UNIQUELY COLOURABLE m-DICHROMATIC ORIENTED GRAPHS

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Abstract

The **dichromatic number** $d_k(D)$ of a digraph D is the minimum number of colours needed to colour V(D) in such a way that no monochromatic directed cycle is obtained. A digraph D is called **uniquely colourable** if any acyclic $d_k(D)$-colouring of V(D) induces the same partition of V(D). In this paper we construct an infinite family of uniquely colourable m-dichromatic oriented graphs for all $m \geq 2$.

1. Introduction and Terminology

An **acyclic m-colouring** of a digraph D is a colouring of the vertices of D with m colours in such a way that no monochromatic directed cycle is obtained. The **dichromatic number** $d_k(D)$ of D is the minimum number m such that there exists an acyclic m-colouring of D. The dichromatic number was introduced independently by Neumann-Lara [5] and Meyniel [4], and has been studied in several papers; see [1,2,3,4,5,6,7].

A digraph D is called **uniquely colourable** if every acyclic $d_k(D)$-colouring of D induces the same partition of V(D). In this paper we study uniquely colourable oriented graphs. We obtain an infinite family of uniquely colourable m-dichromatic oriented graphs for every $m \geq 2$. Some techniques introduced in [7] to study vertex critical m-dichromatic tournaments are used in this paper.

In section 2, we construct two families of uniquely colourable 2-dichromatic oriented graphs. These families are then used in section 3 to generate uniquely colourable r-dichromatic oriented graphs for $r \geq 3$.

Let D be a digraph, V(D) and A(D) will denote the vertex and arc sets of D respectively.

For a vertex $v \in V(D)$, $d^+(v)$, $d^-(v)$, $\Gamma^+(v)$ and $\Gamma^-(v)$ will denote the in and out-degree of v and the in and out-neighborhood of v respectively. The set $\{0,1,2,\ldots,n-1\}$ will be denoted by I_n.

Given a subset $S \subset V(D)$, $D[S]$ will denote the subdigraph of D induced by S. In this paper the word cycle will refer only to directed cycles. All digraphs considered here are oriented graphs.

2. Constructing Uniquely Colourable 2-Dichromatic Oriented Graphs

In this section we will obtain a family of uniquely colourable 2-dichromatic oriented graphs. The following definitions are needed:

For $i_1, i_2, \ldots, i_s \in I_n - \{0\}$, let $\vec{C}_n(i_1, i_2, \ldots, i_s)$ be the digraph with vertex set I_n whose arcs are the ordered pairs $(j, j+i_r)$, $j \in I_n$, $1 \leq r \leq s$, where $j+i_r$ is taken mod n.

Let $H_r = \vec{C}_{2r+1}(1,2,\ldots, r-1, r+1)$. Observe that H_r is a regular tournament with $2r+1$ vertices. In [7] it was proved (see Theorem 2) that $d_k(H_r) = 3$.

Let i be a vertex of H_r, $r \geq 4$. The following properties can be proved.

i) $d^+(i) = d^-(i) = r$.

ii) The subtournaments induced in H_r by $\Gamma^+(i)$ and $\Gamma^-(i)$ respectively contain directed cycles.

iii) Let T_r be an acyclic subtournament of H_r with r vertices. If the source of T_r is vertex i then

$$V(T_r) = \{i, i+1, \ldots, i+r-1\} = S_i \text{ or}$$

$$V(T_r) = S_i \cup \{i+r+1\} - \chi\, i+1\} = S_i'$$

iv) In $H_r[V(H_r) - S_i]$, $r \geq 4$, the in-degree $d^-(v)$ of any vertex is at least 2.

We can now prove the following result.

Theorem 1: $H_r' = H_r - 0$ is uniquely 2-colourable $(r \geq 4)$ with chromatic classes S_1 and S_{r+1}.

