# ALGORITHMS FOR ASYMPTOTICALLY OPTIMAL CONTAINED RECTANGLES AND TRIANGLES

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# ALGORITHMS FOR ASYMPTOTICALLY OPTIMAL CONTAINED RECTANGLES AND TRIANGLES

## (Extended Abstract)

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**Abstract**

We consider the problem of computing asymptotically maximal area rectangles and triangles contained in simple polygons. We exhibit an algorithm which given an $n$-node polygon $P$, and a fixed integer $k$ computes in time $O(n^2k^2 \log n)$ and space $O(n)$ a rectangle $R_{alg}$ contained in the polygon such that $\frac{|R_{alg}|}{|R_{max}|} \geq \left(\frac{k\rho(P)}{k\rho(P)+16\pi}\right)^2 \left(1 - \frac{8}{\rho(P)k}\right)$, where $R_{max}$ is the maximal area rectangle contained in the polygon and $\rho(P)$ is the ratio of the polygon (i.e. the quotient of its length over its width). The same algorithm has time complexity $O(nk^2)$ on convex polygons. A more efficient algorithm with time complexity $O(nk)$ can also be given for objects of the same similarity type (e.g. squares, equilateral triangles, and more generally similar triangles) contained in convex polygons. A similar result with time complexity $O(n^3k^3 \log n)$ can be proved for arbitrary triangles contained in simple polygons. All our algorithms have space complexity $O(n)$.

**1980 Mathematics Subject Classification:** 68U05, 52A43

**CR Categories:** F.2.2, I.3.5

**Key Words and Phrases:** Area, Asymptotically optimal, Convex polygon, Contained, Inscribed, Length, Ratio, Rectangle, Simple polygon, Square, Triangle, Width.

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# 1 Introduction

We have developed new algorithms for the problem of computing an asymptotically maximal area rectangle (respectively, triangle) contained in a simple polygon. More specifically our problem can be formulated as follows. Given a simple polygon $P$ let $R_{max}$ be a maximal area rectangle (respectively, triangle) contained in $P$. Compute a rectangle (respectively, triangle) contained in $P$ such that the quotient $\frac{|R|}{|R_{max}|}$ is asymptotically close to 1, where $|R|, |R_{max}|$ denote the areas of $R, R_{max}$, respectively.

There have been several results in the literature concerning the computation of maximal area polygons contained (or inscribed) in a given polygon. The most general among these is the result of Chang and Yap [4] which gives an $O(n^7)$ algorithm for finding the largest convex polygon inscribed in a given polygon. In addition several variants of this problems are possible. One simply varies the shape of the approximating polygon as well as the shape of the polygon being approximated, e.g. see [2] and [14]. Some additional results related to this problem can also be found in "Unsolved Problems in Geometry" by H. T. Croft, K. J. Falconer and R. K. Guy Problem A16) [6], as well as in "Old and New Unsolved Problems in Geometry and Number Theory" by V. Klee and S. Wagon (Section 11) [11].

More related to the problem we propose to study in this paper are the results of [8] and [12]. DePano, Ke and O'Rourke [8] give an $O(n^2)$ time and $O(n)$ space algorithm for the maximal area square or equilateral triangle inscribed in a convex polygon and an $O(n^3)$ algorithm computing the maximal area equilateral triangle inscribed in a simple polygon. Melissaratos and Souvaine [12] give an $O(n^4)$ algorithm for computing the maximal area triangle contained in a simple polygon.

An interesting feature of the problems previously discussed is their continuous (as opposed to discrete) character. A discrete version of this problem that has been studied in the literature concerns the computation of a maximal area $k$-gon with vertices among a given set of $n$ points. This has been considered by various researchers, e.g. [9] gives an $O(n^2 \log n)$ algorithm for solving this problem. Related papers are also [1] and [7], among others.

It is also worth mentioning the corresponding "dual"* problems arising by considering "enclosure" instead of "containment" conditions. For example, what is the minimal area polygon (taken from a specified class) enclosing a given polygon. There are several interesting results regarding this problem, e.g. see the references in [4].

*Usually, the corresponding containment and enclosure problems are completely unrelated and the term dual should not be interpreted that there is a transformation transforming a solution of one type of problem into a solution of its dual.

## 1.1 Results of the paper

In this paper we improve the results of [8] and [12] by almost a factor $n$ for polygons of constant ratio and asymptotically optimal area rectangles and triangles. Namely, we describe an algorithm which given an $n$-node polygon $P$, and an integer $k$ computes in time $O(n^2k^2 \log n)$ and space $O(n)$ a rectangle $R_{alg}$ contained in the polygon such that

$$\frac{|R_{alg}|}{|R_{max}|} \geq \left(\frac{k\rho(P)}{k\rho(P) + 16\pi}\right)^2 \left(1 - \frac{8}{\rho(P)k}\right), \qquad (1)$$

where $R_{max}$ is the maximal area rectangle contained in the polygon and $\rho(P)$ is the ratio of the polygon (intuitively, this is the ratio of the width to the length of the polygon).

