REPRESENTING PARTIAL ORDERS BY POLYGONS AND CIRCLES IN THE PLANE

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Abstract

The representation of partial orders as containment orders for sets of geometric objects has attracted much attention over the last few years. New results for the cases of polygon (n-gon) and circle orders are presented:

(i) A result which relates the crossing number of a poset to the intersection number of polygon orders is given;

(ii) Two promising schemes for constructing circle orders for three-dimensional posets are shown not to work;

(iii) The Hiraguchi partial order is shown to be a circle order with circles of only two radii;

(iv) The question of which bipartite partial orders are circle orders is examined.

1. Introduction

There are various representations of a partial order which help us to understand it. Perhaps the most familiar are its representations as a graph and as an intersection of linear orders. This paper extends previous studies on the possibility of representing a given partial order as a containment partial order of circles in the plane, and compares such representations to other more familiar ones. To some extent the research is a sequel to [6], though it is largely independent of that paper.

We shall see, among other things, that there are bipartite partial orders which cannot be represented by circles, and that, in a certain sense, there is no "universal" representation of all three-dimensional partial orders by circles; we do not yet know whether every three-dimensional partial order can be represented by circles.

A binary relation $<$ on a set X defines a *partial order* $P(X,<)$ on X if for any $x, y, z \in X$ it satisfies:

(i) $x < y$, $y < z$ implies $x < z$ (transitivity), and
(ii) $x \not< x$ (antisymmetry).

The partially ordered set $P(X,<)$ is a *linear order* if it also satisfies
(iii) $x < y$ or $y < x$ for all $x, y \in X$.

Let $P(X,<)$ be a poset. A *realizer* of P of size k is a collection of linear orders $\mathcal{L} = \{L_1(X,<_1), L_2(X,<_2), \ldots, L_k(X,<_k)\}$ such that $L_1(X,<_1) \cap L_2(X,<_2) \cap \cdots \cap L_k(X,<_k) = P(X,<)$, where the intersection is defined by $x < y \iff x <_i y$ for all i. It can be easily proved that every poset can be obtained as the intersection of a number of linear orders. Dushnik and Miller [1] define the *dimension* of P, denoted by dim P, to be the size of the smallest possible realizer of P. Such a realizer is called a *minimum realizer* of P. Note that if $|X| = \infty$, dim P may not be finite.

Given a realizer $\mathcal{L} = \{\mathrm{L_i(X,<_i)} | 1 \le \mathrm{i} \le \mathrm{k}\}$ for $P(X,<)$ and x,$y \in X$, define $Q(x,y;\mathcal{L}) = \big|\{i \in \{1,\ldots,k\} |$ either $(x <_i y$ and $y <_{i+1} x)$ or $(y <_i x$ and $x <_{i+1} y)\}\big|$, where $L_{k+1}(X,<_{k+1}) \equiv L_1(X,<_1)$. The *crossing number* $\chi(\mathcal{L})$ is defined as the maximum over the set $\{Q(x,y;\mathcal{L}) | x, y \in X\}$. The *crossing number* $\chi(P(X,<))$ of a poset $P(X,<)$ is now defined as $\min\{\chi(\mathcal{L})|\mathcal{L}$ is a realizer of $P\}$.

To understand the reason for the choice of the term crossing number, assume that $X = \{x_1, \ldots, x_m\}$ is finite. Each linear extension $L_i(X, <_i)$ of $P(X, <)$ induces a permutation π_i on $\{1, \ldots, m\}$, $i = 1, \ldots, k$. Consider $k+1$ vertical lines $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_{k+1}$ such that each $\mathbf{L}_i$ is labelled from bottom to top by π_i, $i = 1, \ldots, k$, and $\mathbf{L}_{k+1}$ is labeled by π_1. Given $L_i(X, <_i)$ define the *rank function* $f_i : X \to \{1, \ldots, m\}$ by the relationship $f_i(x) = |\{y \in X : y <_i x \text{ or } y = x\}|$, so $f_i(x_j) = \pi_i^{-1}(j)$. For each $x \in X$, the piecewise linear curve connecting the points $f_i(x)$ on $\mathbf{L}_i$ (see Figure 1.1) depicts the rank of x in each linear order, and the number of crossings of the curve associated with x by the curve associated with y is just $Q(x, y; \mathcal{L})$, since $x <_i y$ is equivalent to $f_i(x) < f_i(y)$. Thus the crossing numbers can be defined in terms of the f_i's, a point of view which is tacitly adopted in section 5.

FIGURE 1.1 ABOUT HERE

The reader should be advised that in some other discussions (e.g. [10]) of crossing numbers, only the lines $\mathbf{L}_1, \ldots, \mathbf{L}_k$ (and the functions $f_1, \ldots, f_k$) are considered. For our purposes, the definition herein helps to simplify the analysis. The reader can easily verify that under our definition, $Q(x, y; \mathcal{L})$ and hence $\chi(\mathcal{L})$ and $\chi(P(X, <))$ are always even numbers.

Let $\Phi = \{P_1, \ldots, P_m\}$ be a family of sets. A partial order $P(X, <)$ on a set $X = \{x_1, \ldots, x_m\}$ is said to represent Φ if $x_i < x_j$ iff P_i is contained in P_j; $P(X, <)$ is called the *containment partial order* of Φ. Conversely, we say that Φ is a *realization* of $P(X, <)$. Every partial order $P(X, <)$ on a set X is the containment partial order of at least one family of sets. This is easily seen by associating with each element $x \in X$ the set $S(x) = \{y \in X | y < x\} \cup \{x\}$. Then $P(X, <)$ is the containment partial order of $\Phi = \{S(x)|x \in X\}$. When the elements of Φ are intervals on the real line, the partial orders thus obtained are exactly all partial orders of dimension 2 (see [2]). In [5] it was proved that partial orders of dimension $2n$ correspond to families of n-boxes in $\Re^{2n}$.

The convex hull of a convex plane curve C will be denoted by $\hat{C}$, and the interior of $\hat{C}$ by C^0, so that $\hat{C} = C^0 \cup C$. If C_x, $x \in X$, is a convex plane curve, then a realization $\Phi = \{\hat{C}_x | x \in X\}$ of $P(X, <)$ is said to be a realization *in convex curves* (technically, Φ is a realization in the convex hull of convex curves). In this case Φ is said to be *normal* if $\cap\{C_x^0 | x \in X\} \neq \emptyset$ and Φ is *regular* if it is normal and $x < y$ if and only if $C_x \subset C_y^0$. If $\Phi = \{\hat{C}_x | x \in X\}$ where each C_x is a circle (n-polygon), $P(X, <)$ is called a circle (n-gon) order.

