<u>Case 1</u>. The graph $S - w$ is connected.

Let $e_i = (x_i, w)$, $i=1,2,3$, be the three edges incident with w. Replace e_i by $e_i' = (x_i, w_i)$, where w_i are new vertices, define $wt(e_i') = wt(e_i)$, $i=1,2,3$, and remove w. Then the resulting graph denoted by S' is connected. See Figure 3.1 for an illustration. Denote $n' = |V(S')|$, $m' = |E(S')|$, $g' = \lceil (m'-n'+1)/2 \rceil$. Then $n' = n+2$, $m' = m$, and

$$g' = \lceil (m'-n'+1)/2 \rceil = \lceil ((m-n+1)/2) - 1 \rceil = g - 1 .$$

By the inductive assumption S' can be divided into subgraphs S'_A and S'_B such that

$$wt(S'_A) \leq (2/3)wt(S), \quad wt(S'_B) \leq (2/3)wt(S),$$

by deleting from S' a set S'_C of no more than g edges. As the degree of w is three, at least one graph among S'_A and S'_B (say S'_A) contains no more than one edge from the set $\{e_1', e_2', e_3'\}$.

If S'_A contains such an edge (say e_1'), then define $M = S'_C \cup \{e_1\}$. Obviously $|M| \leq g+1$ and M divides S into subgraphs $S_A = S'_A - w_1$ and $S_B = S'_B - w_2 - w_3 + w + e_2 + e_3 - (S'_C \cap \{e_2', e_3'\})$. If no such edge exists, then define $M = S'_C$, $S_A = S'_A$ and $S_B = S'_B - w_1 - w_2 - w_3 + w + e_1 + e_2 + e_3 - (S'_C \cap \{e_1', e_2', e_3'\})$. Thus in both cases S_A and S_B are separated by M and $wt(S_A) \le wt(S'_A) \le (2/3)wt(S)$, $wt(S_B) \le wt(S'_B) \le (2/3)wt(S)$.

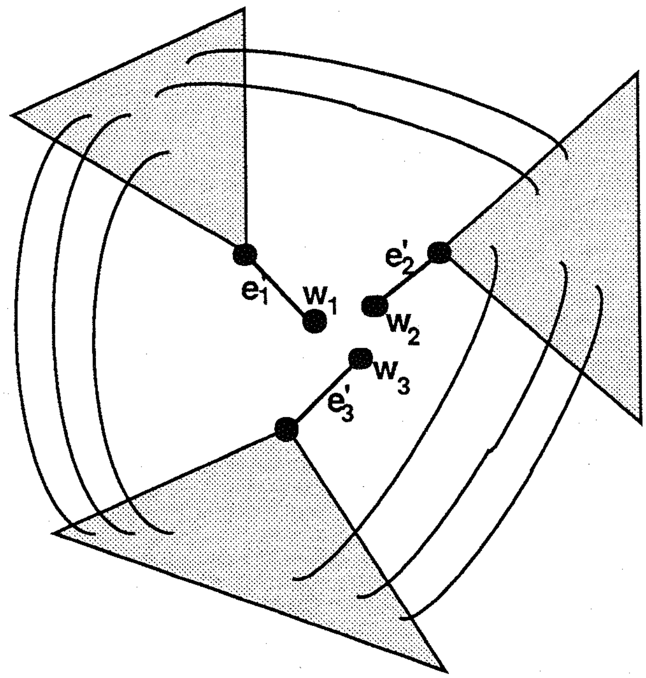

Figure 3.1. Illustration to Case 1 of the proof of Lemma 3.1.

Case 2. The graph $S - w$ is disconnected.

Let K be the component of $S-w$ with the smallest weight. Then $wt(K) \le (1/2)wt(S)$. Let $E(w)$ be the set of edges incident with w and $E_w(K)$ be the set of edges joining w and K. Then $|E(w)|=3$ and $|E_w(K)| \in \{1,2\}$ (since S is connected). If $wt(K)+wt(E_w(K)) \ge 1/3$, then define $M=E_w(K)$. If $wt(K)+wt(E_w(K))<1/3$ and $wt(K)+wt(E(w)) \ge 1/3$, then define $M=E(w) \setminus E_w(K)$.