The main idea of our construction is based on the fact that the ratio of the maximal rectangle contained in the polygon $P$ is $\Omega(\rho(P))$ (thus the constant implied by this lower bound is independent of the polygon $P$). The value of the parameter $k$ specifies the desired accuracy of the rectangle computed by our algorithm as indicated in inequality (1) and also affects its time complexity. For example, assuming $\rho(P) = \Omega(1)$ and choosing $k = O(\log n)$ our algorithm will compute in time $O(n^2 \log^3 n)$ and space $O(n)$ a rectangle $R_{alg}$ such that

$$\frac{|R_{alg}|}{|R_{max}|} \geq \left(\frac{1}{1 + 16\pi/\log n}\right)^2 \left(1 - \frac{8}{\log n}\right).$$

The same algorithm for convex polygons has time complexity $O(nk^2)$ (Theorem 1). For objects of the same similarity type (e.g. squares, equilateral triangles contained in a convex polygon the time complexity is $O(nk)$ (Theorem 2).

The previous best known algorithm for computing the maximal area equilateral triangle or square contained in an $n$-node convex polygon has time complexity $O(n^2)$ and space complexity $O(n)$ [8]. On simple polygons, the best previous result for equilateral triangles has time complexity $O(n^3)$ and is due to [8], while for arbitrary triangles has time complexity $O(n^4)$ and is due to [12]. Here is a table summarizing the time complexities proved in our paper for rectangles and triangles contained in convex or simple polygons. The space complexity of all our algorithms is $O(n)$.

| Type of Polygon | Similar Objects | Arbitrary Rectangles | Arbitrary Triangles |
|---|---|---|---|
| Convex | $O(nk)$ | $O(nk^2)$ | $O(nk^3)$ |
| Simple | $O(n^2k \log n)$ | $O(n^2k^2 \log n)$ | $O(n^3k^3 \log n)$ |

More formally the main theorem of this paper is the following.

THEOREM 1 *There is an algorithm which given a fixed integer $k$ and an arbitrary simple polygon with $n$ nodes computes in time $O(n^2k^2 \log n)$ a rectangle*

*$R_{alg}$ which is contained in the polygon such that inequality (1) is valid, where $R_{max}$ is the maximal area rectangle contained in the polygon and $\rho(P)$ is the ratio of the polygon. If in addition the polygon is convex then the time complexity is $O(nk^2)$.*

A similar result with time complexity $O(n^3k^3 \log n)$ can be proved for arbitrary triangles. The corresponding approximation formula is the following

$$\frac{|T_{alg}|}{|T_{max}|} \geq \left(\frac{k\rho(P)}{k\rho(P) + 8\pi}\right)^2 \left(1 - \frac{2}{\rho(P)k}\right), \tag{2}$$

where $T_{max}$ is the maximal area triangle contained in the polygon and $T_{alg}$ is the output of the algorithm with parameter $k$ (see Theorem 13).

### 1.2 Outline of the paper

Here is a brief outline of the paper. The notion of ratio $\rho(P)$ of the simple polygon will be made precise in sections 2 and 3. We describe the proof of the main theorem (Theorem 1) in three stages. First we give the proof for the simpler case of squares contained in convex polygons. Subsequently we generalize it to rectangles in convex and simple polygons. In the last section we outline the modifications necessary to extend our result to arbitrary triangles. Notice that throughout this paper we will use the notation $|B|$ for the length (respectively, area) of the line segment (respectively, plane region) $B$.

## 2 Convex Polygons

First we consider convex polygons. The case of squares is typical among objects of the same similirity type. Later we introduce the notion of ratio of a convex polygon and show how to extend the proof of the main theorem to the class of arbitrary rectangles.

### 2.1 Objects of the same similarity type

An interesting special case of Theorem 1 is when the objects to be optimized are of a given similarity type, like squares, equilateral triangles, similar triangles and similar rectangles. The case of equilateral triangles on simple polygons and squares on convex polygons has also been considered by [8]. Note that for the case of squares there is no need to refer to the ratio of the polygon. In fact we can prove the following theorem for the case of squares. (The case of the other objects is entirely analogous.)

