REDUCED CONSTANTS FOR SIMPLE CYCLE GRAPH SEPARATION

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Abstract: If G is an n vertex maximal planar graph, then the vertex set of G can be partitioned into three sets A, B, C such that neither A nor B contains more than $(1-\delta)n$ vertices for any $\delta \leq 1/3$, no edge from G connects a vertex in A to a vertex in B, and C is a cycle in G containing no more than $(\sqrt{2\delta}+\sqrt{2-2\delta})\sqrt{n}+O(1)$ vertices. Specifically, when $\delta=1/3$, the separator C is of size $(\sqrt{2/3}+\sqrt{4/3})\sqrt{n}+O(1)$, which is roughly $1.971197\sqrt{n}$. The constant 1.971197 is an improvement over the best known so far result of Miller $2\sqrt{2} \approx 2.828427$. If non-negative weights adding to at most 1 are associated with the vertices of G, then the vertex set of G can be partitioned into three sets A, B, C such that neither A nor B has weight exceeding $1-\delta$, no edge from G connects a vertex in A to a vertex in B, and C is a simple cycle with no more than $2\sqrt{n}+O(1)$ vertices.

1. INTRODUCTION

Let S be a class of graphs closed under the subgraph relation. S is said to satisfy an *f(n)-separator theorem* [8] if there exist constants $\alpha<1$ and $\beta>0$ such that the set of the vertices of any n-vertex graph in S can be partitioned into sets A, B, C with the property that no edge joins a vertex in A to a vertex in B, neither A nor B has more than αn vertices, and the set C, called a *separator,* has no more than $\beta f(n)$ vertices. Separator theorems have been used in algorithms for VLSI imbedding problems, solving sparse systems of linear equations, finding approximate solutions to NP-complete problems, and others. The efficiency of the application algorithms essentially depends on the size of the set C. Thus, once an asymptotically optimal for a class of graphs function f(n) is established, it is important for any given $\alpha<1$ to make the constant β as small as possible.

This paper examines separators for the class of planar graphs. It was proved in [8] that a $\sqrt{n}$-separator theorem holds for the class of planar graphs for any $1>\alpha\geq 1/2$. The constant $\beta=\sqrt{8}$ from [8] for the case of $\alpha=2/3$ was improved to $\sqrt{6}$ [2] and a lower bound of $\sqrt{4\pi\sqrt{3}/3} \approx 1.555$ was found [2]. Separator theorems for graphs of fixed genus were later established in [4,7,6]. Recently, many further improvements and generalizations have been reported [1,3,5,10-13]. It was proved in [9] that, for the case of $\alpha=2/3$, a $\sqrt{n}$-separator theorem for maximal planar graphs holds, where the separator C has the additional property of being a simple cycle of the graph. We show in this paper that, for any $\alpha\geq 2/3$, any maximal planar graph satisfies a $\sqrt{n}$-separator theorem, where the separator is a simple cycle and where the leading constant β tends to $\sqrt{2}$ as α tends to 1. For the well examined case of $\alpha=2/3$ a weighted version of that theorem is proved with a leading constant 2, namely that if non-negative weights adding to at most 1 are associated with the vertices of G, then the set of the vertices of G can be partitioned into three sets A, B, C such that neither A nor B has weight exceeding 2/3, no edge from G connects a vertex in A to a vertex in B, and C is a simple cycle with no more than $2\sqrt{n}+O(1)$ vertices. The best previously known results for $\alpha=2/3$ and the new ones are summarized in Table 1.

[Table 1]

2. Preliminary Results

For any graph G let V(G) and E(G) denote the set of vertices and the set of edges of G respectively. Let non-negative weights adding to 1 are assigned to vertices of G. A *γ-separator* of G, $\gamma\in[1/2,1]$, we call a set C of vertices of G whose removal leaves no connected component of weight greater or equal to γ. If G has a spanning tree T of radius r, then each non-tree edge of G defines a unique simple cycle with the edges of T. Each such cycle, called *fundamental cycle*, contains at most 2r+1 vertices of G, if it contains the root of T, or 2r-1 vertices otherwise.

We start with some preliminary results from [8] and [2], and some technical constructions.

LEMMA 1 [8]: Let G be a maximal planar graph with a spanning tree of radius r, with non-negative weights less than 1/3 on its vertices adding to 1. Then there exists a (2/3)-separator of G that is a fundamental cycle.

For a weighted graph G and a set X of vertices of G denote by w(G) and w(X) the sums of the weights of vertices belonging to G and to X respectively. By |X| we denote the number of vertices of X.