Proof: The sets $S_1 = \{1,2,\ldots,r\}$ and $S_{r+1} = \{r+1,\ldots,2r\}$ induce acyclic subtournaments of H_r', and since H_r' is not acyclic, then $d_k(H_r')=2$. We shall now prove that H_r' is also uniquely 2-colourable. Let C_0 and C_1 be the chromatic classes of an acyclic 2-colouring of H_r'. Then $H_r'[C_0]$ and $H_r'[C_1]$ are acyclic subtournaments of H_r'. Let i be the source of $H_r'[C_0]$. Clearly $C_0 \subset \{i\} \cup \Gamma^+(i, H_r)$. By i) and ii) it follows that $|C_0| \leq r$. Similarly, $|C_1| \leq r$; and since $|C_0|+|C_1| = 2r$, $|C_0|=|C_1| = r$. By iii) it follows that $C_0 = S_i$ or $C_0 = S_i'$.

Suppose that $C_0 = S_i'$. By iv), $\delta^-(H_r[V(H_r) - S_i']) \geq 2$. It follows that $\delta^-(H_r'[C_1]) \geq 1$. Therefore $H_r[C_1]$ is not acyclic. Hence $C_0 = S_i$. Similarly $C_1 = S_j$ for some j. Therefore $\{C_0, C_1\} = \{S_1, S_{r+1}\}$, and theorem 1 follows. [ ]

Corollary 1: $H_r - (r,0)$, $r \geq 4$, is a uniquely colourable 2-dichromatic oriented graph. Furthermore $d_k(H_r - (j,0)) = 3$ for $j \in \{r+2, r+3, \ldots, 2r\}$.

Proof: $S_1 \cup \{0\}$ and S_{r+1} produce an acyclic 2-colouring of $H_r - (r,0)$. Then $d_k(H_r - (r,0)) = 2$. But since $d_k(H_r) = 3$, (by Theorem 2 in [7]) in any acyclic 2-colouring γ of $H_r - (r,0)$ vertices r and 0 receive the same colour. However γ induces an acyclic 2-colouring in $H_r - 0$ whose chromatic classes are, by Theorem 1, S_1 and S_{r+1}. Then the chromatic classes of γ are $S_1 \cup \{0\}$ and S_{r+1}.

Similarly for $j \in \{r+2, r+3, \ldots, 2r\} \subset S_{r+1}$, if $H_r-(j,0)$ were 2-dichromatic, then $S_{r+1} \cup \{0\}$ would be a chromatic class of any acyclic 2-colouring of $H_r - (j,0)$. However $H_r [S_{r+1} \cup \{0\}] - (j,0)$ is not acyclic. [ ]

Remark 1: It should be pointed out that $H_3 - (u,v)$ is uniquely 2-colourable for every $(u,v) \in A(H_3)$ but H_3-0 is not.

3. Constructing Uniquely Colourable r-Dichromatic Oriented Graphs, r≥3.

3.1 The function $\tilde{n}$ (m_0, m_1, m_2)

Let m_0, m_1, m_2 be three non-negative integers. The function $\tilde{n}(m_0,m_1,m_2)$ was defined in [7] as the smallest integer k for which there exist three subsets J_0, J_1, J_2 of I_k such that $|J_i| = m_i$, $0 \leq i \leq 2$ and $\bigcap_{i=0,1,2} J_i = \emptyset$

The following lemma was proved in [7].

Lemma 1: Suppose that $m_0 \leq m_1 \leq m_2$. Then

$$\tilde{n}(m_0,m_1,m_2) = \begin{cases} m_2 & \text{if } m_0 + m_1 \leq m_2 \\ \left\lceil \frac{1}{2}(m_0 + m_1 + m_2) \right\rceil & \text{if } m_0 + m_1 \geq m_2 \end{cases}$$

We say that (m_0,m_1,m_2) is an $\tilde{n}$- **upcritical triple** if $1 \leq m_i$, i=0,1,2, and $\tilde{n}(m_0+1, m_1,m_2) = \tilde{n}(m_0, m_1+1, m_2) = \tilde{n}(m_0, m_1, m_2+1) = \tilde{n}(m_0,m_1,m_2)+1$. The next result follows easily from Lemma 1.