THEOREM 2 *There is an algorithm which given a fixed integer $k$ and an arbitrary simple polygon with $n$ nodes computes in time $O(kn)$ a square $S_{alg}$ which* *is inscribed in the polygon such that*

$$\frac{|S_{alg}|}{|S_{max}|} \geq \frac{1+(4\pi/k)^2}{(1+4\pi/k)^2},$$

*where $S_{max}$ is the maximal area square inscribed in the polygon.*

Before proceeding with the details of the proof of the theorem we give an outline of the algorithm. Suppose we are given an arbitrary $n$-node convex polygon. Then the algorithm is as follows:

- 1. For the given integer $k$ consider $k$ directions, say $D_1, \ldots, D_k$, such that $D_k$ forms an angle $2\pi/k$ with the horizontal line.
- 2. For each direction $D \in \{D_1, \ldots, D_k\}$ compute the maximal area square contained in the polygon and one of whose sides is parallel to direction $D$.
- 3. Show that the square of maximal area among the above $k$ squares has area close to the area of the maximal $S_{max}$.

We now proceed to describe the details of the above construction. First we need a lemma.

LEMMA 3 *There is an algorithm such that given an arbitrary direction $D$ and an arbitrary convex polygon on $n$ nodes computes in time $O(n)$ a maximal square which is contained in the polygon and has a side parallel to direction $D$.*

PROOF of lemma 3. See proof of Lemma 7. ■

LEMMA 4 *Let $\phi$ be a given angle and let $S, S'$ be squares. Assume that $S'$ is the maximal square inscribed in $S$ and whose sides form an angle equal to $\phi$. Then*

$$\frac{|S'|}{|S|} = \frac{1+\tan^2\phi}{(1+\tan\phi)^2}.$$

PROOF of lemma 4. Elementary calculations show that

$$\frac{|S'|}{|S|} = x^2 + (1-x)^2, \tag{3}$$

where $x$ is as in Figure 1. Moreover, $\tan\phi = \frac{x}{1-x} = -1 + \frac{1}{1-x}$, which implies that $1 - x = \frac{1}{1+\tan\phi}$. Substituting these in eqaution (3) we obtain

$$\frac{|S'|}{|S|} = (1-x)^2\left(1+\left(\frac{x}{1-x}\right)^2\right) = \frac{1+\tan^2\phi}{(1+\tan\phi)^2},$$

which proves the lemma. ■

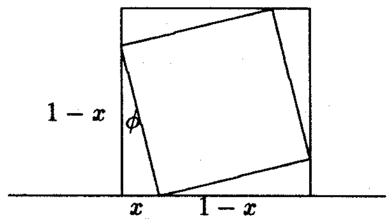


Figure 1: Square $S'$ is inscribed in square $S$.

PROOF of Theorem 2. For the given integer $k$ consider the angles $\phi_i = \frac{2\pi i}{k}$, where $i = 1, \ldots, k$. Each angle determines a direction $D_i$ forming an angle $\phi_i$ with the horizontal. For each $i = 1, \ldots, k$ apply Lemma 3 to compute the maximal area square, say $S_i$, contained in the polygon one of whose sides is parallel to direction $D_i$. Clearly the complexity of computing all these squares is $O(kn)$. Let $S_{max}$ be the maximal area square inscribed in the polygon. Clearly there exists an integer $1 \leq i \leq k$ such that the angle $\psi_i$ formed by a side of the square $S_{max}$ with direction $D_i$ is $\leq \frac{2\pi}{k}$. Let $S'_i$ be the maximal area square cintained in the square $S_{max}$ one of whose sides is parallel to direction $D_i$. Since $S_i$ has maximal area for direction $D_i$ we have that $|S_i| \geq |S'_i|$. Moreover Lemma 4 implies that

$$\frac{|S_i|}{|S_{max}|} \geq \frac{|S'_i|}{|S_{max}|} \geq \frac{1 + \tan^2 \psi_i}{(1 + \tan \psi_i)^2}.$$

Now observe that for $k$ large enough and $x \leq \frac{2\pi}{k}$ we have that $\cos^2 x > \frac{1}{2}$ which is identical with $\frac{1}{\cos x} < 2 \cos x$. Using standard trigonometric inequalities we obtain $\tan x = \frac{\sin x}{\cos x} = \frac{\sin x}{x} \cdot \frac{x}{\cos x} < \frac{1}{\cos x} \cdot \frac{x}{\cos x} < 2 \cos x \cdot \frac{x}{\cos x} = 2x$. Now let $S_{alg}$ be the maximal area square among the $k$ squares $S_1, \ldots, S_k$. Since the function $f(t) = \frac{1+t^2}{(1+t)^2}$ is monotone decreasing we obtain

$$\frac{|S_{alg}|}{|S_{max}|} \geq \frac{1 + \tan^2 \psi_i}{(1 + \tan \psi_i)^2} \geq \frac{1 + (4\pi/k)^2}{(1 + 4\pi/k)^2}.$$