Assume that $wt(K)+wt(E(w))<1/3$. Replace $K+w+E(w)$ by a single edge e' of weight $wt(e')=wt(K)+wt(E(w))$. Let S'' be the resulting graph (Figure 3.2). As K is connected, then $|V(K)| \le |E(K)|+1$. Thus the difference between the number of vertices and the number of edges of S'' is less or equal than the difference between the number of vertices and the number of edges of S, and S'' has less vertices than S. By the inductive assumption there exists a set S''_C of no more than g+1 edges separating S''.

If $e' \notin S''_C$, then S''_C separates also S in the manner required by the lemma. If $e' \in S''_C$, then we replace e' by a single edge from $E(w)$ adding the other edges from $E(w)$ and K to the part of smaller weight. This partition satisfies the lemma as by assumption $wt(K)+wt(E(w)) < (1/3)wt(S)$. □

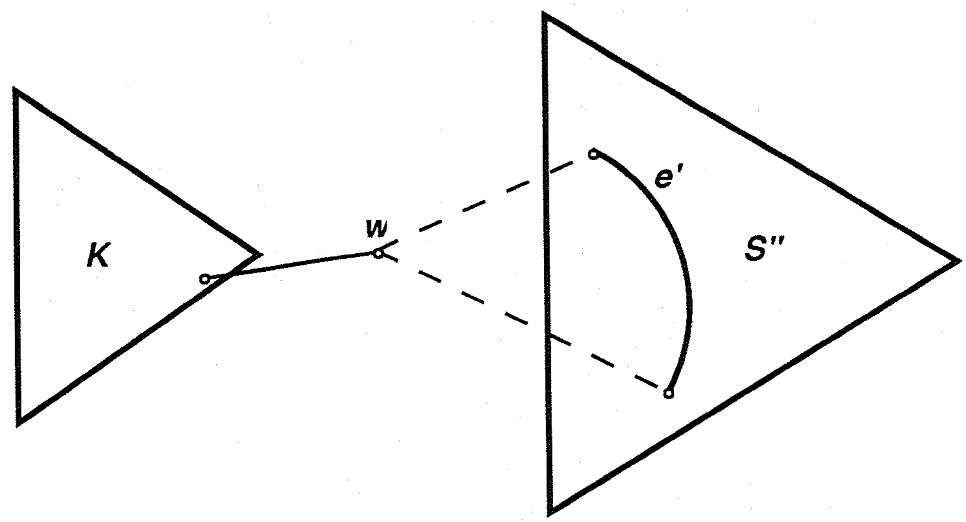

Figure 3.2. Illustration to Case 2 of the proof of Lemma 3.1.

From the results of Section 2 and Lemma 3.1 it follows:

Theorem 3.1. Let G be a maximal graph of orientable genus g and non-negative weights on the vertices and let T be a spanning tree of G. There exists a set of no more than $g+1$ *T-cycles* of G whose removal leaves no component with weight exceeding $(2/3)wt(G)$.

Proof: Construct the separation graph S of G. Find a set of at most $g+1$ edges dividing S as in Lemma 3.1. By Theorem 2.1 removal of the corresponding T-cycles of G leaves no component of G with more than $(2/3)|V(G)|$ vertices. □

The next Lemma 3.2 is a straightforward generalization of a result from [2,3].