LEMMA 2 [2]: Let G be a maximal planar graph with weights less than (1/3)w(G) on its vertices and a spanning tree of radius r. Then the vertices of G can be partitioned into 4 sets $A_1=A_4,A_2,A_3,D$, such that $w(A_i)\leq w(G)/2$ for i=1,2,3, no edge connects a vertex in A_i to a vertex in A_{i+1}, and D is a union of the fundamental cycles induced by 3 non-tree edges forming a triangle.

By Lemma 2, for any i, the vertices of A_i are adjacent to one of the three fundamental cycles forming D. (Note that in a special case it is possible that D degenerates to a simple cycle and one of the sets A_i be empty).

In the sequel we denote by G a maximal planar graph with a weight $w(v)\in[0,1/3)$ on each vertex v and by T a breadth-first spanning tree of G with root a vertex t and with radius r. We assume that the sum of the weights of the vertices is at most one and (without loss of generality) that G is embedded in the plane so that t appears on the outer face. Divide the vertices of G into *levels* according to their distance to t. Denote by L(l) the set of vertices on level l. We will make often use of the fact that for each $l\in[1,r-1]$ the set of vertices of L(l) that are adjacent to vertices of L(l+1) induces a set $\mathbb{C}(l)=\{C_1(l),\ldots,C_{p_l}(l)\}$ of edge-disjoint non-intersecting simple cycles [9].

LEMMA 3: Let $\mathbb{C}=\{C_1,\ldots,C_k\}$ be the set of cycles of level r-1 of G. Assume that for any $i\in[1,k]$ the weight of all vertices inside C_i is less than 1/3. Then for $q\in\{1/2,2/3\}$ there exists a simple-cycle (2/3)-separator C of G such that

(i) $|C|\le q|L(1)|+(1-q)|L(r-1)|+2(r-3)+3$;

(ii) C contains no vertices on level 0 or r.

Proof: We will prove the claim by applying Lemmas 1 and 2 on a modified version of G. The modification will consist of two phases: in the first levels r-1 and r of G will be modified and in the second level 1 will be modified.

As G is triangulated, the subgraph embedded in each cycle is connected. Contract the subgraph embedded inside C_i, $1\le i\le k$, into a new vertex w_i of the same weight as the subgraph. Thus the weight of w_i is less than 1/3. Let w_i be connected to the vertices $(v_1, v_2, \ldots, v_m)$ of C_i. Since G is triangulated, these vertices form a cycle (to be denoted for simplicity of notation also by C_i) around w_i. Introduce a set of new vertices (of zero weight) $(x_1, x_2, \ldots, x_m)$ inside C_i (see Figure 1). Now add the vertex w_i inside this structure and connect it by a single tree edge to $x_{\lfloor m/2 \rfloor}$. The vertex $x_{\lfloor m/2 \rfloor}$ is connected by a single tree edge to $v_{\lfloor m/2 \rfloor}$. Triangulate the structure as shown in Figure 1. This construction is carried out on all cycles C_i, $1\le i\le k$. Denote by G' the resulting modification of G and by T' the modification of T.

In G' the tree path (with respect to T') to t from any vertex of the structure corresponding to the cycle C_i contains no more than $r+1+\lfloor |C_i|/2 \rfloor$ vertices and does not intersect the inside of any other cycle C_j since the cycles are not nested. Furthermore, the fundamental cycles from Lemma 1 or 2 for G' do not contain w_i for any $i \in [1,k]$. (A fundamental cycle of G' containing w_i has weight less than 1/3 and encloses zero weight.)

Having a simple-cycle separator of G', we can construct a corresponding separator of G by collapsing the corresponding pairs of vertices v_j and x_j from level r-1. Then any separator of G' constructed by Lemma 1 or 2 (with respect to T') corresponds to a simple cycle in G that contains no vertices from level r.

[Figure 1]

This concludes the first phase of our modification. In the second, we assign new weights on the vertices of G' as follows: to all vertices from level 1 we assign weight 1 and to all other vertices we assign level 0. Our goal now is, given any fundamental cycle that contains the root of the tree, to reroute the cycle along a suitably small number of vertices of level one.

Apply Lemma 2 to G' with respect to the new weights. Let A_1, A_2, A_3, D be the resulting partition. Then each A_i contains no more than $|L(1)|/2$ vertices from level 1. Suppose we have the nontrivial case where $t \in D$. If Y_i is the set of vertices of $V(G')\backslash A_i$ adjacent to vertices of A_i, i=1,2,3, then by the above arguments Y_i contains no more than $2r+\lfloor |C_s|/2 \rfloor$ vertices for some $s \in [1,k]$. Thus

(1) $|Y_i| \leq 2r+\lfloor |L(r-1)|/2 \rfloor$, i=1,2,3.