This completes the proof of the theorem. ∎

## 2.2 Rectangles

We now proceed to extend the previous theorem to the case of arbitrary rectangles contained in convex polygons. For any convex set $C$ and any vector $\vec{v}$ let $w(\vec{v})$ be the width of $C$ along in the direction parallel to the vector $\vec{v}$. Also, define

| | |
|---|---|
| width of $C$ : | $w(C) = \min\{w(\vec{v}) : \vec{v} \in \mathbb{R}^2\}$ |
| length of $C$ : | $L(C) = \max\{w(\vec{v}) : \vec{v} \in \mathbb{R}^2\}$ |
| ratio of $C$ : | $\rho(C) = w(C)/L(C)$. |

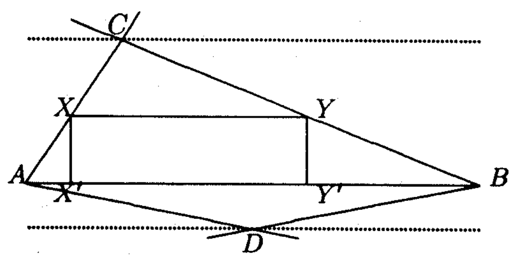


Figure 2: The quadrangle $ACBD$ inscribed in the convex set.

For the case of a rectangle $R$ with sides $a, b$ (where $a \leq b$) we will also make use of the notation $\rho'(R) = \frac{a}{b}$. (Although we will not use this, it is worth noticing that $\rho(R) \geq \sqrt{2}\rho'(R)$). It is also well-known that for convex polygons with $n$ nodes $L(C), w(C)$ can be computed in $O(n)$ time (e.g. see [13][section 4.2]).

The proof requires three lemmas. The first concerns a lower bound on the ratio of the rectangle with maximal area inscribed in the convex polygon.

**Lemma 5** *For any convex set $C$ if $R_{max}$ is a maximal area rectangle contained in $C$ then*

$$\rho'(R_{max}) \geq \frac{\rho(C)}{8}.$$

**Proof** of lemma 5. Let $L := L(C), w := w(C)$. Determine two points $A, B$ on the convex set $C$ such that $L = |AB|$ and consider the points $C, D$ where two lines parallel to $AB$ are tangent to the convex set (see Figure 2).

Consider the quadrangle $ACBD$. It is divided into two triangles $ACB, ADB$. Without loss of generality assume assume that one of them, say $ACB$, satisfies $|ACB| \geq \frac{1}{2}|ACBD|$. Let $X, Y$ be the midpoints of the line segments $AC$ and $CB$, respectively. Project $X, Y$ onto the points $X', Y'$, respectively, which lie on the line $AB$. This forms a rectangle $XYY'X'$. It is easy to see that $|X'Y'| = \frac{1}{2}|AB|$ and

$$\begin{aligned} |XX'| &= \tfrac{1}{2}(\text{height of triangle } ACB) \\ &\geq \tfrac{1}{2}\left(\tfrac{1}{2}w\right) \\ &= \tfrac{1}{4}w. \end{aligned}$$

It follows that the rectangle $R = XX'Y'Y$ satisfies

$$\rho'(R) = \frac{|XX'|}{|X'Y'|} \geq \frac{w/4}{L/2} = \rho(C)/2. \tag{4}$$

and

$$|R| \geq \frac{1}{4}|ACBD| \geq \frac{1}{8}wL. \tag{5}$$

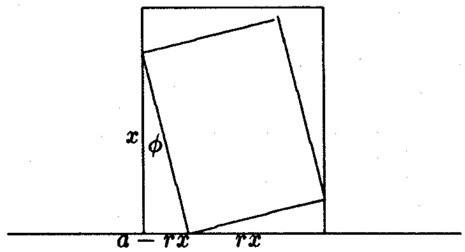


Figure 3: Rectangle $R'$ is inscribed in rectangle $R$.

Now we will use inequalities (4), (5) in order to prove the lemma. Let $R_{max}$ be a maximal area rectangle inscribed in the convex set $C$ and suppose that it has sides $a, b$ with $b \leq a$. Then its ratio is equal to $\rho'(R_{max}) = \frac{b}{a}$. Let $\rho'(R_{max}) = \frac{1}{k}\rho(C)$. The proof of the lemma would be complete if we could show that $k \leq 8$. Indeed from inequalities (4), (5) we have that

$$\begin{aligned} |R_{max}| &= ab \\ &\leq bL \\ &\leq \frac{a}{k}\frac{w}{L}L \\ &= \frac{1}{k}aw \\ &\leq \frac{1}{k}Lw \end{aligned}$$

Using this last inequality and (5) we obtain that

$$\begin{aligned} \frac{1}{8}wL &\leq |R| \\ &\leq |R_{max}| \\ &\leq \frac{1}{k}wL. \end{aligned}$$

