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ENUMERATION PROBLEMS RELATING TO DIRICHLET'S THEOREM*

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Abstract

We consider enumeration problems relating to Dirichlet's theorem on the probability that two integers, taken at random, are relatively prime. We study weighted versions of the asymptotic number of lattice points inside a domain Δ which are visible from the origin. This leads to a uniform approach in the study of the asymptotic behavior of several problems in combinatorial and computational geometry like, the number of lines traversing at least k vertices of a cube or simplex, as well as the number of incidencies between a set of points and lines and the complexity of the edge visibility region.

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*A preliminary version of part of this work appears in [13] and is part of the second author's Ph.D. thesis [17].

1 Introduction

An old theorem of G. Lejeune Dirichlet dating back to the year 1849 states that the probability that two integers, taken at random, are relatively prime is $6/\pi^2$ [12, page 324], [10, page 269]. This theorem is also known in an equivalent geometric formulation: the probability that a vertex of the infinite lattice (of pairs of integers) is visible from the origin is $6/\pi^2$. This equivalence is merely due to the simple observation that in the integer lattice, a vertex $P = (x, y)$ is visible from the origin (i.e. the line segment joinning P and $(0, 0)$ meets no lattice points other than P and $(0, 0)$) if and only if $\gcd(x, y) = 1$ (here we assume that the obstacles are the vertices of the integer lattice).

Dirichlet's theorem admits several interesting generalizations. One of them concerns the probability that two integer square matrices, chosen at random, have relatively prime determinants [9]. Another one concerns the probability that a vertex P is visible from each of the points of a fixed subset S of the lattice. This problem was considered by several researchers. The most important contributions are those of Rearick [18] who solved the case where the points of S are pairwise visible and Rumsey who solved the most general case of arbitrary sets S [19]. A nontrivial application of Rumsey's theorem is in the study of the camera placement problem, which is concerned with the placement of a fixed number of (point) cameras on an integer lattice in order to maximize their visibility [13]. For more related problems the reader should consult [1] and [7].

1.1 Results of the Paper

This paper is concerned with a more geometric formulation of Dirichlet's theorem: if Δ is a bounded region in the plane and if $G(\Delta)$ is the set of points of Δ with integer coefficients which are visible from the origin then

$$|G(\Delta)| \sim \frac{6}{\pi^2} \text{ area}(\Delta)$$

as Δ grows by homothety to the entire plane [10, page 409]. Our subsequent analysis of several geometric problems requires the asymptotic evaluation of multidimensional versions of sums of the form

$$\sum_{P \in G(\Delta)} f(P)$$

as a function of $\int_\Delta f$, where f is a real function which is either monotone or Lipschitzian. Intuitively one can think of f as a function measuring (for an observer sitting at the origin) the visibility of a point P, while the sum $\sum_{P \in G(\Delta)} f(P)$ quantifies the total visibility from the origin.

We give a "weighted version" of Dirichlet's theorem and use it in order to give a unified approach to the study of the asymptotic behavior of several enumeration problems in combinatorial and computational geometry. These problems are

1. the number of lines passing through at least k vertices of the cube $C_n^d = \{ (x_1, \ldots, x_d) \in \mathbb{N}^d \mid 0 \leq x_i \leq n \}$, simplex $S_n^d = \{ (x_1, \ldots, x_d) \in \mathbb{N}^d \mid \sum_i x_i \leq n \}$ and more generally of the cartesian product of such simplexes,
2. the maximal number of incidencies between a set of points and lines (of a given size) in the plane [22],[6, chapitre 6],
3. the complexity of the plane region illuminated by a segment in the presence of other segments [16, pages 219-223].

We show how to calculate asymptotically optimal bounds for the first problem and constants of known lower bounds for the other two. A slightly modified version of the first problem allows us to calculate the average asymptotic length as well as the moments of higher order of a maximal segment in the cube C_n^d. For other similar applications consult [20, 8], as well as problem E 3217 [1987, 549; 1989, 64] in the problem section of the "American Mathematical Monthly".

2 General Results

Let Δ be a convex compact subset of $\mathbb{R}^d$ $(d \geq 2)$ and let f be a real function defined on Δ. Let $G(\Delta)$ be the set of points $P = (x_1, \ldots, x_d) \in \Delta$ with integer coordinates which are visible from the origin, i.e. $\gcd(P) := \gcd(x_1, \ldots, x_d) = 1$. The following two theorems give an asymptotic estimate of the sum $\sum_{P \in G(\Delta)} f(P)$ when f is either monotone (i.e. either non-decreasing or non-increasing with respect to all the variables of f simultaneously) or Lipschitzian, respectively. Recall that $\zeta(d)$ denotes the Riemann zeta function $\sum_{n \geq 1} n^{-d}$.

THEOREM 1 *Let Δ be a convex compact subset of $\mathbb{R}^d$ of diameter δ and let f be a monotone real positive function defined on Δ. Then*

$$\left| \sum_{P \in G(\Delta)} f(P) - \frac{1}{\zeta(d)} \int_\Delta f \right| = O\left(\max f \begin{cases} \delta \log \delta & \text{if } d = 2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right).$$

THEOREM 2 *Let Δ be a convex compact subset of $\mathbb{R}^d$ of diameter δ and let f be a real positive function defined on Δ and satisfying the Lipschitz condition $|f(x) - f(y)| \leq A\,|x - y|$, for some constant $A > 0$. Then*

$$\left| \sum_{P \in G(\Delta)} f(P) - \frac{1}{\zeta(d)} \int_\Delta f \right| = O\left((\delta\, A + \max f) \begin{cases} \delta \log \delta & \text{if } d = 2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right).$$

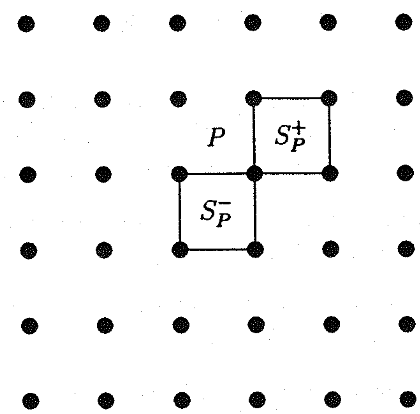

Figure 1: Illustrating the proof of the lemma

We begin with a lemma.

