STAGE-GRAPH REPRESENTATIONS

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Abstract

We consider graph applications of the well-known paradigm "killing two birds with one stone". In the plane, this gives rise to a stage graph as follows: vertices are the points, and $\{u, v\}$ is an edge if and only if the (infinite, straight) line segment joining u to v intersects the stage. Such graphs are shown to be comparability graphs of ordered sets of dimension 2. Similar graphs can be constructed when we have a fixed number k of stages on the plane. In this case, $\{u, v\}$ is an edge if and only if the (straight) line segment uv intersects one of the k stages. We study stage representations of stage graphs and give upper and lower bounds on the number of stages needed to represent a graph.

1980 Mathematics Subject Classification: 68R10, 68U05
CR Categories: F.2.2
Key Words and Phrases: Algorithms, Girth, Ray shooting, Partial Orders, Stage Graphs.
Carleton University, School of Computer Science: SCS-TR-95-07
Note: This technical report is a revised and expanded version of TR-239

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[¶]Research supported in part under an R&D agreement between Carleton University and ALMERCO Inc.

[‖]Work by the author was carried during a stay at Carleton University.

1 Introduction

Suppose we have a flock of birds and wish to kill all birds by throwing stones. As the saying goes, we might be able to kill two birds with one stone. Birds are stationary and our positions may be delimited by certain areas of the plane or the space, possibly disconnected due to the presence of e.g., lakes. The objective is to find shooting positions that will minimize the total number of stones thrown. Geometrical optimization and graph-theoretic interpretations of this problem have been studied in [1].

In this paper we consider a new graph representation which is inspired by this paradigm and study a parameter closely related to this representation, the stage number of graph. Consider a line segment L, called the stage, contained in the x-axis of the plane and a set of points X with positive y-coordinates. We assume that no three points are colinear. We define a graph $G(X, L)$ with vertex set X in which two vertices are adjacent if the (infinite) line connecting them intersects L (see Figure 1). $G(X, L)$ is called a plane stage ray-shooting graph with one stage, or simply a stage graph. We also consider generalizations

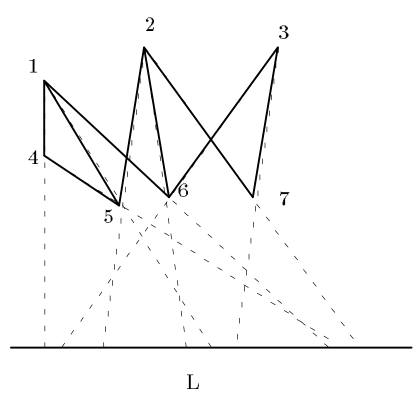

Figure 1: Stage representation of the graph with vertices $1, 2, 3, 4, 5, 6, 7$ and edges $\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{3, 6\}, \{3, 7\}, \{4, 5\}$.

of plane 1-stage ray-shooting graphs to plane ray shooting graphs from several stages. We assume a collection of n points on the plane in general position (i.e. no three co-linear) with positive y-coordinates, as well as k fixed but arbitrary finite, closed, non-intersecting, straight line segments (also called stages) all lying on the x-axis. We define a graph as follows: Vertices are the given points, and $\{u, v\}$ is an edge if and only if the infinite (straight) line segment uv joining the point u to v intersects one of the k stages. We say that the graph is represented (via ray-shooting) by k stages.

Our results make use of a characterization theorem first proved in [1]: it states that a graph G is a stage graph if and only if G is the comparability graph of an order of dimension 2. Comparability graphs are undirected graphs G which can be oriented in such a way that if we have $u \rightarrow v$ and $v \rightarrow w$ in G then $u \rightarrow w$ is also an edge of G. Comparablity graphs of dimension 2 have received a fair amount of attention in the literature (see [15, 10, 3].

1.1 Results of the paper

For fixed n, let $\mathcal{G}_k$ denote the class of graphs (on n points in general position) which can be represented by k stages as above. These classes of graphs are shown to define a hierarchy. We prove upper and lower bounds on the number of stages needed to represented a graph and establish results on separating the classes $\mathcal{G}_k$. In particular, we prove the existence of graphs which require $\Omega(n^2/\log n)$ stages for their representation; furthermore, we show how to construct graphs requiring $\Omega(\sqrt{n})$ stages for their representation. We also study the determination of stage numbers for several common graphs, including lines, cycles, trees and complete bipartite.

