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TR-220, MARCH 1993

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ON MULTI-LABEL LINEAR INTERVAL ROUTING SCHEMES
(Extended Abstract)

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March 24, 1993

Abstract

We consider linear interval routing schemes studied in [3, 5] from a graph theoretic perspective. We examine how the number of linear intervals needed to obtain shortest path routings in networks is affected by the product and join operations on graphs. This approach allows us to generalize some of the results in [3, 5] concerning the minimum number of intervals needed to achieve shortest path routings in certain special classes of networks. We also establish the precise value of the minimum number of intervals needed to achieve shortest path routings in the network considered in [10].

1980 Mathematics Subject Classification: 68R10, 68R05
CR Categories: G.2.2, F.2.2
Key Words and Phrases: Bipartite graph, Complete graph, Join of graphs, Graph operations, Hierarchy of graphs, Interval routing scheme, Linear interval routing scheme, Product of graphs.
Carleton University, School of Computer Science: SCS-TR-220

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1 Introduction

The problem of routing messages in communication networks so that the messages travel along shortest paths has been studied by a number of researchers (see, for example, the list of references at the end of this paper). When the number of nodes in the network is large, it becomes impractical to store large amounts of routing information at a node so as to achieve a shortest path route for each message. Hence methods that use compact routing tables are of considerable practical importance. One such method called **interval routing** was proposed in [11]. The basic idea of this method is the following. Each node of the network is assigned a distinct integer from the set $\{1, 2, \ldots, n\}$, where n is the number of nodes in the network. This assignment is considered as producing a circular ordering of the nodes. Each link is then labeled with a subinterval of the circular interval $[1, n]$. Because of the circular ordering of the nodes, the subinterval assigned to a link may involve a wraparound. The assignment of intervals to links must satisfy the following two conditions:

1. For any node v of the network, the subintervals associated with the links emanating from v are pairwise disjoint.

2. For any node v of the network, the union of the subintervals associated with the links emanating from v covers the interval $[1, n]$.

Given an interval labeling satisfying the above two conditions, routing is carried out as follows. Suppose a message with destination d arrives at a node v. If $d = v$, then the message has reached its destination. Otherwise, the above conditions guarantee that there is exactly one link emanating from v whose associated interval contains the destination node d. Therefore, the node v forwards the message along that link.

An interval labeling I for a network G is **valid** if the above routing method does not cause any message to cycle (i.e., every message is guaranteed to reach its destination). Reference [11] presents a method of obtaining a valid interval labeling for any network. For acyclic networks, they show that the resulting labeling is **optimal**; that is, for any source-destination pair, the labeling routes messages along a shortest path. For other networks, they showed that the scheme routes a message along a path whose length is at most twice the diameter of the network.

Subsequent to [11], a number of other researchers have addressed the problem of obtaining optimal and near-optimal labeling schemes for many classes of networks [6, 7, 8, 12, 13]. Reference [10] presents an example of a network which does not have an optimal interval labeling scheme. More precisely, that network has the property that for every assignment of intervals to the links, there is a pair of nodes for which the scheme produces a path whose length is at least 3/2 times the diameter of the network. References [1, 2] present routing schemes which provide a trade-off between the amount of routing information stored at a node and the quality of the resulting routing (i.e., the maximum ratio of the length of a route produced by the scheme to the length of a shortest path between the endpoints of the route).

An interesting special form of interval routing, called **linear interval routing**, was proposed and studied in [3, 5]. In this scheme, the nodes of the network are arranged in a *linear* order and the interval labeling each link is also restricted to be linear; that is, wraparound is not permitted. In [3, 5] it was shown that certain classes of networks such as hypercubes and k-dimensional meshes for any $k \geq 2$ can be labeled optimally using appropriate linear interval routing schemes. These references provide a complete characterization of the edge weighted networks which admit optimal linear interval routing schemes. They also studied the linear interval variant of the **multi-label** interval routing schemes introduced in [13]. In multi-label linear interval routing schemes, a link may be labeled with a collection of linear intervals satisfying the two conditions listed earlier. Reference [5] presents examples of networks with the following property: for every linear ordering of the nodes, there is at least one link requiring two intervals to achieve a shortest path routing. References [3, 4] study a routing scheme called *prefix routing*, which is a more general form of the linear interval routing scheme.

In this paper, we study multi-label linear interval routing schemes from a graph theoretic perspective. We examine the effect of graph theoretic operations, namely product and join on the number of interval labels needed for an optimal routing. In addition to being of independent interest, our results also generalize some of the results in [3, 5]. For example, optimal linear interval routing schemes with only one interval per link for hypercubes and k-dimensional grids for any $k \geq 2$ follow as corollaries of our results for graph product. We also show that complete r-partite graphs for any $r \geq 2$ admit optimal linear interval routing schemes with only one interval per link. Further, we establish the exact value of the minimum number of linear intervals needed to achieve an optimal routing in the network considered in [10].

