A BRIEF SURVEY OF ART GALLERY PROBLEMS IN INTEGER LATTICE SYSTEMS

Evangelos Kranakis and Michel Pocchiola

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School of Computer Science, Carleton University
Ottawa, Canada, KIS 5B6

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Evangelos Kranakis (1,2) (eva@cwi.nl)

Michel Pocchiola (3) (pocchiol@dmi.ens.fr)

(1) Centrum voor Wiskunde en Informatica (CWI)
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

(2) Carleton University, School of Computer Science
Ottawa, Canada, K1S 5B6

(3) Laboratoire d'Informatique de l'Ecole Normale Supérieure (LIENS)
URA 1327, CNRS, 45 rue d'Ulm, 75230 Paris Cédex 05, France

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Abstract

The camera placement problem concerns the placement of a fixed number of point-cameras on the integer lattice of d-tuples of integers in order to maximize their visibility. We survey some of the combinatorial optimization and algorithmic techniques which have been developed in order to study this and other similar problems in the context of lattices and more generally, in combinations of lattice systems and tilings.

1 Introduction

Visibility and illumination problems are among the most appealing and intuitive research topics of combinatorial geometry. In many cases (though not all) their analysis requires nothing more than basic topics from geometry, number theory and graph theory and as such they are very well suited for a wide audience [2]. In recent years there has been particular emphasis on the algorithmic component of visibility problems in polygonal configurations; as such they have come to be studied under the area of "art gallery (watchman) problems" which in turn lies at the intersection of combinatorial and computational geometry [16].

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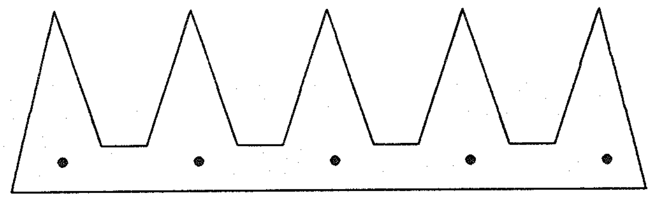

Figure 1: $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary to cover an n-wall art gallery.

Art gallery problems, theorems and algorithms are so named after the celebrated question first posed by V. Klee in 1973: "What is the minimum number of guards sufficient to cover the interior of an n-wall gallery?" The problem was solved soon thereafter first by Chvátal and subsequently also by Fisk (see figure 1.) Since then art gallery problems have successfully emerged as a research area that stresses complexity and algorithmic aspects of visibility and illumination in configurations comprising "obstacles" and "guards". In fact by creating rather idealized situations the theory succeeds in abstracting the algorithmic essence of many visibility problems (like in partitioning theorems, mobile guard configurations, visibility graphs, etc.) thus significantly facilitating the study of their computational complexity.

In the present article we focus on a particular class of art gallery problems, namely those visibility problems which concern configurations of points lying on the vertices of an integer lattice or more generally of a lattice system. By this we assume that we have point obstacles (i.e. lattice points can block the view) and point guards (or cameras) which occupy the vertices of a lattice system. We also assume that the cameras have "full visibility" (i.e. can survey the entire space) and see objects at any distance.

1.1 Some definitions

Before providing an outline of the main themes of investigation we remind the reader of some basic definitions and simple facts. By Λ we denote the d dimensional integer lattice consisting of d-tuples of integers and by Λ_n the complete lattice of d-tuples of integers having absolute value $\leq n/2$. Very important for our subsequent optimization analysis is the notion of density of a set of lattice points. By density of the set $X \subset \Lambda$, denoted by $D(X)$, we understand the limit (if it exists) of the ratio $|X \cap \Lambda_n|/n^d$, as n goes to infinity.