2 Ray shooting from a stage

2.1 Terminology and Definitions

Consider a line segment L contained in the x-axis of the plane and a set of points $X = \{p_1, \ldots, p_n\}$ in general position with positive y-coordinates. We define a graph $G(X, L)$ with vertex set X in which two vertices are adjacent if the line connecting them intersects L (see Figure 1). $G(X, L)$ will be called a plane ray shooting graph.

A binary relation $<$ over a set X defines a partial order $P(X, <)$ on X if it is transitive and antisymmetric. The partially ordered set (or poset, for short) $P(X, <)$ is a linear order if it also satisfies $x < y$ or $y < x$, for all distinct $x, y \in X$. Let $P(X, <)$ be a poset. A realizer of P of size $k+1$ is a collection of linear orders $\{<_0, <_1, \ldots, <_k\}$ on the same set X such that $<_0 \cap <_1 \cap \cdots \cap <_k = <$. It can be proved easily that every poset can be obtained as the intersection of a number of linear orders. The minimal number of linear orders realizing a poset is called its dimension.

2.2 Characterization of ray shooting graphs

In this section we characterize ray shooting graphs. The following Theorem is from [1] and we include only an outline of its proof for completeness.

Theorem 2.1 *A graph G is a plane ray shooting graph if and only if G is the comparability graph of an ordered set of dimension 2.*

Proof (Outline) Consider a set $X = \{p_1, \ldots, p_n\}$ of n points on the plane with y-coordinates greater than 0 and a line segment L contained in the x-axis, with end points p and q. Let $G(X, L)$ be the ray shooting graph of X and L. We orient the edge $\{p_i, p_j\}$ of $G(X, L)$, $p_i \to p_j$ if the y-coordinate of p_i is smaller than that of p_j, otherwise we orient $p_j \to p_i$. It is easy to see that $G(X, L)$ is a comparability graph. This orientation of $G(X, L)$ defines a partial order $P(X, <)$ on X in which $p_i < p_j$ if $p_i \to p_j$. We call this the ray shooting ordering of the graph. To show that $P(X, <)$ has dimension 2, we will produce two linear extensions $<_1$ and $<_2$ of $P(X, <)$ such that $<_1 \cap <_2 = <$. To produce $<_1$ sort the points of X in the counterclockwise direction with respect to p, i.e. $p_i <_1 p_j$ if the slope of the line joining p_i to p is smaller than the slope of the line joining p_j to p. In $<_2$ we now define $p_i <_2 p_j$ if the slope of the line joining p_i to p is greater than the slope of the line joining p_j to q. It now follows that $< = <_1 \cap <_2$.

Conversely, let $P(X, <)$ be an ordered set of dimension 2 and $<_1, <_2$ be two total orders on X such that $< = <_1 \cap <_2$. Choose two points p, q on the x-axis. The line segment L has p, q as endpoints. Let p_i be an element of X. Let $r(i)$ and $s(i)$ be the ranks of p_i in $<_1$ and $<_2$, respectively. Consider a set $\{\lambda_1, \ldots, \lambda_n\}$ of n lines through p sorted in increasing order according to their slopes and a set $\{\beta_1, \ldots, \beta_n\}$ of n lines through q sorted in decreasing order according to their slopes such that each λ_i intersects each β_j at a point with positive y-coordinate, $1 \leq i, j \leq n$. Let us label with p_i the point at which $\lambda_{r(i)}$ and $\beta_{s(i)}$ intersect and identify the points of X with $p_1, \ldots, p_n$ It is now easy to see that the set X of points on the plane labeled $p_1, \ldots, p_n$ and the line segment L are such that $G(X, L)$ is the ray shooting graph of $P(X, <)$. ■

Using the fact that recognition of orders of dimension 2 can be done in $O(n^2)$ time [15], it follows that recognizing 2-ray shooting graphs can be done in $O(n^2)$ time. In fact, our characteriztaion theorem implies an $O(\min\{n^2, n + m \log n\})$ time algorithm to recognize stage graphs with n vertices and m edges.

3 Stage Number of Certain Graphs

In this section we determine the stage number of several simple graphs, including lines, cycles, trees and complete bipartite. For any graph G on n vertices let $\mathrm{ST}(G)$ be the minimum number k of stages needed in order to represent the graph by k stages as a multiple stage graph on n points. In the sequel we consider the size $\mathrm{ST}(G)$ as a function of the number of vertices of the graph.