This proves the lemma. ∎

LEMMA 6 *Let $\phi < \pi/2$ be a given angle, $0 < r < 1$ and let $R$ be a rectangle with sides $a, b$ such that $a \leq b$. Assume that $R'$ is a maximal area rectangle inscribed in $R$ with ratio $r$, i.e. $\rho'(R') = r$, and whose sides form an angle equal to $\phi$ with the sides of $R$. Then*

$$\frac{|R'|}{|R|} \geq \left(\frac{1}{1+\phi/r}\right)^2 \frac{\rho'(R)}{r}.$$

PROOF of lemma 6. Let $x$ be as in figure 3. Then we have that $|R'| = \left(\frac{x}{\cos\phi}\right)^2 r$. Using the obvious idendity $\tan\phi = \frac{a-rx}{x}$ we obtain that $x = \frac{a}{r+\tan\phi}$. Since $\sin\phi \leq \phi$ and $\cos\phi \leq 1$ we have that

$$|R'| = \left(\frac{a}{\cos\phi(r+\tan\phi)}\right)^2 r = \left(\frac{a}{r\cos\phi+\sin\phi}\right)^2 r \geq \left(\frac{a}{r+\phi}\right)^2 r.$$

It follows that

$$\frac{|R'|}{|R|} = \frac{|R'|}{ab} \geq \frac{r\rho'(R)}{(r+\phi)^2} = \left(\frac{1}{1+\phi/r}\right)^2 \frac{\rho'(R)}{r},$$

which proves the lemma. ∎

**LEMMA 7** *There is an algorithm such that given an arbitrary direction $D$, a ratio $r$ and an arbitrary convex polygon on $n$ nodes computes in time $O(n)$ a maximal area rectangle of ratio $r$ which is contained in the polygon and has a side parallel to direction $D$.*

**PROOF (OUTLINE)** of lemma 7. First of all observe that a maximal area rectangle contained in the polygon must have at least two non-adjacent vertices lying on the perimeter of the polygon. Let $D_1, D_2$ be the directions of the two diagonals of the desired rectangle. Then the required rectangle will be found by "sweeping" a line parallel to $D_i$, first for $i = 1$ and subsequently for $i = 2$. From each vertex of the polygon draw a line parallel to the diagonal. This divides the polygon into $n$ slabs. Now it is easy to see that in each slab we can determine in constant time the maximal area rectangle (with the given ratio and direction) contained in the polygon and one of whose diagonals lies within the slab. Since there are $n$ slabs the complexity of this algorithm is $O(n)$. ∎

The idea of the main algorithm is as follows. Compute the width and length of the polygon and determine its ratio. For given $k$, determine a set of $k$ directions and compute a set of $k$ ratios of candidate rectangles. For each direction and ratio compute a maximal area rectangle contained in the polygon and having the specified direction and ratio. Now output the rectangle of maximal area. More formally the main algorithm is as follows.

ALGORITHM FOR MAXIMAL AREA RECTANGLES:
INPUT: A convex polygon $C$ with $n$ nodes.
1. Compute the width $w(C)$, length $L(C)$, and ratio $\rho(C)$ of the convex polygon.
2. Compute the numbers

$$r_j = \frac{\rho(C)}{8} + \frac{j}{k} \cdot \left(1 - \frac{\rho(C)}{8}\right), \text{ for } j = 0, 1, \ldots, k. \tag{6}$$

3. For each $i, j = 1, \ldots, k$ compute the maximal area rectangle $R_{j,i}$ inscribed in the convex polygon having ratio $r_j$ and one of whose sides is in the direction forming an angle $\frac{2\pi i}{k}$ with the horizontal.
OUTPUT:

$$R_{alg} = \max\{R_{j,i} : j, i = 1, \ldots, k\}.$$

(Here max is interpreted as maximum in area.) In the sequel we prove that this algorithm satisfies the conclusion of the theorem.

PROOF of theorem 1 for the case of convex polygons. For the given integer $k$ consider the angles $\phi_i = \frac{2\pi i}{k}$, where $i = 1, \ldots, k$. Each angle determines a direction $D_i$ forming an angle $\phi_i$ with the horizontal. In view of Lemma 5 the ratio of a maximal area rectangle contained in the polygon is bounded below by $\rho(C)/8$. Divide the interval $[\rho(C)/8, 1]$ of possible ratios into $k$ equal parts and define $r_j$ as in (6). Let $R(r, \phi)$ be a maximal area rectangle with ratio $r$ inscribed in the rectangle $R_{max}$ and whose sides form an angle of size $\phi$. In view of Lemma 6 we have that

$$\frac{|R(r,\phi)|}{|R_{max}|} \geq \left(\frac{1}{1+\phi/r}\right)^2 \frac{\rho'(R_{max})}{r}.$$