The remainder of this paper is organized as follows. Section 2 presents some definitions and terminology. Section 3 considers graph theoretic operations and examines how the minimum number of linear intervals needed for optimal routing is affected by these operations. Section 4 establishes the minimum number of intervals needed for the network considered in [10]. Section 5 points out some problems for further research.

2 Preliminary Definitions and Terminology

Throughout this paper, a communication network is represented by a connected undirected graph. The nodes of the graph represent the sites of the network and the edges of the graph represent bidirectional communication links. We assume that the edges are *unweighted*; that is, each edge is of unit cost. Thus the length of a path between a pair of nodes is simply the number of edges in the path. All graphs considered in this paper are simple (i.e., they do not have multi-edges), connected and do not have self-loops. We assume standard graph theoretic terminology as given in [9].

Since an edge $\{u, v\}$ of the graph represents a bidirectional link, an interval routing scheme will assign one or more intervals to the edge at u and one or more intervals at v. The reader may find it convenient to think of an undirected edge $\{u, v\}$ as a pair of oppositely directed edges, namely (u, v) and (v, u). Then the interval(s) assigned to the edge $\{u, v\}$ at u can be thought of as being assigned to the directed edge (u, v) and the interval(s) assigned at v can be thought of as being assigned to the directed edge (v, u). Given a graph G and an interval labeling I for G, the number of intervals used in I is defined to be the maximum number of intervals assigned to an edge at a node. The **LIRS number** of G, denoted by LIRS(G), is the minimum integer ℓ such that there is an optimal linear interval labeling of G using at most ℓ intervals per edge. For example, it is known [5] that for all $k \geq 5$, LIRS(C_k) = 2, where C_k is the simple cycle on k nodes. Also, it follows from the results in [5, 11] that for every tree T, LIRS(T) is at most 2 and that there are trees which require two intervals. As will be seen in Section 4, for any integer ℓ, there is a graph G such that LIRS(G) is at least ℓ.

We now mention some notation for special classes of graphs which will be used in the subsequent sections of this paper.

1. For any positive integer n, K_n denotes the *complete graph* on n nodes.
2. For any positive integer n, L_n denotes the *line graph* on n nodes (i.e., L_n is a simple path on n nodes).

3. For positive integers m and n, $K_{m,n}$ denotes the *complete bipartite graph* with m and n nodes respectively on the two sides of the bipartition.

4. A generalization of the class of complete bipartite graphs is the class of *complete r-partite graphs.* For any integer $r \geq 2$, the complete r-partite graph with $n_i \geq 1$ nodes in group i of the r-partition $(1 \leq i \leq r)$ is denoted by $K_{n_1,n_2,\ldots,n_r}$. In this graph, every pair of distinct groups of nodes is connected together as a complete bipartite graph.

5. For any positive integer n, the graph Q_n denotes the n-dimensional hypercube. This graph has 2^n nodes numbered 0 through $2^n - 1$. There is an edge between nodes numbered i and j iff the binary representations of i and j differ in exactly one bit position.

6. For any positive integer n, the *n-dimensional grid* with dimensions d_1, d_2, ..., d_n, is denoted by $R_{d_1,d_2,\ldots,d_n}$. It is the graph consisting of $\prod_{i=1}^{n} d_i$ nodes, where each node x is of the form $(x_1, x_2, \ldots, x_n)$, with $1 \leq x_i \leq d_i$ for every i $1 \leq i \leq n$. There is an edge between a pair of nodes x and y if and only if there is an i, $1 \leq i \leq n$ such that $x_i = y_i + 1$ or $x_i = y_i - 1$ and $x_j = y_j$ for all $j \neq i$.

3 Bounds on LIRS Numbers for Special Classes of Graphs

In this section, we first present some bounds on the LIRS numbers for certain special classes of graphs. We then examine the effect of graph theoretic operations on the LIRS number.

3.1 A Simple Bound

THEOREM 1 *Let G be a graph with n nodes. Let δ denote the minimum node degree in G. Then* LIRS$(G) \leq \min\{\lceil n/2 \rceil, n - \delta\}$.

PROOF Label the nodes of G in a one-to-one fashion using integers 1, 2, ..., n. Assign intervals to the edges emanating from each node as follows. Consider any node v. Let w_1, w_2, ..., w_p be the labels of the nodes adjacent to v, where $p =$ degree(v). Let x_1, x_2, ..., x_r be the labels of the nodes which are *not* adjacent to v. Clearly, $r = n - p - 1$. For each edge $\{v, w_i\}$ emanating from v $(1 \leq i \leq p)$, assign the interval $[w_i, w_i]$. For each x_j,

$1 \leq j \leq r$, fix a shortest path from v to x_j. Let w_q be the node (adjacent to v) which appears immediately after v in that shortest path. Add the interval $[x_j, x_j]$ to the edge $\{v, w_q\}$.