Let $\Phi = \{\hat{C}_x | x \in X\}$ be a realization of $P(X, <)$ in convex curves. The *intersection number* $I(\Phi)$ is $\max\left|\{C_x \cap C_y | x, y \in X, x \neq y\}\right|$ and may well be infinite, and is 0 by convention if X consists of only one element. Urrutia [10] showed that every poset has a realization in convex polygons in the plane. In Section 2 we shall prove a theorem which relates the concepts of the crossing number of a poset to the intersection number of such realizations. It is easy to see that $\chi(P) = 0$ if and only if P is linear, and this implies that dim $P = 1$; the converse implication holds if X is finite or countable. It has been shown [9] that if X is finite and P has a realization in circles then $\chi(P) \leq 2$; trivially $\chi(P) \leq 2$ if dim $P \leq 3$, so we are led to two conjectures:

Conjecture 1.1. If X is finite and $\chi(P) \leq 2$ then P has a realization in circles.

Conjecture 1.2. If X is finite and dim $P \leq 3$, then P has a realization in circles.

Scheinerman and Wierman [8] proved that the infinite version of Conjecture 1.2 is *not* true, that Z^3 is not a circle order. One approach to attempting to prove that Conjecture 1.2 *is* true is to find a construction which will yield the required circles for any three-dimensional poset. In section 3 we show that a certain "natural" construction is insufficient for the realization of all three dimensional partial orders in circles. In section 4 we show how circles can be used for realization of two well-known posets, while in section 5 it is proved that close relatives of the posets discussed in section 4 *do not* have realizations in circles.

In [9] the following proposition, which will prove useful to us later, is unstated but proved as part of the proof of Theorem 3.2 of that paper:

Proposition 1.3. The intersection number of a circle order is at most 2.

We conclude this section with a brief description of relevant research. Golumbic, Rotem and Urrutia [6] developed the concept of crossing number, so crucial to the theory herein. Fishburn [2] provided a major contribution to the whole area in his book on interval orders. Sidney, Sidney and Urrutia [9] showed that the crossing number of a partial order is at most 2, and used this result to identified a poset of cardinality 14 which is not a circle order. Santoro and Urrutia [7] show applications of the crossing number to the theory of angle orders and n-gon orders. Finally, Fishburn proves that every finite interval order is a circle order [3], and that there are finite circle orders that are not angle orders [4].

2. A Result on Realizations in Convex Curves

Urrutia [10] proved that every finite dimensional poset has a realization in convex polygons in the plane. We extend this result.

Theorem 2.1. If $P(X,<)$ is a poset such that X is at most countable and $\dim(P) = n < \infty$, then P has a regular realization in n-gons if $n \geq 3$, and in triangles if $n \leq 2$. Indeed, $\chi(P)$ is the intersection number of some regular realization of P in n-gons (or triangles, if $n \leq 2$); moreover, if X is finite, then $\chi(P)$ is the smallest intersection number of any regular realization of P in convex curves.

Proof. To prove the theorem, let $\mathcal{L} = \{\mathrm{L_i(X,<_i)} : 1 \leq \mathrm{i} \leq \mathrm{n}\}$ be a realizer of P. First assume $n \geq 3$. For $1 \leq i \leq n$, define $f_i : X \to \Re$ such that $f_i(x) < f_i(y)$ iff $x <_i y$. For example, in the case of finite $|X|$ f_i could be the rank function defined in the previous section.

Now choose a positive number R so large that $(R+1)|cos(2\pi/n)| < R$, and rescale the f_i if necessary so that each has range in the interval $[R, R+1]$. To each $x \in X$ associate the n-gon C_x whose successive vertices are the points $(f_j(x)cos(2\pi j/n), f_j(x)sin(2\pi j/n))$ for $j = 1, \ldots, n$. Then $\Phi = \{\hat{C}_x | x \in X\}$ is a regular realization of P in convex polygons, and if $x \neq y$ then the cardinality of $C_x \cap C_y$ is $Q(x,y;\mathcal{L})$. Hence the intersection number $I(\Phi) = \chi(\mathcal{L})$. Choosing $\mathcal{L}$ so that $\chi(\mathcal{L}) = \chi(\mathrm{P(X,<)})$ gives one half of the theorem for $n \geq 3$.

For the $n = 2$ case, the corresponding result may be obtained by taking a realizer $\mathcal{L}' = \{\mathrm{L_i(X.<_i)} : \mathrm{i} = 1,2\}$ and converting it to a realizer $\mathcal{L} = \{\mathrm{L_i(X,<_i)} : \mathrm{i} = 1,2,3\}$ where $L_3(X,<_3) = L_2(X,<_2)$. The above analysis can then be applied.

Next suppose that X is finite and that $\{C_x : x \in X\}$ is a regular realization of P in convex curves with intersection number $k < \infty$. Choose a point A in $\cap\{C_x^0 : x \in X\}$, choose a ray R_0 starting at A and passing through none of the (finitely many) points in $\cup\{C_x \cap C_y : x \in X,\ y \in X,\ x \neq y\}$, let R_θ $(0 \leq \theta \leq 2\pi)$ be the ray starting at A and making angle θ with R_0, let $\tilde{\theta}_1 < \cdots < \tilde{\theta}_n$ be the values of θ, if any, such that $0 < \theta < 2\pi$ and $R_\theta \cap C_x \cap C_y \neq \emptyset$ for some x and y in X with $x \neq y$, set $\theta_1 = 0$ and, if $n > 1$, choose $\theta_2, \ldots, \theta_n$ so that $\tilde{\theta}_{j-1} < \theta_j < \tilde{\theta}_j$ for $j = 2, \ldots, n$. Let $f_j(x)$ be the distance from A to the point $R_{\theta_j} \cap C_x$, and let $L(X,<_i)$ be defined by the condition $x < y$ if and only if

$f_i(x) < f_i(y)$. It is straightforward to verify that $\mathcal{L} = \{\mathrm{L_i(X, <_i)} : 1 \leq \mathrm{i} \leq \mathrm{n}\}$ is a realizer of $P(X, <)$ and that $\chi(\mathcal{L}) = \mathrm{k}$, completing the proof. For X finite, a little extra effort gives the theorem even if a regular realization must also satisfy the condition: x, y, z distinct implies that $C_x \cap C_y \cap C_z = \emptyset$. Q.E.D.

The above theorem suggests a conjecture previously raised in [10]:

Conjecture 2.2. If X is finite and P has a regular realization in convex curves each pair of which crosses at two points or not at all, then P has a realization in circles.

3. Universal Realizations of Three Dimensional Posets by Circles

Suppose $P(X,<)$ is a three-dimensional poset of cardinality $m < \infty$ and with realizer $\{L_i(X,<_i) | 1 \leq i \leq 3\}$. Let $f_i : X \to \{1,\ldots,m\}$, $1 \leq i \leq 3$, be the previously defined rank functions. The scheme we wish to explore is the following:

> If q is a positive integer or ∞, try to locate sequences $\{\alpha_i | 1 \leq i \leq q\}$, $\{\beta_i | 1 \leq i \leq q\}$ and $\{\gamma_i | 1 \leq i \leq q\}$ in the plane, all these points being distinct, in such a way that if $P(X,<)$ is any finite three-dimensional poset with $|X| = m \leq q$ and $\mathcal{L} = \{L_i(X,<_i\ ) \ |1 \leq i \leq 3\}$ is a realizer of P with rank functions $\{f_1, f_2, f_3\}$, then by associating with each $x \in X$ the circle C_x which passes through the three points $\alpha_{f_1(x)}$, $\beta_{f_2(x)}$ and $\gamma_{f_3(x)}$ we obtain a realization of P in circles.

