CONTAINMENT OF ELEMENTARY

GEOMETRIC OBJECTS

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1. GEOMETRIC CONTAINMENT AND PARETO DOMINANCE

Consider a family **F** of closed curves in the plane; e.g. polygons. Given A, B, ∈ **F**, it might be important to determine whether A fits into B ; that is, whether there exists a rotation and a translation which will move A in the plane so that A and its interior lies within B (A is contained in B). The problem of deciding if A fits into B, for all A,B ∈ **F** is herein called the containment problem for **F**.

For some families of closed curves, it is easy to find a parametric representation of each object in the family in such a way that containment can be solved by testing for vector (or, Pareto) dominance on the representations.

Example: Let **F** be the family of squares (equilateral triangles, circles), and let S_i, $S_j \in$ **F**. Then S_i fits into S_j if the area of S_i is not greater than the area of S_j . That is, vector dominance on one paremeter r solves the containment problem for these families.

Example: Let **F** be the family of ellipses; and, given $E_k \in$ **F**, let a_k and b_k denote the length of the major and minor axes of E_k, respectively. Then, for any E_i, $E_j \in$ **F**, E_i fits in E_j if and only if $(a_i, b_i) \leq (a_j, b_j)$. That is, vector dominance on two parameters solves the containment problem for ellipses.

In view of these facts and of the existence of efficient algorithms with complexity $O(n(\log n)^{d-1})$, for solving vector dominance in $\mathbb{R}^d$, the following question naturally arises:
For what families of geometrical objects can the containment problem

be reduced to a Pareto dominance problem?

Consider the family $\mathbf{R}$ of all rectangles. In this case, the above question becomes: is it possible to characterize each rectangle R_k by a vector of two parameters $P_k = \langle P^1(R_k), P^2(R_k)\rangle$ in such a way that rectangle R_k fits into rectangle R_j iff $P_k \leq P_j$?

Obvious candidates for parameters are: length, width, area, diagonal, etc. The educated guess that any two of them suffice to solve the problem is, unfortunately, an incorrect one as illustrated by the following (counter-)examples.

Example: <Width, Length>, where length is defined as the largest of the two sides. Let L_k and W_k represent the length and width of R_k, respectively, and let $R_i, R_j \in \mathbf{R}$. If $\langle W_i, L_i\rangle \leq \langle W_j, L_j\rangle$ then R_i fits into R_j. However, the converse is not necessarily true: let $W_i = 0$, and $W_j \neq 0$ and $L_i = (W_j^2 + L_j^2)^{1/2}$; then $L_j < L_i$ but R_i fits into R_j.

Example: <Area, Diagonal>. Let A_k and D_k represent the area and the diagonal of R_k, respectively; and let $R_i, R_j \in R$. If R_i fits into R_j, then $\langle A_i, D_i\rangle \leq \langle A_j, D_j\rangle$. However, regardless of A_j, no rectangle $R_i \neq R_j$ with $D_i = D_j$ and $A_i \neq 0$ fits into R_j. Thus, by choosing $0 < A_i < A_j$, a contradiction arises.

The main result of this paper is to show that these negative results are not accidental (e.g., due to poor choice of parameters), but intrinsic to the problem. In fact, it is proved that the containment problem for rectangles <u>cannot be solved</u> with two parameters by vector dominance <u>regardless of the choice of parameters</u>.

The proof is obtained by using the well known result that any set of mutually disjoint open sets in the plane has cardinality at most $\aleph_0$, and by showing that solvability of this problem by vector dominance implies the existence of such a set having cardinality at least $\aleph_1$.

In Section 3, it will be shown how to apply the proof technique to obtain a parametric representation of rectangles such that containment of rectangles into squares can be reduced to Pareto dominance.

2. The Main Result

Associate with each rectangle $R_k \in R$ (with width W_k and length L_k) the point (W_k, L_k) in $\mathbb{R}^2$. Using this mapping, for each R_k there exists a region S_k of $\mathbb{R}^2$ consisting of all points in $\mathbb{R}^2$ whose associated rectangle fits in R_k (see Figure 1). Note that we consider only points in the first quadrant with ordinate (length) $\geq$ abscissa (width).

A rectangle R_i fits <u>tightly</u> into R_k if any increase in W_i or L_i would make it not fit into R_k. More formally, R_i fits tightly into R_k if and only if

a) $R_i \neq R_k$

b) R_i fits into R_k

c) for any rectangle R_j such that $W_i \leq W_j$ and $L_i \leq L_j$, if R_j fits into R_k then $R_i = R_j$.