It is easy to verify that for each node v, the intervals assigned to the edges emanating from v are pairwise disjoint and that the union of these intervals covers the interval $[1, n]$. It is also easy to verify that the resulting labeling achieves shortest path routings between every pair of vertices.

We will now bound the maximum number of intervals assigned to any edge. First notice that no edge can ever be labeled with more than $\lceil n/2 \rceil$ intervals without two intervals combining into a single larger interval. Thus $\text{LIRS}(G) \leq \lceil n/2 \rceil$. For the other part of the bound stated above, consider the node v and the edge $\{v, w_i\}$. The above method assigns at most $r + 1 = n - p = n -$ degree(v) intervals to the edge $\{v, w_i\}$ (one interval for the node w_i and at most r intervals for the nodes $x_1, x_2, \ldots, x_r$). Thus the maximum number N_v of intervals assigned to any edge emanating from v is $n -$ degree(v). Therefore,

$$\begin{aligned} \text{LIRS}(G) &= \max_v \{N_v\} \\ &\leq \max_v \{n - \text{degree}(v)\} \\ &\leq n - \delta \end{aligned}$$

where the last step follows because for every node v, degree$(v) \geq \delta$. This completes the proof. ∎

There are graphs for which the above upper bound is achievable. For example, consider the graph K_n. Here $\delta = n - 1$ and so by the above theorem, $\text{LIRS}(K_n) = 1$.

In general, the bound provided by the above theorem is weak. For example, for the line graph L_n, $\delta = 1$ and so the bound provided by the theorem is $\lceil n/2 \rceil$. It is easy to see that $\text{LIRS}(L_n) = 1$.

3.2 Complete r-Partite Graphs

It was shown in [13] that for every complete bipartite graph there is an optimal interval labeling scheme using only one circular interval per edge. We first show that for every complete bipartite graph, one *linear* interval per edge suffices to obtain optimal routings. We then generalize this result to show that for any $r \geq 2$, one linear interval per edge suffices even for complete r-partite graphs.

THEOREM **2** *For any positive integers m and n,* $\text{LIRS}(K_{m,n}) = 1$.

PROOF Let V_1 and V_2 denote the bipartition of node set of $K_{m,n}$, with $|V_1| = m$ and $|V_2| = n$. Label the nodes of V_1 in a one-to-one fashion using the integers 1, 2, ..., m and the nodes of V_2 also in a one-to-one fashion using the integers $m+1$, $m+2$, ..., $m+n$. Since the numbering of the nodes is one-to-one, in the following discussion we will not distinguish between the name of a node and the integer labeling that node.

The edges emanating from the nodes in V_1 are labeled as follows. For $1 \leq i \leq m$, consider the node i in V_1. For each j, $1 \leq j \leq n$, $K_{m,n}$ contains the edge $\{i, m+j\}$. Label the edge $\{i, m+1\}$ with the interval $[1, m+1]$. For $2 \leq j \leq n$, label the edge $\{i, m+j\}$ with the interval $[m+j, m+j]$.

The edges emanating from the nodes in V_2 are labeled as follows. For $1 \leq j \leq n$, Consider the node $m+j$ in V_2. For $1 \leq i \leq m-1$, label the edge $\{m+j, i\}$ with the interval $[i, i]$. Finally, label the edge $\{m+j, m\}$ with the interval $[m, m+n]$.

Note that the above procedure assigns exactly one linear interval per edge. It is easy to verify that for each node v, the intervals assigned to the edges emanating from v are pairwise disjoint and that the union of these intervals covers the interval $[1, m+n]$.

We now show that the resulting labeling achieves shortest path routings between every pair of vertices. To see this, first consider a pair of nodes on opposite sides of the bipartition. Let i be the node from V_1 and $m+j$ be the node from V_2. Suppose a message is sent from i to $m+j$. If $j = 1$, then the interval $[1, m+1]$ assigned to the edge $\{i, m+1\}$ contains $m+1$ and so the path from i to $m+1$ consists of only one edge. If $2 \leq j \leq n$, then the interval $[m+j, m+j]$ assigned to the edge $\{i, m+j\}$ trivially contains the node $m+j$ and so the path consists of only one edge. A similar proof applies when the message is sent from $m+j$ to i.

Now consider a pair of nodes on the same side of the bipartition. Suppose the two nodes i and j are from V_1 and assume that a message is being sent from i to j. Note that the interval $[1, m+1]$ assigned to the edge $\{i, m+1\}$ contains j. Therefore, the message is first sent from i to $m+1$. If $j < m$, then the interval $[j, j]$ assigned to the edge $\{m+1, j\}$ trivially contains j; if $j = m$, then the interval $[m, m+n]$ assigned to the edge $\{m+1, m\}$ contains j. In either case, the message reaches j using just one edge from $m+1$. Therefore, the length of the route for the message from i to j is 2, which is optimal. A similar proof applies when we consider a pair of nodes from V_2.