Consider again the original weights of G'. For any set of vertices of G' we denote by w(X) the sum of the weights of all vertices of X. We consider the following three cases:

A. *There exists $j \in \{1,2,3\}$ such that $w(A_j) \in [1/3,2/3]$*

Delete t from Y_j and join the vertices from Y_j from level 1 along the path induced by the set of vertices on level 1 in A_j. Denote the resulting cycle by C' and the cycle in G corresponding to C' by C. Then by (1) C satisfies the lemma for q=1/2.

B. *$w(A_i)<1/3$, $i=1,2,3$*

Let $\bar{A}_i:=(A_i \cup Y_i)\setminus t$, $i=1,2,3$. If there exists $i \in \{1,2,3\}$ such that $w(\bar{A}_i) \geq 1/3$, then in the same way as in Case A we construct a simple-cycle separator satisfying the lemma for $q=1/2$. Suppose now that $w(\bar{A}_i)<1/3$, $i=1,2,3$. Without loss of generality assume that $w(\bar{A}_1) \leq w(\bar{A}_2) \leq w(\bar{A}_3) < 1/3$. Note that $w(\bar{A}_i \cup t) < 2/3$, $i=1,2,3$. Moreover

$$w(\bar{A}_1 \cup t) = 1-(w(\bar{A}_2)+w(\bar{A}_3))+w(\bar{A}_1 \cap \bar{A}_2)+w(\bar{A}_2 \cap \bar{A}_3)+w(\bar{A}_3 \cap \bar{A}_1)$$

$$> 1/3+w(\bar{A}_1 \cap \bar{A}_2)+w(\bar{A}_2 \cap \bar{A}_3)+w(\bar{A}_3 \cap \bar{A}_1) > 1/3 .$$

Therefore $w(\bar{A}_i \cup t) \in (1/3,2/3)$ for $i=1,2,3$. Denote by $\bar{Y}_i$ the set of vertices in $V(G)\setminus(A_i \cup t)$ adjacent to vertices in $A_i \cup t$, $i=1,2,3$. If for some $i \in \{1,2,3\}$ A_i contains no vertices at level 1, then $\bar{Y}_i$ is not a simple cycle. In this case for any $j \in \{1,2,3\}\setminus\{i\}$ $\bar{Y}_j$ satisfies the lemma for $q=1/2$. Assume now that for any i A_i contains at least one vertex at level 1. Then $\bar{Y}_i$ is a simple cycle. From (1) and the definition of $\bar{Y}_i$ it follows

$$|\bar{Y}_1|+|\bar{Y}_2|+|\bar{Y}_3| \leq 6(r-3)+|L(r-1)|+3+2|L(1)|+3 .$$

Then there exists $j \in \{1,2,3\}$ such that the cycle in G corresponding to $\bar{Y}_j$ satisfies the lemma for $q=2/3$.

C. *There exists $j \in \{1,2,3\}$ such that $w(A_j)>2/3$*

Delete all vertices of G' not belonging to $A_j \cup Y_j$ and add new non-tree edges connecting pairs of vertices from Y_j in order to triangulate the embedding. Let G" be the resulting graph and A",B",C" be a vertex partition of G" satisfying Lemma 1. As $w(A_j)>2/3$, then

C" is not induced by any of the additional non-tree edges. Without loss of generality assume that $w(A")\geq w(B")$. Then

$$w(A")\leq(2/3)w(G")\leq 2/3 \quad ,$$

$$w(A"\cup C") \geq (1/2)w(G") \geq (1/2)w(A_j)>1/3.$$

Since $A"\cup C"$ is a subset of $A_j\cup Y_j$, then $A"\cup C"$ contains no more than $|L(1)|/2+2$ vertices from level 1. Thus if Z is the set of the vertices at level 1 in A", then the simple cycle in G corresponding to the separator $C=C"\setminus\{t\}\cup Z$ of G' satisfies the lemma for q=1/2. □

At the end of this section we will prove a lemma that will be then used to appropriately transform graphs of arbitrary radius to graphs of small radius on which Lemma 3 can be efficiently applied. The following simple claim will be used in our proof.