(Note: we can either assume α_i, β_j and γ_k are never collinear, or we can allow C_x to be a line, thought of as a degenerate circle; which course we take will be immaterial in the sequel, but if we do permit C_x to be a line, we must specify which half-plane it determines will be C_x^0). Unfortunately, this scheme must fail if $q = \infty$ (Theorem 3.1), and a more specialized version must fail for $q \geq 9$ (Theorem 3.4).

Theorem 3.1. If $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_j\}_{j=1}^{\infty}$, $\{\gamma_k\}_{k=1}^{\infty}$ are any three infinite sequences of points in the plane, no two of the points coinciding, then there are indices i, j, k, i', j', k' with $i < i'$, $j < j'$, $k < k'$ such that the circle (or line) through α_i, β_j, γ_k crosses or coincides with the circle (or line) through $\alpha_{i'}$, $\beta_{j'}$, $\gamma_{k'}$.

Proof. By stereographic projection, we may work on the sphere S instead of the plane. Passing to subsequences if necessary, we may suppose that there are (not necessarily distinct) points λ, μ, ν in S such that $\alpha_i \to \lambda$, $\beta_j \to \mu$, $\gamma_k \to \nu$, and that no α_i, β_j or γ_k coincides with λ, μ, ν. Let $C(\alpha,\beta,\gamma)$ denote the circle through the three distinct points α, β, γ in S.

Suppose $i < j$ and $C(\alpha_i,\beta_i,\nu)$ crosses $C(\alpha_j,\beta_j,\nu)$ at ν. For k large enough, γ_k and γ_{k+1} will be so close to ν that $C(\alpha_i,\beta_i,\gamma_k)$ will cross $C(\alpha_j,\beta_j,\gamma_{k+1})$, completing the argument. Thus we may assume that whenever $i < j$, $C(\alpha_i,\beta_i,\nu)$ and $C(\alpha_j,\beta_j,\nu)$ either coincide or are tangent to one another at ν. Similar reasoning leads to a similar assumption for $C(\alpha_i,\mu,\gamma_i)$ and $C(\alpha_j,\mu,\gamma_j)$, and for $C(\lambda,\beta_i,\gamma_i)$ and $C(\lambda,\beta_j,\gamma_j)$.

Now suppose there are two or more distinct circles among the $C(\alpha_i,\beta_i,\nu)$, say $C(\alpha_i,\beta_i,\nu) \neq$

$C(\alpha_j, \beta_j, \nu)$ and $i < j$. These circles are tangent at ν, so $C(\alpha_i, \beta_j, \nu)$ is not tangent to $C(\alpha_{j+1}, \beta_{j+1}, \nu)$ (which **is** tangent to them, or coincides with one of them). Thus $C(\alpha_i, \beta_j, \nu)$ crosses $C(\alpha_{j+1}, \beta_{j+1}, \nu)$ at ν, so if k is large enough then $C(\alpha_i, \beta_j, \gamma_k)$ will cross $C(\alpha_{j+1}, \beta_{j+1}, \gamma_{k+1})$, again completing the argument. Thus we may assume that all the $C(\alpha_i, \beta_i, \nu)$ coincide, so all the α_i and β_i lie on a single circle. Similarly we may assume that all the β_i and γ_i lie on a single circle, hence all the circles $C(\alpha_i, \beta_j, \gamma_k)$ coincide. Q.E.D.

We do not know whether an analogue of Theorem 3.1 holds for finite sequences of points, but a somewhat specialized version of the universal scheme which had seemed quite promising does not work. The idea is to choose three rays R_α, R_β, R_γ from a single point, the angle between each pair of rays being $2\pi/3$, and to distribute the α_i along R_α, the β_i along R_β, and the γ_i along R_γ in identical fashion. Precisely, for $t > 0$ let $\alpha(t) = (t, 0)$, $\beta(t) = (-t/2, \sqrt{3}t/2)$, $\gamma(t) = (-t/2, -\sqrt{3}t/2)$. For given q, is there an increasing sequence of positive numbers $\{t_i\}_{i=1}^q$ such that our scheme works with $\alpha_i = \alpha(t_i)$, $\beta_i = \beta(t_i)$, $\gamma_i = \gamma(t_i)$? To prove that this must fail if $q \geq 9$, we require two lemmas. If t, u, v are positive numbers let C_{tuv} denote the circle through $\alpha(t)$, $\beta(u)$ and $\gamma(v)$.

Lemma 3.2. If $0 < a < b < c \leq d$ and $4b \leq c$, then C_{caa} crosses C_{ddb}.

Proof. The circle C with center $(2b, 0)$ and radius $2b$ is tangent at $(b, -b\sqrt{3})$ to the line L through $\gamma(b)$ and $(b, -b\sqrt{3})$. If $t > 0$ then C_{ttb} lies strictly above L (except for tangency at $\gamma(b)$), so $(b, -b\sqrt{3}) \notin \hat{C}_{ttb}$, hence $C \not\subset \hat{C}_{ttb}$; but $C \subset \hat{C}_{caa}$ (because $4b \leq c$), hence $C_{caa} \not\subset \hat{C}_{ttb}$. If $t > a$ then $\beta(t) \notin \hat{C}_{caa}$ so $C_{ttb} \not\subset \hat{C}_{caa}$, and C_{caa} crosses C_{ttb}. Taking $t = d$ gives the lemma. Q.E.D.

Lemma 3.3. Let $0 < a < b < c$.

(a) If $b \leq a\sqrt{2}$ and $c/b \geq (b/a)^4$ then C_{aaa} crosses C_{cbb}.

(b) If $c \leq b\sqrt{2}$ and $b/a \geq (c/b)^4$ then C_{ccc} crosses C_{abb}.

Proof. We prove (a). A similar argument can be used to prove (b), or we may observe that (a) and (b) are equivalent via the inversion $(x, y) \to (x/(x^2 + y^2), y/(x^2 + y^2))$.

Rescaling, we may suppose $a = 1$. By direct computation, $(\delta, 0) \in C_{b^5bb}$ where $\delta = -(b^5 + 2b)/(2b^4 + 1)$. Since $1 = a < b \leq a\sqrt{2} = \sqrt{2}$ we have $b - 1 > 0$ and $b + b^2 + b^3 - 1 - b^4 > b + b^2 + b^2 - 1 - b^2 \cdot b^2 = (b - 1) + (2 - b^2)b^2 > 0$, so $\delta - (-1) =$

$(b-1)(b+b^2+b^3-1-b^4)/(2b^4+1) > 0$ and $\delta > -1$. Because $c \geq b(b/a)^4 = b^5$, C_{cbb} contains some point $(\tau, 0)$ with $-1 < \delta \leq \tau < -b/2$, hence $(\tau, 0) \in C^0_{111}$. Thus $C_{111} \not\subset \hat{C}_{cbb}$. On the other hand $(c, 0) \notin \hat{C}_{111}$, so $\hat{C}_{cbb} \not\subset C_{111}$ and therefore C_{111} and C_{cbb} must cross. Q.E.D.