Let $\mathcal{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers, p ranges over $\mathcal{P}$. Two lattice points x and y are p-visible if they are distinct modulo p; two points x, y which are p-visible for all prime $p \in \mathcal{P}$ are visible in the geometric sense, i.e. the line segment joining x and y avoids all the lattice points but x, y (see

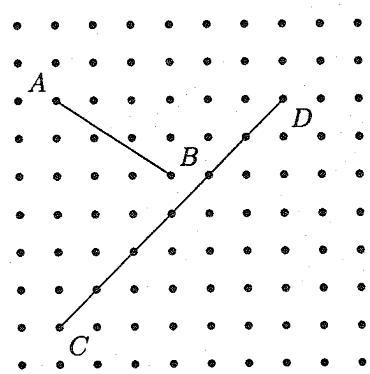

Figure 2: Points A and B are visible; points C and D are p-visible for $p \neq 2, 3$.

figure 2). For $X \subset \Lambda$, X/p denotes the quotient set of X by the negation of the p-visibility relation; we recall that X/p is the set whose elements, called cosets, are the sets $x + p\,\Lambda$ as x ranges over X.

1.2 Related literature

Interesting visibility problems have been studied on integer lattices [5, 10]. Of these we single out two which are relevant for our study.

Rumsey [20] shows that for any finite[1] set S of lattice points, the density of the set of lattice points visible from each point of S is given by the infinite product

$$\prod_{p \in \mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right). \tag{1}$$

The above formula was previously obtained by G. Leujeune Dirichlet for the case $|S| = 1$ ("the probability that d integers chosen at random are relatively prime is $1/\zeta(d)$"[2]) [11, page 324]) and by Rearick [18, 19] for the case where $|S| = 2$ and the points of S are pairwise visible.

An interesting (and in general still open) art gallery problem was posed by Moser [15] in 1966: given a set P of points in the plane how many guards located at points of P are needed to see the unguarded points of P? Abbott [1] studies the case $P = \Lambda_n$ and shows that the minimum number $f(n)$ of guards which are necessary in order to see all the points of Λ_n (see figure 3) verify the inequalities

$$\frac{\ln n}{2 \ln \ln n} < f(n) < 4 \ln n.$$

[1] Rumsey gives a characterisation of the sets S for which the above formula is true.
[2] $\zeta(z)$ denotes the Riemann zeta function, $\sum_{n \geq 1} n^{-z}$, $|z| > 1$.

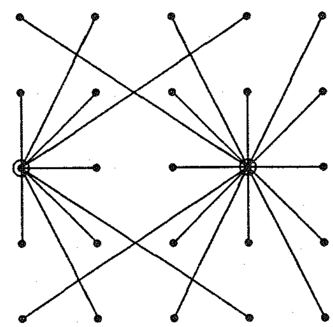

Figure 3: Two guards are enough to cover lattice of d-tuples of integers $\leq n/2$. the points of the finite lattice 5×5.

The lower bound result follows by applying the Chinese remainder theorem and the Prime number theorem. For the upper bound Abbott constructs recursively a sequence $x_1, x_2, \ldots, x_k$ such that for each i, x_{i+1} is a point x in the lattice Λ_n for which the set-theoretic difference $V_n(x) \setminus (V_n(x_1) \cup \cdots \cup V_n(x_i))$, where $V_n(x)$ is the set of points of Λ_n visible from x, is of maximal size and shows that $k = O(\ln n)$ iterations of this procedure suffice in order to cover all the vertices of the lattice. His method however gives no "qualitative" information on the location of these points on the lattice. Nevertheless, he also shows using work of Erdös [4] that there exists a constant $\alpha > 0$ such that for n sufficiently large every point of the lattice Λ_n is visible from the set $\{(1,0)\} \cup \{(0,j) \mid j = 0, 1, \ldots, k\}$, where $k = O(\ln^\alpha n)$. It is straightforward to see that his methods extend easily in order to yield similar results for the d-dimensional lattice Λ_n.

2 Camera Placement Problem

The camera placement problem in multidimensional lattices is the following. We are given s cameras $C_1, \ldots, C_s$ which are supposed to be located on the nodes of the d-dimensional lattice Λ. We are interested in determining a set $S = \{A_1, \ldots, A_s\}$ of positions (lattice points) for these cameras in such a way that if camera C_i is positioned at location A_i, for $i = 1, \ldots, s$, then the density of the lattice points visible by at least one of the cameras is maximized, i.e. under what conditions on the set S of possible camera locations is the quantity

$$u(S) := \sum_{E \subseteq S, E \neq \emptyset} (-1)^{|E|+1} \prod_{p \in \mathcal{P}} \left(1 - \frac{|E/p|}{p^d}\right), \tag{2}$$

which is obtained from (1) using the principle of inclusion/exclusion maximized? Such configurations will be called optimal.