LEMMA 3 *Under the hypothesis of theorem 1 we have*

$$\left| \int_\Delta f - \sum_{P \in \Delta_1} f(P) \right| = O(\delta^{d-1} \max f)$$

where Δ_1 is the set of points of Δ that have integer coefficients.

Proof of the lemma. First it will be necessary to extend f on $\mathbb{R}^d$ by preserving its monotonicity. Without loss of generality we may assume that the function f is non-decreasing. Then we extend f on $\mathbb{R}^d$ by setting $f(x) := \inf\{ f(y) \mid y \in \Delta, x \leq y \}$ with the convention that $\inf \emptyset = \sup_\Delta f$ (by abuse of notation we use the same symbol for the function f and its extension). It is then easy to verify that the extension is still positive, non-decreasing, upper semi-continuous and its supremum does not change, i.e. $\sup_{\mathbb{R}^d} f = \sup_\Delta f$.

The proof that follows generalizes the principle result of Nosarzewska [15]. Let S be the cube with vertices the 2^d points $(x_1, \ldots, x_d)$ where $x_i \in \{0, 1\}$. For each point P let $P \pm S = \{ P \pm Q \mid Q \in S \}$ and define S_P^+ (respectively, S_P^-) to be the cube $P + S$ (respectively, $P - S$). Let $\overline{\Delta}$ be the set of points whose distance from the boundary of Δ is $\leq \sqrt{d}$, i.e. $\overline{\Delta} = \{ P \mid d(P, \partial\Delta) \leq \sqrt{d} \}$ and let $\Delta^+ = \Delta \cup \overline{\Delta}$ and $\Delta^- = \Delta \setminus \overline{\Delta}$. It is obvious that $\Delta^- \subset \Delta \subseteq \Delta^+$ and $\Delta^+ \setminus \Delta^- \subseteq \overline{\Delta}$. Moreover the reader can easily verify that

$$\bigcup_{P \in \Delta_1} S_P^+ \subseteq \Delta^+ \text{ and } \bigcup_{P \in \Delta_1} S_P^- \supseteq \Delta^-.$$

Since the function f is non-decreasing

$$f(P) = \min_{Q \in S_P^+} f(Q) = \max_{Q \in S_P^-} f(Q)$$

from which we can deduce the following inequalities

$$\sum_{P \in \Delta_1} f(P) \leq \sum_{P \in \Delta_1} \int_{S_P^+} f \leq \int_{\bigcup_{P \in \Delta_1} S_P^+} f \leq \int_{\Delta^+} f,$$

$$\sum_{P \in \Delta_1} f(P) \geq \sum_{P \in \Delta_1} \int_{S_P^-} f \geq \int_{\bigcup_{P \in \Delta_1} S_P^-} f \geq \int_{\Delta^-} f.$$

By combining these last two inequalities we obtain

$$\left| \int_\Delta f - \sum_{P \in \Delta_1} f(P) \right| \leq \int_{\Delta^+} f - \int_{\Delta^-} f \leq \int_{\overline{\Delta}} f.$$

Moreover we have[1]

$$\int_{\overline{\Delta}} f \leq \text{area}(\overline{\Delta}) \max_{\overline{\Delta}} f \leq \text{area}(\overline{\Delta}) \max_{\Delta} f.$$

We now show that $\text{area}(\overline{\Delta}) = O(\delta^{d-1})$. In fact since Δ is convex the area of $\overline{\Delta}$ is less than 2 times the area of $\overline{\Delta} \setminus \Delta$. According to the formula of Steiner-Minkowski [3, page 141] or [2, théorème 12.10.6], we can write the area of this last set in the following form

$$\text{area}(\overline{\Delta} \setminus \Delta) = \sum_{i=1}^{d} \ell_i(\Delta)\, d^{\frac{i}{2}}$$

where the functions $\ell_i(.)$ have the property of being bounded on the set of convex subsets of the unit ball and satisfy the following identities $\ell_i(k\Delta) = k^{d-i}\ell_i(\Delta)$. It follows, under the non-restrictive hypothesis $0 \in \Delta$, that

$$\text{area}(\overline{\Delta} \setminus \Delta) = \sum_{i=1}^{d} \delta^{d-i}\, \ell_i(\frac{1}{\delta}\,\Delta)\, d^{\frac{i}{2}} = O(\delta^{d-1}),$$

which completes the proof of our lemma. ∎

Proof of theorem 1. Let Δ_k be the set of points of Δ with integer coordinates all divisible by k. We observe that

$$G(\Delta) = \Delta_1 \setminus \bigcup_{p \in \mathcal{P}} \Delta_p$$

[1] For the sake of simplicity we write area(E) for the d-dimensional Lebesgue measure of the set $E \subseteq \mathbb{R}^d$.

where $\mathcal{P}$ is the set of prime numbers. Using a standard sieve argument (see for example [14]) we can write

$$\sum_{P \in G(\Delta)} f(P) = \sum_{k \geq 1} \mu(k) \sum_{P \in \Delta_k} f(P)$$

where μ is the Möbius function. We now use the lemma in order to estimate the sum $\sum_{P \in \Delta_k} f(P)$. Consider the homothety $h_k(P) = kP$. It follows from the equality $\Delta_k = k(\frac{1}{k}\Delta \cap \mathbb{Z}^d)$ that

$$\sum_{P \in \Delta_k} f(P) = \sum_{P \in \frac{1}{k}\Delta \cap \mathbb{Z}^d} f \circ h_k(P).$$

Therefore by applying the lemma we obtain

$$\left| \sum_{P \in \frac{1}{k}\Delta \cap \mathbb{Z}^d} f \circ h_k(P) - \int_{\frac{1}{k}\Delta} f \circ h_k \right| = O\left((\frac{\delta}{k})^{d-1} \max_{\frac{1}{k}\Delta} f \circ h_k \right).$$