By definition of $r_j$ there exists an integer $j$ such that $|\rho'(R_{max}) - r_j| < 1/k$. Since $r_j \geq \rho(C)/8$ it follows that

$$\left|\frac{\rho'(R_{max})}{r_j} - 1\right| < \frac{1}{kr_j} \leq \frac{8}{\rho(C)k}. \tag{7}$$

Now fix $j$ such that inequality (7) is satisfied. In view of Lemma 7 for each $i, j = 1, \ldots, k$ we can compute the maximal area rectangle $R_{j,i}$, $i = 1, \ldots, k$, inscribed in the convex polygon having ratio $r_j$ and one of whose sides is in direction $D_i$. In view of the choice of the angles $\phi_i = \frac{2\pi i}{k}$ there exists an integer $i$ such that the angle formed by the rectangles $R_{j,i}$ and $R_{max}$, say $\psi_i$, is $\leq \frac{2\pi}{k}$. It follows that

$$\begin{aligned}
\frac{|R_{alg}|}{|R_{max}|} &\geq \frac{|R_{j,i}|}{|R_{max}|} \\
&\geq \frac{|R(r_j,\psi_i)|}{|R_{max}|} \\
&\geq \left(\frac{1}{1+\psi_i/r_j}\right)^2 \frac{\rho'(R_{max})}{r_j} \\
&\geq \left(\frac{k\rho(C)}{k\rho(C)+16\pi}\right)^2 \frac{\rho'(R_{max})}{r_j} \\
&\geq \left(\frac{k\rho(C)}{k\rho(C)+16\pi}\right)^2 \left(1 - \frac{8}{\rho(C)k}\right).
\end{aligned}$$

This completes the proof of the theorem. ■

# 3 Simple Polygons

In the sequel we extend Theorem 1 to the case of simple polygons. In this case our algorithm is more complicated and uses ideas from the shortest path algoritm for triangulated simple polygons as developed by Guibas et al [10].

First we modify the definition of the ratio of a simple polygon $P$ as follows. The length $L(P)$ is the maximal euclidean distance between any two visible points in the polygon. By Guibas et al [10] there is an $O(n)$ algorithm for computing the visible points of a given edge from any other edge of the polygon using the shortest path algorithm. In particular we can compute the visible part

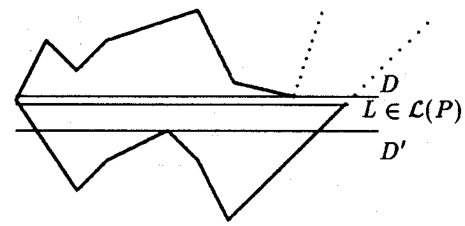


Figure 4: Defining the ratio of a simple polygon.

of an edge from any other edge. This easily implies that we can compute the length $L(P)$ of the polygon in time $O(n^2)$. Let $\mathcal{L}(P)$ be the set of line segments contained in the polygon having length equal to the length $L(P)$ of the polygon. For each segment $L \in \mathcal{L}(P)$ define the width $w_L(P)$ as follows (see Figure 4).

Consider two lines $D, D'$ parallel to $L$ such that no reflex vertex of the polygon lies below $D$ (respectively above $D'$). Then $w_L(P)$ is the distance between $D, D'$. Now let $w(P) = \min\{w_L(P) : L \in \mathcal{L}(P)\}$. As before the ratio $\rho(P)$ of the polygon is the quotient of $w(P)$ over $L(P)$. We are now in a position to prove the generalization of Lemma 5 to simple polygons.

LEMMA 8 *For any simple polygon $P$ if $R_{max}$ is a maximal area rectangle contained in $P$ then*

$$\rho'(R_{max}) \geq \frac{\rho(P)}{8}.$$

PROOF (OUTLINE) of lemma 8. We consider Figure 5 which is an extension of Figure 4. Let $A, B$ be two visible points in the polygon such that $|AB|$ is maximal. Consider the first reflex vertex from $A$ along the perimeter of the polygon which lies above (resp. below) $AB$; let this vertex be $A_1$ (resp. $A_2$). Define the vertices $B_1, B_2$ from vertex $B$ exactly as before. Let $C_1$ (resp. $C_2$) be the point of intersection of the lines $AA_1$ and $BB_1$ (resp. $AA_2$ and $BB_2$). Let $M_1, N_1, M_2, N_2$ be the midpoints of the line segments $AC_1, BC_1, BC_2, BC_2$, respectively. Let $r_1$ (resp. $r_2$) be the reflex vertex of the polygon above (resp. below) $AB$ which also has minimal distance from $AB$. By definition the vertical distance between $r_1$ and $r_2$ is $\geq w(P)$. Hence, one of the two reflex vertices must have distance at least $w(P)/2$ from the line $AB$. Without loss of generality assume it is $r_1$. We now consider two cases depending on whether or not $r_1$ lies above or below the line $M_1N_1$.