The camera placement problem can be thought of as a "qualitative" version of Abbott's problem already stated in the introduction. Despite the fact that Abbott's (and hence Moser's) question still remains open we expect that our investigations will also contribute to a better understanding of this problem. Indeed since the density $u(S)$ depends only on the relation of p-visibility on the cameras, we expect to deduce some qualitative information on the locations of the cameras that achieves the $O(\log n)$ upper bound of Abbott's theorem. In the present paper we outline some of the optimization results developed for the solution of the camera placement problem. No proofs will be given here, but the reader interested in more detailed accounts is advised to consult [17, 13, 12].

2.1 Equivalence with a non-linear integer optimization problem

The difficulty of the optimization problem previously stated is due not only to the way we specify and manipulate the locations of the cameras, but also on the formulation of $u(S)$ as an alternating sum in identity (2). In the sequel it will be necessary to reformulate the problem as a non-linear integer optimization problem. To accomplish this we introduce, for $\mathcal{Q}$ subset of $\mathcal{P}$, the $\mathcal{Q}$-visibility of the configuration S as the density, denoted by $u(\mathcal{Q}, S)$, of the set of lattice points p-visible for all $p \in \mathcal{Q}$ from at least one point of the configuration S. It can be shown that for any prime p the $\mathcal{Q}$-visibility of the configuration S is the mean of the $\mathcal{Q} \setminus \{p\}$-visibility of the p^d sub-configurations $S \setminus c$ where c ranges over the cosets of Λ/p

$$p^d \, u(\mathcal{Q}, S) = \sum_{c \in \Lambda/p} u(\mathcal{Q} \setminus \{p\}, S \setminus c). \tag{3}$$

As a first consequence of this expresion we get that a necessary optimality condition is that

$$|S/p| = \min\{s, p^d\}. \tag{4}$$

This means that the cameras of an optimal configuration must be located in different cosets of Λ/p, for $p^d \geq s$, in such a way that condition (4) is satisfied. The difficulty of the problem is now to determine the "optimal" repartition of the cameras in the cosets of $\Lambda/2, \Lambda/3, \ldots, \Lambda/p_r$ where $p_1 = 2, p_2 = 3, \ldots, p_r$ is the (finite) increasing sequence of prime numbers less than $s^{1/d}$. To each configuration S (satisfying (4)) we associate the family of integers (a_c) indexed by the elements $c = (c_1, \ldots, c_r) \in \mathcal{C} := \Lambda/2 \times \cdots \times \Lambda/p_r$ defined by

$$a_c = |S \cap c_1 \cap c_2 \cap \ldots \cap c_r|.$$

It turns out that this family of numbers determines the density $u(S)$. Conversely, it can be shown that given a family of numbers $(a_c)_{c \in \mathcal{C}}$ there exists a configuration S of $s = \sum_c a_c$ points, which satisfies (4), and to which the family (a_c) is associated by the above described procedure.

Equiped with this new way of specifying a configuration of cameras we give now a new expression for the function $u(S)$ to be maximized. We introduce the *reduced density function*, defined on the subsets E of S by

$$u'(E) := u(\mathcal{P} \setminus \{p_1, \ldots, p_r\}, E),$$

and the family of *reduced configurations* $\mathcal{B}_c \subseteq S$ defined by

$$\mathcal{B}_c = S \setminus \bigcup_{i=1}^{i=r} c_i.$$

Then by a repeated application of (3) we get that the visibility of the configuration S is the mean of the $\mathcal{P} \setminus \{p_1, \ldots, p_r\}$-visibility of the $m^d := p_1^d \ldots p_r^d$ reduced configurations, i.e.