Let G be a connected n-vertex graph whose vertices are divided into levels according to their distance to some vertex t. For any pair l',l'' of levels denote by $C(l',l'')$ the set of vertices on levels l' and l''. Remove $C(l',l'')$ from G and distribute the sets of vertices of the resulting components in sets $A(l',l'')$ and $B(l',l'')$ in order to minimize

$$\max\{|A(l',l'')|, |B(l',l'')|\}.$$

Let G be as above and let $L(l)$ denote the number of vertices on level l. For $\alpha \leq 1$, let l_α denote the lowest level such that

$$\sum_{l=0}^{l_\alpha} L(l) \geq \alpha\, n.$$

Lemma 3.2 [2,3] For any $k > 0$ there exist levels l' and l'', $l' \leq l_{1/3} \leq l_{2/3} \leq l''$, such that either

(3.1) $\quad \max\{|A(l',l'')|,|B(l',l'')|\} \leq (2/3)n$, and $|C(l',l'')| \leq \sqrt{6kn}$,

or

(3.2) $\quad L(l')+L(l'') + 2k(l''-l'-1) \leq \sqrt{6kn}$.

The following theorem establishes an upper bound on the size of the minimum separator for a graph of genus g.

Theorem 3.2. Let G be an n-vertex graph of genus g. The vertices of G can be divided into three sets A, B, C such that no edge joins A and B, $\max(|A|,|B|) \leq (2/3)n$ and $|C| \leq \sqrt{6(g+1)n}$.

Proof: Without a loss of generality assume that G is connected. (If the graph is disconnected, then we need to find only a separator for its largest component; see e.g. [7,2].) Choose a vertex t and assign levels to the vertices of G according to their distance to t. Set $k=g+1$. If a pair l',l'' of levels exists satisfying (3.1) then the theorem follows directly. Assume that no such pair exists. Then a pair l', l'' satisfying (3.2) exists. As $l' \leq l_{1/3}$ and $l'' \geq l_{2/3}$, then the graph induced by the set of the vertices on levels below l' and above l'' has no component of more than $(2/3)n$ vertices.

Contract the subgraph induced by the set of vertices on levels not exceeding l' to a single vertex t^* and remove all vertices on levels l'' and above. If the resulting graph K^* has no more than $(2/3)n$ vertices, then the theorem holds. Assume $|K^*|>(2/3)n$. Imbed K^* on a surface of genus g and add new edges (if necessary) to make the imbedding a triangulation. Let T be a breath-first spanning tree of K^* with root t^*. Assign weights $1/|V(K^*)|$ to the vertices of K^*. By Theorem 3.1 there exists a set of no more than $g+1$ *T-cycles* of G whose removal leaves no component with more than $(2/3)|V(K^*)|$ vertices. As each T-cycle has no more than $2(l''-l'-1)+1$ vertices of K^* one of which is t^*, we obtained a set C of $L(l') + L(l'') + 2(g+1)(l''-l'-1)$ vertices of G whose removal leaves no component with more than $(2/3)n$ vertices. By (3.2) $|C|\leq\sqrt{6(g+1)n}$. Finally combine the components of G-C (each of no more than $(2/3)n$ vertices) to form sets A and B satisfying the theorem (see e.g. [7]). □

Notice that we can also prove in a straightforward way a weighted version of Theorem 3.2 (see [7,3,6]) that improves the best previously known result by a factor of $\sqrt{2}$. Furthermore, given an imbedding of G on the its genus surface, one can find in a linear time a partition A, B, C satisfying the theorem by applying the technique from [7].

4. The non-orientable case

The *non-orientable genus* of a graph G is defined as the minimum genus of a non-orientable surface, called a *non-orientable genus surface of G*, on which G can be imbedded. The Euler formula, applied to graphs imbedded on their non-orientable genus surfaces, is

$$n - m + f = 2c - q,$$

where n, m, f, c, and q are respectively the number of vertices, edges, faces, components, and the non-orientable genus of the imbedding.

The definition of a separation graph and the proof of Theorems 3.1 and 3.2 easily generalize to include the non-orientable case. We need to prove an analogue of Lemma 2.1. Let G be a connected graph of non-orientable genus q and S be a separation graph of G.

Lemma 4.1. Let n and m denote the number of vertices and edges of S respectively. Then $m-n = q-1$.