Theorem 3.1

1. *The line graph L_n on n vertices can be represented with a single stage. Hence $\mathrm{ST}(L_n) = 1$.*

2. *The cycle C_n on n vertices can be represented with a single stage if and only if $n \leq 4$. Moreover, $\mathrm{ST}(C_n) = 2$, for $n \geq 5$.*

3. *A graph with girth ≥ 5 requires at least two stages for its representation.*

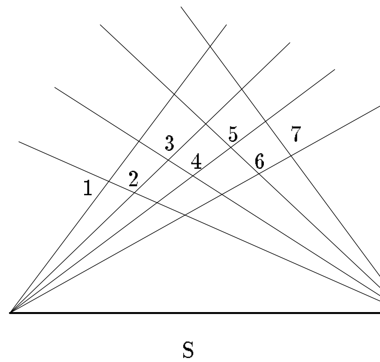

Figure 2: Representing the 7 node line graph with the stage S

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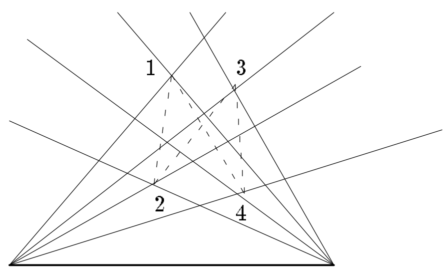

Figure 3: Representing the 4-element cycle

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PROOF

(1) The representation of the line graph L_n for $n = 7$ is depicted in Figure 2. We draw two pencils of lines from the endpoints of the stage S. It is easy to check that the n points can be represented with the numbers $1, 2, 3, \ldots, n$ as appropriate intersections of these lines (see Figure 2).

(2) It is easy to represent with one stage the 2- or 3-element cycles. The representation of the 4-element cycle is given in Figure 3. The edges

$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}$$

of the corresponding cycle are represented by the dashed lines.

To represent an n-cycle with two stages, we first represent the line graph L_n with one stage and then add an extra stage in order to represent the edge joinning its two endpoints.

The fact that an n-cycle with $n \geq 5$ requires at least two stages will be derived from a property about partially ordered sets. Let $(P, <)$ be a strict

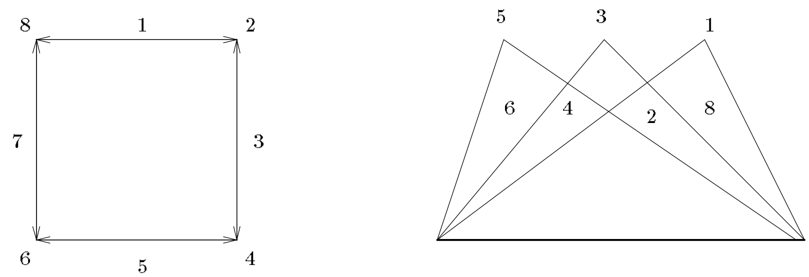

Figure 4: Impossibility of representing an ordered 8-cycle as a single stage graph. Note that any cone that contains vertices 6 and 8 must also contain vertices 2 and 4.

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partially ordered set (finite or infinite). Define a graph on P by joinning $x, y \in P$ if and only if either $x < y$ or $y < x$ [8][9.32]. First we show that no cycle of length ≥ 5 in the graph G can have an odd number of vertices. To see this we argue as follows. Let a cycle in this graph consist of the points $1, 2, \ldots, n$, for some $n \geq 5$, where $\{i, j\}$ is an edge if and only if either $j = i+1$ or $i = 1, j = n$. Without loss of generality let us assume that $1 < 2$ (a similar argument works if $2 < 1$). Since $1, 3$ are $<$-incomparable we must have that $3 < 2$. Since $2, 4$ are incomparable we must also have that $3 < 4$, and so on. Hence we have the following inequalities

$$\begin{array}{l} 3, 5 < 4 \\ 5, 7 < 6 \\ \ldots \\ n-2, n < n-1 \end{array} \tag{1}$$

Now, if $1 < n$ then also $1 < n - 1$, since $n < n - 1$, we obtain a contradiction; if $n < 1$ then also $n < 2$, since $1 < 2$, again yielding a contradiction.