LEMMA 4: For any $q\in(0,1)$ and any positive n_1 and n_2

$$\sqrt{qn_1}+\sqrt{(1-q)n_2}\leq\sqrt{n_1+n_2}.$$

LEMMA 5: If r denotes the highest level of a graph G, define additional levels -1,-2,... and r+1,r+2,... containing no vertices. Let l be any integer. Then there exist levels $l_1^*<l\leq l_2^*$ such that for all $q\in(0,1)$

(2) $$q|L(l_1^*)|+(1-q)|L(l_2^*)|+2(l_2^*-l_1^*-1)\leq 2\sqrt{n}.$$

Proof: Let for any integers $i\leq j$ $s(i,j)$ denote the number of vertices on levels in $[i,j]$. If $i>j$ let $s(i,j)=0$. Fix any $q\in(0,1)$ and define $l_1=l_1(q)$ as the lowest level not exceeding l-1 such that for any $i\in(l_1,l-1]$

(3) $$q|L(i)|+2(l-i-1)>2\sqrt{q}\sqrt{s(0,l-1)}.$$

We will show now that, for each $q\in(0,1)$, $l_1(q)$ is well defined (i.e. $l_1(q)\neq-\infty$) and thus

(4) $$q|L(l_1)|+2(l-l_1-1)\leq 2\sqrt{q}\sqrt{s(0,l-1)}.$$

Replacing in (3) i by $l_1+1,\ldots,l-1$ and summing up the resulting inequalities we get

$$qs(l_1+1,l-1)+a(a-1)>2\sqrt{q}\sqrt{s(0,l-1)}.a,$$

where $a=l-l_1-1$. Consider the function

$$f(x)=x(x-1)-2\sqrt{q}\sqrt{s(0,l-1)}.x+qs(l_1+1,l-1).$$

The maximum value x^* such that f(x) is non-negative for $x\in[0,x^*]$ is

$$x^*\leq\sqrt{q}(\sqrt{s(0,l-1)}-\sqrt{s(0,l-1)-s(l_1+1,l-1)}) \quad .$$

As $a\leq x^*$, then

(5) $\quad l-l_1-1\leq\sqrt{q}\ (\ \sqrt{s(0,l-1)}-\sqrt{s(0,l_1)}\ )$.

Thus we showed that $l_1(q)$ is finite and (4) holds.

Similarly, there exists a level $l_2=l_2(1-q)\geq l$ such that

(6) $\quad (1-q)|L(l_2)|+2(l_2-l)\leq2\sqrt{1-q}\ \sqrt{s(l,r)}$

and

(7) $\quad l_2-l\leq\sqrt{1-q}\ (\sqrt{s(l,r)}-\sqrt{s(l_2,r)}\ )$.

Add (4) and (6):

(8) $$q|L(l_1(q))|+(1-q)|L(l_2(1-q))|+2(l_2(1-q)-l_1(q)-1)$$
$$\leq2(\sqrt{q}\sqrt{s(0,l-1)}+\sqrt{1-q}\sqrt{s(l,r)})\leq2\sqrt{n}.$$

By inequality (8) for any *fixed* $q\in(0,1)$ levels l_1^* and l_2^* exist that satisfy (2). We need to show that levels l_1^* and l_2^* exist that satisfy (2) for *all* q.

Add (5) and (7):

(9) $$l_2-l_1-1\ \leq\ (\sqrt{q}\sqrt{s(0,l-1)}+\sqrt{1-q}\sqrt{s(l,r)})$$
$$-\ (\sqrt{q}\sqrt{s(0,l_1)}+\sqrt{1-q}\sqrt{s(l_2,r)})$$
$$\leq\sqrt{s(0,r)}-\sqrt{s(0,r)-s(l_1+1,l_2-1)}\ .$$

Remove the vertices on levels $l_1+1,\ldots,l_2-1$ from G and denote by G_q the new graph. Define the i-th level of G_q to be the i-th level of G if $i\leq l_1$, or the $(i+l_2-l_1-1)$-th level of G, if $i>l_1$. Let $L_q(l')$ and n_q denote respectively the number of vertices on level l' and the total number of vertices of G_q. Then (9) can be rewritten as

(10) $\quad l_2(1-q)-l_1(q)-1\ \leq\ \sqrt{n}-\sqrt{n_q}$.

As we show next, the lemma follows from (8) and (10). Make the inductive assumption that the lemma is true for all graphs with less than n vertices. Consider the following two cases:

(a) For all $q\in(0,1)$ $l_1(q)=l-1$ and $l_2(1-q)=l$. Then by (8) the lemma holds for $l_1^*=l-1$ and $l_2^*=l$.

(b) There exists $q'\in(0,1)$ such that $l_1(q')<l-1$ or $l_2(1-q')>l$. Then $l_1+1\leq l_2-1$ whence $n_{q'}<n$. By the inductive assumption there exist levels $l_1'<l\leq l_2'$ in $G_{q'}$ such that for any $q\in(0,1)$

(11) $qL_{q'}(l_1')+(1-q)L_{q'}(l_2')+2(l_2'-l_1'-1)\leq 2\sqrt{n_{q'}}$.