Simple calculations show that

$$\max_{\frac{1}{k}\Delta} f \circ h_k = \max_{\Delta} f, \quad \int_{\frac{1}{k}\Delta} f \circ h_k = \frac{1}{k^d} \int_{\Delta} f.$$

By summing on k it follows that

$$\left| \sum_{P \in G(\Delta)} f(P) - \sum_{k \leq \delta} \frac{\mu(k)}{k^d} \int_{\Delta} f \right| = O\left(\sum_{k \leq \delta} (\frac{\delta}{k})^{d-1} \max_{\Delta} f \right)$$

which we simplify to

$$\left| \sum_{P \in G(\Delta)} f(P) - \sum_{k \leq \delta} \frac{\mu(k)}{k^d} \int_{\Delta} f \right| = O\left(\max_{\Delta} f \begin{cases} \delta \log \delta & \text{if } d = 2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right)$$

Using the well-known identities

$$\sum_{k \geq 1} \frac{\mu(k)}{k^d} = \frac{1}{\zeta(d)}, \quad \left| \sum_{k > \delta} \frac{\mu(k)}{k^d} \right| = O\left(\frac{1}{\delta^{d-1}} \right),$$

(see for example [12, exercise 10, section 4.5.2]) and the fact that area$(\Delta) = O(\delta^d)$, the proof of the theorem can be completed without difficulty. ■

To give the proof of theorem 2 it will be necessary to prove a second lemma similar to lemma 3.

LEMMA 4 *Under the hypothesis of theorem 2 we have that*

$$\left| \int_{\Delta} f - \sum_{P \in \Delta_1} f(P) \right| = O\left(\delta^{d-1}\left(A\,\delta + \max f\right)\right),$$

where Δ_1 is the set of points of Δ with integer coordinates.

Proof of the lemma. We extend f on $\mathbb{R}^d$ by preserving the Lipschitz condition. By the compactness of Δ for each $x \in \mathbb{R}^d$ there exists a point $x^* \in \Delta$ such that

$$|x - x^*| = \min_{y \in \Delta} |x - y|. \tag{1}$$

Using the convexity of Δ it can be shown that for each x the point $x^* \in \Delta$, defined as above, is unique (indeed, if both $x^*, x_1^* \in \Delta$ satisfied (1) then so would $tx^* + (1-t)x_1^*$, for all $0 \le t \le 1$) and the mapping $x \in \mathbb{R}^d \longrightarrow x^* \in \Delta$ is non-increasing on distances (i.e. $|x^* - y^*| \le |x - y|$, for all x, y). This guarantees that the function $f(x) := f(x^*)$ (by abuse of notation we use the same symbol for the function f and its extension) is well defined for $x \in \mathbb{R}^d$ and satisfies the same Lipschitz condition.

We use the notations introduced during the course of the proof of the first lemma. For each point P let S_P be the unit cube with center P and put $\Delta' = \bigcup_{P \in \Delta_1} S_P$. We see easily that $(\Delta \setminus \Delta') \cup (\Delta' \setminus \Delta) \subseteq \overline{\Delta}$. It follows that

$$\left| \int_{\Delta} f - \int_{\Delta'} f \right| \le \int_{\overline{\Delta}} f.$$

Due to the Lipshitzian character of f we can write

$$\left| \sum_{P \in \Delta_1} f(P) - \int_{\Delta'} f \right| \le \sum_{P \in \Delta_1} \left| f(P) - \int_{S_P} f \right| \le A \sum_{P \in \Delta_1} 1.$$

Combining these last two inequalities we obtain

$$\left| \sum_{P \in \Delta_1} f(P) - \int_{\Delta} f \right| \le \int_{\overline{\Delta}} f + A \sum_{P \in \Delta_1} 1.$$

Now we use the first lemma in order to write

$$\int_{\overline{\Delta}} f = O(\max_{\Delta} f\, \delta^{d-1}) \text{ and } \sum_{P \in \Delta_1} 1 = O(\delta^d)$$

which allows to complete the proof of the lemma. ∎

Proof of theorem 2. The proof is similar to that of theorem 1. We need only mention that if the function f satisfies the Lipschitz condition with constant

A then the function $f \circ h_k$ satisfies the Lipschitz condition with constant $k\,A$. ∎

Remark. In the two dimensional case $d = 2$ the convexity condition on the domain Δ is not necessary. In fact it is possible to obtain an upper bound on the area of $\overline{\Delta}$ under the hypothesis that the boundary of Δ is rectifiable. Let $\ell(\Delta)$ be its length. Partition $\partial\Delta$ in r arcs $A_1, \ldots, A_r$, the first $r-1$ having length $\sqrt{2}$ and the last A_r having length $\leq \sqrt{2}$. Then we have

$$r \leq [\ell(\Delta)/\sqrt{2}] + 1 \leq \frac{\ell(\Delta)}{\sqrt{2}} + 1.$$

On the other hand since $\partial\Delta$ is contained in the union of r discs each of radius $\leq \sqrt{2}$, the domain $\overline{\Delta}$ is contained in the union of r discs each of radius $\leq 2\sqrt{2}$. Consequently

$$\text{area}(\overline{\Delta}) \leq 8\,\pi\,r \leq \frac{8\,\pi}{\sqrt{2}}\,\ell(\Delta) + 8\,\pi.$$

Hence the proof of the first lemma is valid in this case as well.

We use our theorems on relatively complicated functions f. Here we only mention the case where the function f is a monomial and the summation is over the cube $\{(x_1, \ldots, x_d) \mid 0 \leq x_i \leq n, i = 1, \ldots, d\}$.

THEOREM 5 *For fixed d we have*

$$\sum_{\substack{0 \leq x_1, \ldots, x_d \leq n \\ \gcd(x_1, \ldots, x_d) = 1}} x_1^{a_1} \cdots x_d^{a_d} \sim \frac{1}{\zeta(d)} \frac{n^{a_1+1}}{a_1+1} \cdots \frac{n^{a_d+1}}{a_d+1}.$$

asymptotically in n. ∎

In particular, if the a_i's are equal to zero we get the d-dimensional version of the previously mentioned theorem of Dirichlet.