If we apply this result to the partially ordered set induced via ray shooting from a single stage on the given points of the plane we derive that the only possible cycles must be of length n, for some even $n \geq 6$. Let us assume without loss of generality that $1 < 2$ (the case $2 < 1$ is entirely analogous). As in (1) we can show that

$$\begin{array}{l} 1, 3 < 2 \\ 3, 5 < 4 \\ 5, 7 < 6 \\ \ldots \\ n-3, n-1 < n-2 \\ n-1, 1 < n. \end{array} \tag{2}$$

But it is straightforward to check (see Figure 4) that this is an impossible configuration in the ray shooting ordering.

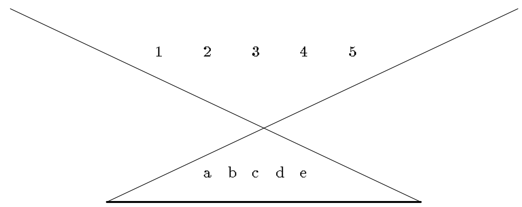

Figure 5: Representing $K_{5,5}$

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(3) This follows immediately from part (2) of the Theorem. ■

Theorem 3.2 *The complete bipartite graph $K_{m.n}$ can be represented with one stage.*

PROOF The idea for the representation is depicted in Figure 5. To represent $K_{m,n}$ we choose two sets A, B of sizes m, n, respectively. Say, $A = \{1, 2, \ldots\}$ and $B = \{a, b, \ldots\}$. Place the sets of points parallel to each other and to the stage as in Figure 5. ■

3.1 Trees as stage graphs

Next we study the representation of trees as stage graphs. A special case of trees are *caterpillars* which have the property that the elimination of a caterpillar's leaves results in a line graph.

Theorem 3.3 *Caterpillars are precisely the trees representable with one stage.*

PROOF First of all we show that caterpillars are representable with one stage. The representation is depicted in Figure 6. The stage is S. We arrange the points of the body of the caterpillar on the two dashed lines. On the top dashed line we place the points $1, 3, 5, 7, 9, \ldots$ and on the bottom dashed line the points $2, 4, 6, 8, \ldots$. Each of the these points has its "legs" located on the dotted line. The odd (respectively, even) points have their legs placed in the region below (respectively, above) them and delimited by the two dashed lines. It is clear that in this way we can represent all caterpillars.

If a tree is not a caterpillar then it must contain the tree depicted in the left-hand side of Figure 7 as a subtree. However it can be shown that this tree can not be represented with one stage. The idea is the following. If we represented the subgraph consisting of the nodes $1, 2, 3, 4, 5$ using a single stage, say S, as in the right-hand side of Figure 7, then it can be seen that the nodes $6, 7$ can only be placed inside the region marked with R. But then it is clear that they would both have to be adjacent to 3, which is a contradiction. Hence this tree is not representable with one stage. ■

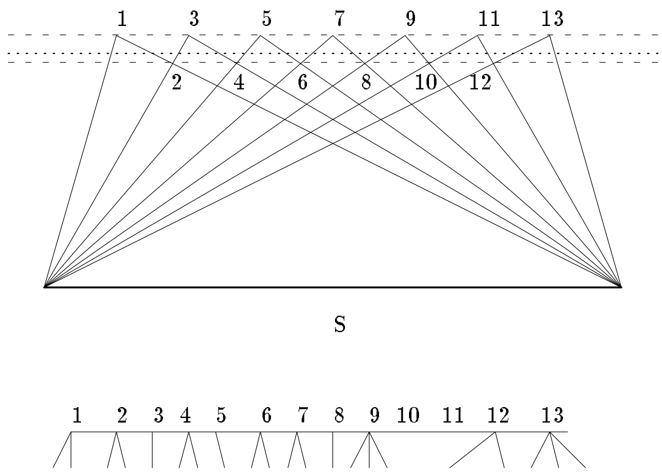

Figure 6: A caterpillar and its representation
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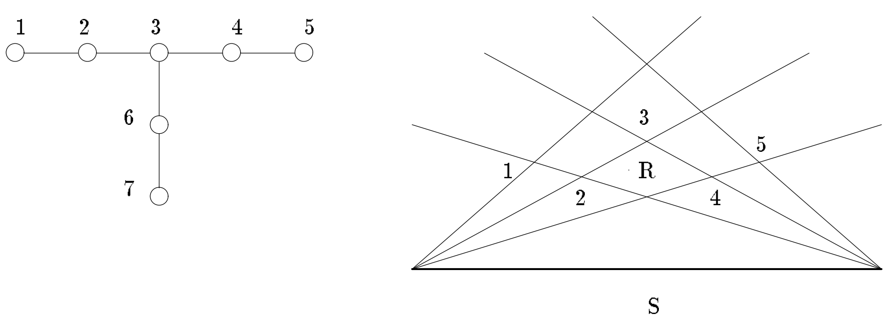