Moreover from (10)

(12) $l_2(1-q')-l_1(q')-1 \leq \sqrt{n}-\sqrt{n_{q'}}$.

From (11) and the definition of $L_{q'}$ we have (for any $q\in(0,1)$)

$$q|L(l_1')|+(1-q)|L(l_2'+l_2(1-q')-l_1(q')-1)|+2(l_2'-l_1'-1)\leq 2\sqrt{n_{q'}} .$$

Define $l_1^*=l_1'$ and $l_2^*=l_2'+l_2(1-q')-l_1(q')-1$. Multiplying (12) by 2 and adding it to the last inequality we get (2). □

3. NONWEIGHTED SEPARATION

Now we show how to apply Lemmas 3 and 5 in order to prove a separator theorem for nonweighted planar graphs. Before stating the main result of this section we will introduce some notations.

Consider a graph G with no original weights on its vertices. Assign a weight $1/n$ to each vertex of G. Assume that the root of the breadth-first spanning tree T of G is on the outer face of the embedding. For any $l>0$, the deletion from G of vertices at levels not exceeding l leaves components each of which is surrounded by a cycle, called the *leading curve* of that component, that consists only of vertices at level l. Let $\delta\in(0,1/3]$ and $l(\delta)$ be the farthest level (in T) from the root such that when vertices at levels $\leq l(\delta)$ are deleted from G, there still exists a connected component with weight at least δ. Call one such component $G(l(\delta))$. For all $i>0$, inductively define the component $G(l(\delta)-i)$ as that component which is formed by deleting all vertices at levels not exceeding $l(\delta)-i$ from G, and which contains $G(l(\delta)-i+1)$ as a subgraph. Let $C(l(\delta)-i)$ be the leading curve of $G(l(\delta)-i)$ for all $i\geq 0$. Then $C(l(\delta)-i)$ is nested inside $C(l(\delta)-i-1)$ for all $i\geq 0$. Let j be the largest non-negative value of i for which $C(l(\delta)-i)$ still embeds vertices with total weight not exceeding $1-\delta$ on its inside. If such a j exists, then define $l^-(\delta):=l(\delta)-j$. Then, the number of vertices embedded between $C(l^-(\delta))$ and $C(l(\delta))$ is at most $(1-2\delta)n$. See Figure 2.

[Figure 2]

THEOREM 1: Let G be an n vertex maximal planar graph, and $\delta \leq 1/3$. Then the vertex set of G can be partitioned into three sets A, B, C such that neither A nor B contains more than $(1-\delta)n$ vertices, no edge from G connects a vertex in A to a vertex in B, and C is a simple cycle with no more than $(\sqrt{2\delta}+\sqrt{2-2\delta})\sqrt{n}+O(1)$ vertices.

Proof: Construct a breadth-first spanning tree T of G with root on the outer face of the given planar embedding of G. Assign a weight $1/n$ to each vertex of G. Let the level $l(\delta)$ be as defined before.

First we get an estimation of the difference $l(\delta)-l^-(\delta)$. Suppose that the level $l^-(\delta)$ is undefined. This can only happen if the largest component obtained by deleting all the vertices at levels not exceeding $l(\delta)$ also happens to have weight $>1-\delta$. In this case, we set $l^-(\delta)=l(\delta)$, whence $l(\delta)-l^-(\delta)=0$.

Now suppose that the level $l^-(\delta)$ is well defined. Then $l^-(\delta)\leq l(\delta)$. If there exists a k ($l(\delta)\geq k\geq l^-(\delta)$) such that the leading curve C(k) of the component G(k) as defined before contains $\leq c\sqrt{n}$ vertices for some c to be specified, then pick the cycle C(k) to be the separator C, the vertices inside C to be set A, and the vertices outside C to be set B. Assume then that there exists no such level k. As the total number of vertices at levels l, $l(\delta)\geq l>l^-(\delta)$, inside $C(l^-(\delta))$ does not exceed $(1-2\delta)n$ we have

(13) $$l(\delta)-l^-(\delta) < (1-2\delta)n/(c\sqrt{n}) \leq (1-2\delta)\sqrt{n}/c.$$

Remove levels $l^-(\delta)-1, l^-(\delta), \ldots, l(\delta)$ from G. The resulting graph G' has $n'\leq 2\delta n$ vertices. Apply Lemma 5 to G'. By the lemma, there exist levels l_1^* and l_2^* in G' such that for any $q\in(0,1)$ the corresponding levels l_1 and l_2 in G satisfy

(14) $$q|L(l_1)|+(1-q)|L(l_2)|+2[(l_2-l_1-1)-(l(\delta)-l^-(\delta)+2)] \leq 2\sqrt{n'} \leq 2\sqrt{2\delta n}$$
$$l_1<l^-(\delta)-1<l(\delta)<l_2 .$$

By the definition of $l^-(\delta)$, some connected component, say G*, obtained from G by deleting all vertices at levels l_1 and below, has weight exceeding $1-\delta\geq 2/3$.