Theorem 3.4. Let $0 < t_1 < \cdots < t_9$. Then there are indices i,j,k,i',j',k' with $i < i'$, $j < j'$, $k < k'$ such that $C_{t_i t_j t_k}$ crosses $C_{t_{i'} t_{j'} t_{k'}}$.

Proof. By Lemma 3.2 with $a = t_1$, $b = t_2$, $c = t_8$, $d = t_9$ we may suppose $t_8 < 4t_2$. By Lemma 3.3(a) with $a = t_1$, $b = t_2$, $c = t_8$ we may suppose $t_2/t_1 > (t_8/t_2)^{1/4}$. By Lemma 3.3(b) with $a = t_2$, $b = t_8$, $c = t_9$ we may suppose $t_9/t_8 > (t_8/t_2)^{1/4}$. We have

$$4 > t_8/t_2 = \prod_{i=2}^{7}(t_{i+1}/t_i)$$

so for some $i \in \{2, \ldots, 7\}$

$$1 < t_{i+1}/t_i = \delta \leq (t_8/t_2)^{1/6} < 4^{1/6} < \sqrt{2}.$$

Since

$$(t_8/t_{i+1}) \cdot \delta \cdot (t_i/t_2) = t_8/t_2 \geq \delta^6$$

either

$$t_8/t_{i+1} \geq \delta^{5/2} \tag{3.5}$$

or

$$t_i/t_2 \geq \delta^{5/2}. \tag{3.6}$$

Suppose (3.5) holds. Then

$$t_9/t_{i+1} = (t_9/t_8)(t_8/t_{i+1}) > (t_8/t_2)^{1/4} \cdot \delta^{5/2} \geq (\delta^6)^{1/4} \cdot \delta^{5/2} = \delta^4 = (t_{i+1}/t_i)^4,$$

so by Lemma 3.3(a) with $a = t_i$, $b = t_{i+1}$, $c = t_9$ we see that $C_{t_i t_i t_i}$ crosses $C_{t_9 t_{i+1} t_{i+1}}$. If (3.6) holds, arguing similarly, by Lemma 3.3(b) with $a = t_1$, $b = t_i$, $c = t_{i+1}$ we get that $C_{t_{i+1} t_{i+1} t_{i+1}}$ crosses $C_{t_1 t_i t_i}$. Q.E.D.

4. Realizations in Circles for Two Well-Known Posets

We now define the three bipartite posets G_n, H_n and K_n. For $n \geq 2$ let $A_n = \{1, \ldots, n\}$, let W_n be the set of all non-empty subsets S of A_n (where for each i the subset $\{i\}$ is to be distinguished from the element i of A_n), let U_n consist of the subsets of cardinality $n-1$, and if $n > 2$ let V_n consist of the subsets of cardinality $n-1$ and $n-2$; thus $U_n \subset V_n \subset W_n$. Then $G_n = (A_n \cup W_n, <)$ where the only instances of $<$ are given by $S < i$ if $i \in S$. By restricting $<$ to $A_n \cup U_n$ and (if $n > 2$) to $A_n \cup V_n$ we get $H_n = (A_n \cup U_n, <)$ and $K_n = (A_n \cup V_n, <)$. H_n is the "Hiraguchi partial order" on $2n$ elements; it is a sub-order of G_n and, if $n > 2$, H_n is a sub-order of K_n and K_n is a sub-order of G_n. G_n is (essentially) the most complicated bipartite poset with only n elements in one of the two subsets of its "bipartite partition", and it is well known that $\dim(G_n) = n$.

In [10] it is pointed out that H_n is a circle order for all values of n. Since $\dim(H_n) = n$, it follows that there are partial orders of dimension n which are circle orders, for every value of n. Theorem 4.1 provides a stronger version of this result on H_n.

Theorem 4.1. H_n has a realization in circles, using only two different radii.

Proof. For $j \in A_n$ associate with j the circle C_j with center $(cos(2\pi j/n), sin(2\pi j/n))$ and radius 2, and with $\bar{j} = A_n \setminus \{j\}$ associate the circle $C_{\bar{j}}$ with center $(-cos(2\pi j/n), -sin(2\pi j/n))$ and radius ϵ. For sufficiently small positive ϵ, it is easy to verify that $\{C_x : x \in A_n \cup U_n\}$ is a realization of H_n in circles; and we note that $0 < \epsilon \leq 2 - \sqrt{2(1 + cos(2\pi/n)}$ is the precise requirement. If a regular realization is required, the addition to each radius of a sufficiently large constant will produce the desired result. Q.E.D.

In view of Conjecture 1.2, it is plausible to conjecture that either every bipartite poset with X finite has a realization in circles, or G_4, perhaps the most complicated such 4-dimensional poset, does not. Therefore it may come as a surprise that G_4 has a realization in circles, while for $n \geq 5$, G_n — indeed K_n — does not. We present the first result now, and leave the second one for the next section.

Theorem 4.2. G_4 has a realization in circles.

Proof. In Table 4.1 we list for each element x of $A_4 \cup W_4$ the center and radius of a circle C_x such that $\{C_x : x \in A_4 \cup W_4\}$ is a regular realization of G_4. The verification is left to the reader. Q.E.D.

Element x of $A_4 \cup W_4$	Center of C_x	Radius of C_x
1	(128,104)	192
2	(-80,44)	100
3	(-72-32)	120
4	(-12,40)	52
{1}	(128,104)	8
{2}	(-112,128)	8
{3}	(0,-72)	8
{4}	(-12,40)	50
{1,2}	(-48,120)	8
{1,3}	(32,-24)	8
{1,4}	(32,56)	4
{2,3}	(-80,-16)	8
{2,4}	(-32,56)	26
{3,4}	(-12,24)	35
{1,2,3}	(-56,80)	4
{1,2,4}	(4,84)	4
{1,3,4}	(28,12)	2
{2,3,4}	(-60,28)	2
{1,2,3,4}	(-8,-8)	3

Table 4.1. A Regular Realization of G_4 in Circles

5. Bipartite Posets Without Realizations in Circles

With Proposition 1.3 in mind, we devote this section to proving:

Theorem 5.1. If $n \geq 5$ then $I(K_n) \geq 4$, so $I(G_n) \geq 4$; in particular, K_n and G_n are n - dimensional bipartite posets which cannot be represented by circles, and $I(K_5) = I(G_5) = 4$.

This should, of course, be contrasted with the situation when $n = 4$, which was treated in Theorem 4.2. It suffices to prove Theorem 5.1 for $n = 5$, since K_5 is a sub-order of K_n for $n > 5$.

To prove this result we assume that K_5 has a realizer $\mathcal{L} = \{L_i(X, <_i) | 1 \leq i \leq m < \infty\}$ such that $\chi(\mathcal{L}) \leq 2$. Letting $\bar{f} = \{f_i | 1 \leq i \leq m\}$ be the corresponding set of rank functions, we deduce various properties of $\bar{f}$ which lead to a contradiction.