This completes the proof of Theorem 2. ∎

We now present a generalization of the above theorem.

THEOREM 3 *For any integer $r \geq 2$ and positive integers $n_1, n_2, \ldots, n_r$,* LIRS $(K_{n_1,n_2,\ldots,n_r}) = 1$.

PROOF (OUTLINE) Let G be the given complete r-partite graph. Let V_1, V_2, …, V_r denote the node sets of G, where $|V_i| = n_i$, $1 \leq i \leq r$. For convenience, let $n_0 = 0$ and for $1 \leq i \leq r$, define $t_i = \sum_{j=0}^{i-1} n_j$. Number the nodes in V_i using the integers $t_i + 1$ through t_{i+1}, $1 \leq i \leq r$. To assign an interval to each edge, first consider the edges from the nodes in V_1 to the nodes in the other $r-1$ node sets. For each such edge $\{j, p\}$, if $p = n_1 + 1$ (note that the node $n_1 + 1$ is in V_2), assign the interval $[1, n_1 + 1]$; otherwise, assign the interval $[p, p]$. Now consider each V_i, for $2 \leq i \leq r$. Note that the nodes in V_i are numbered from $t_i + 1$ through t_{i+1}. For each edge $\{j, p\}$, if $p = t_i$ (note that t_i is the largest numbered node in V_{i-1}), assign the interval $[t_i, t_{i+1}]$; otherwise, assign the interval $[p, p]$. Using an argument similar that of Theorem 2, it is possible to show that the resulting labeling is optimal. ■

3.3 Product of Graphs

The following definition is from [9]. Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, the **product** $G_1 \times G_2$, has the node set V' and edge set E' where

$$\begin{aligned} V' &= V_1 \times V_2 \quad \text{and} \\ E' &= \{\{u, v\} : u = (u_1, u_2), v = (v_1, v_2), \text{ and} \\ &\qquad [(u_1 = v_1 \text{ and } \{u_2, v_2\} \in E_2) \text{ or } (u_2 = v_2 \text{ and } \{u_2, v_2\} \in E_2)]\}. \end{aligned}$$

In understanding the proofs of the theorems in this section, the reader will find it helpful to think of $G_1 \times G_2$ as being obtained in the following manner. Let $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ and let $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$, where $n_1 = |V_1|$ and $n_2 = |V_2|$. Consider the node set $V_1 \times V_2$. For each i, $1 \leq i \leq n_1$, construct an isomorphic copy of G_2 on the set of nodes $\{(x_i, y_1), (x_i, y_2), \ldots, (x_i, y_{n_2})\}$ (using the obvious bijection between this set of nodes and V_2). Further, for each j, $1 \leq j \leq n_2$, construct an isomorphic copy of G_1 on the set of nodes $\{(x_1, y_j), (x_2, y_j), \ldots, (x_{n_1}, y_j)\}$. The resulting graph is $G_1 \times G_2$.

We start by considering the product of a graph G with the line graph L_r. For expository purposes, we first present the result for the product of a graph G with L_2.

THEOREM 4 *For any graph G,* LIRS $(L_2 \times G) \leq$ LIRS (G).

PROOF (OUTLINE) Given a linear interval scheme for G, we will show how to construct a linear interval scheme for $L_2 \times G$ using at most LIRS(G) intervals per edge.

Let n be the number of nodes in G. Visualize $L_2 \times G$ as being formed by taking two copies of G and adding an edge between each of the n pairs of corresponding nodes in the two copies of G. Number the nodes in the first copy of G with integers 1 through n using the given linear ordering for the nodes of G. Number the nodes in the second copy of G with integers $n+1$ through $2n$, again using the given linear ordering for the nodes of G. After this numbering step, note that corresponding pairs of nodes have numbers p and $n+p$, $1 \leq p \leq n$.

For each edge $\{i, j\}$ in $L_2 \times G$, carry out the assignment of intervals as follows.

1. If both i and j are from the first copy of G (i.e., $i \leq n$ and $j \leq n$), then assign all the intervals assigned to the edge $\{i, j\}$ in G.

2. If both i and j are both from the second copy of G, (i.e., $n+1 \leq i \leq 2n$ and $n+1 \leq j \leq 2n$), then consider the intervals assigned to the edge $\{i-n, j-n\}$ in G. For each such interval $[x, y]$, assign the interval $[x+n, y+n]$.

3. If i is from the first copy of G and j is from the second copy of G, then assign the interval $[n+1, 2n]$.

4. If i is from the second copy of G and j is from the first copy of G, assign the interval $[1, n]$.

Clearly, the above scheme uses at most LIRS(G) intervals per edge of $L_2 \times G$. It is easy to verify that the scheme provides shortest path routes for each pair of nodes. ∎

The following is an interesting corollary of this theorem.