$$m^d\, u(S) = \sum_{c \in \mathcal{C}} u'(\mathcal{B}_c). \tag{5}$$

It turns out, under the assumption that the configuration S satisfies (4), that the reduced density function $u'(E)$ depends only on the size $|E|$ of the set E and it can be verified that $u'(e) := u'(E)$, where $e = |E|$, is absolutely monotone[3]; furthermore the cardinal b_c of the reduced configuration $\mathcal{B}_c$ can be expressed as a function of the family of integers a_c by the relation

$$b_c = \sum_{h(c,c')=r} a_{c'}$$

where $h(c, c')$, the Hamming distance, is defined as the number of i such that $c_i \neq c'_i$. We have reformulated our optimization problem in terms of the following non-linear integer optimization problem

$$\max \left\{ \sum_{I \in \mathcal{I}} u'(b_I) \mid b_I = \sum_{h(I,J)=r} a_J,\ \sum_I a_I = s,\ a_I \in \mathbb{N} \right\} \tag{6}$$

where $\mathcal{I} = [1..p_1^d] \times \cdots \times [1..p_r^d]$ is a set of multi-indices, h is the Hamming distance and the function u' is an absolutely monotone function.

The concavity of u' ($\Delta^2 u' < 0$) and the fact that the terms b_I sum to a constant ($= s \prod_{i=1}^{i=r}(p_i^d - 1)$) suggest that for an optimal configuration the numbers b_I must "differ from each other by a minimum amount". A classical measure of this deviation is the sum $\sum_I b_I^2$ of the squares of numbers b_I, which we will call the *variance* of the configuration S.

The previous considerations enable us to conjecture that an optimal configuration must be of minimal variance. All our subsequent considerations are guided by this conjecture which we confirm for the case of "almost all" configurations of $s \leq 5^d$ cameras. Moreover we have the following characterization.

[3]Using the standard notation of the calculus of finite difference: $\Delta^1 f(x) = f(x+1) - f(x), \Delta^{n+1} f = \Delta^1(\Delta^n f)$ this means that $(-1)^{n+1}\Delta^n u'(e) > 0$ for all integers $n \geq 1$.

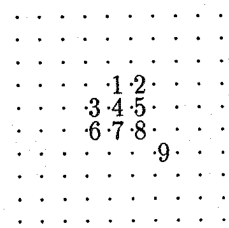

Figure 4: An optimal configuration of $s \leq 9$ guards in the plane.

Theorem 2.1 *The variance of a configuration S is minimal if and only if for every square free integer n and every c and $c' \in \Lambda/n$ the cardinals of $c \cap S$ and $c' \cap S$ differ by at most one. In other words the cameras have to be clustered in cosets of approximatively equal size.* ∎

2.2 Optimization for $s \leq 5^d$ cameras

The previous transformations make it possible to give elegant characterizations of optimal configurations of $s \leq 3^d$ cameras.

Theorem 2.2 *A configuration S of size $\leq 3^d$ is optimal if and only if its variance is minimal.* ∎

Thus, a configuration S of size $\leq 2^d$ is optimal if and only if its cameras are pairwise p-visible for all primes p, while a configuration S of size $\leq 3^d$ is optimal if and only if its cameras are pairwise p-visible for all primes $p \geq 3$, and for all $c, c' \in \Lambda/2$ $||S \cap c| - |S \cap c'|| \leq 1$.

For $3^d < s \leq 5^d$ the problem is much more difficult. Let $L_1, \ldots, L_{2^d}$ be the 2^d cosets of $\Lambda/2$, $C_1, \ldots, C_{3^d}$ the 3^d cosets of $\Lambda/3$ and let us use the abbreviations $l_i = |L_i \cap S|$, $c_j = |C_j \cap S|$, $a_{i,j} = |L_i \cap C_j \cap S|$. Now recall that our optimization problem has been transformed to the following one

$$\max \left\{ \sum_{i,j} u'(b_{i,j}) \mid b_{i,j} = \sum_{k \neq i, l \neq j} a_{k,l}, \sum_{i,j} a_{i,j} = s, \, a_{i,j} \in \mathbb{N} \right\}. \qquad (7)$$

Then we can show (and this is not easy [17]) the following conditions on optimal configurations