Proof: Let m', n' and f' be the number of edges, vertices, and faces of G. By the Euler formula we have $n'-m'+f' = 2-q$. By the definition of S it holds $n = f'$ and $m = m'-(n'-1)$. Thus $m-n = m'-n'+1-f' = q-1$. □

By Lemma 4.1 the separation graph S of a graph of non-orientable genus q is a tree plus q non-tree edges. For instance, if $q=1$, by Lemma 4.1 S contains a unique simple cycle (Figure 4.1).

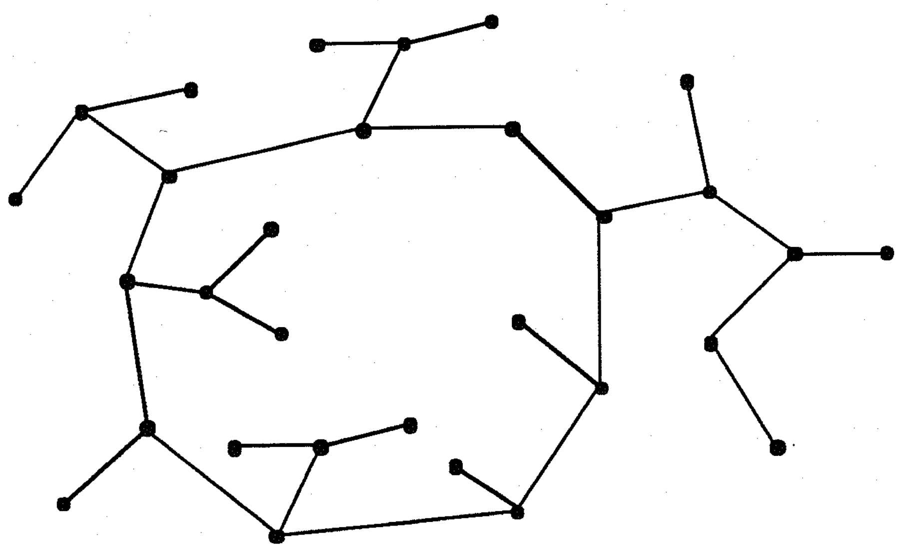

Figure 4.1. A separation graph for $q=1$

The following theorems hold:

Theorem 4.1. Let G be a maximal graph of non-orientable genus q and non-negative weights on the vertices and let T be a spanning tree of G. There exists a set of no more than $\lceil q/2 \rceil+1$ *T-cycles* of G whose removal leaves no component with weight exceeding $(2/3)wt(G)$.

Theorem 4.2. Let G be an n-vertex graph of non-orientable genus q. The vertices of G can be divided into three sets A, B, C such that no edge joins A and B, $\max(|A|,|B|)\leq(2/3)n$ and $|C|\leq\sqrt{(3q+9)n}$.

The proof of Theorems 4.1 and 4.2 is similar to the proof of Theorem 3.1 except that Lemma 4.1 instead of Lemma 2.1 is used.

5. A lower bound for toroidal graphs

If G is toroidal (g=1), then by Theorem 3.2 the separator C contains no more than $\sqrt{12n}$ vertices. We will obtain an estimation on the lowest constant that can replace $\sqrt{12} \approx 3.4641$ using the idea from [2].

Intuitively, the approach of [2] (for the case of planar graphs) is the following. We choose a sphere E_0 as a representative of surfaces of genus zero. An n-vertex graph H_n imbedded on E_0 is defined whose vertices are "uniformly distributed" on E_0. Then we relate the problem of finding a separator of H_n, for n sufficiently large, to the problem of dividing E_0 into two disjoint regions whose areas s_1 and s_2 have ratio $s_1/s_2 = 1/2$. The same approach, however, can not be directly applied to toroidal graphs due to the more complicated structure of the torus compared to that of the sphere. We are going to deal with the problem by implicitly introducing a special (non-Eucledean) metric on the torus.

We will make use of the following geometric fact:

Lemma 5.1. Among all simple closed curves in the plane surrounding a region with unit area the circle has a minimum length.