Continue the labeling of the f_j periodically so that if j_1, j_2 are any integers, then $f_{j_1} = f_{j_2}$ whenever $j_1 \equiv j_2 \pmod m$. Let Z denote the set of positive integers. For each $j \in Z$ let $s(j)$ and $s'(j)$ be the distinct elements of A_5 which satisfy:

$$f_j(s(j)) < f_j(i) \qquad \forall i \in A_5 \setminus \{s(j)\}$$

$$f_j(s'(j)) < f_j(i) \qquad \forall i \in A_5 \setminus \{s(j), s'(j)\}$$

Intuitively, $s(j)$ $(s'(j))$ is the element of A_5 which occurs first (second) in f_j. For any fixed $i \in A_5$ we have (since $i \notin A_5 \setminus \{i\}$) the existence of at least one $j = j_0$ such that

$$f_{j_0}(i) < f_{j_0}(A_5 \setminus \{i\}).$$

If $i' \in A_5 \setminus \{i\}$ then

$$f_{j_0}(A_5 \setminus \{i\}) < f_{j_0}(i').$$

These two facts show that $s(j_0) = i$, so

$$\{s(j) | j = 1, \ldots, m\} = \{s(j) | j \in Z\} = \{1, \ldots, 5\}, \tag{5.2}$$

and $m \geq 5$. The following fact is fundamental.

Lemma 5.3. Suppose that $\chi(\mathcal{L}) \leq 2$ and that $j_1 < j_2 < j_3 < j_4 < j_1 + m$. Then it is impossible that $s(j_1) = s(j_3) \neq s(j_2) = s(j_4)$.

Proof. Otherwise, $Q(s(j_1), s(j_2); \mathcal{L}) \geq 4$. Q.E.D.

By a *block* we mean a non-empty set B of finitely many consecutive integers $j_1, \ldots, j_2$ such that, for some $s(B) \in A_5$,

$$s(j) = s(B), \quad \forall j \in \{j_1, \ldots, j_2\},$$

$$s(j_1 - 1) \neq s(B) \neq s(j_2 + 1).$$

If B is a block, so is $B + m = \{j + m | j \in B\}$, and $s(B + m) = s(B)$. After shifting indices if necessary, we may suppose that $s(1) \neq s(m)$ and that the numbering of the f_j's has been chosen so that 1 is the smallest element of a block B_1; then the blocks into which Z is partitioned can be labeled as B_k for $k \in Z$ in such a way that

$$j \in B_k, \ j' \in B_{k'}, \ k < k' \text{ imply } j < j'.$$

In particular,

$$s(B_k) \neq s(B_{k+1}). \tag{5.4}$$

The set $\{1, \ldots, m\}$ is partitioned into blocks $B_1, \ldots, B_r$; always

$$B_{k+r} = B_k + m, \quad s(B_{k+r}) = s(B_k),$$

and by (5.2)

$$\{s(B_k) | k = 1, \ldots, r\} = \{s(B_k) | k \in Z\} = A_5. \tag{5.5}$$

Choosing $j_i \in B_{k_i}$ in Lemma 5.3 yields

Lemma 5.6. Suppose that $\chi(\mathcal{L}) \leq 2$ and that

$$k_1 < k_2 < k_3 < k_4 < k_1 + r.$$

Then it is false that

$$s(B_{k_1}) = s(B_{k_3}) \neq s(B_{k_2}) = s(B_{k_4}).$$

An integer k (and the block B_k) is a *singleton* if $s(B_{k'}) = s(B_k)$ implies $k' \equiv k \pmod r$.

Lemma 5.7. Suppose that $\chi(\mathcal{L}) \leq 2$, that $k_1 < k_2$, and that $s(B_{k_1}) = s(B_{k_2})$. Then there is a singleton u which satisfies $k_1 < u < k_2$.

Proof. Let

$$E = \{k \in Z | k_1 < k \leq k_2 \text{ and } \exists k',\ k_1 \leq k' < k, \text{ such that } s(\mathrm{B_k}) = s(\mathrm{B_{k'}})\}.$$

E contains k_2, so E has a smallest number t, and $t > k_1 + 1$ by (5.4). Let $u = t - 1$, so that

$$k_1 < u < t \leq \min\{k_1 + r, k_2\},$$

and it remains to check that u is a singleton. Suppose $s(B_h) = s(B_u)$. We may assume $k_1 \leq h < k_1 + r$, and we must prove $h = u$. If $h < u$ then $u \in E$ and $u < t$, which is impossible, so $h \geq u$. If $h > u$, choosing w so that $k_1 \leq w < t$ and $s(B_w) = s(B_t)$, we obtain a contradiction to Lemma 5.6 with the roles of k_1, k_2, k_3, and k_4 played by w, u, t, and h respectively. Hence, $h = u$. Q.E.D.

Lemma 5.8. Suppose that $\chi(\mathcal{L}) \leq 2$ and that u, w are singletons which satisfy $1 < w - u < r$. Then either there is a singleton v which satisfies $u < v < w$, or the integers $s(B_k)$ for $u < k < w$ are all distinct and there is a (necessarily strictly decreasing) sequence of integers $\left(l(k)\right)_{k=u+1}^{w-1}$ such that, for all $k \in \{u+1, \ldots, w-1\}$,

$$w < l(k) < u + r \text{ and } s(B_{l(k)}) = s(B_k).$$

Proof. Assume no singleton v with $u < v < w$ exists. By Lemma 5.7 the numbers $s(B_k)$, $u < k < w$, must be distinct. Since k is not a singleton there exists a number $l(k) \not\equiv k (\text{mod } r)$ such that $s(B_{l(k)}) = s(B_k)$. In particular, we may choose $l(k)$ so that $w < l(k) < u + r$, since the sequence of blocks is fully represented every r consecutive blocks.

Suppose that $u < k < k' < w$. Then $w < l(k) < l(k') < u + r$ would yield a contradiction to Lemma 5.6 with the roles of k_1, k_2, k_3 and k_4 played by k, k', $l(k)$ and $l(k')$ respectively. Thus $l(k') > l(k)$ and it follows that $\left(l(k)\right)_{k=u+1}^{w-1}$ is strictly decreasing. Q.E.D.

For distinct members i_1 and i_2 of A_5 let

$$E(i_1, i_2) = E(i_2, i_1) = A_5 \setminus \{i_1, i_2\},$$

so that

$$f_j(E(i_1,i_2)) < f_j(i),\ \forall j \text{ if } i \in A_5 \setminus \{i_1,i_2\}, \tag{5.9}$$

and there is at least one integer $j(i_1,i_2)$ (which we fix up to congruence modulo m) such that

$$f_{j(i_1,i_2)}(i_1) < f_{j(i_1,i_2)}(E(i_1,i_2)). \tag{5.10}$$

In all cases if possible we take $j(i_1,i_2) = j(i_2,i_1)$, in which case we note that $\{i_1,i_2\} = \{s(j(i_1,i_2)), s'(j(i_1,i_2))\}$. Suppose that $j(i_1,i_2)$ is in block B_v and that $j(i_2,i_1)$ is in block B_w, where v and w are determined up to congruence modulo r. There are two possibilities:

$$s(B_v) = i_1 \neq i_2 = s(B_w), \tag{5.11}$$

or

$$j(i_1,i_2) = j(i_2,i_1) = \hat{j} \text{ and } \{s(\hat{j}), s'(\hat{j})\} = \{i_1,i_2\} \tag{5.12}$$

(as previously stated, we take (5.12) whenever possible).