Theorem 2.3 *If S is an optimal configuration then*

$$|l_i - l_j| \leq 1 \quad \textit{and} \quad |c_i - c_j| \leq 1.$$

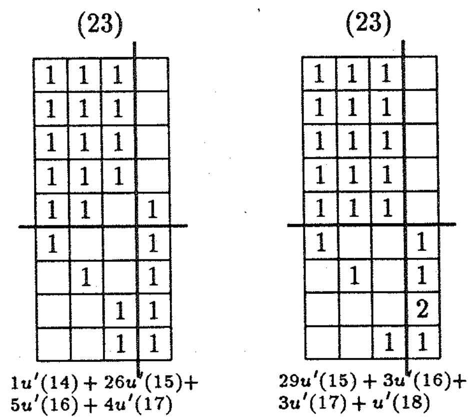

Figure 5: The two candidates to optimality when $s = 23$

Furthermore, after any permutation of the indices which insures that the sequences l_i and c_j are decreasing, we must have

$$a_{i,j} = \begin{cases} \delta \quad or \quad \delta+1 & if\ (i,j) > (i_0, j_0) \\ 0 \quad or \quad 1 & otherwise \end{cases}$$

where $i_0 = s \bmod 2^d$ and $j_0 = s \bmod 3^d$ and where $\delta \leq a = o(s)$ is an indetermined integer. ∎

The above caracterisation fails, in general, to give the exact value of δ. There exist examples for which $a \neq 0$. Nevertheless it can be shown that for almost all values of s (the ratio is at least $(1 - (5/6)^d)$) we have $a = 0$. In that case the above constraints are equivalent to the minimality of the variance of the configuration. So, in general, we are forced to conduct a search for a number of configurations which are "candidates" to optimality, one for each value of δ between 0 and a. The following theorem shows that this search is equivalent to a linear integer optimization problem which is solvable in polynomial time.

Theorem 2.4 *Optimal configurations among the ones satisfying the constraints expressed in theorem 2.3 are characterized by the condition $\sum_{i>i_0, j>j_0} a_{i,j}$ is maximal. Furthermore this last optimization problem can be solved in time $O(s \log s)$ for the entire set of values of δ.* ∎

For example in two dimensions there is only one candidate to optimality for every value of $s \leq 5^2 = 25$ except for $s = 23$ where there are two candidates; these candidates are depicted in Figure 5. The configurations are represented by matrices of size $2^d \times 3^d$ where the columns and the rows represent the cosets of $\Lambda/2$ and $\Lambda/3$ while the boxes represent the cosets of $\Lambda/6$. The various entries of the matrix give a complete description of the repartition of the cameras in the cosets of $\Lambda/2, \Lambda/3$ and $\Lambda/6$.

To decide between the candidates to optimality we are faced with the numerical evaluation of infinite products of the form $d'(k) = \prod_{p\neq 2,3}\left(1 - \frac{k}{p^d}\right)$ which converge very slowly (a power of $1/N$, if we take N terms). Using a technique developed by Vardi and Flajolet [6] efficient evaluation can be done.

2.3 Locations of the cameras

In the previous section we have given a description of the optimal configurations of size $s \leq 5^d$ in terms of the p-visibility relations on the set $\{C_1, \ldots, C_s\}$ of cameras. To be complete it remains to locate the cameras in the lattice Λ i.e. to determine a set $\{A_1, \ldots, A_s\}$ of s points of Λ such that

$$A_i \not\equiv A_j \bmod p \iff \text{the cameras } C_i \text{ and } C_j \text{ are } p\text{-visible.} \tag{8}$$

Such an s tuple of cameras will be called an $\mathcal{E}$-configuration, where $\mathcal{E}$ denotes the family of p-visibility relations[4] on the set $\mathcal{C}$ of cameras. The following theorem confirms the existence of such a solution.