Next we define a sequence $G_1,G_2,\ldots$ of graphs such that $|V(G_n)|=n^2$ and the genus of G_n is at most one and prove that the minimum separator of G_n has size at least cn for some constant c to be specified below.

Introduce a coordinate system in the plane P whose axes have unit length and angle between them $\pi/3$.

Define an infinite graph G_∞ such that the vertices of G_∞ are all points in P with integer coordinates and for each vertex $v = (x,y)$ the set of its neighbors is

$$\{(x,y+1),(x,y-1),(x+1,y),(x-1,y),(x+1,y-1),(x-1,y+1)\}.$$

Let R_n be the rhombus in P bounded by the axes and the lines $\{(x,y)\colon x = n\}$ and $\{(x,y)\colon y = n\}$ and G_n^* be the (imbedded) subgraph of G_∞ induced by the set of vertices of G_∞ in R_n (Figure 5.1).

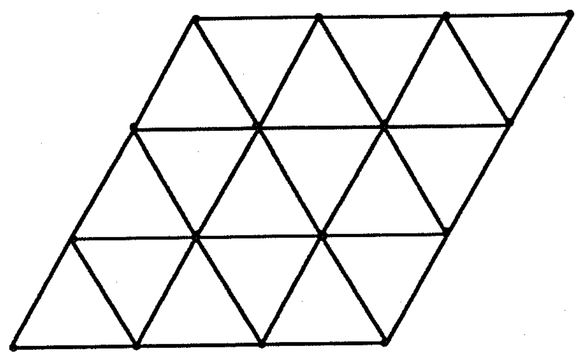

Figure 5.1. The graph G_n^*

Definition. Let u and v be vertices of a graph G. By *merging* u and v we mean the operation of deleting v from G and adding edges (u,w) for all vertices $w \neq u$ that are adjacent to v and not adjacent to u.

Merge vertices (n,i) with $(0,i)$ and (i,n) with $(i,0)$ in G_n^*, for $i=0,\ldots,n$. The resulting graph which we denote by G_n can be imbedded on the torus (with an imbedding consistent with that of G_n^*).

Definition. We say that the curve γ consisting of a finite number of simple closed non-intersecting curves *divides* a region R in the plane into subregions R_1 and R_2, if $R_1 \cup R_2 \cup \gamma = R$ and any continuous curve joining a point in R_1 with a point in R_2 intersects γ.

Lemma 5.2. Let the curve γ divides R_n into two subregions R' and R" with areas $s(R') \leq s(R'')$. Then the length of γ, which we denote by $l(\gamma)$, satisfies $l(\gamma) \geq 2\sqrt{\pi s(R')}$.

Proof: Suppose that R' is a connected region. Then R' will consist of a region D(R') homeomorphic to a disc and a finite number of holes inside D(R'). Let γ^* be the boundary of D(R'). Then by Lemma 5.1 we have

$$l(\gamma) \geq l(\gamma^*) \geq 2\sqrt{\pi s(D(R'))} \geq 2\sqrt{\pi s(R')} \tag{5.1}$$

If R' is not connected, let $R'_1, \ldots, R'_k$ be all connected regions of R'. Applying (5.1) to each R'_j, $j = 1,\ldots,k$ we obtain

$$l(\gamma) \geq \sum_{j=1}^{k} 2\sqrt{\pi s(R'_j)} \geq 2\sqrt{\pi s(R')}.$$

Theorem 5.1. The best constant that can replace $\sqrt{12}$ from Theorem 3.2 is no smaller than $\sqrt{2\pi/\sqrt{3}} \approx 1.9046$.

Proof: Let n be fixed and C be a non-weighted minimal separator for G_n. Denote as usual by A and B ($|A| \leq |B| \leq 2n^2/3$) the corresponding partition of $V(G_n) \backslash C$. Denote by $R(A)$ the subregion of R_n containing all triangles t of the imbedding of G_n such that at least one vertex of t belongs to A and let $R(B)$ contains all other triangles of the imbedding. Denote by C^* the boundary of $R(A)$. It is clear that C^* separates $R(A)$ and $R(B)$ and consists of vertices from C and edges between them. We need the following fact.