3 Applications

In this section we give applications of our main theorems to the calculation of the number of lines traversing vertices of domains, and analyze the incidence and illumination problems.

3.1 Calculating the Number of Lines

As a first application of theorem 1 we give an asymptotic evaluation of the number of lines traversing at least $k+1$ points of the cube

$$C_m^d = \{(x_1, \ldots, x_d) \in I\!N^d \mid 0 \leq x_i < m\},$$

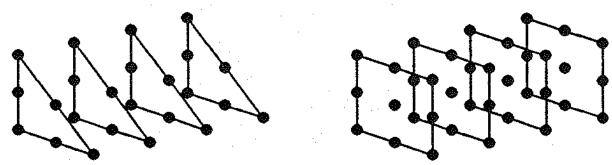

Figure 2: Examples of products of simplexes.

of the simplex

$$S_m^d = \{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid 0 \le \sum_{i=1}^{d} x_i < m\}$$

and more generally of a cartesian product of such simplexes. We formalize this as follows. Let $\mathcal{J}$ be a partition of $\{1, \ldots, d\}$ and let $n = (n_I)_{I \in \mathcal{J}}$ be a function of $\mathcal{J}$ into $\mathbb{Z}$. Put

$$\mathcal{D}(n) = \left\{ (x_1, \ldots, x_d) \in \mathbb{N}^d \mid 0 \le \sum_{i \in I} x_i < n_I,\ \forall I \in \mathcal{J} \right\}.$$

For example,

$\mathcal{D}(\{\{i\} \mid i \le d\} \to m)$ is the cube C_m^d, while
$\mathcal{D}(\{\{1, \ldots, d\}\} \to m)$ is the simplex S_m^d.

In general $\mathcal{D}(n)$ is the cartesian product of $|\mathcal{J}|$ simplexes of dimension $|I|$ and size n_I for $I \in \mathcal{J}$,

$$\mathcal{D}(n) = \prod_{I \in \mathcal{J}} S_{u_I}^{|I|}.$$

Let $\delta(n, k)$ be the number of lines with positive slope passing through at least $k + 1$ points of the domain $\mathcal{D}(n)$. The following theorem gives an asymptotic evaluation of the function $\delta(n, k)$.

THEOREM 6 *Let $\mathcal{J}$ be a partition of $\{1, \ldots, d\}$ and let $n = (n_I)_{I \in \mathcal{J}}$ be a function of $\mathcal{J}$ into $\mathbb{Z}$. The number $\delta(n, k)$ of lines with positive slope traversing at least $k + 1$ points of the domain $\mathcal{D}(n)$ is given by the formula*

$$\delta(n, k) = \frac{1}{\zeta(d)} \prod_{I \in \mathcal{J}} \frac{n_I^{2|I|}}{(2\,|I|)!} \left\{ \frac{1}{k^d} - \frac{1}{(k+1)^d} \right\} + O\left(\begin{cases} \frac{|n|^3}{k^2} \log \frac{|n|}{k} & \text{if } d = 2 \\ \frac{|n|^{2d-1}}{k^d} & \text{otherwise} \end{cases} \right)$$

where $|n| = \max_I n_I$.

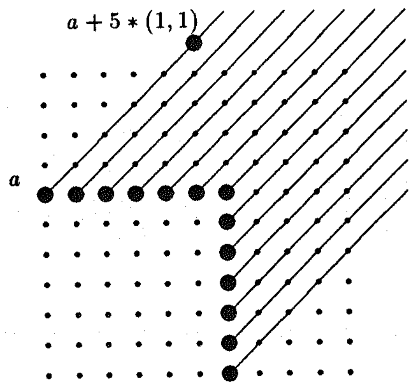

Figure 3: Lines with slope (1, 1) passing through at least 5 points of a cube.

Proof. Let $p = (p_1, \ldots, p_d) \in \mathbb{N}^d$ be a positive slope, i.e. a d-tuple of positive integers such that $\gcd(p_1, \ldots, p_d) = 1$. We denote $g_k(p, n)$ the cardinal of the set, denoted $G_k(p, n)$, of the lines with positive slope p each traversing at least $k + 1$ points of the domain $\mathcal{D}(n)$. By partitioning the set of lines according to the value of their slope the following equality is obtained

$$\delta(n, k) = \sum_{\gcd(p)=1} g_k(p, n), \tag{2}$$

which immediately indicates a possible application of the general theorems in the previous paragraph. We show that $g_k(p, n)$ is a polynomial in the coordinates $p_1, \ldots, p_d$ of p. The proof is divided into two lemmas.

LEMMA 7 *$g_k(p, n) = |\mathcal{D}(n - k\,p)| - |\mathcal{D}(n - (k + 1)\,p)|$, where $(n - k\,p)_I = n_I - k \sum_{i \in I} p_i$.*

Proof. The mapping which associates to each line of $G_k(p, n)$ the unique point a such that

$$a, a + p, a + 2p, \ldots, a + kp \in \mathcal{D}(n) \text{ and } a + (k + 1)p \notin \mathcal{D}(n)$$

is a bijection of $G_k(p, n)$ on the set E of points a of the domain $\mathcal{D}(n)$ such that

$$a + k\,p \in \mathcal{D}(n) \text{ and } a + (k + 1)\,p \notin \mathcal{D}(n).$$

Let E_i be the set of points a of the domain $\mathcal{D}(n)$ such that $a + i\,p \in \mathcal{D}(n)$. Since $E_i = \mathcal{D}(n - i\,p)$ and $E_{i+1} \subseteq E_i$ we obtain $g_k(p, n) = |E_k| - |E_{k+1}|$, which gives the desired expression. ∎

LEMMA 8

$$|\mathcal{D}(n)| = \prod_{I \in \mathcal{J}} \frac{1}{|I|!} \prod_{i=0}^{|I|-1} (n_I + i)$$