Figure 7: A tree requiring two stages
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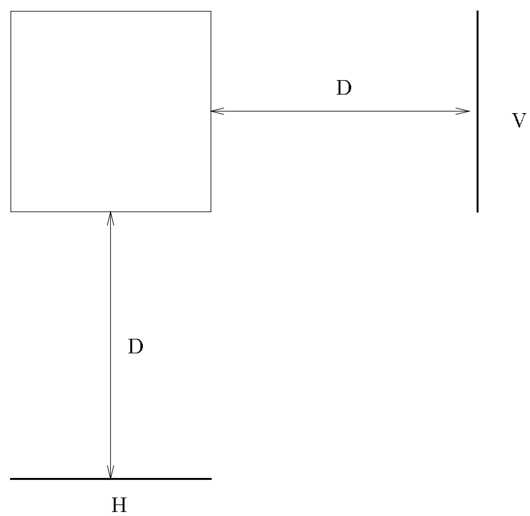

Figure 8: Representing a tree with two stages. The points representing the tree vertices are placed inside the square. There are two stages V, H, one vertical and one horizontal, respectively, at distance D from the square. D is assumed to be sufficiently large.

Theorem 3.4 *Every tree can be represented with at most two stages.*

PROOF (OUTLINE) The representation of an arbitrary tree with two stages is depicted in Figure 8 and is achieved as follows. There is horizontal stage H and a vertical stage V. The vertices of the tree are represented as points on the plane and are all placed inside the square depicted in Figure 8. The stages are placed at distance D from the square, where D is chosen sufficiently large.

The main component of the construction are the cones depicted in Figure 9. There are two types of cones: horizontal and vertical. For the vertical cones the base is the entire horizontal stage H while for the horizontal cones the base is the entire vertical stage. Each cone consists of two parts: the "primary" (which determines the main cone) and the "secondary" which is a line whose slope is such that its extension at infinity does not intersect either of the two stages. We position the points by alternating horizontal and vertical cones according to their tree height.

To represent the nodes of the tree we now place points within these cones. Points are placed within the square depicted in Figure 8. The root is placed at the left-top corner of the square. Its children are placed within a vertical cone whose top is this root. The children of the root are placed within this cone and on its secondary part. Each child is now the top of a horizontal cone as depicted in Figure 10. The grandchildren are placed on the secondary parts of these cones. Moreover we place the chaldren in increasing x-coordinate.

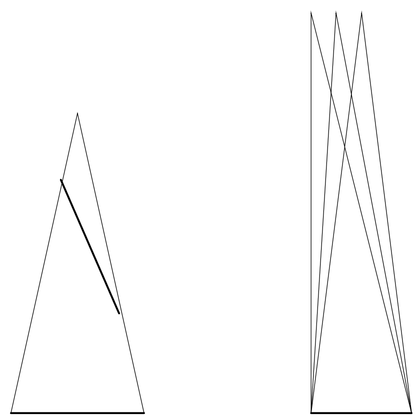

Figure 9: Representing the primary and secondary cones. The parent is placed at the top of the cone and its children are placed on the slanted thick line of the secondary cone. The right-hand side of the picture represents the vertical cones formed by three children and the horizontal stage. A similar representation holds for the horizontal case.

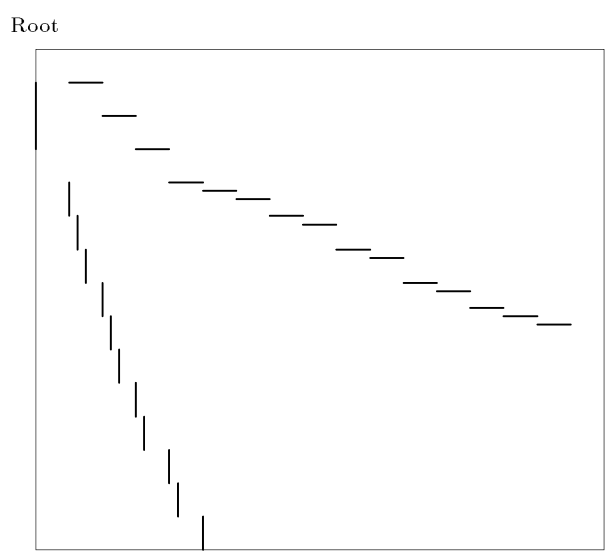

Figure 10: Representing tree vertices inside the square.