Lemma 5.3. The curve C^* consists of a finite number of simple closed non-intersecting curves.

Proof: For the sake of simplicity let C^* denote the imbedded subgraph of G_n corresponding to the curve C^*. The context will make it clear whether the subgraph or the curve is considered. We will prove that all vertices of C^* have degrees two. Consider the following cases:

1. There exists a vertex v of degree 0 in C^*. Deleting v from C and adding it to A contradicts to the minimality of C.

2. There exists a vertex v of degree 1 in C^*. At least one of the vertices adjacent to v belongs to A. As no edge of G joins A and B then all vertices adjacent to v belong to A. Then v can be deleted from C and added to A.

3. There exists a vertex v of degree 3 in C^*. Then at least one edge e of C^* is incident either with two triangles from $R(A)$ or with two triangles from $R(B)$. In any such case e does not belong to the boundary of $R(A)$ which violates the definition of C^*.

4. There exists a vertex v of degree 4 in C^*. Using the fact that the triangles t_1 and t_2 adjacent to a edge from C^* have to belong to $R(A)$ and $R(B)$ respectively, it is easy to see that the only possibility in this case is the one illustrated on Figure 5.2.

In this situation deleting v from C and its adding to A again leads to a contradiction.

5. There is a vertex of degree 5 or 6. This case is impossible since there can not exist two triangles from $R(B)$ adjacent to an edge of C^*.

Thus the only remaining possibility is that all vertices of C^* have degree 2. □

To complete the proof of Theorem 5.1 assume that $|C| < 2n$ (otherwise the theorem follows immediately). Consider the case where C^* is connected.

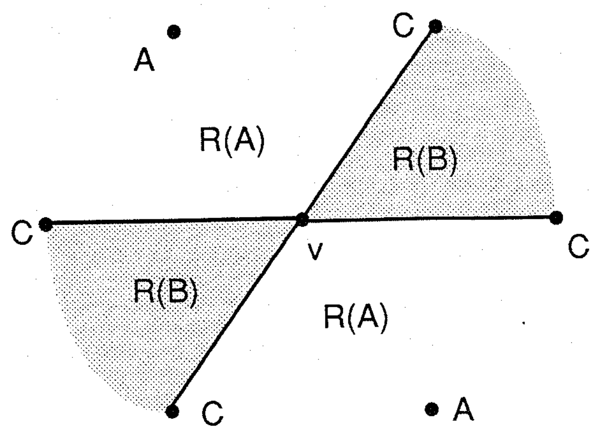

Figure 5.2. Illustration for Case 4.

Since $|V(C^*)| \le |C|$ and the closed curve C^* separates the imbedding of G_n, there exist integers x_0 and y_0 such that for any vertex $(x,y) \in C^*$, $x \ne x_0$ and $y \ne y_0$. Without a loss of generality assume that $x_0 = y_0 = n$. By cutting the imbedding of G_n along lines $x_0 = n$ and $y_0 = n$ we obtain a planar imbedding homeomorphic to the rhombus R_n. Then the curve C^* separates R_n into two regions containing $|A|$ and $n^2 - |A| - |V(C^*)|$ vertices of G_n respectively.

Since each vertex from A is incident to six triangles and each triangle is incident to no more than three vertices of A, then $s(R(A)) \ge \sqrt{3}|A|/2$. Thus we have

$$|C| \ge |V(C^*)| = l(C^*) \ge 2\sqrt{\pi s(R(A))} \ge \sqrt{2\sqrt{3}\pi|A|} \ge$$
$$\ge \sqrt{2\sqrt{3}\pi(n^2/3 - |C|)} \ge \sqrt{2\pi\sqrt{3}(n^2/3 - 2n)} =$$
$$= \sqrt{2\pi/\sqrt{3}}\, n - o(n).$$

The case where C^* consists of more than one cycle can be considered as in the proof of Lemma 5.2. □

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