Remark 1: If R_j fits tightly into R_k, then in any "immersion" of R_j into R_k, the intersection of the boundaries of both rectangles consists of the four vertices of R_j.

Remark 2: If R_i fits into R_j and R_j fits tightly into R_k, $(R_i \neq R_j \neq R_k)$ then R_i does not fit tightly into R_k; i.e. R_i

fits "loosely" into R_k .

This is equivalent to saying that (W_i, L_i) lies in the interior of S_k. (See Figure 2.)

Remark 3: For any rectangle R_i there is an infinite number of rectangles R_j that fit tightly into R_i.

Theorem 1: The containment problem for rectangles cannot be solved by vector dominance in the plane.

Proof: Assume that the containment problem for rectangles can be solved by vector dominance on two parameters. Let P^1 and P^2 be such parameters, and $P(R_k) = (P^1(R_k), P^2(R_k))$ be the point in associated to R_k by using these parameters. Then for any $R_i, R_j \in$ **R**, R_i fits in R_j if and only if $P(R_i) \leq P(R_j)$; i.e. $P^1(R_i) \leq P^1(R_j)$ and $P^2(R_i) \leq P^2(R_j)$.

<u>Claim 1</u>: If R_i and R_j fit tightly into R_k , then by Remark 2, neither $P(R_i)<P(R_j)$ nor $P(R_i)>P(R_j)$.

<u>Proof of Claim 1:</u> Let R_i and R_j fit tightly into R_k and assume $P(R_i) < P(R_j)$. It follows that R_i fits into R_j. But, by Remark 2, would follow that R_i does not fit tightly into R_k yielding a contradiction. [ ]

<u>Claim 2</u>: Given R_k , there exists an $\bar{R}_k$ that fits tightly in R_k and $P^1(\bar{R}_k) < P^1(R_k)$ and $P^2(\bar{R}_k) < P^2(R_k)$.

<u>Proof of Claim 2:</u> By Remark 3 for any R_k with non-zero area there exist at least three distinct rectangles R_1, R_2 and R_3 which fit tightly into R_k. By Claim 1 these three rectangles must be mapped to three non comparable elements $P(R_1)$, $P(R_2)$ and $P(R_3)$. Let $(X_i, Y_i) = (P^1(R_i), P^2(R_i))$, i = 1,2,3. If the claim is not true, then

each vector (X_i,Y_i) ties $(P^1(R_k),\ P^2(R_k))$ in at least one coordinate. Therefore, for at least two of the three vectors, say (X_1,Y_1) and (X_2,Y_2), must tie in one coordinate; thus either $P(R_1) > P(R_2)$ or $P(R_2) > P(R_1)$. This would contradict Claim 1 since both R_1 and R_2 fit tightly into R_k. [ ]

Associate to R_k the open set $O(R_k) = \{(x,y) \in \mathbb{R}^2 : P^1(\bar{R}_k) < x < P^1(R_k),\ P^2(\bar{R}_k) < Y < P^2(R_k)\}$. (See Figure 3.)

<u>Claim 3</u>: For any R_k, there exists no rectangle R_j such that $P(R_j) \in O(R_k)$.

<u>Proof of Claim 3:</u> Notice that if such a rectangle R_j exists, then $P(\bar{R}_k) < P(R_j) < P(R_k)$, i.e. $\bar{R}_k$ fits into R_j which in turn fits into R_k. This would contradict the fact that $\bar{R}_k$ fits tightly into R_k.[ ]

From these claims it follows that to any rectangle R_k we can associate an open set $O(R_k)$ such that no rectangle R_j is mapped by P_1 and P^2 into $O(R_k)$. In particular it follows that

<u>Claim 4</u> If R_i fits into R_j then $O(R_i) \cap O(R_j) = \emptyset$.

Consider now the set of rectangles $L = \{R_i : 2\ W_i = L_i\}$ whose cardinality is obviously $\aleph_1$. Let $O(L) = \{O(R_i) : R_i \in L\}$; the cardinality of $O(L)$ is $\aleph_1$, and by Claim 4 its elements are mutually disjoint. But any set of mutually disjoint open sets in the plane has at most $\aleph_0$ elements. This contradication concludes the proof of the Theorem. [ ]

3. A Containment Problem That is Reducible

In view of the negative result of the previous section, it is natural to ask under what restrictions the rectangle containment problem can be reduced to Pareto dominance.

Consider the subfamily $\mathbf{Q} \subseteq \mathbf{R}$ of all squares. It is trivial to determine containment of squares into rectangles using only one parameter. Here, it is shown that containment of elements in $\mathbf{R}$ into elements in $\mathbf{Q}$ can be reduced to vector dominance with two parameters. Associate again with each $R_k \in \mathbf{R}$ the point (W_k, L_k) in $\mathbb{R}^2$.