Theorem 2.5 *Let $\mathcal{E} = (\neg \sim_p)_{p\in\mathcal{P}}$ be a family of p-visibility relations on the set of cameras $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$. The density of the set of $\mathcal{E}$-configurations is given by the infinite product*

$$\prod_p \frac{(p^d)_{|\mathcal{C}/\sim_p|}}{p^{sd}} \tag{9}$$

where $(x)_y = x(x-1)(x-2)\ldots(x-y+1)$ is the descent factorial. Furthermore an $\mathcal{E}$-configuration exists if and only if the above density is non null. ∎

As long as we are concerned with an optimal configuration the above expression of the density of $\mathcal{E}$-configurations depends only on s; indeed using condition (4) on optimal configurations the product (9) can be rewritten

$$\prod_{k=1}^{s-1}\prod_{p\in\mathcal{P}}\left(1 - \frac{\min\{k, p^d - 1\}}{p^d}\right).$$

The size of the above expression prohibits the use of unsophisticated random sampling in order to get $\mathcal{E}$-configurations. This is because the probability that a random s-camera configuration is an $\mathcal{E}$-configuration is $\leq 1/\zeta(d)^{s-1}$. Thus on the average it will be necessary to randomly sample at least $\zeta(d)^{s-1}$ times (which

[4] A p-visibility relation on the set of cameras is simply the negation of an equivalence relation.

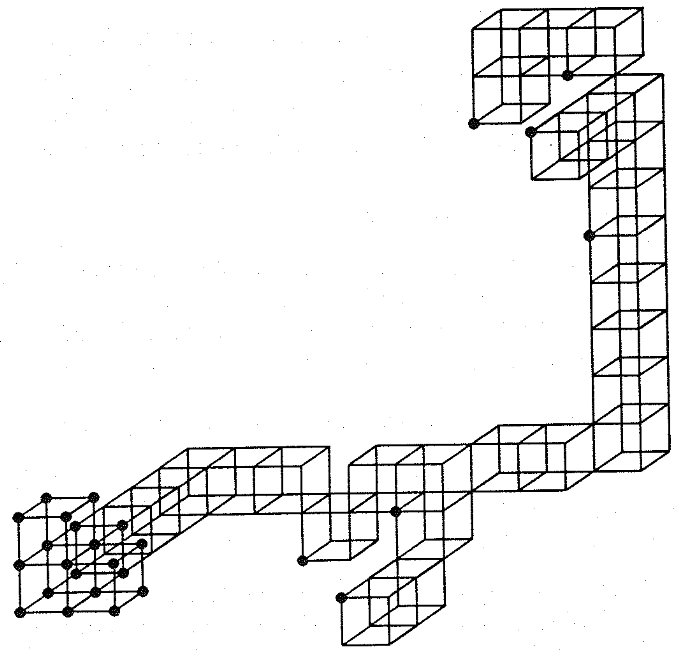

Figure 6: Optimal Configuration of 27 cameras in dimension 3

is exponential in s) before one succeeds in obtaining an $\mathcal{E}$-configuration. In [17] a simple randomized algorithm for doing this has expected time complexity $e^{O(s^{1/d})}$ (for d fixed). This raises the question of whether any iterative techniques starting from an arbitrary configuration will lead to an optimal one. Simulated annealing offers such an effective technique [21] but it is not known whether convergence to an optimal configuration can be achieved in polynomial time.

In order to return to the original problem solved by Abbott, it will be interesting to determine the minimal size ($\max\{\|A_i - A_j\|\}$) of an optimal configuration and to determine the pattern of visible and non-visible points around the cameras.

3 Superposition of Lattices

Another interesting problem concerns generalizations of the camera placement problem to more general lattice systems, like tilings or even more generally of point configurations obtained by superposing lattices and/or tilings [8, 9].

Preliminary investigations [14] show that the problem reduces to the following three subproblems:

1. give a number theoretic characterization of the visibility relation among points of the given tiling system,
2. extend Rumsey's theorem; in particular, it is necessary to determine the density of the visibility sets $V(S)$ in arbitrary tiling systems,
3. investigate combinatorial optimization techniques in order to construct optimal configurations.