COROLLARY 5 *For any integer $n \geq 2$,* LIRS$(Q_n) = 1$.

PROOF It is well known that $Q_2 = L_2$ and $Q_i = Q_{i-1} \times L_2$ for $i \geq 3$. Since LIRS$(L_2) = 1$, the corollary follows immediately. ∎

We now generalize Theorem 4 to any line graph L_r as follows.

THEOREM 6 *For any integer $r \geq 2$ and graph G,* LIRS$(L_r \times G) \leq$ LIRS(G).

PROOF (OUTLINE) Let n be the number of nodes in G. Visualize $L_r \times G$ as being formed by taking r copies of G and creating a copy of L_r with each of the n sets of the corresponding nodes in the r copies of G.

For $1 \leq p \leq r$, number the nodes in the p^{th} copy of G with integers $(p-1)n+1$ through pn using the given linear ordering for the nodes of G.

Assign intervals to the edges of $L_r \times G$ as follows. Consider any edge $\{i, j\}$ in $L_r \times G$.

1. Suppose i and j are both from the same copy, say copy t, of G. Let $i_1 = i - (t-1)n$ and $j_1 = j - (t-1)n$. Examine the collection of intervals assigned to the edge $\{i_1, j_1\}$ in G. For each such interval $[x, y]$, assign the interval $[x + (t-1)n, y + (t-1)n]$ to the edge $\{i, j\}$.

2. Suppose i and j are both from different copies, say copies t_i and t_j of G. Since we are considering the product of G with L_r, it must be the case that either $t_j = t_i + 1$ or $t_i = t_j + 1$. If $t_j = t_i + 1$, then assign the interval $[nt_i + 1, nr]$ to the edge $\{i, j\}$; if $t_i = t_j + 1$, then assign the interval $[1, nt_j]$ to the edge $\{i, j\}$.

It is not difficult to verify that the above is an optimal labeling which uses at most LIRS(G) intervals per edge. ∎

The above theorem has the following corollary.

COROLLARY 7 *For every n-dimensional grid* $R_{d_1, d_2, \ldots, d_n}$,

$$\text{LIRS}(R_{d_1, d_2, \ldots, d_n}) = 1.$$

PROOF It is well known that the n-dimensional grid can be generated by the product of an appropriate sequence of line graphs. ∎

By extending the ideas used in the proofs of Theorems 4 and 6, we can prove a general result for the LIRS number of the product $G_1 \times G_2$ of two arbitrary graphs G_1 and G_2. We begin with a lemma which points out how the lengths of shortest paths in $G_1 \times G_2$ are related to the lengths of the shortest paths in G_1 and G_2. The proof of the lemma is straightforward.

LEMMA 8 *Let* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be two graphs, where* $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ *and* $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. *Let* $d_1(x_i, x_j)$ *denote the length of a shortest path between* x_i *and* x_j *in* G_1 *and let* $d_2(y_i, y_j)$ *denote the length of a shortest path between* y_i *and* y_j *in* G_2. *Then for any pair of nodes* $u = < x_{i_1}, y_{j_1} >$ *and* $v = < x_{i_2}, y_{j_2} >$ *in* $G_1 \times G_2$, *the length* $d(u, v)$ *of a shortest path between* u *and* v *in* $G_1 \times G_2$ *is given by* $d(u, v) = d_1(x_{i_1}, x_{i_2}) + d_2(y_{j_1}, y_{j_2})$.
∎

THEOREM 9 *For graphs G_1 and G_2,*

$$\text{LIRS}(G_1 \times G_2) \leq 1 + \max\{\text{LIRS}(G_1), \text{LIRS}(G_2)\}.$$

PROOF (OUTLINE) Given linear interval schemes for G_1 and G_2, we show how to construct a linear interval scheme for $G_1 \times G_2$ using at most $1 + \max\{\text{LIRS}(G_1), \text{LIRS}(G_2)\}$ intervals per edge.

Let f_1 and f_2 be functions specifying the linear orderings of the nodes of G_1 and G_2. Visualize $G_1 \times G_2$ as being obtained by having n_1 copies of G_2 and joining the corresponding vertices in these copies to create a copy of G_1. The copies are considered to be ordered according to f_1. In each copy of G_2, nodes are numbered using the ordering given by f_2 except that the nodes in copy i are numbered using the integers $(i-1)n_2 + 1$ through in_2, $1 \leq i \leq n_1$.

We can think of the above linear ordering of the nodes of $G_1 \times G_2$ as assigning an ordered pair $< p, q >$ of integers to each node v of $G_1 \times G_2$, where p (which satisfies the condition $1 \leq p \leq n_1$) is the number of the copy of G_2 that v belongs to and q (which satisfies the condition $1 \leq q \leq n_2$) is the position of v among the nodes in that copy. The following two-part observation provides an easy way to translate an ordered pair into a position in the linear order of $G_1 \times G_2$ and vice versa.