Consider the connected components $G_1, G_2, \ldots, G_g$ formed when the vertices at levels l_2 and below are deleted from G^*. Each such component is embedded inside the leading curve of G^* and by the definition of $l(\delta)$ has weight below δ since $l_2 > l(\delta)$. (The collection $\{C_1, C_2, \ldots, C_k\}$ of cycles represented by level l_2-1 in G_i for all i, $1 \leq i \leq g$, corresponds to $\mathbb{C}$ from Lemma 3.).

Finally contract the m-vertex subgraph embedded outside the leading curve of G^* into a new root vertex of weight $m/n<1/3$. Apply Lemma 3 on this contracted graph to obtain a cycle C which partitions G into components each with weight $\leq 1-\delta$ (with $\leq(1-\delta)n$ vertices), and which has size not exceeding

$$
\begin{aligned}
& q|L(l_1)|+(1-q)|L(l_2)|+2(l_2-l_1-1)+3 \\
\leq\ & q|L(l_1)|+(1-q)|L(l_2)|+2[(l_2-l_1-1)-(l(\delta)-l^-(\delta)+2)] \\
& \quad +2(l(\delta)-l^-(\delta))+7 \\
\leq\ & 2\sqrt{2\delta n}+2(1-2\delta)\sqrt{n}/c+7 \text{ (by (13) and (14)).}
\end{aligned}
$$

Then the size of the separator in all cases of the proof does not exceed $\max\{ c\sqrt{n}, 2\sqrt{2\delta n}+2(1-2\delta)\sqrt{n}/c+7 \}$. Now pick c so that $c\sqrt{n} = 2\sqrt{2\delta n}+2(1-2\delta)\sqrt{n}/c$, that is c should be $\sqrt{2\delta}+\sqrt{2-2\delta}$. □

When $\delta=1/3$, we obtain the following statement.

COROLLARY 1: Let G be an n vertex maximal planar graph. Then the vertex set of G can be partitioned into three sets A, B, C such that neither A nor B contains more than (2/3)n vertices, no edge from G connects a vertex in A to a vertex in B, and C is a cycle with no more than $(\sqrt{2/3}+\sqrt{4/3})\sqrt{n}+O(1)$ vertices.

The constant $\sqrt{2/3}+\sqrt{4/3}$ is approximately 1.971197. Theorem 1 clearly yields an asymptotic leading constant of $\sqrt{2}$ as δ tends to zero. For small δ this result is not as good as the $\sqrt{32\delta n}$ bound obtained in [5]. (The separator in [5] is, however, not necessarily a simple cycle.) By combining both results we obtain the next theorem.

THEOREM 2: Let G be an n vertex maximal planar graph, and $\delta \le 1/3$. Then the vertex set of G can be partitioned into three sets A, B, C such that neither A nor B contains more than $(1-\delta)n$ vertices, no edge from G connects a vertex in A to a vertex in B, and C contains no more than $(\sqrt{2\delta}+\sqrt{2-2\delta})\sqrt{n}+O(1)$ vertices, if $0.1 \le \delta \le 1/3$, and no more than $\sqrt{32\delta n}$ vertices, if $0 \le \delta \le 0.1$.

4. WEIGHTED SEPARATION

In this section we will consider the more general case where the graph has arbitrary non-negative vertex weights summing to one.

THEOREM 3: Let G be an n vertex maximal planar graph with non-negative vertex weights adding to one. Then there exists a simple cycle (2/3)-separator of G of no more than $2\sqrt{n}+O(1)$ vertices.

