ON ZIGZAG PERMUTATIONS AND COMPARISONS OF ADJACENT ELEMENTS

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Abstract. Each permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ determines a sequence of '<' and '>' relations determined by the relations holding between adjacent values (a_i, a_{i+1}). A new and elementary algorithm is given which, for every such pattern of '<', '>' relations, computes the number of permutations with that pattern. The algorithm enables one to calculate (in bits) the amount of information gained by comparing all adjacent pairs of elements in a list. It also has a simple extension to circular patterns of relations.

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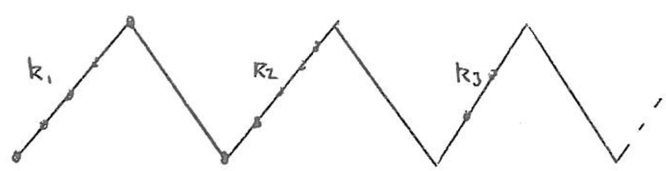
1. Introduction

An up-down permutation $(a_1, a_2, \ldots, a_n)$ of the symbols $1,2,\ldots,n$ is one in which $a_1 < a_2 > a_3 < a_4 > \ldots$ An old result of Andre [1] states that, if z_n denotes the number of such permutations, then

$$\sum \frac{z_n x^n}{n!} = \sec x + \tan x$$

A neat proof of this may be found in [3]. From the properties of secant and tangent coefficients [4] one may then deduce that z_n is asymptotic to $n!\, 2^{n+2}/\pi^{n+1}$.

Several generalisations of this problem have been studied by Carlitz [2,3 and loc.cit.]. In particular he studied permutations in which $a_1, \ldots, a_n$ are ordered according to any fixed zigzag pattern



that is, the rising segments have fixed lengths $k_1, k_2, \ldots$ and the falling segments have length 1. He found an explicit formula for the number of permutations of this type and remarked that "it is unfortunately rather complicated". I shall study the more general case of permutations in which $a_1, \ldots, a_n$ are ordered according to any fixed pattern

Foulkes [4] has given a formula for the number of permutations associated with such a pattern but, since it requires numbers derived from the representation theory of the symmetric group, it is not easy to use. I shall not give a simple formula; instead I shall describe a remarkably simple algorithm (suitable for back of an envelope calculations) which delivers the result in about $n^2/2$ additions. In particular the algorithm gives a handy way of computing the secant and tangent coefficients. The approach also demonstrates the intuitively 'obvious' fact that the up-down pattern of Andre is the pattern associated with the most permutations. Finally I shall discuss the number of permutations which are associated with arbitrary fixed circular zigzags (in which there is also an order relation between a_n and a_1).

Apart from their algorithmic content these questions have an application to a comparison problem in computer science. If one is given a list of distinct values $x_1, \ldots, x_n$ and one compares the $n-1$ adjacent pairs (x_i, x_{i+1}) it is interesting to know how much information is given by the outcomes of the comparisons. The comparison outcomes define a zigzag pattern and the amount of information is $\log_2 n! - \log_2 t$ bits where t is the number of permutations associated with the pattern.

2. The algorithm

Any zigzag pattern P_n on n nodes is obtained by adding an 'up' link or a 'down' link to a pattern P_{n-1} on $n-1$ nodes. Write $z(P_n,r)$ for the number of permutations associated with P_n with last symbol equal to r (that is, $a_n = r$). The algorithm is based on the following recurrence.

LEMMA (a) If the last link of P_n is an up link then

$$z(P_n,r) = \sum_{s=1}^{r-1} z(P_{n-1},s)$$

(b) If the last link of P_n is a down link then

$$z(P_n,r) = \sum_{s=r}^{n-1} z(P_{n-1},s)$$

Proof. Let $(a_1, \ldots, a_n)$ be any of the $z(P_n,r)$ permutations associated with P_n in which $a_n = r$. In case (a) a_{n-1} is a number s with $1 <= s < r$. For any such s, $(a_1, \ldots, a_{n-1})$ is a permutation of $1,2, \ldots, r-1, r+1, \ldots, n$ and the correspondence

$$1 \leftrightarrow 1,\ 2 \leftrightarrow 2,\ \ldots,\ r-1 \leftrightarrow r-1,\ r \leftrightarrow r+1,\ \ldots,\ n-1 \leftrightarrow n$$

maps it bijectively to a permutation $(b_1, \ldots, b_{n-1})$ of $1,2, \ldots, n-1$ associated with P_{n-1} and having $b_{n-1} = s$. Thus

$$z(P_n,r) = \sum_{s=1}^{r-1} z(P_{n-1},s).$$

For case (b) a_{n-1} is a number $s+1$ with $r <= s <= n-1$, and a similar argument applies, but here $a_{n-1} = s+1 > r$ implies $b_{n-1} = s$.