We now alternate the cones. We place the great grandchildren in vertical cones of decreasing y-coordinate. The reason for this last requirement is to guararantee that the of "their" children do not intersect within the square of Figure 8. And so on.

It is clear that by moving the stage sufficiently far we can guarantee that no two cones intersect within the square. This completes the proof of the representation of trees by two stages. ■

4 Bounds on the Number of Stages

In the sequel we establish upper and lower bounds on the number of stages needed to represent a graph.

4.1 Upper Bounds

It is easy to see (see proof of Theorem 4.1) that any graph with m edges can be represented with at most m stages. In general, the following theorem gives an upper bound on the number of stages.

Theorem 4.1 *Every n vertex graph can be represented with at most $\lfloor n(n-1)/4 \rfloor$ stages.*

Proof Let G be a graph with n vertices and m edges. Represent the graph on a set of n points in the plane which are in general position such that for all points u, v in the given set the line segment uv is not parallel to the x-axis. Further assume all the points lie above the x-axis.

For each edge e of the graph locate a "small" stage S_e (i.e. a closed interval) on the x-axis in such a way that the infinite line segment determined by e intersects S_e, but for no other edge of the graph does the corresponding infinite line segment intersect S_e. Without loss of generality we may assume the line segments S_e, for e an edge of the graph, are pairwise nointersecting.

Color every such stage S_e "blue". In addition, add a "red" stage for each non-edge of the graph in such a way that the red and blue stages are pairwise non-intersecting. There are m blue stages. Let r be the number of red stages. Notice that

$$m + r = \frac{n(n-1)}{2}.$$

Now traverse the x-axis from $-\infty$ to $+\infty$ and join adjacent stages of the same color into single stages of that same color. It is clear that the number s of the resulting blue stages is $\leq \min\{m, r\}$. It follows that

$$2s \leq 2\min\{m, r\} \leq m + r = \frac{n(n-1)}{2},$$

which proves that $s \leq \lfloor n(n-1)/4 \rfloor$, as desired. ■

4.2 Lower Bounds

In the sequel we give two lower bound proofs. The first result proves the existence of graphs which require $n/\log n$ stages without giving any indication on how to construct them.

Theorem 4.2 *There exist graphs $G \in \mathcal{G}$ requiring at least $\Omega(n/\log n)$ stages for their representation.*

Proof As was shown in section 2, a 1-stage graph is a comparibility graph of dimension 2 and therefore it may be represented using a permutation π on $\{1, 2, \ldots, n\}$ as follows:

1. $V = \{(i, \pi(i)) : i = 1, \ldots, n\}$,
2. $G = \{\{(i, \pi(i)), (j, \pi(j))\} : i < j, \pi(i) < \pi(j)\}$.

Thus every such graph can be encoded with a single permutation. Hence a graph which is representable with k stages can be encoded with $2k-1$ permutations. In turn, each of these $2k-1$ permutations can be encoded with $n \log n$ bits, for a total of $(2k-1)n \log n$ bits.

Now assume that k is such that every graph is representable with k stages. There are at least $2^{\Omega(n^2)}$ possible graphs and they can all be encoded with $(2k-1)n \log n$ bits. It follows that $(2k-1)n \log n \geq \Omega(n^2)$. This proves the required lower bound. ■

In the sequel we use a result of H. Warren [17] on the number of sign patterns of a set of polynomials in order to prove the existence of graphs whose representation requires superlinear number of stages.

Let $p_1, p_2, \ldots, p_m$ be polynomials in r variables and for $x = (x_1, \ldots, x_r)$ let the sign-pattern at x be the vector $(\text{sgn } p_1(x), \ldots, \text{sgn } p_m(x))$ consisting of $+1$ and -1. Let $s(p_1, \ldots, p_m)$ be the number of different sign-patterns for all values of $x \in \mathbf{R}^r$.

Our main theorem makes use of the following result of H. Warren [17].