Proof. Since the set $\mathcal{D}(n)$ is a cartesian product of simplexes, it suffices to show the result for a simplex. Let $a(d,m)$ be the cardinal of the set of d-tuples of positive integers $(x_1,\ldots,x_d)$ such that $0 \le \sum_i x_i < m$. Using, for example, the recurrence relations

$$a(1,m) = m, \text{ and } a(d+1,m) = \sum_{i=0}^{m} a(d,i)$$

it is easily shown that

$$a(d,n) = \frac{\prod_{i=0}^{d-1}(n+i)}{d!}. \qquad \blacksquare$$

Now we are in a position to give the expression for $g_k(p,n)$; the two preceding lemmas show that

$$g_k(p,n) = \prod_{I\in\mathcal{J}} \frac{1}{|I|!} \prod_{i=0}^{|I|-1} (n_I - k\sum_{j\in I} p_j + i)^+ - \prod_{I\in\mathcal{J}} \frac{1}{|I|!} \prod_{i=0}^{|I|-1} (n_I - (k+1)\sum_{j\in I} p_j + i)^+,$$

where t^+ is equal to t if $t \ge 0$ and 0 otherwise. In particular, $g_k(p,n)$ is zero if for a certain I, $n_I \le k\sum_I p_j$. We apply theorem 2 with the function $p \in \mathbb{R}_+^d \longrightarrow g_k(p,n) \in \mathbb{R}_+$ on the convex domain $\mathcal{D}(\frac{n}{k})$. Recall that this domain is defined by the system of inequalities

$$\sum_{j\in I} p_j \le \frac{n_I}{k}$$

and note that its diameter is $\frac{|n|}{k}$. In the following lemma we compute precisely the maxima of the functions $p \longrightarrow g_k(p,n)$ and $p \longrightarrow \frac{\partial g_k(p,n)}{\partial p_i}$.

Lemma 9

$$\max_{p\in\mathbb{R}^d} g_k(p,n) = O(\frac{|n|^d}{k+1}) \tag{3}$$

$$\max_{p\in\mathbb{R}^d} \frac{\partial g_k(p,n)}{\partial p_i} = O(|n|^{d-1}). \tag{4}$$

Proof. Introduce the function $f(t_1,\ldots,t_d)$ which is defined by

$$f(t_1,\ldots,t_d) = \prod_{i=1}^{d}(1-\alpha t_i + \epsilon_i) - \prod_{i=1}^{d}(1 - t_i + \epsilon_i)$$

where ϵ_i and α are parameters which we compute in the sequel. By factoring the n_I's in the expression of $g_k(p,n)$ as given by equation (3.1) it follows that

$$g_k(p,n) = \left(\prod_{I\in\mathcal{J}} \frac{n_I^{|I|}}{|I|!}\right) f(t_1,\ldots,t_d)$$

where the t_i, α, ϵ_i are given by

$$t_i = \frac{k+1}{n_I} \sum_{j \in I} p_j \in [0,1], \alpha = \frac{k}{k+1}, \epsilon_i = O(d/n_I),$$

where I is the equivalence class of i. Hence taking derivatives with respect to p_i we obtain

$$\frac{\partial g_k(p,n)}{\partial p_{i_0}} = \left(\frac{k+1}{n_{I_0}} \prod_{I \in \mathcal{J}} \frac{n_I^{|I|}}{|I|!} \right) \sum_{j \in I_0} \frac{\partial f(t_1, \ldots, t_d)}{\partial t_j}.$$

Consequently our lemma is a consequence of the affiliation of the sizes of the maxima of f and $\frac{\partial f}{\partial t_i}$ to $O(\frac{1}{k+1})$. The proof of this result is given in the next lemma. ∎

Lemma 10 *The maxima of the function $f(t_1, \ldots, t_d)$ and its partial derivatives are in*

$$\frac{1}{k+1} + O(\frac{1}{\min_I n_I}).$$

Proof. Let $f^*(t_1, \ldots, t_d)$ be the function obtained by substituting in the expression of $f(t_1, \ldots, t_d)$ the ϵ_i by 0. Since the function $f(t_1, \ldots, t_d)$ is bounded on the unit cube we have

$$f(t_1, \ldots, t_d) = f^*(t_1, \ldots, t_d) + O(\max_i \epsilon_i).$$

Similarly,

$$\frac{\partial f(t_1, \ldots, t_d)}{\partial t_i} = \frac{\partial f^*(t_1, \ldots, t_d)}{\partial t_i} + O(\max_i \epsilon_i).$$

The expressions of f^* and $\frac{\partial f^*}{\partial t_i}$ are sufficiently simple in order to allow the calculation of the respective maxima. Since the calculations are elementary we give directly the results. The search for an extremum on the open unit cube, by setting to zero the partial derivatives, shows the existence of a unique extremum obtained for $t_i = t_j$. We denote by τ_1 the extremum of f^* and by τ_2 the extremum of $\frac{\partial f^*}{\partial t_i}$, respectively; their values are

$$\tau_1 = \frac{(1-\alpha)^d}{(1-\alpha^{\frac{d}{d-1}})^{d-1}}, \text{ and } \tau_2 = \alpha(1-\alpha) \left\{ \frac{1-\alpha}{1-\alpha^{d+1/d-1}} \right\}^{d-1}.$$

Using the fact that the function $a \to \alpha^a$ is non-decreasing it can be verified that τ_1 and τ_2 are $\leq 1-\alpha$. By examining the values of the functions on the boundary of the domain it follows that their maximum is exactly $1 - \alpha = \frac{1}{k+1}$. ∎

In order to complete the proof of our theorem it remains to calculate the integral $\int_{\mathbb{R}^d} g_k(p,n)dp$. Since this calculation is elementary we leave it to the reader to verify the following two lemmas.