If (5.11) holds the (unordered) pair $\{i_1,i_2\}$ is *normal* and *exits* at positions $j(i_1,i_2)$ and $j(i_2,i_1)$ and at blocks B_v and B_w; in this case $j(i_1,i_2) \neq j(i_2,i_1)$ and $B_v \neq B_w$. If (5.12) holds $\{i_1,i_2\}$ is *non-normal* and *exits* at position $\hat{j}$ and at block $B_v = B_w$. (The "exit" terminology here is motivated by the pictures that arise in a direct proof that K_n cannot be represented by circles).

Lemma 5.13. Suppose that $\chi(\mathcal{L}) \leq 2$ and that

$$j_1 < j_2 < j_3 < j_4 < j_1 + m.$$

Let $i_1 = s(j_1)$, $i_2 = s(j_2)$, $i_3 = s(j_3)$, and $i_4 = s(j_4)$. Then none of the following can be true:

(a) i_1, i_2, i_3 and i_4 are distinct, $\{i_1,i_3\}$ is normal and exits at j_1 and j_3, and $\{i_2,i_4\}$ is normal and exits at j_2 and j_4.

(b) i_1, i_2 and i_3 are distinct, $i_2 = i_4$, and $\{i_1,i_3\}$ is non-normal and exits at j_1.

(c) i_1, i_2, i_3 and i_4 are distinct, $\{i_1,i_3\}$ is non-normal and exits at j_1, and $\{i_2,i_4\}$ is non-normal and exits at j_2.

Proof. (a) would make $Q(E(i_1, i_3), E(i_2, i_4); \mathcal{L}) \geq 4$ and (b) and (c) would make $Q(i_3, i_4; \mathcal{L}) \geq 4$. Q.E.D.

Lemma 5.13, which carries a large part of the burden in the proof of Theorem 5.1, can be interpreted diagrammatically. Let B_j, $u \leq j < u + r$, be r successive blocks, arranged cyclically (Figure 5.1). Connections are made between distinct blocks B_v and B_w as follows:

(a) with a straight line if $\{s(B_v), s(B_w)\}$ is normal and exits at blocks B_v and B_w;

(b) with a curvy line if $\{s(B_v), s(B_w)\}$ is non-normal and exits at either block B_v or B_w;

(c) with a dotted line if $s(B_v) = s(B_w)$.

Figure 5.1 illustrates some of the possible connections that could occur for $r = 8$.

FIGURE 5.1 ABOUT HERE

The intersection of two straight lines represents an instance of condition (a) of Lemma 5.13 (henceforth to be called "condition (a)"); the intersection of two curvy lines represents an instance of "condition (c)" (proof of condition (c) may require modular renumbering); the intersection of a curvy line with a dotted line represents an instance of "condition (b)" (again, proof may require modular renumbering). These diagramming conventions will be used to simplify the exposition of the proof of Theorem 5.1, which we now present.

Recall that we assume that $\bar{f} = (f_1, \ldots, f_m)$ are the rank functions for $\mathcal{L} = \{L_i(X, <_i) | 1 \leq i \leq m\}$ such that $\chi(\mathcal{L}) \leq 2$, and assume there are r distinct blocks, where r is necessarily ≥ 5. The proof is divided into cases corresponding to the number of singletons in $\{B_j | 1 \leq j \leq r\}$.

Case 1: There are 0 or 1 singletons. Let B_{k_1} and B_{k_2} be successive occurrences of a non-singleton block. By Lemma 5.7, there exist indices u and v with $k_1 < u < k_2 < v < k_1 + r$ such that B_u and B_v are singletons. Contradiction.

Case 2: There are two singletons. In this case, let B_u be a singleton. By lemma 5.8, $r = 8$, B_{u+4} is also a singleton, and $s(B_{u+1}) = s(B_{u+7})$, $s(B_{u+2}) = s(B_{u+6})$, $s(B_{u+3}) = s(B_{u+5})$, as in Figure 5.2. For the remainder of this proof, let $i(j)$ denote $s(B_j)$. To avoid instances of condition (b), the pairs $\{i(u), i(u+2)\}$, $\{i(u), i(u+3)\}$, $\{i(u), i(u+4)\}$, $\{i(u+4), i(u+6)\}$ and $\{i(u+4), i(u+7)\}$ must all be normal. Without loss of generality, $\{i(u), i(u+2)\}$

exits at B_u and B_{u+2}. The reader can easily verify that the normal pairs illustrated in Figure 5.2 are implied.

FIGURE 5.2 ABOUT HERE

It now follows that if $\{i(u+1), i(u+3)\}$ is normal (non-normal) then condition (a) (condition (b)) is violated.

Case 3: There are three singletons. Lemma 5.8 implies there are (subject to modular transformation) three possible cyclical configurations of blocks, as illustrated in Figure 5.3. The notation shown in Figure 5.3 will be used.

FIGURE 5.3 ABOUT HERE

For Figure 5.3(a), the relationships (lines) shown are implied in order as follows:

(i) The dotted lines: $i(u+1) = i(u+6)$ and $i(u+2) = i(u+5)$.

(ii) To avoid condition (b), $\{i(u), i(u+3)\}$ and $\{i(u), i(u+4)\}$ are normal.

(iii) To avoid condition (b), $\{i(u+1), i(u+3)\} = \{i(u+3), i(u+6)\}$ is normal. To avoid condition (a), $\{i(u+1), i(u+3)\}$ exits at B_{u+1} and B_{u+3}. Similarly, $\{i(u+4), i(u+6)\}$ is normal and exits at B_{u+4} and B_{u+6}.

The above implications appear diagramatically in Figure 5.3(a). It follows directly that if $\{i(u), i(u+2)\} = \{i(u), i(u+5)\}$ is normal (non-normal), an instance of condition (a) (condition (b)) will occur.

For Figure 5.3(b), the flow of implications is as follows:

(i) The dotted lines: $i(u+1) = i(u+6)$ and $i(u+2) = i(u+4)$.

(ii) To avoid condition (b), $\{i(u), i(u+3)\}$, $\{i(u), i(u+5)\}$ and $\{i(u+3), i(u+5)\}$ must be normal.

(iii) To avoid an instance of condition (b), $\{i(u+1), i(u+3)\}$ must be normal. To avoid an instance of condition (a), $\{i(u+1), i(u+3)\}$ must exit at B_{u+1} and B_{u+3}.

The above implications appear diagramatically in Figure 5.3(b). It now follows that if $\{i(u), i(u+2)\}$ is normal (non-normal) an instance of condition (a) (condition (b)) will occur.

For Figure 5.3(c), the elimination of B_{u+6} yields a structure isomorphic to that in

Figure 5.3(b), and for which the analysis for this previous case applies.

This completes the discussion of Case 3.