For any $R_i \in \mathbf{R}$ define two parameters:

$$P^1(R_i) = \begin{cases} (L_i + W_i)W_i/L_i & \text{if } W_i/L_i \le \sqrt{2} - 1 \\ \sqrt{2}\ W_i & \text{otherwise} \end{cases}$$

$$P^2(R_i) = \begin{cases} W_i + L_i & \text{if } W_i/L_i \le \sqrt{2} - 1 \\ \sqrt{2}\ L_i & \text{otherwise} \end{cases}$$

and let $P(R_i) = (P^1(R_i), P^2(R_i))$

Lemma: For all $R_i \in \mathbf{R}$ and $Q_k \in \mathbf{Q}$, R_i fits into Q_k if and only if $P(R_i) \le P(Q_k)$.

Proof: Consider the region S_k of $\mathbb{R}^2$ consisting of all points $(x,y) \in \mathbb{R}^2$, $x \le y$, whose associated rectangle fits into $Q_k \in \mathbf{Q}$. It is not difficult to see that $(x,y) \in S_k$ if and only if $(x+y) \le 2\ L_k$ or $y \le L_k$. (See Figure 4.) By the definitions of P^1 and P^2 the result follows. [ ].

Hence, the following theorem holds:

Theorem 2: Containment of rectangles into squares is reducible to vector dominance in the plane.

4. Open Problems and Concluding Remarks

It has been shown that the relations of geometric containment and vector dominance in the plane are not equivalent for the family of rectangles.

Using the proof technique of Theorem 1, it can be shown that the same negative result holds also for the family **I** of isosceles triangles.

We conclude with the following two conjectures:

(Weak) <u>Conjecture</u>: Containment of rectangles is not reducible to vector dominance in $\mathbb{R}^k$, for any k, using continuous or piece-wise continuous parameters.

(Strong) <u>Conjecture</u>: The containment problem for families of polygons with k degrees of freedom cannot be solved by Pareto dominance in $\mathbb{R}^m$.

Addendum:

Some recent results, based on a different proof technique, have shown the weak conjecture to be true [4]. It is still not known whether the strong conjecture holds.

Acknowledgement:

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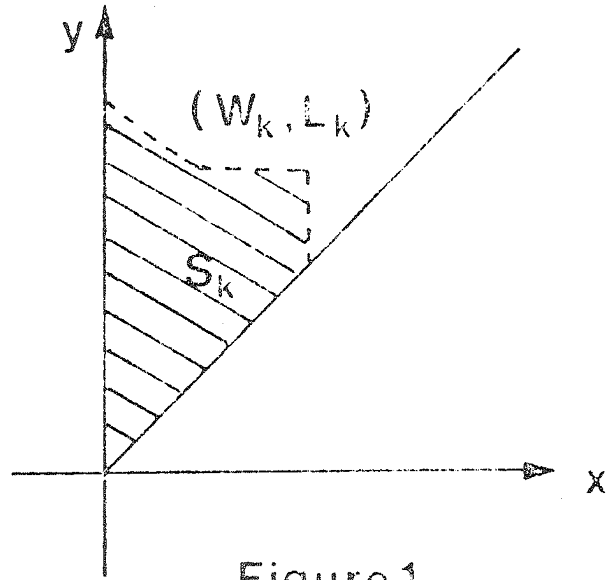