Observation:

(a) Suppose a node v in $G_1 \times G_2$ is assigned the ordered pair $< p, q >$, where $1 \leq p \leq n_1$ and $1 \leq q \leq n_2$. The position of v in the linear order of $G_1 \times G_2$ is $(p-1)n_2 + q$.

(b) Suppose t is the position of a node v in the linear ordering for $G_1 \times G_2$. The ordered pair for v is $< p, q >$, where

$$p = \lceil t/n_2 \rceil \text{ and } q = t - (p-1)n_2.$$

Let us now consider how intervals can be assigned to the edges of $G_1 \times G_2$. Consider the edge $\{a, b\}$ where a and b are the numbers of the two nodes in the linear ordering for $G_1 \times G_2$. Let $< p_a, q_a >$ and $< p_b, q_b >$ denote the ordered pairs corresponding to a and b respectively. (Given a and b, the above observation can be used to find the values of p_a, p_b, q_a and q_b). There are two cases.

Case 1: $p_a = p_b$.

In this case, nodes a and b are in the same copy, namely copy p_a, of G_2. We find the collection of intervals assigned to the edge $\{q_a, q_b\}$ in the linear interval scheme for G_2. For each interval $[x, y]$ in the collection, we add the interval $[(p_a - 1)n_2 + x, (p_a - 1)n_2 + y]$ to the edge $\{a, b\}$. Thus the number of intervals assigned to the edge $\{a, b\}$ is equal to the number of intervals assigned to the edge $\{q_a, q_b\}$ in the given scheme for G_2.

Case 2: $p_a \neq p_b$.

Here, nodes a and b are in different copies of G_2. (By the definition of product, $q_a = q_b$). For this case, we assign intervals to the edge $\{a, b\}$ by considering the intervals assigned to the edge $\{p_a, p_b\}$ in the given linear interval scheme for G_1.

Suppose $[x, y]$ is an interval assigned to the edge $\{p_a, p_b\}$ in the given scheme for G_1. If the integer p_a does *not* appear in the interval $[x, y]$, we add the interval $[(x - 1)n_2 + 1, yn_2]$ to the edge $\{a, b\}$ of $G_1 \times G_2$. If the integer p_a appears in the interval $[x, y]$, there are three * possibilities:

(a) If $x = p_a$, then we add the interval $[p_a n_2 + 1, yn_2]$ to the edge $\{a, b\}$ of $G_1 \times G_2$.

(b) If $y = p_a$, then we add the interval $[(x - 1)n_2 + 1, (y - 1)n_2]$ to the edge $\{a, b\}$ of $G_1 \times G_2$.

(c) If $x < p_a < y$, then we add two intervals, namely $[(x-1)n_2+1, (p_a-1)n_2]$ and $[p_a n_2 + 1, yn_2]$ to the edge $\{a, b\}$ of $G_1 \times G_2$.

We note that in the given linear interval scheme for G_1, among all the intervals assigned to the edges emanating from the node labeled p_a, the integer p_a appears in *exactly one* of the intervals. If p_a appears in the middle of that interval, then step (c) given above applies and so the number of intervals assigned to the edge $\{a, b\}$ in $G_1 \times G_2$ is one more than the number of intervals assigned to the edge $\{p_a, p_b\}$ in G_1. Thus $\text{LIRS}(G_1 \times G_2) \leq 1 + \max\{\text{LIRS}(G_1), \text{LIRS}(G_2)\}$.

The above scheme for $G_1 \times G_2$ routes a message from a node a to b in the following manner. From a, it first routes the message to the copy of G_2 which contains the node b. (This route uses the edges of G_1). Then, the message is routed to the node b in that copy. (This route uses the edges of G_2). Using Lemma 8, it is easy to verify that the routing is optimal. ∎

*If $x = y = p_a$, then the interval $[x, y]$ assigned to the edge $\{p_a, p_b\}$ in the given linear interval scheme for G_1 is of no use in routing. We assume that the given scheme for G_1 does not contain such redundant intervals.

3.4 Join of Graphs

The following definition is also from [9]. Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, the **join** of G_1 and G_2, denoted by $G_1 + G_2$, has the node set V' and edge set E' where

$$\begin{aligned} V' &= V_1 \cup V_2 \quad \text{and} \\ E' &= E_1 \cup E_2 \cup \{\{v, w\} : v \in V_1,\ w \in V_2\}. \end{aligned}$$

Informally, $G_1 + G_2$ is obtained by taking an isomorphic copy of G_1 along with an isomorphic copy of G_2, and adding an edge between every node of G_1 and every node of G_2. We note that, in general, the structure of shortest paths in $G_1 + G_2$ is very different from those of G_1 and G_2. This is because in $G_1 + G_2$, there is a path of length at most 2 between every pair of nodes. Therefore, the LIRS number of $G_1 + G_2$ may not be related to the LIRS numbers of G_1 and G_2. However, it is possible to bound the LIRS number of $G_1 + G_2$ using some parameters of G_1 and G_2 as shown by the following theorem.