Case 4: There are four singletons. Two cases are considered (Figure 5.4).

Figure 5.4(a) depicts the case in which $r = 6$ and there exist indices u and $u + 2$ such that B_u and B_{u+2} are not singletons. We have these implications:

(i) The dotted line: $i(u) = i(u + 2)$.

(ii) To avoid condition (b), $\{i(u + 1), i(v)\}$ is normal for $v = u + 3,\ u + 4,\ u + 5$.

(iii) To avoid condition (a), $\{i(u + 3), i(u + 5)\}$ is non- normal.

These relationships are seen in Figure 4(a). Clearly, if $\{i(u), i(u + 4)\}$ is normal (non-normal), an instance of condition (a) (condition (c)) will occur.

FIGURE 5.4 ABOUT HERE

Figure 5.4(b) depicts the case in which there exist indices $u < x < u + r$ such that B_u, B_x and B_{u+r} are non-singletons, and there exist singletons z_1, z_2, z_3 and z_4 such that $u < z_1 < z_2 < x < z_3 < z_4 < u + r$. The dotted line represents the condition $i(u) = i(x)$. To avoid condition (b), the pairs $\{i(u + 1), i(x + 1)\}$ and $\{i(x - 1), i(u + r - 1)\}$ must be normal, thus creating an instance of condition (a).

Case 5: There are five singletons. Figure 5.5 depicts this case. Obviously, not all pairs $\{i(x), i(x+2)\}$ can be non-normal, or condition (c) would occur. Without loss of generality, assume $\{i(u), i(u + 2)\}$ is normal. To avoid condition (a), then condition (c), and then condition (a) again, $\{i(u + 1), i(u + 3)\}$, $\{i(u + 2), i(u + 4)\}$ and $\{i(u + 3), i(u)\}$ must be, respectively, non- normal, normal, and non-normal. These relationships are pictured in Figure 5. It follows that if $\{i(u + 1), i(u + 4)\}$ is normal (non- normal), an instance of condition (a) (condition (c)) will occur.

FIGURE 5.5 ABOUT HERE

This completes the proof for all five cases, and hence of Theorem 5.1. Q.E.D.

6. Conclusion

One of the powerful results which deals with the realization of partial orders in circles is the well-known and easily proved theorem that circle orders have crossing numbers no greater than two. Unfortunately, it is not a trivial task to establish the crossing number of a partial order. More generally, it is still a very challenging task to identify the border between partial orders which have realizations in circles and those which do not.

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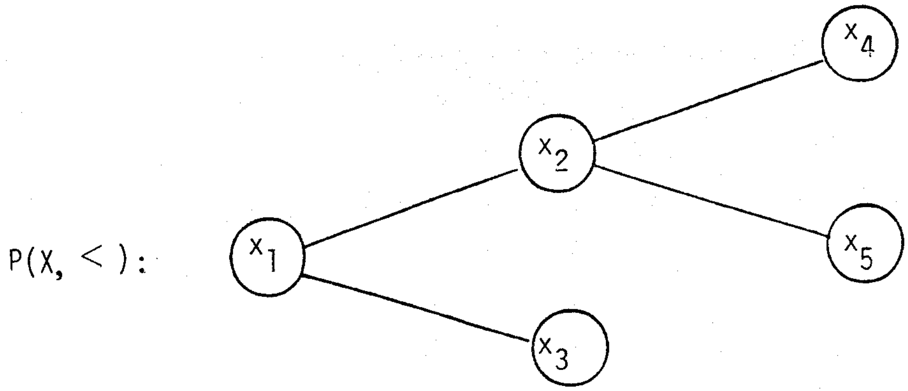
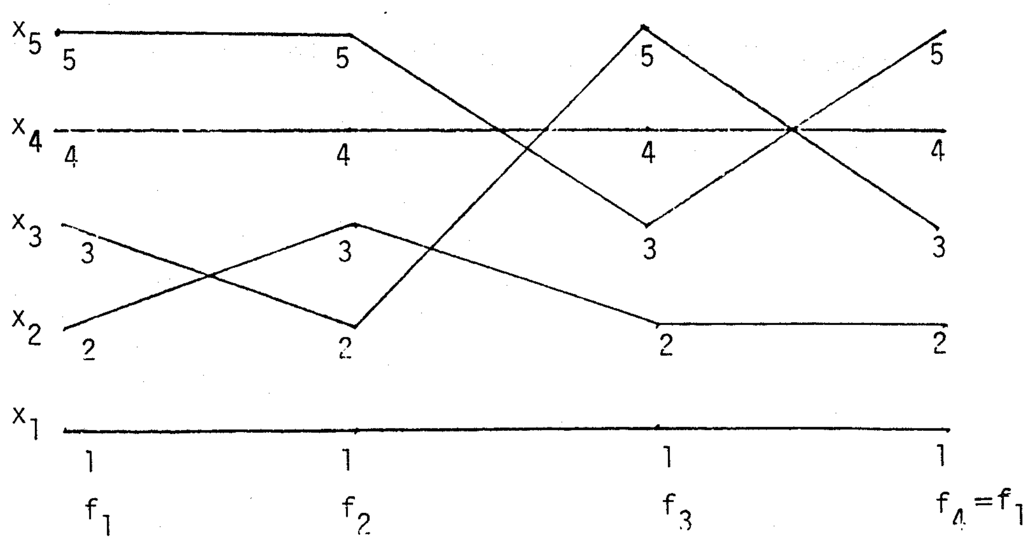


$L_1(X,<_1) = L_4(X,<_4)$ $\quad (x_1, x_2, x_3, x_4, x_5)$

$L_2(X,<_2)$ $\quad (x_1, x_3, x_2, x_4, x_5)$

$L_3(X,<_3)$ $\quad (x_1, x_2, x_5, x_4, x_3)$

x	$f_1(x)$	$f_2(x)$	$f_3(x)$
x_1	1	1	1
x_2	2	3	2
x_3	3	2	5
x_4	4	4	4
x_5	5	5	3



$Q(x_2,x_3;\mathcal{L}) = Q(x_3,x_4;\mathcal{L}) = Q(x_3,x_5;\mathcal{L}) = Q(x_4,x_5;\mathcal{L}) = 2$

$Q(x_i,x_j;\mathcal{L}) = 0$ for all other cases.