Proof: Let $\delta=1/3$. Construct the cycle C as in the proof of Theorem 1. Let n_1 denote the number of vertices on levels not exceeding $l^-(\delta)-1$, n_3 - the number of vertices on levels greater or equal to $l(\delta)+1$, and n_2 - the number of vertices on levels $l^-(\delta)$ through $l(\delta)$ inside $C(l^-(\delta))$. Then we have

(15) $$l(\delta)-l^-(\delta) \le n_2/(c\sqrt{n})$$

(16) $$q|L(l_1)|+(1-q)|L(l_2)|+2[(l_2-l_1-1)-(l(\delta)-l^-(\delta)+1)] \le 2\sqrt{n_1+n_3}\ .$$

Notice that we have to use now the (weaker) inequality (15) instead of (13) because in a weighted graph the weight of a component is not necessarily proportional to the number of its vertices. By Lemma 3, (15), and (16)

$$\begin{aligned}|C| &\le q|L(l_1)|+(1-q)|L(l_2)|+2(l_2-l_1-1)+2\\ &\le q|L(l_1)|+(1-q)|L(l_2)|+2[(l_2-l_1-1)-(l(\delta)-l^-(\delta)+1)]\\ &\quad +2(l(\delta)-l^-(\delta))+4\\ &\le 2\sqrt{n_1+n_3}+n_2/(c\sqrt{n})+4.\\ &= 2\sqrt{n-n_2}+n_2/(c\sqrt{n})+4.\end{aligned}$$

If $c \geq 1$, then the last expression is a monotonically decreasing function of n_2 for $n_2 \geq 0$ and therefore has its maximum value at $n_2 = 0$. Then the size of C does not exceed $2\sqrt{n}+O(1)$. □

The existence of O(n) time algorithms that finds the vertex partitions from Lemmas 1 and 2 (see [8] and [2]) and the constructiveness of our proofs directly lead to O(n) time algorithms that find the separators from Theorems 1-3 and Corollary 1.

As a final remark, we note that our proofs can be modified to handle arbitrary face sizes as in [9], but in order to keep the discussion direct, we omit the tedious details.

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Constants before $\sqrt{n}$			
nw	*nss*	2.422	Djidjev [3]
w	*nss*	$\sqrt{6} \approx 2.449$	Venkatesan [12]
w	*ss*	$\sqrt{8} \approx 2.828$	Miller [9]
nw	*ss*	1.971	here
w	*ss*	2	here

w=weighted, *nw*=non-weighted, *ss*=simple cycle, *nss*=non-simple cycle

Table 1. Summary of known results on separation of planar graphs for $\alpha=2/3$

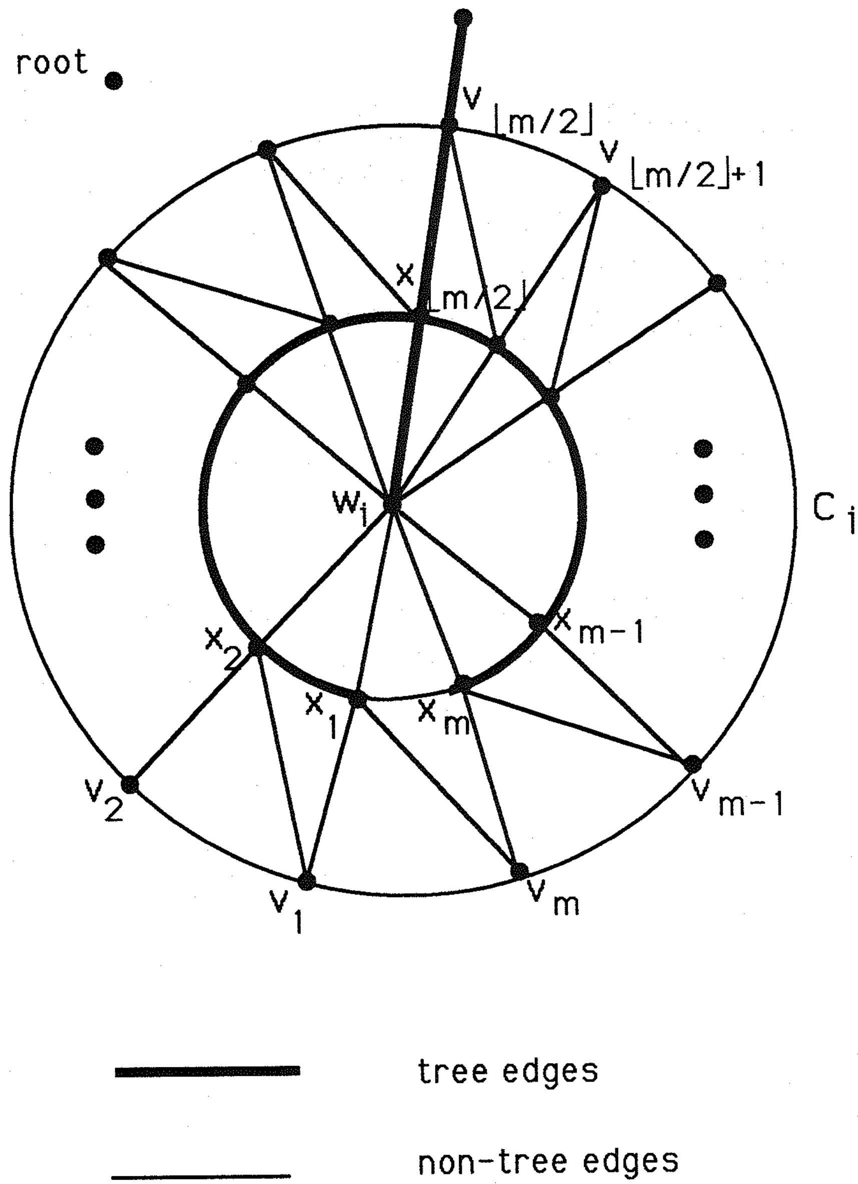
root
v ⌊m/2⌋
v ⌊m/2⌋+1
x ⌊m/2⌋
w i
C i
x m-1
x 2
x 1
x m
v 2
v m-1
v 1
v m
tree edges
non-tree edges