LEMMA 11

$$\prod_{i=0}^{a}(u+i) = a!\sum_{i=0}^{a} u^{i+1}\, d_i(a)$$

where $d_i(a) = \sum_{1\le a_1<\cdots<a_i\le a} 1/(a_1\ldots a_i)$. ∎

LEMMA 12

$$\int_{\mathbb{R}^d} |\mathcal{D}(n-k\,p)|dp = \prod_{I\in\mathcal{J}} \frac{1}{|I|!}\frac{n_I^{2|I|}}{k^{|I|}}\,\rho(|I|-1, n_I)$$

where $\rho(a,b) = \sum_{i=0}^{i=a} d_i(a)\frac{a!\,(i+1)!}{(a+i+2)!}\frac{1}{b^{a-i}}$. ∎

This last lemma allows us to write

$$\int_{\mathbb{R}^d} g_k(p,n)dp = \prod_{I\in\mathcal{J}} \frac{n_I^{2|I|}}{(2|I|)!}\left\{\frac{1}{k^d} - \frac{1}{(k+1)^d}\right\} + O(\frac{|n|^{2d-1}}{k^d}).$$

By combining the preceding lemmas we obtain the complete proof of the theorem. ∎

The following result generalizes the preceding theorem to the case where the lines are assigned weights by a function which is homogeneous with respect to their slopes. The particular form of this generalization concerns the study of the length of segments of the cube C_n^d. The proof of the following theorem resembles the proof of the theorem just derived. Since the technical dificulties were resolved during the proof of the previous theorem we only state the result leaving it to the interested reader to verify the $O(\frac{n^{a+d-1}}{k^a})$-Lipshitzian character on the ad hoc domain of the function $p \to h(p)\,|G_k(p)|$.

THEOREM 13 *Let h be a real function which is homogeneous of degree $a \ge 1$ and of class C^1 on $(\mathbb{R}_+^d)^*$. Let $G_k(p)$ be the set of lines of positive slope $p = (p_1,\ldots,p_d) \in \mathbb{N}^d$ traversing at least $k+1$ points of the cube C_n^d. The number*

$$\delta(h,n,k) = \sum_{\gcd(p)=1} h(p)\,|G_k(p)|$$

is given by the formula

$$\delta(h,n,k) = \frac{n^{a+2d}}{\zeta(d)}\left(\frac{1}{k^{a+d}} - \frac{1}{(k+1)^{a+d}}\right)\omega(h) + O\left(\begin{cases} \frac{n^{a+3}}{k^{a+2}}\log\frac{n}{k} & \text{if } d=2 \\ \frac{n^{a+2d-1}}{k^{a+d}} & \text{otherwise}\end{cases}\right)$$

where $\omega(h) = \int_{[0,1]^d}\prod_i(1-x_i)h(x_1,\ldots,x_d)dx_1\ldots dx_d$. ∎

As an example of an application of theorem 13 we can calculate the average length l_n and standard deviation σ_n of the maximal segments of the cube C_n^d. A segment of the cube with slope $p \in \mathbb{N}^d$ and endpoints A_1 and A_2 is called maximal if for each i one of the points $A_i \pm p$ is not a point on the cube. Thus using theorem 13 we can show that these quantities are given by

$$l_n \sim \frac{2^{2d}}{2^d-1}\omega(\|\;\|)\,n,$$

$$\sigma_n{}^2 \sim \frac{2^{2d}}{2^d-1}\left((2\zeta(d+2)-1)\omega(\|\;\|^2) - \frac{2^{2d}}{2^d-1}\omega(\|\;\|)^2\right)n^2.$$

where $\|\;\|$ is a norm on $\mathbb{R}^d$. For example in two dimensions and using the euclidean norm $\sqrt{x^2+y^2}$ we get $l_n \sim (0.695\ldots)n$ and $\sigma_n \sim (0.185\ldots)n$.

3.2 Analysis of the Incidence Problem

In [22] it is shown that the maximal number of incidencies $I(m,n)$ between m points and n lines in the plane is less than $10^{60}(m^{2/3}n^{2/3}+m+n)$. With a different approach [4, 5] reduces the constant and shows that

$$I(m,n) \le 3\sqrt[3]{6}m^{2/3}n^{2/3} + 25n + 2m.$$

The above bound is tight in the sense that $I(m,n) = \Omega(m^{2/3}n^{2/3}+m+n)$. Only the case where $m^{2/3}n^{2/3}$ is the preponderous term (that is $m = o(n^2)$ and $n = o(m^2)$ is difficult. The example of [6, theorem 6.18], used to achieve the lower bound, is based on the set of points and lines traversing points of a square grid. Theorem 1 allows us to make exact calculations which we summarize in the following theorem.

THEOREM 14 *Assume that* $m = o(n^2)$ *and* $n = o(m^2)$. *Then*

$$\liminf_{n,m\to\infty} \frac{I(m,n)}{m^{2/3}n^{2/3}} \ge \sqrt[3]{6/\pi^2}.$$

Proof. Let ℓ be a line in the grid of size $p \times p$. Denote by $contr(\ell)$ the number of points of the grid lying on the line ℓ. Let L be the set of lines of the grid with positive slope $\le \alpha(p,p)$ and n_α the number of such lines. Put $contr(L) = \sum_{\ell\in L} contr(\ell)$. It is clear that $contr(L)$ is a lower bound for $I(p^2, n_\alpha)$. To calculate precisely the numbers n_α and $contr(L)$ we proceed as follows. Let $u_k(P) = \prod_1^2(p - k\,p_i)^+$ and $g_k(P) = u_k(P) - u_{k+1}(P)$ the number of lines of slope $P = (p_1,p_2)$, with $\gcd(p_1,p_2) = 1$, each traversing at least $k+1$ points of the grid of size p (see equation (2)). Then we have

$$n_\alpha = \sum_{\gcd(P)=1, P\le\alpha p} g_1(P)$$

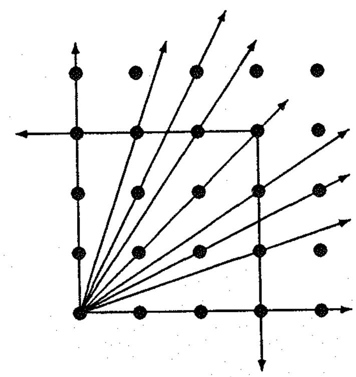

Figure 4: Grid of size $p = 6$ and the slopes $\leq (3,3)$.