CASE 1: $r_1$ lies above the line $M_1N_1$. In this case consider the projections $M_1', N_1'$ on $AB$ of the points $M_1, N_1$, respectively. Consider the rectangle

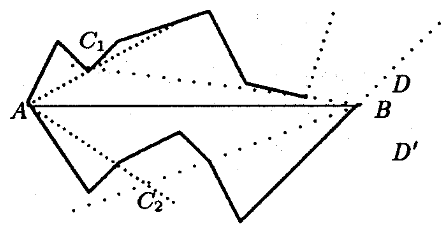


Figure 5: Proving $\rho'(R_{max}) \geq \frac{1}{8}\rho(P)$.

$M_1M_1'N_1'N_1$ and repeat the proof of Lemma 5 with this rectangle replacing the rectangle $XX'Y'Y$. It follows that in this case $\rho'(R_{max}) \geq \frac{\rho(P)}{8}$, as desired.

CASE 2: $r_1$ lies below the line $M_1N_1$. Again in this case consider the projections $M_1', N_1'$ on $AB$ of the points $M_1, N_1$, respectively. Consider the subrectangle, say $R$, of $M_1M_1'N_1'N_1$ which lies below the reflex vertex $r_1$. For this rectangle we have that

$$\rho'(R) \geq \frac{w(P)/2}{|M_1'N_1'|} \geq \frac{\rho(P)}{2}.$$

and

$$|R| \geq \frac{w(P)}{2}\frac{L(P)}{2} = \frac{w(P)L(P)}{4}.$$

Hence again we can repeat the proof of Lemma 5 and show that in this case $\rho'(R_{max}) \geq \frac{\rho(P)}{4}$, as desired. This completes the proof of the lemma. ∎

The next important lemma is the analogue of Lemma 7 to simple polygons.

LEMMA 9 *There is an algorithm such that given an arbitrary direction $D$, a ratio $r$ and an arbitrary simple polygon on $n$ nodes computes in time $O(n^2 \log n)$ a maximal area rectangle of ratio $r$ which is contained in the polygon and has a side parallel to direction $D$.*

PROOF (OUTLINE) of lemma 9. First of all observe that a maximal area rectangle contained in the polygon must have at least two non-adjacent edges incident to the perimeter of the polygon. This easily implies that a maximal area rectangle must be in one of the four configurations depicted in Figure 6. Without loss of generality we may assume that the desired rectangle has fixed ratio, say it is a square and that one of the sides of the square is parallel to a given direction, say $D$. We give the desired algorithm in each of these cases.

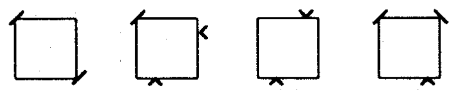


Figure 6: The four cases of the algorithm.

CASE 1. We want to compute the maximal area square contained in the polygon assuming that it has two non-adjacent vertices which are incident to edges of the polygon (see part (1) of Figure 6). Let $L$ be the direction of the diagonal of the square. Divide the polygon into $n$ slabs by drawing lines parallel to $L$ from each vertex of the polygon. Now we compute the maximal square assuming that its diagonal sweeps the polygon within the range of such a slab. As the diagonal of the square sweeps the slab the two vertices of the square lying on the other diagonal traverse a straight line (which is easily determined in constant time). It is now easy to see that in time $O(\log n)$ we can determine the maximal area square if it exists. Since there are $n$ such slabs the complexity of the algorithm in this case is $O(n \log n)$.

CASE 2. We want to compute a maximal area square contained in the polygon such that two addjacent edges are incident to reflex vertices of the polygon while the vertex non-incident to these edges is incident to an edge of the polygon (see part (2) of Figure 6). For any pair $r, r'$ of reflex vertices draw two lines $L, L'$ which are perpendicular to each other, pass through $r, r'$ respectively and one of which is parallel to the given direction $D$. Let $A$ be the point of intersection of $L$ and $L'$. Then the vertex $A'$ of the (desired maximal area) square which is non-adjacent to either $L$ or $L'$ can be found easily in time $O(\log n)$. Since there are $O(n^2)$ pairs $(r, r')$ of reflex vertices the complexity of the algorithm in this case is $O(n^2 \log n)$.

CASE 3. We want to compute a maximal area square such that two non-adjacent edges are incident to reflex vertices of the polygon (see part (3) of Figure 6). For any pair $r, r'$ of reflex vertices we draw two lines, say $L, L'$ parallel to $D$ and passing through $r$ and $r'$, respectively. Now in time $O(\log n)$ compute the right-most as well the left-most point, say $V, V'$ respectively, of the polygon which lies between the two lines $L, L'$. Let $h$ (resp. $h'$) be the distance between $L, L'$ (resp. $V, V'$). It is then clear that the desired square exists only if $h' \geq h$. Since there are $O(n^2)$ pairs $(r, r')$ of reflex vertices the complexity of the algorithm in this case is $O(n^2 \log n)$.