Figure 1

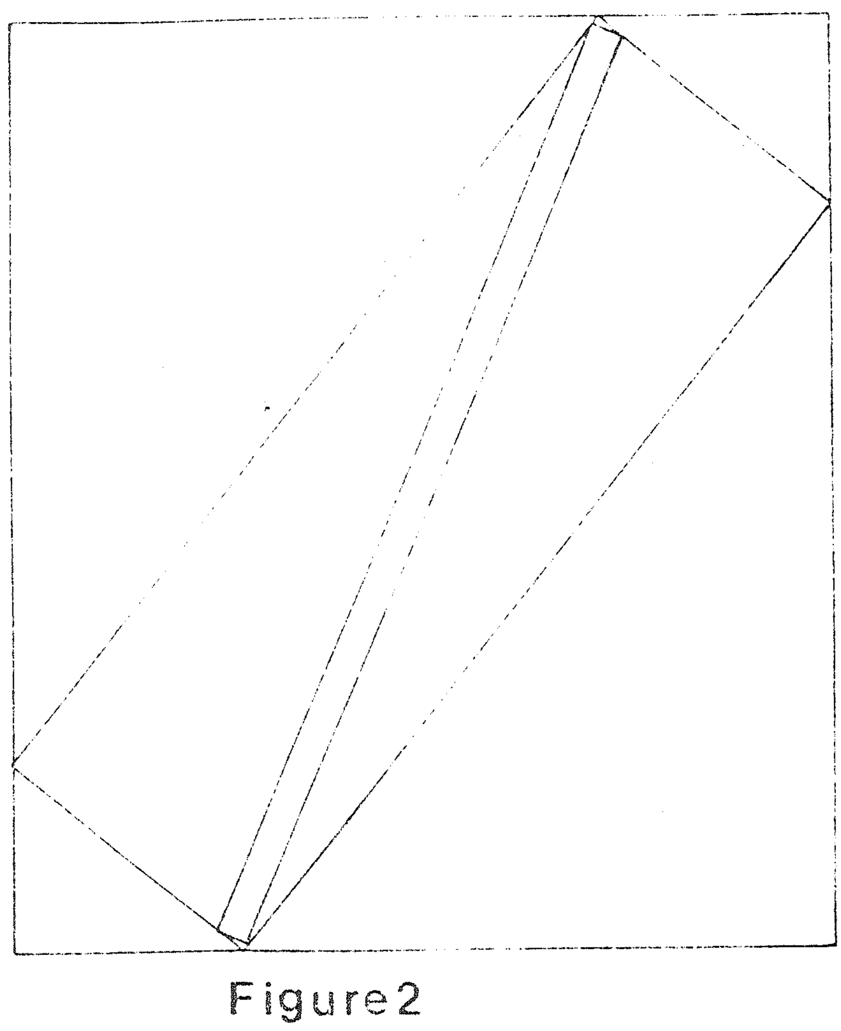
Figure 2

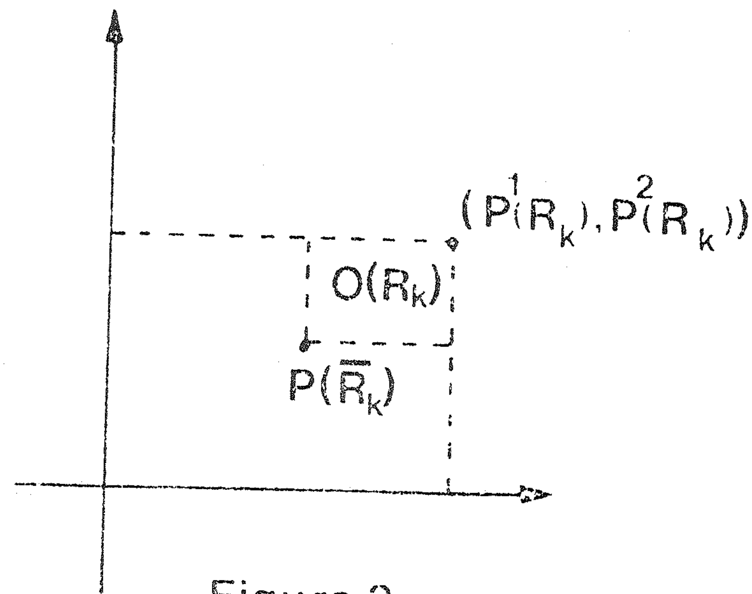

Figure 3

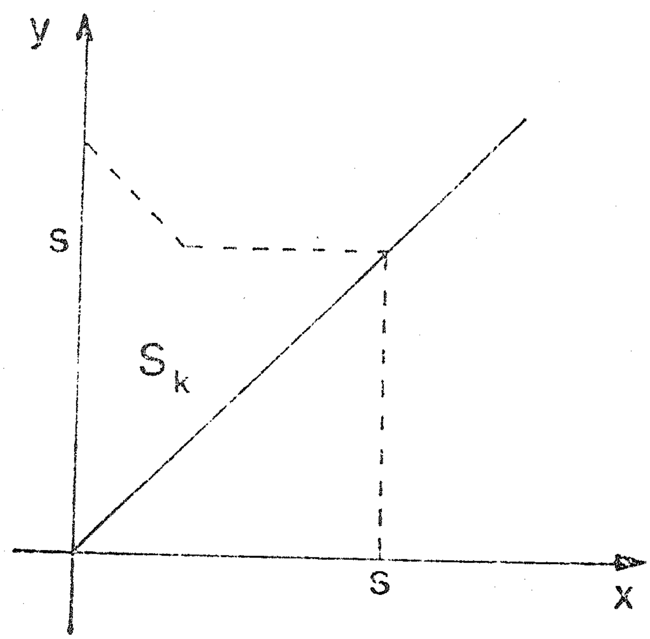

Figure 4

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