Theorem 4.3 *If $p_1, \ldots, p_m$ are polynomials in r variables with degree $\leq d$ then the number of sign-patterns is*

$$s(p_1, \ldots, p_m) \leq \left(\frac{4edm}{r}\right)^r.$$

■

Theorem 4.4 *There exist graphs which require $\Omega(n^2/\log n)$ stages for their representation.* ■

Proof If $P_1, \ldots, P_n$ are points in the plane and $[Q_1, R_1], \ldots, [Q_k, R_k]$ are the stages then the resulting graph is determined by the signs of the following set of polynomials. For any three points P, Q, R the sign of the determinant

$$D(P, Q, R) = \det((P_x, P_y, 1), (Q_x, Q_y, 1), (R_x, R_y, 1))$$

tells us whether point R is to the left or to the right of the directed line PQ. Now the line determined by P_i and P_j hits stage $[Q_l, R_l]$ if and only if

$$D(P_i, P_j, Q_l)D(P_i, P_j, R_l) < 0.$$

There are $m = \binom{n}{2}k$ polynomial conditions of degree $d = 4$ in $r = 2n + 4k$ variables. By the previous Theorem the number of sign-patterns is bounded by

$$\left(\frac{n^2 kC}{2n + 4k}\right)^{2n+4k}$$

where C is a constant.

In order to represent all possible graphs this quantity must be at least $2^{\binom{n}{2}}$. Taking logarithms and ignoring lower order terms this means that

$$k = \Omega(n^2/\log n).$$

This completes the proof of the Theorem. ■

A more careful analysis of the proof indicates that a similar result is valid for any sufficiently large class $\mathcal{G}$ of graphs. For example, for an arbitrary class $\mathcal{G}$ we have that

$$k \geq \frac{\log|\mathcal{G}| - 2n\log e - 6n\log n}{4\log(en^3)}.$$

Thus, we obtain the following result as a corollary.

Theorem 4.5 *For any class $\mathcal{G}$ of graphs on n vertices such that $\log|\mathcal{G}| \geq cn\log n$, for some constant $c > 6$, there exist graphs $G \in \mathcal{G}$ requiring at least $\Omega(\log|\mathcal{G}|/\log n)$ stages for their representation.* ■

By using standard results on the number of graphs of specific type (e.g. regular, bipartite etc) it is possible to determine lower bounds for such classes of graphs [5][Chapter 15], [4].

4.3 Constructive lower bounds

Nevertheless, Theorems 4.2, 4.4 and 4.5 still give no indication on how to construct graphs requiring a large number of stages. To give such a construction we use the previous observation that every cycle with 5 or more nodes requires at least two stages for its representation. This means that graphs which are representable with a single stage must have girth ≤ 4. We take advantage of this fact in order to prove the following result.

Theorem 4.6 *Every graph G with minimal degree d and girth ≥ 5 requires at least $\lfloor d/2 \rfloor$ stages for its representation via ray-shooting.*

PROOF Assume on the contrary that G can be represented with less than $\lfloor d/2 \rfloor$ stages, say s. Let G_i be the subgraph of G corresponding to the ith stage. Let e, e_i be the number of edges of the graphs G, G_i, respectively. Observe that

$$\begin{aligned} 2\sum_{i=1}^{s} e_i &\geq 2e \\ &= \sum_{u \in V} \deg_G(u) \\ &\geq nd. \end{aligned}$$

It follows that for some $i \leq s$ we must have that $e_i \geq nd/2s \geq n$. This implies that the graph G_i must have a cycle. However since the girth of the graph G is ≥ 5 so is the girth of the graph G_i. This means that the graph G_i cannot be representable with one stage, which contradicts its very definition. ■

What is the best lower bound that can be achieved via the construction implied by Theorem 4.6? In other words, for a given d what is the smallest possible number of nodes n of a regular graph of degree d? A well-known theorem of Erdös and Tutte [12] (see also [8]10.11]) gives an indication on the number of stages required by n-node graphs with girth ≥ 5. For completeness we give its simple proof.