Item (1) is a nontrivial problem. Research in progress [14] shows that if the set of points is $\mathcal{O} = \Lambda \setminus \bigcup_{i=1}^{t} G_i$, where G_i are sublattices of Λ then for $x, y \in \mathcal{O}$ which are $\mathcal{O}$-visible (i.e. the segment $]x, y[$ avoid any points of $\mathcal{O}$) we must have that $\gcd(x - y) \leq 2^t$ (the proof uses a result of [3]).[5] For item (2) one requires proving stronger density theorems for visibility sets comparable to Rumsey's theorem [20]. For item (3) a reasonable approach is to refine and extend the algorithmic techniques and reduction theorems we have already developed for the case of $s \leq 5^d$ cameras. For more details see [14]. We illustrate the problems with an example.

Example. Let the set of points be $\mathcal{O} = \{ x \in \Lambda \mid x \not\equiv 0 \bmod 3 \}$. In that case a point x is visible from a point a if and only if $x \not\equiv a \bmod 3, 5, 7, \ldots$, and $(x \not\equiv a \bmod 2)$ or $(x = a \bmod 2$ and $x = -a \bmod 6)$. We get then that the density of the set of points visible from each point of a finite subset S of $\mathcal{O}$ is given by the infinite product

$$\frac{(2^d - |S/2|)(3^d - |S/3| - 1) + \omega}{6^d} \prod_{p>3} \left(1 - \frac{|S/p|}{p^d}\right)$$

where ω is the cardinal of the set of cosets c of $\mathcal{O}/6$ such that

$$\begin{aligned} \forall a \in S \quad & c \not\equiv a \bmod 3 \\ \exists a \in S \quad & c = a \bmod 2 \\ \forall a \in S \quad & c = a \bmod 2 \Rightarrow c = -a \bmod 6. \end{aligned}$$

What can we say for the camera placement problem? Much of the previous optimization analysis holds in this case as well. We can show that

- the optimality condition (4) holds for $p = 3, 5, \ldots$,
- for the s-camera placement problem ($s < 5^d$) the optimization problem becomes

$$\max \left\{ \sum_{I \in \mathcal{O}/6} u'(b_I) \mid b_I = |I \cap (-S)| + \sum_{d(I,J)=2} |J \cap S| \right\},$$

[5] Determining whether or not $\mathcal{O}$ is empty is an NP-complete-problem [7].

where u' is the reduced density function.

Then we can give a complete characterisation of the optimal configurations of size $s \leq 2^d$; for example an optimal 2-camera configuration $\{a, b\}$ is characterized by a and b are visible in Λ and $a + b = 0 \bmod 3$.

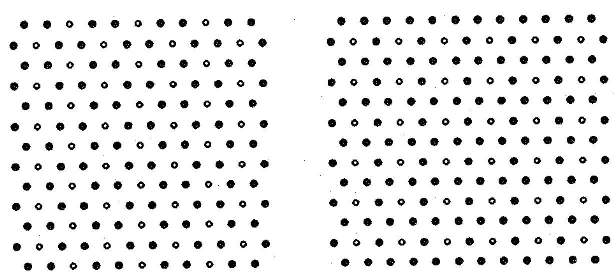

Figure 7: The Honeycomb (left), Hexagonal (right) Lattices

Another interesting class of visibility problems is also obtained when we assume that the set of "obstacle" points (i.e. which block the visibility) and the set of possible locations of the cameras make a partition of Λ. Examples of such tiling systems are depicted in figure 7 where we assume that the $\bullet$ are the obstacle-points and the $\circ$ are the possible locations of the cameras. We mention the following example illustrating this problem.
Example. Let F be a subgroup of L/n where $n = p_1 \ldots p_r$ is a square-free number and let F' be its complement. Let the set of obstacles be $\mathcal{O} = \{ x \in L^d \mid \exists f \in F'(x = f \bmod n) \}$ and the set of possible locations of the guards be $\mathcal{G} = \{ x \in L^d \mid \exists f \in F(x = f \bmod n) \}$. In this instance it can be shown that two lattice points x, y are visible if and only if they are p-visible for all primes $p \neq p_1, \ldots, p_r$. Moreover the density of obstacles visible from a finite set S of guards is given by the infinite product

$$\prod_{\substack{p \neq p_i \\ p \in \mathcal{P}}} \left(1 - \frac{|S/p|}{p^d}\right).$$

All the optimization analysis developed for the camera placement problem in Λ holds here as well except that we have to "forget" completely the p-visiblity relation for $p = p_1, \ldots, p_r$. This allows us to control the value of the reduced density function and give some insight on the validity of our conjecture (minimal variance) about optimal configurations (for more details see [17]).