CASE 4. We want to compute a maximal area square such that one of its edges is incident to a reflex vertex $r$ of the polygon and the two vertices $x, y$ non-incident to this edge are incident to edges of the polygon (see part (4) of Figure 6). To do this we argue as follows. Draw a line parallel to $D$ and passing through the vertex $r$. Let $A$ (resp. $B$) be the closest to the left (resp. right) point to $r$ of the polygon lying on the extension of this line. Clearly, the points $A, B$ can be computed in time $O(\log n)$ [10]. Now the points $x$ and $y$ must be visible from the line segment $AB$. Therefore the shortest paths inside the polygon $P$ from $A$ to $x$ and from $B$ to $y$ must be inward convex chains [10]. Thus we need consider the $O(n)$ edges of the shortest path trees from $A$ and from $B$. It follows that in time $O(n)$ we can compute the maximal area square with the desired characteristics. Since there are $O(n)$ reflex vertices the complexity of the algorithm in this case is $O(n^2)$. This completes the proof of Lemma 9. ∎

PROOF of Theorem 1 for the case of simple polygons. This is similar to the proof in subsection 2.2. The algorithms in the four cases previously considered show that in time $O(n^2 \log n)$ we can compute a maximal area rectangle contained in the polygon having a fixed ratio and one of whose sides is parallel to a given direction. Now using Lemma 8 we can repeat the proof of Theorem 1 as given in subsection 2.2. Details are left to the reader. This completes the proof of Theorem 1.

# 4 Arbitrary Triangles

The results of the previous sections can be easily extended to arbitrary triangles. In the sequel we give only an outline of the necessary steps. Complete proofs will appear in the full paper. For the case of a triangle $T$ with sides $a_1, a_2, a_3$ and corresponding heights $h_1, h_2, h_3$ (where $h_i$ is perpendicular to $a_i$) we define

$$\rho'(T) = \min\left\{\frac{h_1}{a_1}, \frac{h_2}{a_2}, \frac{h_3}{a_3}\right\}.$$

For the case of triangles Lemma 8 has an even stronger lower bound.

LEMMA 10 *For any simple polygon if $T_{max}$ is a maximal area tringle contained in $P$ then*

$$\rho'(T_{max}) \geq \frac{\rho(P)}{2}. \blacksquare$$

The next important result is the analogue of Lemma 9 to arbitrary triangles. Using the techniques of Melissaratos and Souvaine [12] we can show that

LEMMA 11 *There is an algorithm such that given an arbitrary direction $D$, a similarity type of a triangle and an arbitrary simple polygon on $n$ nodes computes in time $O(n^3 \log n)$ a maximal area triangle with the given similarity type, contained in the polygon and having a side parallel to direction $D$.* ∎

The analogue of Lemma 6 to arbitrary triangles is the following.

**Lemma 12** *Let $\phi < \pi/8$ be a given angle, $0 < r < 1$ and let $T$ be a triangle. Assume that $T'$ is a maximal area triangle inscribed in $T$ with ratio $r$, i.e. $\rho'(T') = r$, and whose base forms an angle equal to $\phi$ with the base of $T$. Then*

$$\frac{|T'|}{|T|} \geq \left(\frac{\rho'(T)}{\rho'(T)+4\phi}\right)^2 \frac{r}{\rho'(T)}. \blacksquare$$

As a consequence we improve the main result of Melissaratos and Souvaine [12] by a factor of almost $n$. The main theorem for arbitrary triangles is the following.

**Theorem 13** *There is an algorithm which given an integer $k$ and an arbitrary simple polygon with $n$ nodes computes in time $O(n^3k^3 \log n)$ a triangle $T_{alg}$ which is contained in the polygon such that inequality (2) is valid, where $T_{max}$ is the maximal area triangle contained in the polygon and $\rho(P)$ is the ratio of the polygon. If in addition the polygon is convex then the time complexity is $O(nk^3)$.* $\blacksquare$

# 5 Conclusion

We have designed efficient algorithms for the problem of constructing maximal area rectangles and triangles contained in simple polygons. We used approximation techniques which are based on the fact that the ratio of the maximal rectangle (or triangle) contained in the polygon $P$ is $\Omega(\rho(P))$. The main open problem remaining would be whether our approximation technique can be used in conjunction with the technique of Bentley, Faust and Preparata [3] (see also [13][§4.1.2]) for approximating convex hulls in order to improve the $O(n^7)$ algorithm of Chang and Yap [4] for computing the largest convex polygon contained in a given polygon. Another interesting problem would be the investigation of all these questions in three dimensional space or higher.

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