and

$$contr(L) = \sum_{\substack{P \leq_\alpha p \\ \gcd(P)=1}} \sum_{k=1}^{\infty} (k+1)\{g_k(P) - g_{k+1}(P)\} = \sum_{\substack{P \leq_\alpha p \\ \gcd(P)=1}} \{2\,u_1(P) - u_2(P)\}.$$

Trivial calculations show that $\max g_1(P) = O(\alpha p^2)$ and $\max(2\,u_1 - u_2) = O(p^2)$. According to theorem 1 and assuming that $\alpha p \to \infty$ we obtain

$$n_\alpha = \frac{1}{\zeta(2)} p^4 f_1(\alpha) + O(\alpha^2 p^3 \log \alpha p) \tag{5}$$

and

$$contr(L) = \frac{1}{\zeta(2)} p^4 f_2(\alpha) + O(\alpha p^3 \log \alpha p) \tag{6}$$

where $f_1(\alpha) = \alpha^3(1 - \frac{3}{4}\alpha)$, $f_2(\alpha) = \alpha^2(1 - \frac{1}{2}\alpha^2)$ for $\alpha \leq \frac{1}{2}$ and $f_1(\alpha) = \alpha^2(1 - \frac{1}{2}\alpha)^2 - \frac{1}{16}$, $f_2(\alpha) = 2\alpha^2(1 - \frac{1}{2}\alpha)^2 - \frac{1}{16}$ for $\alpha \geq \frac{1}{2}$. Combining these last two equations we obtain

$$contr(L) = \zeta(2)^{-1/3} n_\alpha^{2/3} p^{4/3} \frac{f_2(\alpha)}{f_1(\alpha)^{2/3}} \left\{ 1 + O\left(\frac{\log \alpha p}{\alpha p} \right) \right\}. \tag{7}$$

Let now m, n be as in the theorem and let $p = \lfloor \sqrt{m} \rfloor$. Since $n_{1/p} \approx p \approx \sqrt{m}$ and $n_1 \approx p^4 \approx m^2$ and since n_α is an increasing function of α we get an $0 < \alpha < 1$ such that $n_\alpha \leq n < n_{\alpha+1/p}$. Furthermore we can easily verify that $\alpha p \to \infty$ and that $\alpha \to 0$ from which we deduce that $n_{\alpha+1/p} \sim n_\alpha \sim n$. Then according to (7) we can write $contr(L) \sim \zeta(2)^{-1/3} n^{2/3} m^{2/3}$ and we use the fact that $contr(L)$ is a lower bound for $I(m,n)$ to conclude. ∎

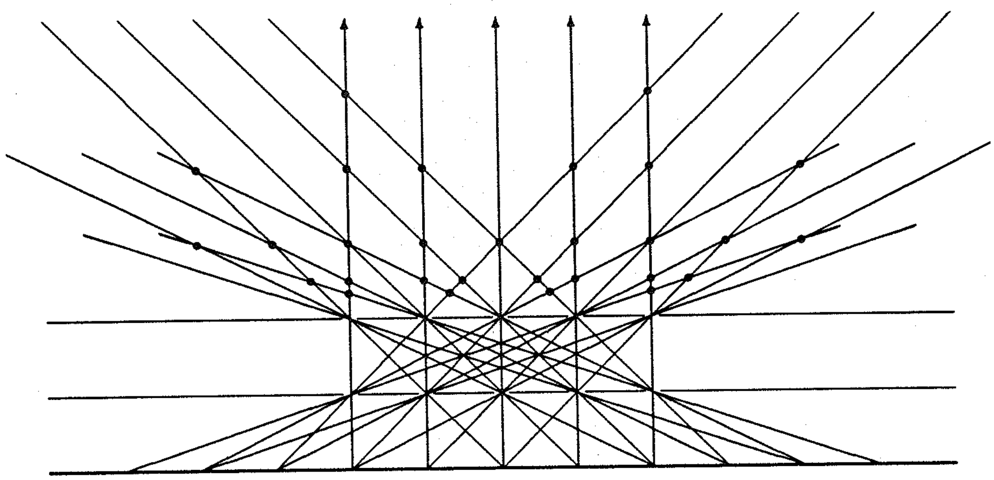

Figure 5: The Visibility Region

3.3 Analysis of the Edge Visibility Region

The problem is to calculate the plane region illuminated by a segment in the presence of other segments. Suri and O'Rourke [21] have established an $\Omega(n^4)$ lower bound on the complexity of any algorithm that calculates explicitely the boundary of the illuminated region. The example used by [16] is illustrated in figure 3.3. It consists of a horizontal luminescent edge ($y = 0$); at the vertices $(i, 0)$, $-n < i < n$ light sources are located emitting light in all directions. Above and parallel to this edge place two rows ($y = 1$, $y = 2$) each consisting of n closely spaced line segments, thus permitting $\Theta(n^2)$ beams of light to emerge above them. Since these beams intersect in $\Theta(n^4)$ points above the second row we obtain a region with $\Theta(n^4)$ vertices and edges. The mathematical analysis that determines the $\Omega(n^4)$ lower bound is based on the evaluation of the number $N(n)$ of distinct intersections located on the half-plane $y > 2$ between lines passing through the points $(i, 1)$ and $(j, 2)$ for $-n < i, j < n$. Theorem 6 can be used to give an asymptotic evaluation of the number $N(n)$. Indeed by using the duality which maps the line passing through the points $(x, 1)$ and $(y, 2)$ to the point (x, y) (see figure 6) we see that the number $2\,N(n)$ is also the number of lines with positive slope passing through at least two points of the grid of size n. Hence we get using theorem 6

$$N(n) \sim \frac{1}{\zeta(2)} \frac{3}{32} n^4.$$

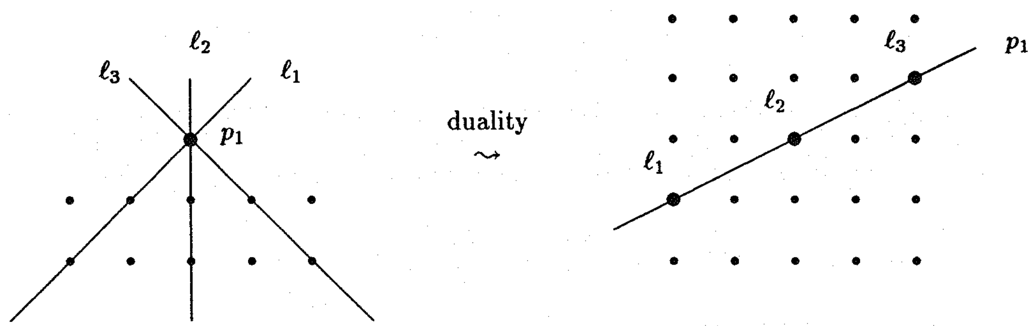

Figure 6: The duality

For generalizations and further applications of this duality to computational geometry the reader may consult [11].