4 Conclusion

The main challenge is to derive caracterisations of optimal configurations for $s > 5^d$ cameras that leads to a polynomial number of candidates to optimality. In the case $s \leq 7^d$ this means to determine the complexity of the following integer optimisation problem

$$\text{Maximise} \sum_{i,j,k} u'(b_{i,j,k})$$

under the constraints

$$b_{i,j,k} = \sum_{i' \neq i, j' \neq j, k' \neq k} a_{i',j',k'}$$

$$\sum_{i,j,k} a_{i,j,k} = s, \quad a_{i,j,k} \in \mathbb{N}$$

where u' is an absolute monotone function.

References

[1] H. L. Abbott. Some results in combinatorial geometry. *Discrete Mathematics*, 9:199–204, 1974.

[2] V. Boltjansky and I. Gohberg. *Results and Problems in Combinatorial Geometry*. Cambridge University Press, 1985.

[3] R. B. Crittenden and C. L. Vanden Eynden. Any n arithmetic progressions covering the first 2^n integers covers all integers. *Proceedings of the American Mathematical Society*, 24:475 – 481, 1970.

[4] P. Erdös. On the integers relatively prime to n and on a number theoretic function considered by Jacobsthal. *Math. Scand.*, 10:163 – 170, 1962.

[5] P. Erdös, P. M. Gruber, and J. Hammer. *Lattice Points*, volume 39 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific and Technical, 1989.

[6] P. Flajolet and I. Vardi. Numerical evaluation of euler products, 1990. Unpublished manuscript.

[7] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the theory of NP-completeness*. W. H. Freeman, 1979.

[8] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers*, volume 37 of *North Holland Mathematical Library*. North Holland, 1987. Second Edition.

[9] B. Grünbaum and G. C. Shephard. *Tilings and Patterns.* W. H. Freeman and Company, 1987.

[10] J. Hammer. *Unsolved Problems Concerning Lattice Points.* Research Notes in Mathematics. Pitman, 1977.

[11] D. Knuth. *The Art of Computer Programming: Seminumerical Algorithms.* Computer Science and Information Processing. Addison Wesley, second edition, 1981. 688 pages.

[12] E. Kranakis and M. Pocchiola. Enumeration and visibility problems in integer lattices. In *Proceedings of the 6th Annual ACM Symposium on Computational Geometry*, 1990.

[13] E. Kranakis and M. Pocchiola. Camera placement in integer lattices. In *Proceedings of 14th International Symposium on Mathematical Programming, Amsterdam, August 5-9*, 1991.

[14] E. Kranakis and M. Pocchiola. Visibility in tilings, 1991. in preparation.

[15] W. O. J. Moser. Problems on extremal properties of a finite set of points. In Goodman et al, editor, *New York Academy of Sciences*, pages 52 – 64, 1985.

[16] J. O'Rourke. *Art Gallery Theorems and Algorithms.* International Series of Monographs on Computer Science. Oxford University Press, 1987. 282 pages.

[17] M. Pocchiola. *Trois Thèmes sur la Visibilité: Enumération, Optimization et Graphique 2D.* PhD thesis, Laboratoire d'Informatique de l'Ecole Normale Supérieure, Paris, October 1990.

[18] D. F. Rearick. Ph.D. Thesis, California Institute of Technology, 1960.

[19] D. F. Rearick. Mutually visible lattice points. *Norske Vid Selsk Fork (Trondheim)*, 39:41 – 45, 1966.

[20] H. Rumsey Jr. Sets of visible points. *Duke Mathematical Journal*, 33:263–274, 1966.

[21] P. J. M. van Laarhoven. *Theoretical and Computational Aspects of Simulated Annealing.* CWI Tract, No 51, 1988.

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