Lemma 2: Let $m_0 \leq m_1 \leq m_2$. Then the triple (m_0, m_1, m_2) is $\tilde{n}$-upcritical if and only if $m_0 + m_1 \geq m_2$ and $m_0 + m_1 + m_2$ is even.

Let D_0, D_1 and D_2 be three mutually disjoint digraphs. We denote by $t(D_0, D_1, D_2)$ the digraph whose vertex set is $\bigcup_{i=0}^{2} V(D_i)$ with arc set $\bigcup_{i=0}^{2} A(D_i) \cup \{(u,v) \mid u \in V(D_i),\ v \in V(D_{i+1}),\ 0 \leq i \leq 2$, where $i+1$ is taken mod 3$\}$. Notice that D_0, D_1 and D_2 are induced subdigraphs of $t(D_0, D_1, D_2)$.

In [7] it was proved that $d_k(t(D_0, D_1, D_2)) = \tilde{n}(m_0, m_1, m_2)$ where $d_k(D_i) = m_i$, $m_i \geq 1$, $i = 0,1,2$.

Lemma 3: If (m_0, m_1, m_2) is an $\tilde{n}$ - upcritical triple, then any acyclic $\tilde{n}(m_0, m_1, m_2)$ - colouring of $t(D_0, D_1, D_2)$ induces an acyclic m_i-colouring of D_i, $i = 0,1,2$.

Proof: Let $m = \tilde{n}(m_0, m_1, m_2)$ and let γ be an acyclic m-colouring $D = t(D_0, D_1, D_2)$. For each D_i let J_i be the set of colours used by γ in $V(D_i)$. Clearly $\bigcap_{i=0} J_i = \phi$ and $|J_i| \geq m_i$, $i = 0,1,2$. If $|J_i| > m_i$ for at least one $i \in \{0,1,2\}$, then $\left| \bigcup_{i=0}^{2} J_i \right| > \tilde{n}(m_0, m_1, m_2)$ since (m_0, m_1, m_2) is $\tilde{n}$-upcritical. This is a contradiction. [ ]

Lemma 4: Let (m_0, m_1, m_2) be $\tilde{n}$-upcritical triple and γ and γ' two $\tilde{n}(m_0, m_1, m_2)$-colourings of $t(D_0, D_1, D_2)$ using the same set of colours; J_i and J_i' the sets of colours occuring in D_i in γ and γ' respectively $i=0,1,2$. If $J_i = J_i'$ for two values of i then $J_i = J_i'$, $i=0,1,2$.

Proof: We can suppose w.l.o.g. that $J_0 = J_0'$ and $J_1 = J_1'$. Since $\bigcap_{i=0}^{2} J_i = \bigcap_{i=0}^{2} J_i' = \phi$, we conclude that $J_0 \cap J_1 \cap (J_2 \cup J_2') = \phi$. It follows that

$|J_2 \cup J_2'| = m_2 = |J_2| = |J_2'|$ since (m_0, m_1, m_2) is $\tilde{n}$-upcritical. Therefore $J_2 = J_2'$. [ ]

3.2 Construction of $D^{(\ell)}$.

Given a digraph D, let $D^{(\ell)}$ be the digraph defined by $V(D^{(\ell)}) = V(D) X I_\ell$ and $((u,i), (v,j)) \in A(D^{(\ell)})$ if and only if $(u,v) \in A(D)$ and $|i-j| \leqslant 1$.

Remark 2: Notice that if $f: V(D) \to I_\ell$ is any function such that $|f(u)-f(v)| \leqslant 1$ for every $u,v \in V(D)$, the subdigraph of $D^{(\ell)}$ induced by $\{(u,f(u)) | u \in V(D)\}$ is isomorphic to D. In particular $D^{(\ell)} [V(D) X \{i\}]$ is isomorphic to D.