4 Conclusion

We have given two general theorems which facilitate calculations of asymptotic evaluations on the number of incidencies between points (with integer coordinates) and lines of a cube and more generally of a product of simplexes. Two natural questions arise. Is it possible to generalize our results to more general classes of convex sets, like spheres and polyhedra? What is the number of incidencies between points with integer coordinates and d-dimensional subspaces, $(d \geq 2)$?

It is clear that our results generalize to convex sets C for which it is possible to express "simply" (as a function of the slope and the integer k) the number of points with integer coordinates included in the domain $C \cap (-k\,p + C) \setminus (-(k+1)\,p + C)$. However, the class of convex sets for which this is possible still remains to be determined. The answer to the second question seems to be more delicate since a subspace of dimension ≥ 2 is not uniquely defined by a "slope" as is in the case of lines.

A natural generalization of our theorems concerns the asymptotic evaluation of sums over visibility sets. If $V(S)$ is the set of lattice points which are visible from each of the points of S then define

$$s(f, S, \Delta) := \sum_{M \in \Delta \cap V(S)} f(M).$$

Rumsey [19] treats the case where the "weight measure" f is constant and the set S is fixed: the density of the set $V(S)$ is given by the infinite product

$$\prod_{p\in\mathcal{P}}\left(1-\frac{|S/p|}{p^d}\right),$$

where S/p denotes the set of equivalence classes of S modulo the congruence relation $\bmod p$. Using techniques similar to those developped in section 2 it can be shown (under ad hoc hypothesis on the function f and for S a fixed finite set) that

$$s(f,S,\Delta)\sim\prod_{p\in\mathcal{P}}\left(1-\frac{|S/p|}{p^d}\right)\int_{\Delta}f.$$

Eventual applications of this result as well as its generalization to the case of "variable" sets S (for example when it is assumed that the points of S are located on the boundary of the domain Δ, as Δ grows by homothety) are still to be explored.

References

[1] H. L. Abbott. Some results in combinatorial geometry. *Discrete Mathematics*, 9:199–204, 1974.

[2] M. Berger. *Géométrie: Convexes et Polytopes, Polyèdres Réguliers, Aires et Volumes*, volume 3. Cedic/Fernand Nathan, 1978. nouvelle édition, 1990.

[3] Y. D. Burago and V. A. Zalgaller. *Geometric Inequalities.* Springer Verlag, 1988. Translated from the Russian.

[4] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Wezl. Combinatorial complexity bounds and arrangements of curves and surfaces. Technical Report R88/1470, University of Illinois at Urbana-Champaign, November 1988.

[5] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Wezl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete and Computational Geometry*, 5(2), 1990.

[6] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, 1987.

[7] P. Erdös, P. M. Gruber, and J. Hammer. *Lattice Points*, volume 39 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman Scientific and Technical, 1989.

[8] B. Ghosh. Random distances within a rectangle and between two rectangles. *Bulletin of the Calcutta Mathematical Society*, 43:17 – 24, 1951.

[9] J. L. Hafner, K. McCurley, and P. Sarnak. Relatively prime values of polynomials. Technical report, IBM, Almaden Research Center, 1990.

[10] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford Science Publications, 1979. 355 pages.

[11] A. Inselberg, T. Chomut, and M. Reif. Convexity algorithms in parallel coordinates. *Journal of the Association for Computing Machinery*, 34(4):767 – 801, 1987.

[12] D. Knuth. *The Art of Computer Programming: Seminumerical Algorithms.* Computer Science and Information Processing. Addison Wesley, second edition, 1981. 688 pages.

[13] E. Kranakis and M. Pocchiola. Enumeration and visibility problems in integer lattices. In *Proceedings of the 6th Annual ACM Symposium on Computational Geometry*, 1990.

[14] W. Narkiewicz. *Number Theory.* World Scientific, second edition, 1983. 371 pages.

[15] M. Nosarzewska. Evaluation de la différence entre l'aire d'une région plane convexe et le nombre des points aux coordonnées entières couverts par elle. *Colloq. Math.*, 1, 1948.

[16] J. O'Rourke. *Art Gallery Theorems and Algorithms.* International Series of Monographs on Computer Science. Oxford University Press, 1987. 282 pages.

[17] M. Pocchiola. *Trois Thèmes sur la Visibilité: Enumération, Optimization et Graphique 2D.* PhD thesis, Laboratoire d'Informatique de l'Ecole Normale Supérieure, Paris, October 1990.

[18] D. F. Rearick. Mutually visible lattice points. *Norske Vid Selsk Fork (Trondheim)*, 39:41 – 45, 1966.

[19] H. Rumsey Jr. Sets of visible points. *Duke Mathematical Journal*, 33:263–274, 1966.

[20] L. A. Santaló. *Integral Geometry and Geometric Probability.* Addison Wesley, 1976.

[21] S. Suri and J. O'Rourke. Worst-case optimal algorithms for constructing visibility polygons with holes. In *Proceedings of the 2nd Annual ACM Symposium on Computational Geometry*, 1986.

[22] E. Szemerédi and W. Trotter Jr. Extremal problems in discrete geometry. *Combinatorica*, 3:381–392, 1983.

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