Theorem 4.7 *Every graph G with minimal degree d and girth ≥ 5 must have more than d^2 vertices.*

PROOF Let $u \in V$ be an arbitrary but fixed vertex of the graph. Let V_i be the set of vertices at distance exactly i from u, where $i = 0, 1, 2$. Notice that since the girth of the graph is ≥ 5 every vertex $v \in V_i$ has exactly one edge to a vertex of V_{i-1}. This means that

$$|V_0| = 1, |V_1| \geq d, |V_2| \geq (d-1)|V_1|.$$

It follows that $n \geq |V_0| + |V_1| + |V_2| \geq d^2 + 1$, as desired ■

There are constructions in the literature of d-regular graphs with girth 5. For example, see [12, 13, 14, 9] as well as [2] and the inductive construction in [16], [8][10.12]. An interesting construction of a regular bipartite graph of degree $p+1$, p^2+p+1 nodes and girth 6, p prime, is the projective plane over the Galois Field on p elements, with $p+1$ lines each line containing exactly $p+1$ points [8][10.15]. It is clear from Theorem 4.6 that this last graph requires at least $(p+1)/2$ stages for its representation. This gives a graph G on n vertices and $\Theta(n^{3/2})$ edges such that $\text{ST}(G) = \Omega(\sqrt{n})$. It is also known [7][Theorem 4.2] that a graph with $n > 2$ vertices, girth ≥ 5 can have at most $\frac{1}{2}n\sqrt{n-1}$ edges. Hence, $\Theta(\sqrt{n})$ is the highest possible stage number for a graph obtained by Theorem 4.6.

5 Conclusion and Open Problems

The notion of stage number, as a graph theoretic parameter, seems to be interesting in its own right. This suggests, the search for tighter (constructive or not) upper and lower bounds on the stage number of specific classes of graphs.

It would be interesting to know whether planar graphs can be represented with a constant number of stages. Another interesting problem is determining the complexity of the recognition problem $G \in \mathcal{G}_k$, both for fixed as well variable k. For $k = 1$ graphs in $\mathcal{G}_1$ can be recognized in $O(n^2)$ time. It has been shown in [1] that in single stage graphs maximum matchings can be computed in time $O(n \log^3 n)$. It would be interesting to know if there is a similar "maximum matching" theorem for the graphs in $\mathcal{G}_k$ that takes into account k as a parameter. Our representation of stage graphs asumes that the stages lie on the x-axis and the points have positive y-coordinates. If we drop this assumption and the stages and points are allowed to intermingle then our bounds are no longer valid.

References

[1] F. Bauernöpel, E. Kranakis, D. Krizanc, A. Maheshwari, M. Noy, J.-R. Sack, J. Urrutia, "Optimal Shooting: Characterizations and Applications", In proceedings of ICALP 95, Springer Verlag LNCS, to appear.

[2] B. Bollobás, "Extremal Graph Theory", Academic Press, 1978.

[3] M.C. Golumbic, "Comparability graph recognition and coloring", Computing 18, 1977, pp. 199-208.

[4] I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration", John Wiley & Sons, New York, 1983.

[5] F. Harary, "Graph Theory", Addison-Wesley Publishing Company, 1969.

[6] D. König, "Theorie der endlichen und unendlichen Graphen", Leipzig, 1936 (reprinted by Chelsea, New York, 1950).

[7] J. H. van Lint and R. M. Wilson, "A Course in Combinatorics", Cambridge University Press, 1992.

[8] L. Lovász, "Combinatorial Problems and Exercises", North Holland Publishing Company, 1979.

[9] V. Neumann-Lara, "k-Hamiltonian Graphs with Given Girth", in: Infinite and Finite Sets, Colloquia Mathematica Societatis János Bolyai, Keszthely, Hungary, 1973, pp. 1133 - 1142.

[10] A. Pnueli, S. Even and A. Lempel, "Transitive orientation of graphs and identification of permutation graphs", Canad. J. Math 23, 1971, pp. 160-175.

[11] F. P. Preparata and M. I. Shamos, "Computational Geometry: An Introduction", Springer-Verlag, New York, 1985.

[12] H. Sachs, "Einführung in die Theorie der endlichen Graphen", Teil I - II, Teubner, 1970 - 1972.

[13] H. Sachs, "Regular Graphs with Given Girth and Restricted Circuits", Journal of the London Mathematical Society, 38 (1963) 423 - 429.

[14] H. Sachs, "On Regular Graphs with Given Girth", in: Theory of Graphs and its Applications, (M. Fiedler, ed.), Academic Press, New York, 1965, pp. 91 - 97.

[15] J. Spinrad, "On Comparability and Permutation Graphs", SIAM J. Comp. 14, 658-670, 1985.

[16] H. Walther and H.-J. Voß, "Über Kreise in Graphen", VEB Deutscher Verlag der Wissenschaften, 1974.

[17] H. Warren, "Lower Bounds for Approximation by Nonlinear Manifolds", Transactions of the AMS 133(1968), 167-178.