Remark 3: Any acyclic r-colouring of D with chromatic classes C_1, $C_2, \ldots, C_r$ induces an acyclic r-colouring of $D^{(\ell)}$ with chromatic classes $C_i X I_\ell$, $1 \leqslant i \leqslant r$. Thus $d_k(D) = d_k (D^{(\ell)})$.

Lemma 5: Let D_i be a uniquely colourable m_i-dichromatic digraph and $D = t(D_1, D_2, D_3)$. If (m_0, m_1, m_2) is an $\tilde{n}$-upcritical triple then in any acyclic m-colouring of $D^{(\ell)}$ with $m = d_k (D^{(\ell)})$ the sets $\{v\} \times I_\ell$, $v \in V(D)$ are monochromatic.

Proof: Let $m = \tilde{n}(m_0, m_1, m_2)$, γ an acyclic m-colouring of $D^{(\ell)}$ and $v \in V(D)$. Suppose w.l.o.g. that $v \in V(D_0)$. It is sufficient to prove that vertices (v,i) and $(v,i+1)$, $0 \leqslant i < \ell-1$, receive the same colour. Let H_0 be the subdigraph $D^{(\ell)} [V(D) x \{i\}]$ of D which by Remark 2 is isomorphic to D. Using γ and H_0 we can induce an acyclic m-colouring γ_7 of D in which vertex $u \in V(D)$ receives the same colour as vertex (u,i) in γ. Similarly using γ

and the subdigraph H_1 of $D^{(e)}$ induced by $((V(D) - \{v\}) \times \{i\}) \cup \{(v,i+1)\}$, $0 \leq i < l-1$, we can induce a second acyclic m-colouring γ_2 of D. Since $V(H_0) - (v,i) = V(H_1) - (v,i+1)$, γ_1 and γ_2 are equal in all vertices of D except possibly in vertex v.

However by Lemma 4 , γ_1 and γ_2 induce two acyclic m_0-colourings of D_0 using the same set of colours, and since D_0 is uniquely colourable, v must receive the same colour in γ_1 and γ_2. Therefore vertices (v,i) and (v,i+1) receive the same colour in γ. [ ]

By using similar arguments the following result can be proved.

Theorem 2: If D is a uniquely r-colourable oriented graph then $D^{(\ell)}$ is also a uniquely r-colourable oriented graph.

3.3 Construction of $D^{(\ell)}(\Lambda)$.

In what follows we shall suppose that D_i is a uniquely colourable m_i-dichromatic oriented graph, $0 \leq i \leq 2$, and $D = t(D_0,D_1,D_2)$. We also assume that (m_0,m_1,m_2) is always an $\tilde{n}$-upcritical triple and $m=\tilde{n}(m_0,m_1,m_2)$. Since D_i is uniquely m_i-colourable, any acyclic m_i-colouring of D_i induces the same partition Π_i of $V(D_i)$, i=0,1,2. Let $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$ and Λ any partition of V(D) induced by an acyclic m-colouring of D.

For each $\alpha \in \Pi$ choose two different vertices X_α, $Y_\alpha \in \alpha$ (This is possible because in a uniquely colourable r-dichromatic oriented graph, $r \geq 2$, each chromatic class contains at least two elements).

If α, $\beta \in \Pi$ are not contained in the same class of Λ, let $Q(\alpha,\beta)$ be the directed square defined by:

$$V(Q(\alpha,\beta)) = \{X'_\alpha, Y'_\alpha, X''_\alpha, Y''_\beta\}$$
$$A(Q(\alpha,\beta)) = \{(X'_\alpha, X''_\beta), (X''_\beta, Y'_\alpha), (Y'_\alpha, Y''_\beta), (Y''_\beta, X'_\alpha)\}$$

where $Z' = (Z,0)$, $Z'' = (Z,\ell-1)$ for $Z \in V(D)$.

Let us define finally

$D^{(\ell)}(\Lambda) = D^{(\ell)} \cup (\bigcup_{\alpha,\beta} Q(\alpha,B))$

<u>Theorem 3:</u> For $\ell \geq 2$ $D^{(\ell)}(\Lambda)$ is a uniquely colourable m-dichromatic oriented graph.