THEOREM **10** *Suppose G_1 and G_2 are graphs with n_1 and n_2 nodes and minimum degrees δ_1 and δ_2 respectively. Then*

$$\text{LIRS}(G_1 + G_2) \le 1 + \max\{\lceil (n_1 - \delta_1 - 1)/n_2 \rceil,\ \lceil (n_2 - \delta_2 - 1)/n_1 \rceil\}.$$

PROOF (OUTLINE) The proof uses an approach similar to that of Theorem 1. For a node v in G_i, let $\text{degree}_i(v)$ denote the degree of v in G_i, $i = 1, 2$.

Label the nodes of G_1 in a one-to-one fashion using integers 1, 2, ..., n_1, and the nodes of G_2 also in a one-to-one fashion using integers $n_1 + 1$, $n_1 + 2$, ..., $n_1 + n_2$. Assign intervals to the edges emanating from each node of $G_1 + G_2$ as follows. Consider a node labeled v in G_1. Let w_1, w_2, ..., w_p be the labels of the nodes adjacent to v in G_1, where $p = \text{degree}_1(v)$. Let x_1, x_2, ..., x_r be the labels of the nodes which are *not* adjacent to v in G_1. Clearly, $r = n - p - 1$. For each edge $\{v, w_i\}$, assign the interval $[w_i, w_i]$ $(1 \le i \le p)$. Note that $G_1 + G_2$ contains the edge $\{v, n_1 + j\}$ for $1 \le j \le n_2$. For each edge $\{v, n_1 + j\}$, assign the interval $[n_1 + j, n_1 + j]$, $1 \le j \le n_2$. For each x_i, $1 \le i \le r$, distribute the r intervals $[x_1, x_1]$, $[x_2, x_2]$, ..., $[x_r, x_r]$ among the n_2 edges $\{v, n_1 + 1\}$, $\{v, n_1 + 2\}$, ..., $\{v, n_1 + n_2\}$ such that each of these n_2 edges receives at most $\lceil r/n_2 \rceil$ intervals. (These intervals allow us to set up a path of length 2 between v and x_i, for $1 \le i \le r$.) Thus the number of intervals assigned to an edge from a node v is at most $1 + \lceil r/n_2 \rceil$. Since $r = n - \text{degree}_1(v) - 1$ and $\text{degree}_1(v) \ge \delta_1$, it follows that the number of intervals assigned to an edge from any node of G_1 is at most $1 + \lceil (n_1 - \delta_1 - 1)/n_2 \rceil$.

In a similar manner, we can show that for any edge emanating from a node in G_2, the number of intervals assigned is at most $1+\lceil (n_2-\delta_2-1)/n_1 \rceil$. The bound on the LIRS number of $G_1 + G_2$ follows.

It is straightforward to verify that the routing provided by the above scheme is optimal. ∎

We remark that since $G_1 + G_2$ is a graph with $n_1 + n_2$ nodes, the above bound can be refined slightly using Theorem 1. In other words, an upper bound on LIRS $(G_1 + G_2)$ is the minimum of $\lceil (n_1 + n_2)/2 \rceil$ and the quantity specified in the statement of the above theorem.

4 A Hierarchy

In this section we construct a hierarchy of graphs whose linear interval labeling scheme requires arbitrarily large number of intervals.

The globe graphs G_s^n are constructed by joining the endpoints of s line segment graphs each consisting of n nodes. More formally we have the following definition. Take s line segment graphs $A_1, A_2, \ldots, A_s$ each consisting of n nodes. Let the endpoints of A_i be a_i and a_i', respectively, and let $e_i = \{a_i, b_i\}, e_i' = \{a_i', b_i'\}$ be the edges of A_i adjacent to a_i and a_i', respectively. The globe graph G_s^n has the vertex set

$$V = \bigcup_{i=1}^{s} (V_i \setminus \{a_i, a_i'\}) \cup \{a, a'\},$$

where a, a' are two new nodes and the edge set

$$E = \bigcup_{i=1}^{s} (E_i \setminus \{e_i, e_i'\}) \cup \{\{a, b_i\}, \{a', b_i'\}\}$$

(see Figure 1). Clearly, G_s^n has $N := |V| = s(n-2) + 2$ nodes and $|E| = s(n-1)$ edges. The globe graphs were first considered by Ružička [10] who proved that LIRS $(G_s^n) \geq 3$, for $n \geq 3, s \geq 14$. Here we give the precise value of LIRS (G_s^n). The main theorem of this section is the following.