FIGURE 1

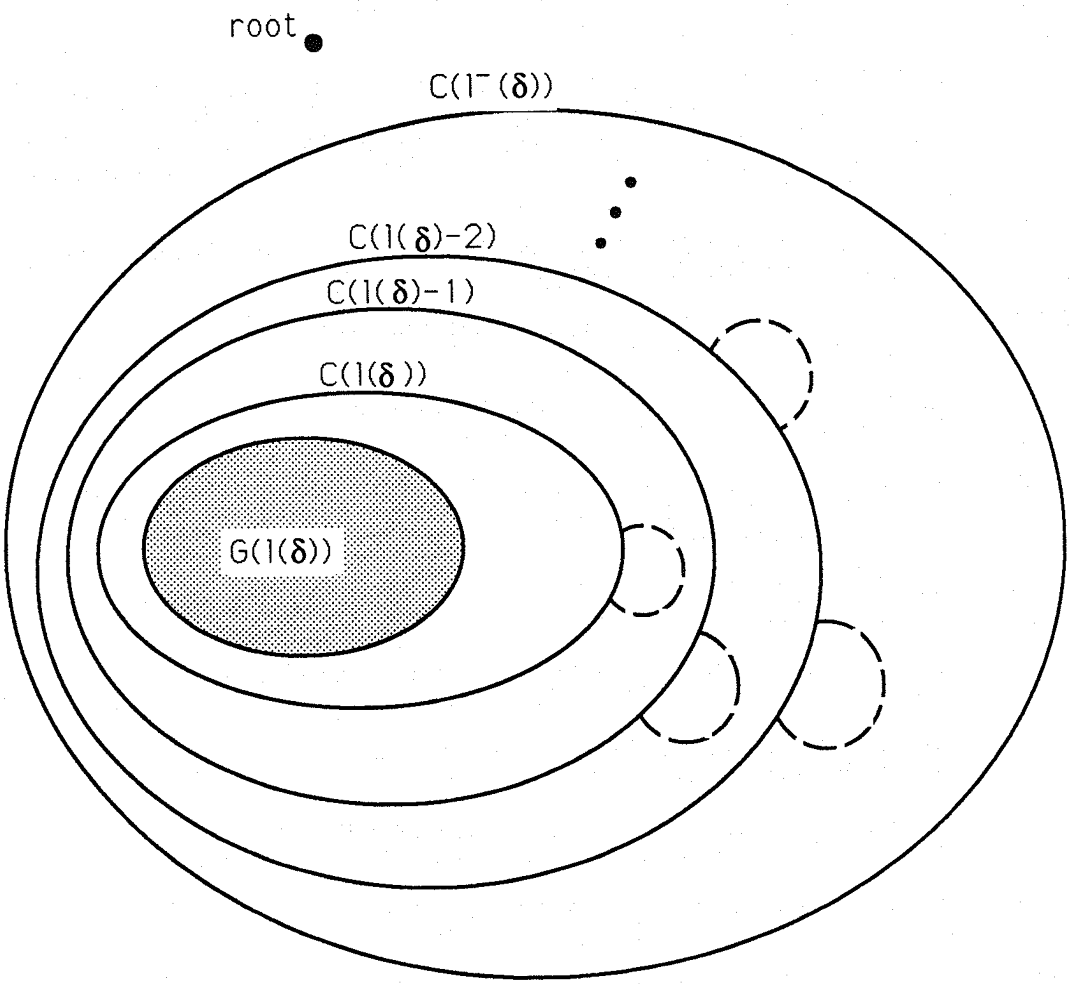

FIGURE 2

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