Figure 1.1. A Poset and Some of Its Descriptors

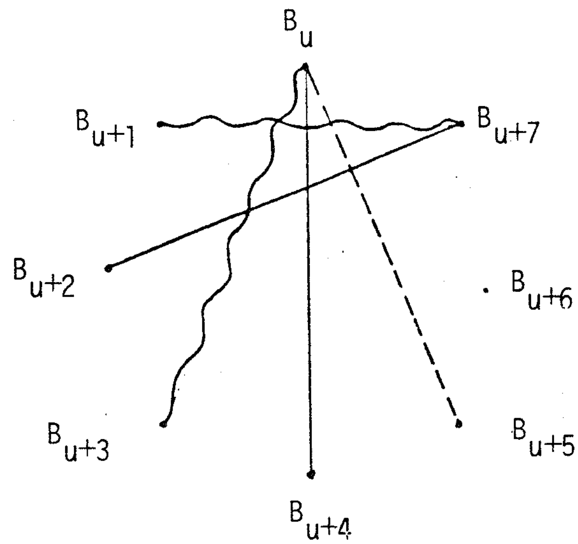


Figure 5.1. Illustration of Diagram Convention for Lemma 5.13

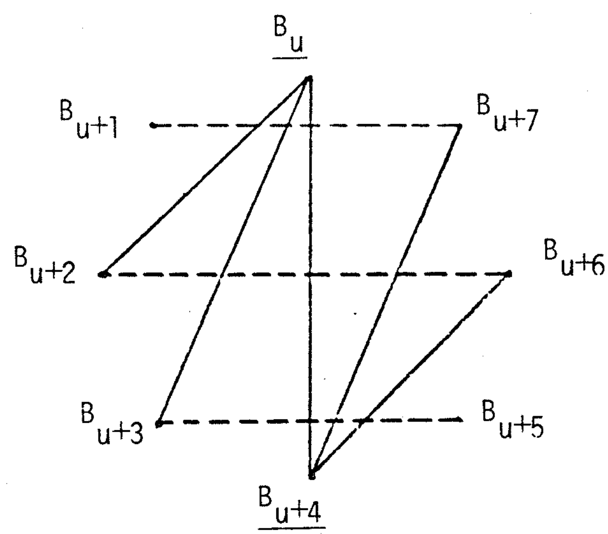
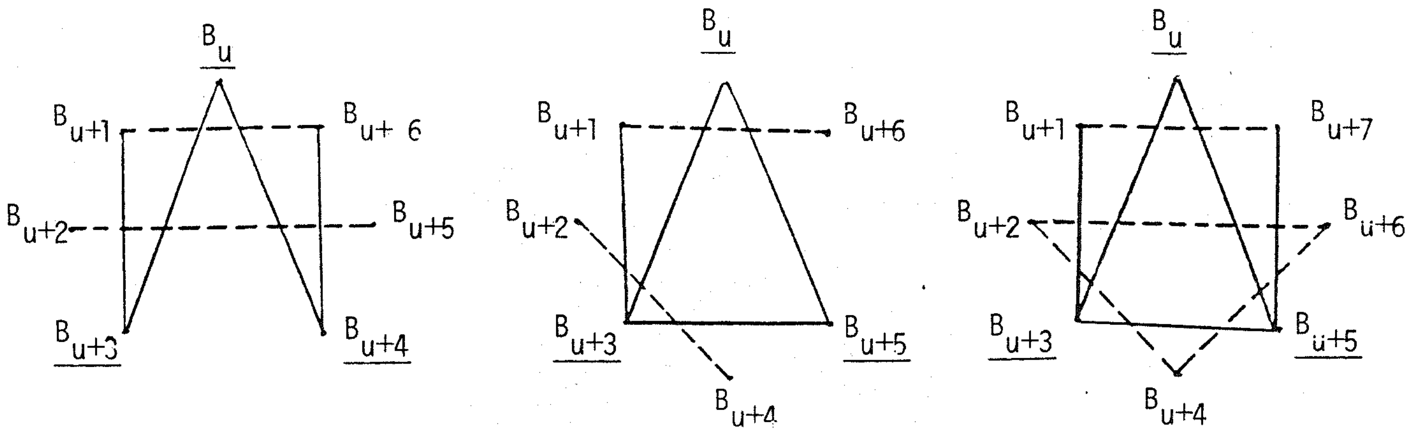


Figure 5.2. Case of Two Singletons (Singletons Underlined)



$i(u+1) = i(u+6)$
$i(u+2) = i(u+5)$

(a)

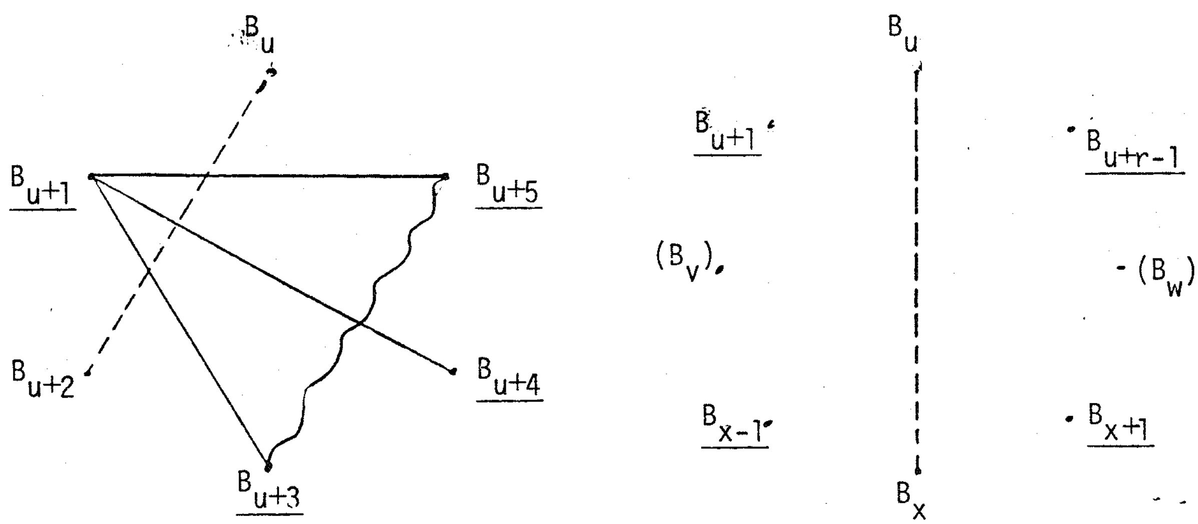
$i(u+1) = i(u+6)$
$i(u+2) = i(u+4)$

(b)

$i(u+1) = i(u+7)$
$i(u+2) = i(u+4) = i(u+6)$

(c)

Figure 5.3. Case of Three Singletons (Singletons Underlined)



$r = 6$
$i(u) = i(u+2)$

(a)

$6 \leqslant r \leqslant 8$
$i(u) = i(x)$
If B_v exists, then $i(v) = i(u)$
If B_w exists, then $i(w) = i(u)$
Values of x and r adjusted to reflect existence of B_v and/or B_w

(b)

Figure 5.4. Case of Four Singletons (Singletons Underlined)

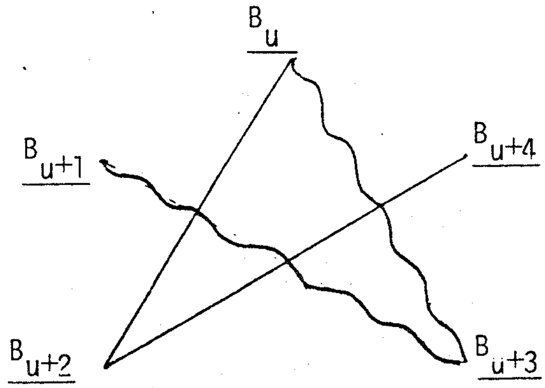


Figure 5.5. Case of Five Singletons

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