THEOREM 11 *Assuming that* $n > 4s^2$ *and* $s \geq 2$, LIRS $(G_s^n) = \lfloor s/2 \rfloor + 1$.

PROOF (OUTLINE) Let the nodes of the segments A_i be $a_{i,2}, \ldots, a_{i,n-1}$, where $i = 1, \ldots, s$. First we prove LIRS $(G_s^n) \leq \lfloor s/2 \rfloor + 1$. Label the nodes

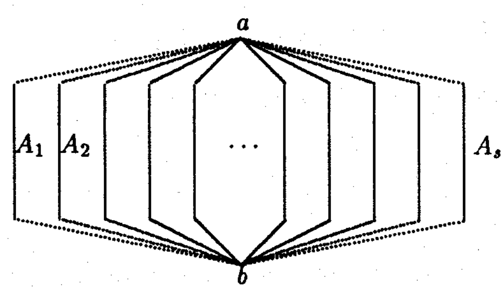

Figure 1: The globe graph G^n_s.

of the arcs $A_1, \ldots, A_s$ as follows. Label A_1 from top to bottom, next A_2 from bottom to top, next A_3 from top to bottom, etc.

Next we prove LIRS $(G^n_s) \geq \lfloor s/2 \rfloor + 1$. Assume on the contrary LIRS $(G^n_s) \leq \lfloor s/2 \rfloor$, i.e. there is node numbering of the globe graph G^n_s such that the interval labeling of each edge has at most $\lfloor s/2 \rfloor$ intervals.

Now consider the edges e'_i, e''_i adjacent to the nodes a'_i, $i = 1, \ldots, s$, respectively; let e'_i be the edge below a'_i and e''_i the edge above it. The shortest path to the nodes in the arc A_i is through edge e'_i while the shortest path to the nodes in $[1, N] \setminus A_i$ is through edge e''_i. It follows that each A_i is the union of $\leq \lfloor s/2 \rfloor$ pairwise disjoint intervals.

Arrange these $\lfloor s/2 \rfloor$ arcs horizontally one above the other. It is easy to see that there exists a vertical straight line such that for each $i > 2$ the vertical line traverses an interval of A_i at a point which is at a distance ≥ 2 from each of its endpoints. To see this we argue as follows. Call a vertical line bad for A_i, if for every interval of A_i all points in the interval are either to the left of $l + 1$ or to the right of $l - 1$ inclusive. It is then clear that at most $\lfloor s/2 \rfloor$ lines are bad for A_i. Therefore at most $s\lfloor s/2 \rfloor$ lines are bad for all the A_is. Therefore assuming $n > 4s\lfloor s/2 \rfloor$ there exists a vertical line which is not bad for any of the arcs. Without loss of generality assume the vertical line traverses a point which is at a location $l < n/2$ and let the corresponding intervals be $I_1, \ldots, I_s$.

Now we can derive the desired contradiction. Consider a vertex, say v, on the arc A_1 which is at a distance $l + n/2$ from the node a. The intervals labeling the two edges adjacent to node v divide the nodes of the graph into two parts, the upper part, say U, and the lower part, say L. By assumption there exist pairwise disjoint intervals such that

$$U = U_1 \cup \cdots \cup U_{\lfloor s/2 \rfloor}, L = L_1 \cup \cdots \cup L_{\lfloor s/2 \rfloor}.$$

and $[1, N] = U \cup L$. However it is clear that

$$\bigcup_{i=1}^{s} I_i \subseteq U_1 \cup \cdots \cup U_{\lfloor s/2 \rfloor} \cup L_1 \cup \cdots \cup L_{\lfloor s/2 \rfloor}.$$

But this is a contradiction since none of the intervals I_i can be contained in any of the intervals U_j and L_j. This completes the proof of the theorem. ■

5 Problems for Future Research

There are several directions for further research on interval routing schemes. For example, given a graph G and an integer k, the complexity of deciding whether LIRS(G) is at most k is open. A characterization for $k = 1$ for weighted graphs has been obtained in [3, 5], but to the best of our knowledge, the problem is open for graphs with unit cost links, even when we allow the nodes of G to be circularly ordered and allow intervals to have wraparound. A related question is that of obtaining bounds on the LIRS numbers of other special classes of graphs. References [6, 7, 8] address this question for planar graphs, graphs of small genus and graphs with constant separators. Another open problem is the following. Reference [11] presents an interval routing scheme which guarantees that every message route is of length at most twice the diameter of the given graph, and reference [10] presents an example of a graph for which every interval routing scheme has at least one pair of nodes requiring a message route of length at least 1.5 times the diameter of the graph. It will be interesting to close the gap between these two bounds.

Acknowledgement

S. S. Ravi would like to express his sincere thanks to the faculty and staff of the School of Computer Science at Carleton University and to David and Nancy Fraser of Ottawa for their hospitality during his visits to Ottawa.

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