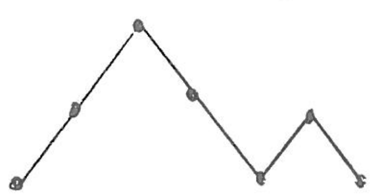
Since the total number of permutations $z(P_n)$ associated with a pattern P_n is $\sum_r z(P_n,r)$ we obtain the following algorithm to calculate this number. Construct a triangular array similar to Pascal's triangle in that the first row is a single 1 and each subsequent row is formed from the partial sums of the previous row. However if the i th link of P_n is an up link one forms partial sums in the forward direction, while if it is a down link one forms partial sums in the backward direction. The final row is then summed to give the result.

For the simple up-down pattern of Andre the triangular array is

						1						
					0		1					
				1		1		0				
			0		1		2		2			
		5		5		4		2		0		
	0		5		10		14		16		16	
61		61		56		46		32		16		0

....

whose row sums 1,1,2,5,16,61,272,... are the interleaved tangent and secant numbers (and obviously the non-zero numbers on the arms of the triangle are, respectively, the tangent and secant numbers). For the less regular zigzag



we would have

						1						
					0		1					
				0		0		1				
			1		1		1		0			
		3		2		1		0		0		
	0		3		5		6		6		6	
26		26		23		18		12		6		0

whose final row sums to 111.

3. Consequences

Each row of a table formed by the rules above is an increasing sequence either when read from left to right or when read from right to left. The next row will also be increasing in one of these directions and is formed by the partial sums of the current row in either the increasing or decreasing directions. Notice that, for the up-down pattern of Andre, the summations are always in the decreasing direction. Since partial summation in the decreasing direction produces a bigger sequence than partial summation in the increasing direction it follows that, for any n, $z(P_n)$ is largest for Andre's pattern.

These investigations were originally motivated by the following computational problem. Suppose that n processors are connected in a chain

and that each processor can communicate only with its two (or one) neighbours. Suppose also that the i th processor contains a value v_i and information about ordering between these values is required but, because of the connections, the only possible comparisons are between adjacent values.

LEMMA The $n-1$ comparisons between adjacent values result in at least $n \log_2 \pi/2 + \log_2 \pi/4 - o(n)$ bits of information.

Proof. The comparisons establish some pattern P_n associated with the values $v_1, \ldots, v_n$, and the information gained is

$$\log_2 n! - \log_2 z(P_n) \text{ bits.}$$

This is minimal when P_n is Andre's pattern. In this case it is known [5] that

$$z(P_n) = n! \frac{2^{n+2}}{\pi^{n+1}} \{1 - 3^{-n-1} + 5^{-n-1} - 7^{-n-1} + \ldots\} \text{ if n is even}$$

and

$$z(P_n) = n! \frac{2^{n+2}}{\pi^{n+1}} \{1 + 3^{-n-1} + 5^{-n-1} + 7^{-n-1} + \ldots\} \text{ if n is odd.}$$

The result now follows since the infinite sums approach 1 as n increases.

These results can all be extended to circular zigzag patterns where the ordering between a_n and a_1 is prescribed. In any permutation of $1, \ldots, n$ associated with a circular zigzag pattern the symbol n must be attached to one of the local maxima. Then clearly the number of permutations associated with the pattern which have n attached to a certain local maximum will be equal to the number of permutations of $1, \ldots,$

n-1 associated with the ordinary zigzag pattern obtained by deleting the two edges of the circular pattern incident with this local maximum. Hence to find the number of permutations associated with a circular zigzag pattern it suffices to compute $z(P_{n-1})$ (where the summation is over all patterns P_{n-1} obtained by deleting a local maximum and its incident edges from the circular zigzag pattern).

As an example, consider the permutations of 1, ..., 8 satisfying $a_1 < a_2 < a_3 > a_4 < a_5 > a_6 < a_7 < a_8 > a_1$.
This pattern has local maxima at a_3, a_5 and a_8 and hence the numbers of permutations associated with the following three patterns must be calculated :

$$x_1 < x_2 > x_3 < x_4 < x_5 > x_6 < x_7$$
$$x_1 < x_2 < x_3 > x_4 < x_5 < x_6 > x_7$$
$$x_1 < x_2 < x_3 > x_4 < x_5 > x_6 < x_7.$$

By following the algorithm in the previous section these numbers are easily computed as 169, 99, 155. Their sum, 423, is the number of permutations associated with the given circular pattern.

Finally notice that the following pretty formula is an immediate consequence of the results above.

PROPOSITION If n is even the number of permutations of 1, ..., n for which $a_1 < a_2 > a_3 < a_4 > \dots < a_n > a_1$ is

$$n! \frac{2^n - 1}{\pi^n} \zeta(n)$$

where $\zeta(n)$ is the Riemann zeta function.

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