Proof: Clearly $d_k(D^{(\ell)}(\Lambda)) \geqslant m$. By Remark 3, Λ induces an acyclic m-colouring of $D^{(\ell)}(\Lambda)$ with chromatic classes $\lambda \times I_\ell$, $\lambda \in \Lambda$. It follows that $d_k(D^{(\ell)}(\Lambda)) = m$. Let γ be an acyclic m-colouring of $D^{(\ell)}(\Lambda)$.

By Lemma 5 the sets $\{v\} \times I_\ell$ are monochromatic in γ. Let γ' be the m-colouring of D in which vertex $v \in V(D)$ receives the same colour as vertex (v,i) in γ. Denote by Λ' the partition induced by γ' in $V(D)$ and suppose that $\Lambda' \neq \Lambda$. Since $|\Lambda'| = |\Lambda| = m$, Λ' is not a refinement of Λ and therefore we can choose $\lambda' \in \Lambda'$ which is not contained in any class of Λ. Let λ_1, $\lambda_2 \in \Lambda$, such that $\lambda_1 \cap \lambda'$ and $\lambda_2 \cap \lambda'$ are not empty. Take $v_i \in \lambda_i \cap \lambda'$, $i=1,2$ and let $\bar{v}_1$ and $\bar{v}_2$ be the classes of Π containing v_1 and v_2 respectively. Notice that $\bar{v}_i \subset \lambda_i \cap \lambda'$, $i=1,2$, by lemma 3. Notice also that $\lambda_1 \neq \lambda_2$. Therefore the square $Q(\bar{v}_1, \bar{v}_2)$ is defined and is a subdigraph of $D^{(\ell)}(\Lambda)[\lambda' \times I_\ell]$ which is monochromatic in γ. This gives a contradiction.
[ ]

Theorem 4: For every $r \geqslant 2$ there exists an infinite family of uniquely colourable r-dichromatic oriented graphs.

Proof: By induction over r. The case r=2 follows from theorem 1. Let us assume that the result has been proved for $2 \leqslant r \leqslant r_0$. Take three mutually disjoint uniquely colourable oriented graphs D_i, i=0,1,2, with dichromatic numbers $m_0=2$, $m_2=m_3=r_0$. Notice that $(2,r_0,r_0)$ is a $\tilde{n}$-upcritical triple and that $\tilde{n}(2,r_0,r_0)= r_0+1$. By theorem 3, $D^{(\ell)}(\Lambda)$ is a uniquely colourable r_0+1-dichromatic oriented graph for $\ell \geqslant 2$. [ ]

Some general properties of uniquely colourable r-dichromatic oriented graphs are being studied [8].

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References

[1] P. Edrös, Problems and results in number theory and graph theory, Proc. Ninth Manitoba Conference on Numerical Math. and Computing (1979) 3-21.

[2] P. Edrös and V. Neumann-Lara, On the dichromatic number of graph, in preparation.

[3] Y.O. Hamidoune, On the decomposition of a minimally strongly h-connected digraph into h+1 acircuitic subgraphs, Discrete Math. 31(1980) 89-90.

[4] H. Maynel, Extension du nombre chromatique et du nombre de stabilité, Preprint.

[5] V. Neumann-Lara, The Dichromatic Number of a Digraph. Journal of Combinatorial Theory, Series B 33 (1982), 265-270.

[6] V. Neumann-Lara, The generalized dichromatic number of a digraph, Publications Preliminares, No. 32, Instituto de Matematicas. Universidad Nacional Autonoma de Mexico (1981).

[7] V. Neumann-Lara and J. Urrutia. Vertex Critical r-Dichromatic Tournaments. Discrete Mathematics 40 (1984) 83-87.

[8] V. Neumann-Lara and J. Urrutia. On Uniquely Colourable r-Dichromatic Digraphs. In preparation.